3.1.3 The gradient expansion

In considering a single degree of freedom, u, we noted that simple reasonings based on continuity, small amplitudes, and slow variations, lead to typical force of the form

$$F(u) = f_0 + f_1 u + f_2 \frac{u^2}{2} + \cdots \approx -Ju\,. \tag{3.1.9}$$

For deformations around a stable equilibrium we must have $f_0 = 0$, and $f_1 < 0$, and can ignore higher order terms and small amplitudes, as implemented in the second part of the equation ($J > 0$).

We anticipate that the time evolution of the field at each position is governed by a force density F that depends on its value at that position, as well as values at nearby locations as in Eq. (3.1.5). Assuming smooth variations over scales larger than any underlying "microscopic" scale then suggests that the force density can be expanded in terms of derivatives of the field at that location, as in Eq. (3.1.5). Indeed, the generalized Taylor expansion for the force density at position x takes the form

$$\mathcal{F}(u(x), u'(x), u''(x), \cdots) = f_0 + f_1 u + g_1 u' + h_1 u'' + \cdots + f_2 \frac{u^2}{2} + kuu' + g_2 \frac{u'^2}{2} + \cdots\,, \tag{3.1.10}$$

where primes indicate derivatives with respect to x. The various terms in this series expansion can be ordered according to powers of u and powers of the gradient.

Once more, we can rely on various arguments to focus on important terms in the series:

- If expanding around an equilibrium conditions, $u^*(x) = 0$, $f_0 = 0$, and $f_1 < 0$.

- The term g_1 is absent in Eq. (3.1.6). This is because of an implicit inversion symmetry $x \to -x$ for the rubber band. Indeed most of the examples we shall encounter have this symmetry.

- For deformations proportional to $\sin(qx)$, representing a sinusoidal distortion, the second derivative term generates a restoring force proportional to $-q^2 h_1$. For the field to be stable against such sinusoidally modulated deformations we need $h_1 > 0$.[1]

- For a deformation of characteristic range λ, the contribution of a term involving m factors of u and n derivatives scales as u^m/λ^n. The leading terms for small amplitude, long wavelength deformations around a steady state of a system with reflection symmetry thus take the generic form

$$\mathcal{F}(x) = -Ju + K\frac{d^2u}{dx^2}\,. \tag{3.1.11}$$

[1]Strictly speaking, this constraint applies to the highest order derivative term included in the equation. In general, to ensure stability we must check that the restoring force is negative for any wavelength.

- With the expansion terminated with the two terms in Eq. (3.1.11), the force density can be obtained from gradient descent in a functional

$$V[u(x)] = \int dx \left[\frac{J}{2} u^2 + \frac{K}{2} \left(\frac{du}{dx} \right)^2 \right] . \tag{3.1.12}$$

  While higher order terms in powers of u can also be represented by corresponding terms in the above functional, this is no necessarily the case for higher order terms involving derivatives. For example, you can check that the term $g_2 u'^2$ in Eq. (3.1.10) cannot be generated as a simple functional derivative.

The force density in Eq. (3.1.6) is even simpler than Eq. (3.1.11) in that the coefficient J of the linear term in u is also absent. Indeed, $J = 0$ is required for the rubber band by yet another symmetry: The energy density stored in the rubber band is not modified under the transformation $u(x) \to u(x) + c$ for any uniform displacement c. This is most obvious in the ring geometry, as addition of c corresponds to rotating a distorted ring without changing any of the local distortions. From another perspective, the equilibrium condition requires the beads on the chain to be equidistant, but (barring pinning at the boundaries) does not specify an actual location. Adding a constant c corresponds to a shift of the chosen equilibrium state. The symmetry $u(x) \to u(x) + c$ thus forbids the term proportional to J in Eq. (3.1.12).