3.1.4 Partial Differential Equations

For the single coordinate u, the force $F(u)$ governs time evolution according to

$$\gamma\dot{u} + m\ddot{u} + \cdots = F(u) \approx -Ju + \cdots \,. \tag{3.1.13}$$

For small deformations around a stable equilibrium point, this equation has exponential solutions describing decay to equilibrium, or oscillations around it. We can ask if there are natural generalizations of Eq. (3.1.13), and corresponding forms of time evolution, for a continuous field $u(x)$. The natural extension of Eq. (3.1.13) takes the form

$$\eta\frac{\partial u}{\partial t} + \rho\frac{\partial^2 u}{\partial t^2} = \mathcal{F}(u) \approx -Ju + K\frac{\partial^2 u}{\partial x^2} + \cdots \,. \tag{3.1.14}$$

- We have used ∂u, rather than du to emphasize that Eq. (3.1.14) is a *partial differential equation* (PDE). The function $u(x,t)$ depends on two arguments; partial derivatives are taken with respect to one argument with the other treated as a constant.

- For $\eta = 0$, the PDE has time reversal symmetry. As in the single particle case of Eq. (1.1.6), the time invariant equation conserves an energy functional

  $$E[u(x,t)] = \int dx \left[\frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{J}{2}u^2 + \frac{K}{2}\left(\frac{\partial u}{\partial x}\right)^2\right] \,. \tag{3.1.15}$$

  Note that the application of chain rule gives

  $$\frac{dE}{dt} = \int dx \left[\rho\frac{\partial^2 u}{\partial t^2} + Ju - K\frac{\partial^2 u}{\partial x^2}\right]\frac{\partial u}{\partial t} \,, \tag{3.1.16}$$

  where the difference in sign between the first and third terms is because only the third term involves derivatives with respect to the argument of the integration, and using integration by parts yields $\int(\partial_x u)(\partial_x\partial_t u) = -\int(\partial_x\partial_x u)(\partial_t u)$. For $\eta = 0$ in Eq. (3.1.14) the square brackets is zero and hence $dE/dt = 0$.

- For $\eta \neq 0$, substituting from Eq. (3.1.14) into Eq. (3.1.16) gives

  $$\frac{dE}{dt} = -\eta\int dx \left(\frac{\partial u}{\partial t}\right)^2 \,, \tag{3.1.17}$$

  quantifying the loss of energy due to friction.

The term proportional to J in Eq. (3.1.14) is absent in many physical systems due to symmetry or other constraints, leading to two of the most commonly encountered PDEs:

- **The diffusion equation** corresponds to the case of $\rho = 0$ (with symmetries appropriate to $J = 0$), and is generally written as

  $$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \,, \tag{3.1.18}$$

  where D is the *diffusion coefficient*.

– If we integrate both sides of the equation with respect to x over an interval from a to b, we find

$$\frac{d}{dt}\int_a^b dx u(x,t) = D\int_a^b dx\frac{\partial^2 u}{\partial x^2} = D\left[\left.\frac{\partial u}{\partial x}\right|_b - \left.\frac{\partial u}{\partial x}\right|_a\right], \tag{3.1.19}$$

i.e. the change in the total amount of u in the interval comes only from the *flux* of u into and out of the interval at its edges.

– The diffusion equation can thus be regarded as describing the dynamics of a *conserved* quantity, through a continuity equation

$$\frac{\partial u}{\partial t} = -\frac{\partial J}{\partial x}, \quad \text{with a flux (current)} \quad J = -D\frac{\partial u}{\partial x}. \tag{3.1.20}$$

- **The wave equation** describes the opposite limit of $\eta = 0$ in a system constrained by time reversal symmetry, and usually written as

$$\frac{\partial^2 u}{\partial t^2} = v^2\,\frac{\partial^2 u}{\partial x^2}, \tag{3.1.21}$$

where v is the *wave velocity*. The wave equation admits solutions of the form $u(x,t) = u_\pm(x \pm vt)$ (i.e. a function only of either $x \pm vt$, rather than x and t separately) in which a deformation is simply translated forward or backward along space with speed v.