# Functional Analysis Review

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–slides courtesy of Andre Wibisono

9.520: Statistical Learning Theory and Applications

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1. Vector Spaces
2. Hilbert Spaces
3. Functionals and Operators (Matrices)
4. Linear Operators

# Vector Space

- A **vector space** is a set $V$ with binary operations

$$+ : V \times V \to V \quad \text{and} \quad \cdot : \mathbb{R} \times V \to V$$

such that for all $a, b \in \mathbb{R}$ and $v, w, x \in V$:

1. $v + w = w + v$
2. $(v + w) + x = v + (w + x)$
3. There exists $0 \in V$ such that $v + 0 = v$ for all $v \in V$
4. For every $v \in V$ there exists $-v \in V$ such that $v + (-v) = 0$
5. $a(bv) = (ab)v$
6. $1v = v$
7. $(a + b)v = av + bv$
8. $a(v + w) = av + aw$

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- Example: $\mathbb{R}^n$, space of polynomials, space of functions.

# Basis

- $B = \{v_1, \ldots, v_n\}$ is a **basis** of $V$ if every $v \in V$ can be uniquely decomposed as

$$v = a_1 v_1 + \cdots + a_n v_n$$

for some $a_1, \ldots, a_n \in \mathbb{R}$.

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- An orthonormal basis is a basis that is orthogonal ($\langle v_i, v_j \rangle = 0$ for $i \neq j$) and normalized ($\|v_i\| = 1$).

# Norm

- Can define norm from inner product: $\|v\| = \langle v, v \rangle^{1/2}$.

# Norm

- A **norm** is a function $\|\cdot\|: V \to \mathbb{R}$ such that for all $a \in \mathbb{R}$ and $v, w \in V$:
  1. $\|v\| \geqslant 0$, and $\|v\| = 0$ if and only if $v = 0$
  2. $\|av\| = |a| \, \|v\|$
  3. $\|v + w\| \leqslant \|v\| + \|w\|$
- Can define norm from inner product: $\|v\| = \langle v, v \rangle^{1/2}$.

# Metric

- Can define metric from norm: $d(v, w) = \|v - w\|$.

# Metric

- A **metric** is a function $d \colon V \times V \to \mathbb{R}$ such that for all $v, w, x \in V$:
    1. $d(v, w) \geqslant 0$, and $d(v, w) = 0$ if and only if $v = w$
    2. $d(v, w) = d(w, v)$
    3. $d(v, w) \leqslant d(v, x) + d(x, w)$
- Can define metric from norm: $d(v, w) = \|v - w\|$.

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- Given $W \subseteq V$, we have $V = W \oplus W^{\perp}$, where $W^{\perp} = \{ v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W \}$.
- Cauchy-Schwarz inequality: $\langle v, w \rangle \leqslant \langle v, v \rangle^{1/2} \langle w, w \rangle^{1/2}$.

# Hilbert Space, overview

- Goal: to understand Hilbert spaces (complete inner product spaces) and to make sense of the expression

$$f = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle \phi_i, \quad f \in \mathcal{H}$$

- Need to talk about:
  1. Cauchy sequence
  2. Completeness
  3. Density
  4. Separability

# Hilbert Space

- A **Hilbert space** is a complete inner product space.

# Completeness

- A normed vector space $V$ is **complete** if every Cauchy sequence converges.

# Cauchy Sequence

- Recall: $\lim_{n\to\infty} x_n = x$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x - x_n\| < \epsilon$ whenever $n \geqslant \mathbb{N}$.

# Cauchy Sequence

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- $(x_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ whenever $m, n \geqslant N$.

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- $(x_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ whenever $m, n \geqslant N$.
- Every convergent sequence is a Cauchy sequence (why?)

# Completeness

- A normed vector space $V$ is **complete** if every Cauchy sequence converges.
- Examples:
  1. $\mathbb{Q}$ is not complete.
  2. $\mathbb{R}$ is complete (axiom).
  3. $\mathbb{R}^n$ is complete.
  4. Every finite dimensional normed vector space (over $\mathbb{R}$) is complete.

# Hilbert Space

- A **Hilbert space** is a complete inner product space.
- Examples:
  1. $\mathbb{R}^n$
  2. Every finite dimensional inner product space.
  3. $\ell_2 = \{(a_n)_{n=1}^{\infty} \mid a_n \in \mathbb{R},\ \sum_{n=1}^{\infty} a_n^2 < \infty\}$
  4. $L_2([0,1]) = \{f\colon [0,1] \to \mathbb{R} \mid \int_0^1 f(x)^2\, dx < \infty\}$

# Orthonormal Basis

- A Hilbert space has a countable orthonormal basis if and only if it is separable.
- Can write:

$$f = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle \phi_i \quad \text{for all } f \in \mathcal{H}.$$

# Density

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- Examples:
  1. $\mathbb{Q}$ is dense in $\mathbb{R}$.
  2. $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.
  3. Weierstrass approximation theorem: polynomials are dense in continuous functions (with the supremum norm, on compact domains).

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- Examples:
  1. $\mathbb{R}$ is separable.
  2. $\mathbb{R}^n$ is separable.
  3. $\ell_2$, $L_2([0,1])$ are separable.

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- Examples:
  1. Basis of $\ell_2$ is $(1, 0, \dots, ), (0, 1, 0, \dots), (0, 0, 1, 0, \dots), \dots$
  2. Basis of $L_2([0, 1])$ is $1, 2\sin 2\pi n x, 2\cos 2\pi n x$ for $n \in \mathbb{N}$

# Maps

Next we are going to review basic properties of maps on a Hilbert space.

- functionals: $\Psi : \mathcal{H} \to \mathbb{R}$
- linear operators $A : \mathcal{H} \to \mathcal{H}$, such that $A(af + bg) = aAf + bAg$, with $a, b \in \mathbb{R}$ and $f, g \in \mathcal{H}$.

# Representation of Continuous Functionals

Let $\mathcal{H}$ be a Hilbert space and $g \in \mathcal{H}$, then

$$\Psi_g(f) = \langle f, g \rangle, \qquad f \in \mathcal{H}$$

is a continuous linear functional.

## Riesz representation theorem

The theorem states that every continuous linear functional $\Psi$ can be written uniquely in the form,

$$\Psi(f) = \langle f, g \rangle$$

for some appropriate element $g \in \mathcal{H}$.

# Matrix

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- If $A \in \mathbb{R}^{m\times n}$, the transpose of $A$ is $A^\top \in \mathbb{R}^{n\times m}$ satisfying

$$\langle Ax, y\rangle_{\mathbb{R}^m} = (Ax)^\top y = x^\top A^\top y = \langle x, A^\top y\rangle_{\mathbb{R}^n}$$

for every $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

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for every $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.
- $A$ is symmetric if $A^\top = A$.

# Eigenvalues and Eigenvectors

- Let $A \in \mathbb{R}^{n \times n}$. A nonzero vector $v \in \mathbb{R}^n$ is an eigenvector of $A$ with corresponding eigenvalue $\lambda \in \mathbb{R}$ if $Av = \lambda v$.

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- Symmetric matrices have real eigenvalues.
- **Spectral Theorem:** Let $A$ be a symmetric $n \times n$ matrix. Then there is an orthonormal basis of $\mathbb{R}^n$ consisting of the eigenvectors of $A$.
- Eigendecomposition: $A = V \Lambda V^\top$, or equivalently,

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^\top.$$

# Singular Value Decomposition

- Every $A \in \mathbb{R}^{m \times n}$ can be written as

$$A = U \Sigma V^\top,$$

where $U \in \mathbb{R}^{m \times m}$ is orthogonal, $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal, and $V \in \mathbb{R}^{n \times n}$ is orthogonal.

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- Singular system:

$$Av_i = \sigma_i u_i \qquad AA^\top u_i = \sigma_i^2 u_i$$
$$A^\top u_i = \sigma_i v_i \qquad A^\top A v_i = \sigma_i^2 v_i$$

# Matrix Norm

- The spectral norm of $A \in \mathbb{R}^{m \times n}$ is

$$\|A\|_{\text{spec}} = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(AA^\top)} = \sqrt{\lambda_{\max}(A^\top A)}.$$

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- The Frobenius norm of $A \in \mathbb{R}^{m \times n}$ is

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2} = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2}.$$

# Positive Definite Matrix

A real symmetric matrix $A \in \mathbb{R}^{m \times m}$ is positive definite if

$$x^t A x > 0, \quad \forall x \in \mathbb{R}^m.$$

A positive definite matrix has positive eigenvalues.

Note: for positive semi-definite matrices $>$ is replaced by $\geqslant$.

# Linear Operator

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$$\|Lf\|_{\mathcal{H}_2} \leqslant C\|f\|_{\mathcal{H}_1} \quad \text{for all } f \in \mathcal{H}_1.$$

# Linear Operator

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- A linear operator is continuous if and only if it is bounded.

# Adjoint and Compactness

- The adjoint of a bounded linear operator $L\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator $L^*\colon \mathcal{H}_2 \to \mathcal{H}_1$ satisfying

$$\langle Lf, g\rangle_{\mathcal{H}_2} = \langle f, L^* g\rangle_{\mathcal{H}_1} \quad \text{for all } f \in \mathcal{H}_1, g \in \mathcal{H}_2.$$

- $L$ is self-adjoint if $L^* = L$. Self-adjoint operators have real eigenvalues.

# Adjoint and Compactness

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- $L$ is self-adjoint if $L^* = L$. Self-adjoint operators have real eigenvalues.
- A bounded linear operator $L : \mathcal{H}_1 \to \mathcal{H}_2$ is compact if the image of the unit ball in $\mathcal{H}_1$ has compact closure in $\mathcal{H}_2$.

# Spectral Theorem for Compact Self-Adjoint Operator

- Let $L\colon \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator. Then there exists an orthonormal basis of $\mathcal{H}$ consisting of the eigenfunctions of $L$,

$$L\phi_i = \lambda_i \phi_i$$

and the only possible limit point of $\lambda_i$ as $i \to \infty$ is 0.

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- Eigendecomposition:

$$L = \sum_{i=1}^{\infty} \lambda_i \langle \phi_i, \cdot \rangle \phi_i.$$

# Fourier Transform

- Integral transform from time to frequency domain,

$$\hat{f}(\omega) = \int exp^{-2\pi i \omega t} f(t) dt.$$

- Invertible on $L^1(\mathbb{R}^n)$, with inverse transform

$$f(t) = \int \exp^{2\pi i \omega t} \hat{f}(\omega) d\omega.$$

# Fourier Transform

- Examples:

$$\int exp^{-2\pi i\omega t} f(t)dt = \hat{f}(\omega)$$

$$\int exp^{-2\pi i\omega t} f'(t)dt = (2\pi i\omega)\hat{f}(\omega)$$

$$\int exp^{-2\pi i\omega t} f(t-a)dt = exp^{-2\pi ia\omega}\hat{f}(\omega)$$

$$\int \exp^{-2\pi i\omega t} \exp^{-\sigma t^2}\, dt = \sqrt{\frac{\pi}{\sigma}} \exp^{-\frac{(\pi\omega)^2}{\sigma}}$$