Math Camp 2: Probability Theory

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σ-algebra

A σ-algebra Σ over a set Ω is a collection of subsets of Ω that is closed under countable set operations:

1. $\emptyset \in \Sigma$.

2. $E \in \Sigma$ then so is the complement of E.

3. If F is any countable collection of sets in Σ, then the union of all the sets E in F is also in Σ.

Measure

A measure μ is a function defined on a σ-algebra Σ over a set Ω with values in $[0, \infty]$ such that

1. The empty set has measure zero: $\mu(\emptyset) = 0$

2. Countable additivity: if E_1, E_2, E_3, ... is a countable sequence of pairwise disjoint sets in Σ,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

The triple (Ω, Σ, μ) is called a *measure space*.

Lebesgue measure

The *Lebesgue measure* λ is the unique complete translation-invariant measure on a σ-algebra containing the intervals in $\mathbb{R}$ such that $\lambda([0,1]) = 1$.

Probability measure

Probability measure is a positive measure μ on the measurable space (Ω, Σ) such that $\mu(\Omega) = 1$.

(Ω, Σ, μ) is called a *probability space*.

A *random variable* is a measurable function $X : \Omega \mapsto \mathbb{R}$.

We can now define probability of an event

$$P(\text{event A}) = \mu\left(\{x : I_{A(x)} = 1\}\right).$$

Expectation and variance

Given a random variable $X \sim \mu$ the expectation is

$$\mathbb{E}X \equiv \int X d\mu.$$

Similarly the variance of the random variable $\sigma^2(X)$ is

$$\text{var}(X) \equiv \mathbb{E}(X - \mathbb{E}X)^2.$$

Convergence

Recall that a sequence x_n converges to the limit x

$$x_n \rightarrow x$$

if for any $\epsilon > 0$ there exists an N such that $|x_n - x| < \epsilon$ for $n > N$.

We say that the sequence of random variables X_n converges to X *in probability*

$$X_n \xrightarrow{P} X$$

if

$$P\left(|X_n - X| \geq \varepsilon\right) \rightarrow 0$$

for every $\epsilon > 0$.

Convergence in probability and almost surely

Any event with probability 1 is said to happen **almost surely**. A sequence of real random variables X_n converges almost surely to a random variable X iff

$$P\left(\lim_{n\to\infty} X_n = X\right) = 1.$$

Convergence almost surely implies convergence in probability.

Law of Large Numbers. Central Limit Theorem

Weak LLN: if $X_1, X_2, X_3, ...$ is an infinite sequence of i.i.d. random variables with finite variance σ^2, then

$$\overline{X}_n = \frac{X_1 + \cdots + X_n}{n} \xrightarrow{P} \mathbb{E}X_1$$

In other words, for any positive number ϵ, we have

$$\lim_{n \to \infty} \mathsf{P}\left(\left|\overline{X}_n - \mathbb{E}X_1\right| \geq \varepsilon\right) = 0.$$

CLT:

$$\lim_{n \to \infty} \Pr\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \leq z\right) = \Phi(z)$$

where Φ is the cdf of $N(0,1)$.

Useful Probability Inequalities

Jensen's inequality: if ϕ is a convex function, then

$$\phi(\mathbb{E}(X)) \leq \mathbb{E}(\phi(X)).$$

For $X \geq 0$,

$$\mathbb{E}(X) = \int_0^\infty \Pr(X \geq t)dt.$$

Markov's inequality: if $X \geq 0$, then

$$\Pr(X \geq t) \leq \frac{\mathbb{E}(X)}{t},$$

where $t \geq 0$.

Useful Probability Inequalities

Chebyshev's inequality (second moment): if X is arbitrary random variable and $t > 0$,

$$\Pr(|X - \mathbb{E}(X)| \geq t) \leq \frac{var(X)}{t^2}.$$

Cauchy-Schwarz inequality: if $\mathbb{E}(X^2)$ and $\mathbb{E}(Y^2)$ are finite, then

$$|\mathbb{E}(XY)| \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}.$$

Useful Probability Inequalities

If X is a sum of independent variables, then X is better approximated by $\mathbb{E}(X)$ than predicted by Chebyshev's inequality. In fact, it's exponentially close!

Hoeffding's inequality:

Let $X_1, ..., X_n$ be independent bounded random variables, $a_i \leq X_i \leq b_i$ for any $i \in 1...n$. Let $S_n = \sum_{i=1}^{n} X_i$, then for any $t > 0$,

$$\Pr(|S_n - \mathbb{E}(S_n)| \geq t) \leq 2exp\left(\frac{-2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

Remark about sup

Note that the statement

$$\text{with prob. at least } 1-\delta\ ,\ \forall f \in \mathcal{F},\ |\mathbb{E}f - \frac{1}{n}\sum_{i=1}^{n} f(z_i)| \le \epsilon$$

is different from the statement

$$\forall f \in \mathcal{F},\ \text{with prob. at least } 1-\delta\ ,\ |\mathbb{E}f - \frac{1}{n}\sum_{i=1}^{n} f(z_i)| \le \epsilon.$$

The second statement is an instance of CLT, while the first statement is more complicated to prove and only holds for some certain function classes.

Playing with Expectations

Fix a function f, loss V, and dataset $S = \{z_1, ..., z_n\}$. The empirical loss of f on this data is $I_S[f] = \frac{1}{n}\sum_{i=1}^{n} V(f, z_i)$. The expected error of f is $I[f] = \mathbb{E}_z V(f, z)$. What is the expected empirical error with respect to a draw of a set S of size n?

$$\mathbb{E}_S I_S[f] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_S V(f, z_i) = \mathbb{E}_S V(f, z_1)$$