HEDGING DERIVATIVE SECURITIES AND INCOMPLETE MARKETS: AN ϵ-ARBITRAGE APPROACH

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Given a European derivative security with an arbitrary payoff function and a corresponding set of underlying securities on which the derivative security is based, we solve the optimal-replication problem: Find a self-financing dynamic portfolio strategy—involving only the underlying securities—that most closely approximates the payoff function at maturity. By applying stochastic dynamic programming to the minimization of a mean-squared error loss function under Markov-state dynamics, we derive recursive expressions for the optimal-replication strategy that are readily implemented in practice. The approximation error or "ϵ" of the optimal-replication strategy is also given recursively and may be used to quantify the "degree" of market incompleteness. To investigate the practical significance of these ϵ-arbitrage strategies, we consider several numerical examples, including path-dependent options and options on assets with stochastic volatility and jumps.

1. INTRODUCTION

One of the most important breakthroughs in modern financial economics is Merton's (1973) insight that under certain conditions, the frequent trading of a small number of long-lived securities can create new investment opportunities that would otherwise be unavailable to investors. These conditions—now known collectively as *dynamic spanning* or *dynamically complete markets*—and the corresponding asset-pricing models on which they are based have generated a rich literature and an even richer industry, in which complex financial securities are synthetically replicated by sophisticated trading strategies involving considerably simpler instruments (see Cox and Ross 1976, Duffie and Huang 1985, Harrison and Kreps 1979, and Huang 1985a, 1985b, for further details of dynamic spanning). This approach is the basis of the celebrated Black and Scholes (1973) and Merton (1973) option-pricing formula, the arbitrage-free method of pricing, and more importantly, hedging other derivative securities, and the martingale characterization of prices and dynamic equilibria.

The essence of dynamic spanning is the ability to replicate exactly the payoff of a complex security by a dynamic portfolio strategy of simpler securities that is *self-financing*, i.e., no cash inflows or outflows, except at the start and at the end. If such a *dynamic-hedging* strategy exists, then the initial cost of the portfolio must equal the price of the complex security, otherwise an arbitrage opportunity exists. For example, under the assumptions of Black and Scholes (1973) and Merton (1973), the payoff of a European call-option on a non-dividend-paying stock can be replicated exactly by a dynamic-hedging strategy involving only stocks and riskless borrowing and lending.

But the conditions that guarantee dynamic spanning are nontrivial restrictions on market structure and price dynamics (see, for example, Duffie and Huang 1985), hence there are situations in which exact replication is impossible, e.g., suppose that stock price volatility σ in the Black and Scholes (1973) framework is stochastic. These instances of *market incompleteness* are often attributable to institutional rigidities and market frictions—transactions costs, periodic market closures, and discreteness in trading opportunities and prices—and while the pricing of complex securities can still be accomplished in some cases via equilibrium arguments (see, for example, Breeden 1979; Duffie 1987; Duffie and Shafer 1985, 1986; Föllmer and Sonderman 1986; and He and Pearson 1991), this still leaves the question of optimal replication unanswered. Perfect replication is impossible in dynamically incomplete markets, but how close can one come, and what does the optimal-replication strategy look like?

In this paper we answer these questions by applying optimal control techniques to the optimal-replication problem: Given an arbitrary payoff function and a set of fundamental securities, find a self-financing dynamic portfolio strategy involving only the fundamental securities that most closely

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approximates the payoff in a mean-squared sense. The initial cost of such an optimal strategy can be viewed as the "production cost" of the option, i.e., it is the cost of the best dynamic approximation to the payoff function given the set of fundamental securities traded. Such an interpretation is more than a figment of economic imagination—the ability to synthesize options via dynamic trading strategies is largely responsible for the growth of the multitrillion-dollar over-the-counter derivatives market. Moreover, in contrast to exchange-traded options, such as equity puts and calls, over-the-counter derivatives are considerably more illiquid. If investment houses were unable to synthesize them via dynamic trading strategies, they would have to take the other side of every option position that their clients' wish to take (net of offsetting positions among the clients themselves). Such risk exposure would dramatically curtail the scope of the derivatives business, limiting both the size and type of contracts available to end users.

Of course, the nature of the optimal-replication strategy depends on how we measure the closeness of the payoff and its approximation. For tractability and other reasons (see §2.5), we choose a mean-squared-error loss function, and we denote by ϵ the root-mean-squared-error of an optimal-replication strategy. In a dynamically complete market, the approximation error ϵ is identically zero, but when the market is incomplete, we propose ϵ as a measure of the "degree" of incompleteness. Although from a theoretical point of view, dynamic spanning either holds or does not hold, a gradient for market completeness seems more natural from an empirical and a practical point of view. We provide examples of stochastic processes that imply dynamically incomplete markets, e.g., stochastic volatility, and yet still admit ϵ-arbitrage strategies for replicating options to within ϵ, where ϵ can be evaluated numerically.

In this respect, our contributions complement the results of Schweizer (1992, 1995) in which the optimal-replication problem is also solved for a mean-squared error loss function. Schweizer considers more general stochastic processes than we do—we focus only on vector-Markov price processes—and uses variational principles to characterize the optimal-replication strategy. Although our approach can be viewed as a special case of his, the Markov assumption allows us to obtain considerably sharper results and yields an easily implementable numerical procedure (via dynamic programming) for determining the optimal-replication strategy and the replication error ϵ in practice.

Our results also complement the burgeoning literature on option pricing with transactions costs, e.g., Leland (1985), Hodges and Neuberger (1989), Bensaid et al. (1992), Boyle and Vorst (1992), Davis et al. (1993), Edirisinghe et al. (1993), Henrotte (1993), Avellaneda and Paras (1994), Neuberger (1994), Whalley and Wilmott (1994), Grannan and Swindle (1996), and Toft (1996) (see, also, the related papers of Hutchinson et al. 1994, Brandt 1998, and Bertsimas et al. 2000). In these studies, the existence of transactions costs induces discrete trading intervals, and the optimal replication is solved for some special cases, e.g., call and put options on stocks with geometric Brownian motion or constant-elasticity-of-variance price dynamics. In this paper, we solve the more general problem of optimally replicating an arbitrary derivative security where the underlying asset is driven by a vector Markov process.

To demonstrate the practical relevance of our optimal-replication strategy, even in the simplest case of the Black and Scholes (1973) model, where an explicit optimal-replication strategy is available, Table 1 presents a comparison of our optimal-replication strategy with the standard Black-Scholes "delta-hedging" strategy for replicating an at-the-money put option on 1,000 shares of a \$40-stock over 25 trading periods for two simulated sample paths of a geometric Brownian motion with drift $\mu = 0.07$ and diffusion coefficient $\sigma = 0.13$ (rounded to the nearest \$0.125).

V_t^* denotes the period-t value of the optimal replicating portfolio, θ_t^* denotes the number of shares of stock held in that portfolio, and V_t^{BS} and θ_t^{BS} are defined similarly for the Black-Scholes strategy.

Despite that both sample paths are simulated geometric Brownian motions with identical parameters, the optimal-replication strategy has a higher replication error than the Black-Scholes strategy for path A and a lower replication error than Black-Scholes for path B. Specifically, for path A:

$$V_{25}^* - 1000 \times \max[0,\ \$40 - P_{25}] = \$199.1,$$
$$V_{25}^{\mathrm{BS}} - 1000 \times \max[0,\ \$40 - P_{25}] = \$172.3,$$

and for path B:

$$V_{25}^* - 1000 \times \max[0,\ \$40 - P_{25}] = -\$40.3,$$
$$V_{25}^{\mathrm{BS}} - 1000 \times \max[0,\ \$40 - P_{25}] = -\$299.2.$$

That the optimal-replication strategy underperforms the Black-Scholes strategy for path A is not surprising, since the optimal-replication strategy is optimal only in a mean-squared sense (see §2.1), not path by path (these two sample paths were chosen to be illustrative, not conclusive). In a more extensive simulation study, in which 250,000 sample paths were generated, the average replication error of the Black-Scholes strategy is \$248.0 and the average error of the optimal-replication strategy is \$241.2. That the Black-Scholes strategy underperforms the optimal-replication strategy for path B is also not surprising, since the former is designed to replicate the option with continuous trading, whereas the optimal-replication strategy is designed to replicate the option with 25 trading periods

For sample path A, the differences between the optimal-replication strategy and the Black-Scholes are not great—V_t^* and θ_t^* are fairly close to their Black-Scholes counterparts. However, for sample path B, where there are two large price movements, the differences between the two replication strategies and the replication errors are substantial.

Table 1. Comparison of optimal-replication strategy and Black-Scholes delta-hedging strategy for replicating an at-the-money put option on 1,000 shares of a $40-stock over 25 trading periods.

	Sample Path A					Sample Path B				
Period t	P_t	V_t^*	θ_t^*	V_t^{BS}	θ_t^{BS}	P_t	V_t^*	θ_t^*	V_t^{BS}	θ_t^{BS}
0	40.000	1461.0	−474.8	1466.4	−481.7	40.000	1461.0	−474.8	1466.4	−481.7
1	40.750	1104.9	−394.2	1105.1	−400.8	39.875	1520.4	−491.8	1526.6	−495.9
2	42.125	562.9	−252.1	554.0	−264.0	40.250	1336.0	−452.4	1340.6	−454.3
3	41.375	751.9	−318.6	752.0	−331.8	36.500	3032.3	−762.4	3044.3	−845.8
4	42.000	552.8	−256.1	544.7	−267.2	36.875	2746.4	−744.1	2727.1	−822.1
5	43.125	264.7	−157.8	244.0	−169.5	36.500	3025.4	−777.9	3035.4	−858.3
6	43.250	245.0	−145.6	222.8	−155.1	37.000	2636.5	−750.1	2606.3	−824.5
7	42.250	390.6	−216.7	377.9	−229.5	39.875	479.9	−407.8	235.8	−500.5
8	43.000	228.2	−149.7	205.8	−160.6	39.875	479.9	−412.1	235.8	−501.4
9	41.750	415.2	−250.3	406.6	−267.9	40.125	376.9	−384.8	110.4	−468.4
10	42.000	352.7	−221.1	339.6	−235.5	39.500	617.3	−466.9	403.2	−556.1
11	42.625	214.5	−157.2	192.4	−168.9	41.250	−199.7	−227.2	−570.0	−315.0
12	41.750	352.1	−233.8	340.2	−248.5	40.625	−57.7	−300.8	−373.1	−394.7
13	41.500	410.5	−258.4	402.3	−271.0	39.875	167.9	−403.7	−77.0	−506.9
14	42.625	119.8	−128.5	97.5	−141.7	39.375	369.8	−482.4	176.4	−590.1
15	42.875	87.7	−100.5	62.1	−110.7	39.625	249.2	−452.1	28.9	−552.9
16	42.875	87.7	−91.5	62.1	−99.2	39.750	192.7	−439.2	−40.2	−534.3
17	43.125	64.8	−65.5	37.3	−70.4	39.250	412.3	−533.4	226.9	−632.3
18	43.000	73.0	−62.3	46.1	−65.4	39.500	278.9	−500.5	68.8	−592.6
19	43.000	73.0	−50.8	46.1	−51.7	39.750	153.8	−461.4	−79.3	−546.5
20	41.875	130.2	−121.8	104.3	−128.2	39.750	153.8	−472.5	−79.3	−552.5
21	41.125	221.5	−209.8	200.4	−219.8	39.875	94.8	−452.8	−148.4	−526.6
22	41.375	169.1	−137.5	145.5	−140.7	39.625	208.0	−538.7	−16.7	−610.2
23	40.625	272.2	−263.5	251.0	−271.2	39.875	73.3	−476.8	−169.3	−542.8
24	40.000	436.9	−475.7	420.5	−496.3	40.000	13.7	−432.2	−237.1	−496.3
25	40.500	199.1	0.0	172.3	0.0	40.125	−40.3	0.0	−299.2	0.0

*For two simulated sample paths of a geometric Brownian motion with parameters $\mu = 0.07$ and $\sigma = 0.13$.

Even in such an idealized setting, the optimal-replication strategy can still play an important role in the dynamic hedging of risks.

In §2 we introduce the optimal-replication problem and propose a solution based on stochastic dynamic programming. The scope of the ϵ-arbitrage approach is illustrated in §3 and §4 analytically and numerically for several examples including path-dependent options and options on assets with mixed jump-diffusion and stochastic-volatility price dynamics. The sensitivity of the replication error to price dynamics is studied in §5, and we conclude in §6.

2. ϵ-ARBITRAGE STRATEGIES

In this section, we formulate and propose a solution approach for the problem of replicating a derivative security in incomplete markets. In §2.1, we introduce the optimal-replication problem and the principle of ϵ-arbitrage, and provide examples in §2.2 of the types of incompleteness that our framework can accommodate. In §2.3 and §2.4, we propose stochastic dynamic programming algorithms in discrete and continuous time, respectively, that solve the optimal-replication problem.

2.1. The Optimal-Replication Problem

Consider an asset with price P_t at time t, where $0 \leqslant t \leqslant T$, and let $F(P_T, \mathbf{Z}_T)$ denote the payoff of some European derivative security at maturity date T, which is a function of P_T and other variables $\mathbf{Z}_T$ (see below). For expositional convenience, we shall refer to the asset as a *stock* and the derivative security as an *option* on that stock, but our results are considerably more general.

As suggested by Merton's (1973) derivation of the Black-Scholes formula, the optimal-replication problem is to find a dynamic portfolio strategy—purchases and sales of stock and riskless borrowing and lending—on $[0, T]$ that is self-financing and comes as close as possible to the payoff $F(P_T, \mathbf{Z}_T)$ at T. To formulate the optimal-replication problem more precisely, we begin with the following assumptions:

ASSUMPTION 1. *Markets are frictionless, i.e., there are no taxes, transactions costs, shortsales restrictions, and borrowing restrictions.*

ASSUMPTION 2. *The riskless borrowing and lending rate is* 0.

ASSUMPTION 3. *There exists a finite-dimensional vector* $\mathbf{Z}_t$ *of state variables whose components are not perfectly correlated with the prices of any traded securities, and* $[P_t \;\; \mathbf{Z}_t]'$ *is a vector Markov process.*

ASSUMPTION 4. *Trading takes place at known fixed times $t \in \mathcal{T}$. If $\mathcal{T} = \{t_0, t_1, \ldots, t_N\}$, trading is said to be discrete. If $\mathcal{T} = [0, T]$, trading is said to be continuous.*

Note that Assumption 2 entails no loss of generality since we can always renormalize all prices by the price of a zero-coupon bond with maturity at time T (see, for example, Harrison and Kreps 1979).

At time 0, consider forming a portfolio of stocks and riskless bonds at a cost V_0 and as time progresses, let θ_t, B_t, and V_t denote the number of shares of the stock held, the dollar value of bonds held, and the market value of the portfolio at time t, respectively, $t \in \mathcal{T}$, hence,

$$V_t = \theta_t P_t + B_t. \tag{2.1}$$

In addition, we impose the condition that after time 0, the portfolio is *self-financing*, i.e., all long positions in one asset are completely financed by short positions in the other asset, so that the portfolio experiences no cash inflows or outflows:

$$P_{t_{i+1}}(\theta_{t_{i+1}} - \theta_{t_i}) + B_{t_{i+1}} - B_{t_i} = 0, \quad 0 < t_i < t_{i+1} \leqslant T. \tag{2.2}$$

This implies that

$$V_{t_{i+1}} - V_{t_i} = \theta_{t_i}(P_{t_{i+1}} - P_{t_i}), \tag{2.3}$$

and in continuous time,

$$dV_t = \theta_t dP_t. \tag{2.4}$$

We seek a self-financing portfolio strategy $\{\theta_t\}$, $t \in \mathcal{T}$, such that the terminal value V_T of the portfolio is as close as possible to the option's payoff $F(P_T, \mathbf{Z}_T)$. Of course, there are many ways of measuring "closeness," each giving rise to a different optimal-replication problem. For reasons that will become clear (see §2.5), we choose a mean-squared-error loss function. Other recent examples of the use of mean-squared-error loss functions in related dynamic-trading problems include Duffie and Jackson (1990), Duffie and Richardson (1991), Schäl (1994), and Schweizer (1992, 1995). For such a loss function, our version of the optimal-replication problem becomes

$$\min_{\{\theta_t\}} \mathrm{E}\Big[V_T - F(P_T, \mathbf{Z}_T)\Big]^2, \tag{2.5}$$

subject to the self-financing condition (2.3) or (2.4), the dynamics of $[P_t \ \ \mathbf{Z}_t]'$, and the initial wealth V_0, where the expectation E is conditional on information at time 0.

Note that we have placed no constraints on $\{\theta_t\}$, hence it is conceivable that for certain replication strategies, V_T is negative with positive probability. Imposing constraints on $\{\theta_t\}$ to ensure the nonnegativity of V_T would render the optimal-replication problem (2.5) intractable. However, negative values for V_T are not nearly as problematic in the context of the optimal-replication problem as they are for the optimal consumption and portfolio problem of, for example, Merton (1971). In particular, V_T does not correspond to an individual's wealth, but is the terminal value of a portfolio designed to replicate a particular payoff function. See Dybvig and Huang (1988) and Merton (1992, Chapter 6) for further discussion.

A natural measure of the success of the optimal-replication strategy is the square root of the mean-squared replication error (2.5), evaluated at the optimal $\{\theta_t\}$, hence we define

$$\epsilon(V_0) \equiv \sqrt{\min_{\{\theta_t\}} \mathrm{E}\left\{\left[V_T - F(P_T, \mathbf{Z}_T)\right]^2\right\}}. \tag{2.6}$$

We shall show below that $\epsilon(V_0)$ can be minimized with respect to the initial wealth V_0 to yield the *least-cost* optimal-replication strategy and a corresponding measure of the minimum replication error ϵ^*:

$$\epsilon^* \equiv \min_{\{V_0\}} \epsilon(V_0). \tag{2.7}$$

In the case of Black and Scholes (1973) and Merton (1973), there exists optimal-replication strategies for which $\epsilon^* = 0$, hence we say that *perfect* arbitrage pricing holds.

However, there are situations—dynamically incomplete markets, for example—where perfect arbitrage pricing does not hold. Particularly, in Assumption 3, the presence of state variables $\mathbf{Z}_t$ that are not perfectly correlated with the prices of any traded securities is the source of market incompleteness in our framework. While this captures only one potential source of incompleteness—and does so only in a "reduced-form" sense—nevertheless, it is a particularly relevant source of incompleteness in financial markets. Of course, we recognize that the precise nature of incompleteness, e.g., institutional rigidities, transactions costs, technological constraints, will affect the pricing and hedging of derivative securities in complex ways. For example, "structural" models in which institutional sources of market incompleteness are captured, e.g., transactions costs, shortsales constraints, and undiversifiable labor income, have been developed by Aiyagari (1994), Aiyagari and Gertler (1991), He and Modest (1995), Heaton and Lucas (1992, 1996), Lucas (1994), Scheinkman and Weiss (1986), Telmer (1993), and Weil (1992). Also, Magill and Quinzii (1996) present a comprehensive analysis of market incompleteness in a very general setting.

Nevertheless, how well one security can be replicated by sophisticated trading in other securities does provide one measure of the degree of market incompleteness, even if it does not completely characterize it. In much the same way that the Black and Scholes (1973) and Merton (1973) models focus on the *relative* pricing of options—relative to the exogenously specified price dynamics for the underlying asset—we hope to capture the degree of *relative* incompleteness, relative to an exogenously specified set of Markov state variables that are not completely hedgeable.

In some of these cases, we shall show in §2.3 and §2.4 that ϵ-arbitrage pricing is possible, i.e., it is possible to derive a mean-square-optimal replication strategy that is able to approximate the terminal payoff $F(P_T, \mathbf{Z}_T)$ of an option to within ϵ^*. But before turning to the solution of the optimal-replication problem, we provide several illustrative examples that delineate the scope of our framework.

2.2. Examples

Despite the restrictions imposed by Assumptions 1–4, our framework can accommodate many kinds of market incompleteness and various types of derivative securities as the following examples illustrate:

(a) Stochastic Volatility. Consider a stock price process that follows a diffusion process with stochastic volatility, e.g., Hull and White (1987) and Wiggins (1987). The stock price and stock-price volatility are assumed to be governed by the following pair of stochastic differential equations:

$$dP_t = \mu P_t\,dt + \sigma_t P_t\,dW_{Pt},$$

$$d\sigma_t = g(\sigma_t)\,dt + \kappa\sigma_t\,dW_{\sigma t},$$

where W_{Pt} and $W_{\sigma t}$ are Brownian motions with mutual variation $dW_{Pt}\,dW_{\sigma t} = \rho\,dt$. This stochastic volatility model is included in our framework by defining $Z_t = \sigma_t$. Then, clearly the vector process $[P_t\ \ \mathbf{Z}_t]'$ is Markov.

(b) Options on the Maximum. In this and the next two examples, we assume that $\mathcal{T} = \{t_0, t_1, \ldots, t_N\}$ and that the stock price P_t process is Markov for expositional simplicity. The payoff of the option on the maximum stock price is given by

$$F\left(\max_{i=0,\ldots,N} P_{t_i}\right). \tag{2.8}$$

Define the state variable

$$Z_{t_i} \equiv \max_{k=0,\ldots,i} P_{t_k}.$$

The process $[P_{t_i}\ \ Z_{t_i}]'$ is Markov, since the distribution of $P_{t_{i+1}}$ depends only on P_{t_i} and

$$Z_{t_{i+1}} = \max[Z_{t_i}, P_{t_{i+1}}], \quad Z_0 = P_0.$$

The payoff of the option can be expressed in terms of the terminal value of the state variables (P_T, Z_T) as $F(Z_T)$.

(c) Asian Options. The payoff of "Asian" or "average-rate" options is given by

$$F\left(\frac{1}{N+1}\sum_{i=0}^{N} P_{t_i}\right).$$

Let Z_{t_i} be the following state variable:

$$Z_{t_i} \equiv \frac{1}{i+1}\sum_{k=0}^{i} P_{t_k}$$

and observe that the process $[P_{t_i}\ \ Z_{t_i}]'$ is Markov, since the distribution of $P_{t_{i+1}}$ depends only on P_{t_i} and

$$Z_{t_{i+1}} = \frac{Z_{t_i}(i+1) + P_{t_{i+1}}}{(i+2)}, \quad Z_0 = P_0.$$

As before, the payoff of the option can be written as $F(Z_T)$.

(d) Knock-Out Options. Given a knock-out price $\overline{P}$, the payoff of a knock-out option is $\beta_T h(P_T)$, where $h(\cdot)$ is a function of the terminal stock price and

$$\beta_T = \begin{cases} 1 & \text{if } \max_{i=0,\ldots,N} P_{t_i} \leqslant \overline{P} \\ 0 & \text{if } \max_{i=0,\ldots,N} P_{t_i} > \overline{P}. \end{cases}$$

Define the state variable Z_t:

$$Z_0 = \begin{cases} 1 & \text{if } P_0 \leqslant \overline{P} \\ 0 & \text{if } P_0 > \overline{P}, \end{cases}$$

$$Z_{t_{i+1}} = \begin{cases} 1 & \text{if } P_{t_{i+1}} \leqslant \overline{P} \text{ and } Z_{t_i} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that resulting process $[P_{t_i}\ \ Z_{t_i}]'$ is Markov, $Z_T = \beta_T$. The payoff of the option is given by $F(P_T, Z_T) = Z_T h(P_T)$.

2.3. ϵ-Arbitrage in Discrete Time

In this section, we propose a solution for the optimal-replication problem (2.5) in discrete time via stochastic dynamic programming. To simplify notation, we adopt the following convention for discrete-time quantities: time subscripts t_i are replaced by i, e.g., the stock price P_{t_i} will be denoted as P_i and so on. Under this convention, we can define the usual *cost-to-go* or *value* function J_i as

$$J_i(V_i, P_i, \mathbf{Z}_i) \equiv \min_{\substack{\theta(k, V_k, P_k, \mathbf{Z}_k),\\ i\leqslant k\leqslant N-1}} \mathrm{E}\left[[V_N - F(P_N, \mathbf{Z}_N)]^2 \,|\, V_i, P_i, \mathbf{Z}_i\right], \tag{2.9}$$

where V_i, P_i, and $\mathbf{Z}_i$ are the state variables, θ_i is the control variable, and the self-financing condition (2.3) and the Markov property (Assumption 3) make up the law of motion for the state variables. By applying Bellman's principle of optimality recursively (see, for example, Bertsekas 1995),

$$J_N(V_N, P_N, \mathbf{Z}_N) = [V_N - F(P_N, \mathbf{Z}_N)]^2, \tag{2.10}$$

$$J_i(V_i, P_i, \mathbf{Z}_i) = \min_{\theta(i, V_i, P_i, \mathbf{Z}_i)} \mathrm{E}\big[J_{i+1}(V_{i+1}, P_{i+1}, \mathbf{Z}_{i+1}) \mid V_i, P_i, \mathbf{Z}_i\big] \quad i = 0, \ldots, N-1, \tag{2.11}$$

the optimal-replication strategy $\theta^*(i, V_i, P_i, \mathbf{Z}_i)$ can be characterized and computed (Schweizer 1995 provides sufficient conditions for the existence of the optimal hedging strategy). In particular, we have the following result (see the appendix for proofs):

THEOREM 1. *Under Assumptions* 1–4 *and* (2.3), *the solution of the optimal-replication problem* (2.5) *for* $\mathcal{T} = \{t_0, t_1, \ldots, t_N\}$ *is characterized by the following*:

(*a*) *The value function* $J_i(V_i, P_i, \mathbf{Z}_i)$ *is quadratic in* V_i, *i.e., there are functions* $a_i(P_i, \mathbf{Z}_i)$, $b_i(P_i, \mathbf{Z}_i)$, *and* $c_i(P_i, \mathbf{Z}_i)$

such that

$$J_i(V_i, P_i, \mathbf{Z}_i) = a_i(P_i, \mathbf{Z}_i)\cdot\left[V_i - b_i(P_i, \mathbf{Z}_i)\right]^2 + c_i(P_i, \mathbf{Z}_i), \quad i = 0, \ldots, N. \tag{2.12}$$

(*b*) *The optimal control* $\theta^*(i, V_i, P_i, \mathbf{Z}_i)$ *is linear in* V_i, *i.e.,*

$$\theta^*(i, V_i, P_i, \mathbf{Z}_i) = p_i(P_i, \mathbf{Z}_i) - V_i q_i(P_i, \mathbf{Z}_i) = (p_i - q_i b_i) - q_i(V_i - b_i). \tag{2.13}$$

(*c*) *The functions* $a_i(\cdot)$, $b_i(\cdot)$, $c_i(\cdot)$, $p_i(\cdot)$, *and* $q_i(\cdot)$, *are defined recursively as*

$$a_N(P_N, \mathbf{Z}_N) = 1, \tag{2.14}$$

$$b_N(P_N, \mathbf{Z}_N) = F(P_N, \mathbf{Z}_N), \tag{2.15}$$

$$c_i(P_N, \mathbf{Z}_N) = 0, \tag{2.16}$$

and for $i = N-1, \ldots, 0$

$$p_i(P_i, \mathbf{Z}_i) = \frac{\mathrm{E}\left[a_{i+1}(P_{i+1}, \mathbf{Z}_{i+1})\cdot b_{i+1}(P_{i+1}, \mathbf{Z}_{i+1})\cdot(P_{i+1}-P_i)|P_i, \mathbf{Z}_i\right]}{\mathrm{E}\left[a_{i+1}(P_{i+1}, \mathbf{Z}_{i+1})\cdot(P_{i+1}-P_i)^2|P_i, \mathbf{Z}_i\right]}, \tag{2.17}$$

$$q_i(P_i, \mathbf{Z}_i) = \frac{\mathrm{E}\left[a_{i+1}(P_{i+1}, \mathbf{Z}_{i+1})\cdot(P_{i+1}-P_i)\,|\,P_i, \mathbf{Z}_i\right]}{\mathrm{E}\left[a_{i+1}(P_{i+1}, \mathbf{Z}_{i+1})\cdot(P_{i+1}-P_i)^2|P_i, \mathbf{Z}_i\right]}, \tag{2.18}$$

$$a_i(P_i, \mathbf{Z}_i) = \mathrm{E}\left[a_{i+1}(P_{i+1}, \mathbf{Z}_{i+1})\cdot\left(1 - q_i(P_i, \mathbf{Z}_i)\cdot(P_{i+1}-P_i)\right)^2|P_i, \mathbf{Z}_i\right], \tag{2.19}$$

$$b_i(P_i, \mathbf{Z}_i) = \frac{1}{a_i(P_i, \mathbf{Z}_i)}\mathrm{E}\left[a_{i+1}(P_{i+1}, \mathbf{Z}_{i+1})\cdot\left(b_{i+1}(P_{i+1}, \mathbf{Z}_{i+1}) - p_i(P_i, \mathbf{Z}_i)\cdot(P_{i+1}-P_i)\right)\cdot\left(1 - q_i(P_i, \mathbf{Z}_i)\cdot(P_{i+1}-P_i)\right)|P_i, \mathbf{Z}_i\right], \tag{2.20}$$

$$c_i(P_i, \mathbf{Z}_i) = \mathrm{E}\left[c_{i+1}(P_{i+1}, \mathbf{Z}_{i+1})\,|\,P_i, \mathbf{Z}_i\right] + \mathrm{E}\left[a_{i+1}(P_{i+1}, \mathbf{Z}_{i+1})\cdot\left(b_{i+1}(P_{i+1}, \mathbf{Z}_{i+1}) - p_i(P_i, \mathbf{Z}_i)\cdot(P_{i+1}-P_i)\right)^2|\,P_i, \mathbf{Z}_i\right] - a_i(P_i, \mathbf{Z}_i)\cdot b_i(P_i, \mathbf{Z}_i)^2. \tag{2.21}$$

(*d*) *Under the optimal-replication strategy* θ^*, *the minimum replication error as a function of the initial wealth* V_0 *is*

$$J_0(V_0, P_0, \mathbf{Z}_0) = a_0(P_0, \mathbf{Z}_0)\cdot\left[V_0 - b_0(P_0, \mathbf{Z}_0)\right]^2 + c_0(P_0, \mathbf{Z}_0), \tag{2.22}$$

hence the initial wealth that minimizes the replication error is $V_0^* = b_0(P_0, \mathbf{Z}_0)$, *the least-cost optimal-replication strategy is the* $\{\theta^*(i, V_i, P_i, \mathbf{Z}_i)\}$ *that corresponds to this initial wealth, and the minimum replication error over all* V_0 *is*

$$\epsilon^* = \sqrt{c_0(P_0, \mathbf{Z}_0)}, \tag{2.23}$$

where the inequalities $a_i(P_i, \mathbf{Z}_i) > 0$ *and* $c_i(P_i, \mathbf{Z}_i) \geqslant 0$ *follow by induction.*

Theorem 1 shows that the optimal-replication strategy θ^* has a particularly simple structure (2.13) that can be related to the well-known "delta-hedging" strategy of the continuous-time Black-Scholes/Merton option-pricing model. In particular, if prices P_t follow a geometric Brownian motion, it can be shown that the first term of the right side of (2.13) corresponds to the derivative security's "delta" (the partial derivative of the security's price with respect to P_t), and the second term vanishes in the continuous-time limit (see §3.2 for further discussion). This underscores the fact that standard delta-hedging strategies which are derived from continuous-time models, need not be optimal when applied in discrete time, and the extent to which the continuous-time and discrete-time replication strategies differ is captured by the second term of (2.13).

The fact that both the optimal control function (2.13) and the value function (2.12) are defined recursively is especially convenient for computational purposes. Finally, because the value function is quadratic in V_i, it possesses a global minimum as a function of the initial wealth V_0, and this global minimum and the initial wealth that attains it can be computed easily.

2.4. ϵ-Arbitrage in Continuous Time

For the continuous-time case $\mathcal{T} = [0, T]$, let $[P_t\ \ \mathbf{Z}_t]'$ follow a vector Markov diffusion process

$$dP_t = \mu_0(t, P_t, \mathbf{Z}_t)P_t\,dt + \sigma_0(t, P_t, \mathbf{Z}_t)P_t\,dW_{0t}, \tag{2.24}$$

$$dZ_{jt} = \mu_j(t, P_t, \mathbf{Z}_t)Z_{jt}\,dt + \sigma_j(t, P_t, \mathbf{Z}_t)Z_{jt}dW_{jt} \qquad j = 1, \ldots, J, \tag{2.25}$$

where W_{jt}, $j = 0, \ldots, J$ are Wiener processes with mutual variation

$$dW_{jt}\,dW_{kt} = \rho_{jk}(t, P_t, \mathbf{Z}_t)\,dt.$$

The continuous-time counterpart of the Bellman recursion is the Hamilton-Jacobi-Bellman equation (see, for example, Fleming and Rishel 1975), and this yields the following:

THEOREM 2. *Under Assumptions* 1–4 *and* (2.4), *the solution of the optimal-replication problem* (2.5) *for* $\mathcal{T} = [0, T]$ *is characterized by the following*:

(*a*) *The value function* $J(t, V_t, P_t, \mathbf{Z}_t)$ *is quadratic in* V_t, *i.e., there are functions* $a(t, P_t, \mathbf{Z}_t)$, $b(t, P_t, \mathbf{Z}_t)$, *and* $c(t, P_t, \mathbf{Z}_t)$ *such that*

$$J(t, V_t, P_t, \mathbf{Z}_t) = a(t, P_t, \mathbf{Z}_t)\cdot[V_t - b(t, P_t, \mathbf{Z}_t)]^2 + c(t, P_t, \mathbf{Z}_t), \quad 0 \leqslant t \leqslant T. \tag{2.26}$$

(*b*) *For* $t \in [0, T]$, *the functions* $a(t, P_t, \mathbf{Z}_t)$, $b(t, P_t, \mathbf{Z}_t)$, *and* $c(t, P_t, \mathbf{Z}_t)$ *satisfy the following system of partial differential equations* (*we omit the arguments of* $a(\cdot)$, $b(\cdot)$, *and* $c(\cdot)$ *in* (2.27)–(2.29) *to economize on notation*):

$$\frac{\partial a}{\partial t}+\sum_{j=0}^{J}\mu_j Z_j\frac{\partial a}{\partial Z_j}+\frac{1}{2}\sum_{i,j=0}^{J}\sigma_i\sigma_j Z_i Z_j\rho_{ij}\frac{\partial^2 a}{\partial Z_i\partial Z_j} = \left(\frac{\mu_0}{\sigma_0}\right)^2 a + 2\sum_{j=0}^{J}\frac{\sigma_j}{\sigma_0}\mu_0\rho_{0j}Z_j\frac{\partial a}{\partial Z_j} + \frac{1}{a}\sum_{i,j=0}^{J}\sigma_i\sigma_j Z_i Z_j\rho_{0i}\rho_{0j}\frac{\partial a}{\partial Z_i}\frac{\partial a}{\partial Z_j} \tag{2.27}$$

$$\frac{\partial b}{\partial t}+\sum_{j=0}^{J}\mu_j Z_j\frac{\partial b}{\partial Z_j}+\frac{1}{2}\sum_{i,j=0}^{J}\sigma_i\sigma_j Z_i Z_j\rho_{ij}\frac{\partial^2 b}{\partial Z_i\partial Z_j} = \sum_{j=0}^{J}\frac{\partial b}{\partial Z_j}Z_j\left(\frac{\sigma_j}{\sigma_0}\mu_0\rho_{0j}-\frac{1}{2}\sum_{i=0}^{J}\frac{\sigma_i\sigma_j Z_i}{a}\cdot(\rho_{0i}\rho_{0j}-\rho_{ij})\frac{\partial a}{\partial Z_i}\right) \tag{2.28}$$

$$\frac{\partial c}{\partial t} + \sum_{j=0}^{J}\mu_j Z_j\frac{\partial c}{\partial Z_j} + \frac{1}{2}\sum_{i,j=0}^{J}\sigma_i\sigma_j Z_i Z_j\rho_{ij}\frac{\partial^2 c}{\partial Z_i\partial Z_j} = a\sum_{i,j=0}^{J}\sigma_i\sigma_j Z_i Z_j\frac{\partial b}{\partial Z_i}\frac{\partial b}{\partial Z_j}(\rho_{0i}\rho_{0j}-\rho_{ij}), \tag{2.29}$$

with boundary conditions:

$$a(T, P_T, \mathbf{Z}_T)=1,\ b(T, P_T, \mathbf{Z}_T)=F(P_T, \mathbf{Z}_T),\quad c(T, P_T, \mathbf{Z}_T)=0, \tag{2.30}$$

where Z_i *denotes the ith component of* $\mathbf{Z}_t$ *and* $Z_0 \equiv P_t$.

(*c*) *The optimal control* $\theta^*(t, V_t, P_t, \mathbf{Z}_t)$ *is linear in* V_t *and is given by*

$$\theta^*(t, V_t, P_t, \mathbf{Z}_t) = \sum_{j=0}^{J}\rho_{0j}\frac{\sigma_j Z_j}{\sigma_0 Z_0}\frac{\partial b}{\partial Z_j}-(V_t-b) \cdot\left(\sum_{j=0}^{J}\rho_{0j}\frac{\sigma_j Z_j}{a\sigma_0 Z_0}\frac{\partial a}{\partial Z_j}+\frac{\mu_0}{\sigma_0^2 Z_0}\right). \tag{2.31}$$

(*d*) *Under the optimal-replication strategy* θ^*, *the minimum replication error as a function of the initial wealth* V_0 *is*

$$J(0, V_0, P_0, \mathbf{Z}_0)=a(0, P_0, \mathbf{Z}_0)[V_0-b(0, P_0, \mathbf{Z}_0)]^2 + c(0, P_0, \mathbf{Z}_0), \tag{2.32}$$

hence, the initial wealth that minimizes the replication error is $V_0^* = b(0, P_0, \mathbf{Z}_0)$; *the least-cost optimal-replication strategy is the* $\{\theta^*(t, V_t, P_t, \mathbf{Z}_t)\}$ *that corresponds to this initial wealth*; *and the minimum replication error over all* V_0 *is* (*it can be shown that* $a(t, P_t, \mathbf{Z}_t) > 0$ *and* $c(t, P_t, \mathbf{Z}_t) \geqslant 0$)

$$\epsilon^* = \sqrt{c(0, P_0, \mathbf{Z}_0)}. \tag{2.33}$$

2.5. Interpreting ϵ^* and V_0^*

Theorems 1 and 2 show that the optimal-replication problem (2.5) can be solved for a mean-squared-error measure of replication error under Markov state dynamics. In particular, the optimal-replication strategy $\theta^*(\cdot)$ is a dynamic trading strategy that yields the minimum mean-squared replication error $\epsilon(V_0)$ for an initial wealth V_0. The fact that $\epsilon(V_0)$ depends on V_0 should come as no surprise, and the fact that $\epsilon(V_0)$ is quadratic in V_0 underscores the possibility that delta-hedging strategies can be under- or overcapitalized, i.e., there exists a unique V_0^* that minimizes the mean-squared replication error. One attractive feature of our approach is the ability to quantify the impact of capitalization V_0 on the replication error $\epsilon(V_0)$.

V_0^* Is Not a Price. In this sense, V_0^* may be viewed as the minimum production cost of replicating the payoff $F(P_T, \mathbf{Z}_T)$ as closely as possible, to within ϵ^*. However, because we have assumed that markets are dynamically incomplete—otherwise, ϵ^* is 0 and perfect replication is possible—V_0^* cannot be interpreted as the price of a derivative security with payoff $F(P_T, \mathbf{Z}_T)$ unless additional economic structure is imposed. In particular, in dynamically incomplete markets, derivatives cannot be priced by arbitrage considerations alone—we must resort to an equilibrium model in which the prices of all traded assets are determined by supply and demand.

To see why V_0^* cannot be interpreted as a price, observe that two investors with different risk preferences may value $F(P_T, \mathbf{Z}_T)$ quite differently, and will therefore place different valuations on the replication error ϵ^*. While both investors may agree that V_0^* is the minimum cost for the optimal-replication strategy $\theta^*(\cdot)$, they may differ in their willingness to pay such a cost for achieving the replication error ϵ^*. For example, Duffie and Jackson (1990) and Duffie and Richardson (1991) develop replication strategies under specific preference assumptions. Moreover, some investors' preferences may not be consistent with a symmetric loss function, e.g., they may value negative replication errors quite differently than positive replication errors.

More to the point, an asset's price is the outcome of a market equilibrium in which investors' preferences, budget dynamics, and information structure interact through the imposition of market-clearing conditions, i.e., supply equals demand. In contrast, V_0^* is the solution to a simple dynamic optimization problem that does not typically incorporate any notion of economic equilibrium.

Why Mean-Squared Error? In fact, there are many possible loss functions, each giving rise to a different set of optimal-replication strategies, hence a natural question to ask in interpreting Theorems 1 and 2 is why use mean-squared?

An obvious motivation is, of course, tractability. We showed in §2.3 and §2.4 that the optimal-replication problem can be solved via stochastic dynamic programming for a mean-squared error loss function and Markov state dynamics, and that the solution can be implemented as an exact and efficient recursive algorithm. In §3 and §4, we apply this algorithm to a variety of derivative securities in incomplete markets and demonstrate its practical relevance analytically and numerically.

Another motivation is that a symmetric loss function is the most natural choice when we have no prior information about whether the derivative to be replicated is being purchased or sold. Indeed, when a derivatives broker is asked by a client to provide a price quote, the client typically does not reveal whether he is a buyer or seller until after receiving both bid and offer prices. In such cases, asymmetric loss functions are inappropriate, since positive replication errors for a long position become negative replication errors for the short position.

Of course, in more structured applications such as Duffie and Jackson (1990) in which investors' preferences, budget dynamics, and information sets are specified, it is not apparent that mean-squared-error optimal-replication strategies are optimal from a particular investor's point of view. However, even in these cases, a slight modification of the mean-squared-error loss function yields optimal-replication strategies that have natural economic interpretations. In particular, by defining mean-squared error with respect to an equivalent martingale measure, the minimum production cost V_0^* associated with this loss function can be interpreted as an equilibrium market price that, by definition, incorporates all aspects of the economic environment in which the derivative security is traded.

The difficulty with such an interpretation is the multiplicity of equivalent martingale measures in incomplete markets—it is only when markets are dynamically complete that the equivalent martingale measure is unique (see Harrison and Kreps 1979). It may be possible to compute upper and lower bounds for ϵ^* over the entire set of equivalent martingale measures, but without additional structure these bounds are likely to be extremely wide and of little practical relevance.

Nevertheless, ϵ^* is a useful metric for the degree of market incompleteness, providing an objective measure of the difficulty in replicating a derivative security. For example, we shall see in §5 that although stochastic volatility and mixed jump-diffusion processes both imply market incompleteness, our ϵ-arbitrage strategy shows that for certain parameter values, the former is a more difficult type of incompleteness to hedge than the latter.

3. ILLUSTRATIVE EXAMPLES

To illustrate the scope of the ϵ-arbitrage approach to the optimal-replication problem, we apply the results of §2 to four specific cases for the return-generating process: state-independent returns (§3.1), geometric Brownian motion (§3.2), a jump-diffusion model (§3.3), and a stochastic volatility model (§3.4).

3.1. State-Independent Returns

Suppose that stock returns are state-independent so that

$$P_i = P_{i-1}(1+\phi_{i-1}), \tag{3.1}$$

where ϕ_{i-1} is independent of the current stock price and all other state variables. This, together with the Markov Assumption 3, implies that returns are statistically independent (but not necessarily identically distributed) through time. Also, let the payoff of the derivative security $F(P_T)$ depend only on the price of the risky asset at time T.

In this case, there is no need for additional state variables $\mathbf{Z}_i$ and the expressions in Theorem 1 simplify to

$$a_N = 1, \quad b_N(P_N) = F(P_N), \quad c_N(P_N) = 0, \tag{3.2}$$

and for $i = N-1, \dots, 0$,

$$a_i = a_{i+1}\frac{\sigma_i^2}{\sigma_i^2+\mu_i^2}, \tag{3.3}$$

$$\begin{aligned} b_i(P_i) = {} & \mathrm{E}\big[b_{i+1}\big(P_i(1+\phi_i)\big)|P_i\big] \\ & - \frac{\mu_i}{\sigma_i^2}\mathrm{Cov}\big[\phi_i, b_{i+1}\big(P_i(1+\phi_i)\big)|P_i\big], \end{aligned} \tag{3.4}$$

$$\begin{aligned} c_i(P_i) = {} & \mathrm{E}\big[c_{i+1}\big(P_i(1+\phi_i)\big) \mid P_i\big] \\ & + \frac{a_{i+1}}{\sigma_i^2}\big\{\sigma_i^2 \mathrm{Var}\big[b_{i+1}\big(P_i(1+\phi_i)\big) \mid P_i\big] \\ & - \mathrm{Cov}\big[\phi_i, b_{i+1}\big(P_i(1+\phi_i)\big) \mid P_i\big]^2\big\}, \end{aligned} \tag{3.5}$$

$$p_i(P_i) = \frac{\mathrm{E}\big[\phi_i b_{i+1}\big(P_i(1+\phi_i)\big) \mid P_i\big]}{(\sigma_i^2+\mu_i^2)P_i}, \tag{3.6}$$

$$q_i(P_i) = \frac{\mu_i}{(\sigma_i^2+\mu_i^2)P_i}, \tag{3.7}$$

where $\mu_i = \mathrm{E}[\phi_i]$ and $\sigma_i^2 = \mathrm{Var}[\phi_i]$.

3.2. Geometric Brownian Motion

Let the stock price process follow the geometric Brownian motion of Black and Scholes (1973) and Merton (1973). We show that the ϵ-arbitrage approach yields the Black-Scholes/Merton results in the limit of continuous time, but in discrete time there are important differences between the optimal-replication strategy of Theorem 1 and the standard Black-Scholes/Merton delta-hedging strategy.

For notational convenience, let all discrete time intervals $[t_i, t_{i+1})$ be of equal length $t_{i+1} - t_i = \Delta t$. The assumption of geometric Brownian motion then implies

$$P_{i+1} = P_i \cdot (1+\phi_i), \tag{3.8}$$

$$\log(1+\phi_i) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}z_i, \tag{3.9}$$

$$z_i \sim \mathcal{N}(0, 1). \tag{3.10}$$

Recall that for $\Delta t \ll 1$ (a large number of time increments in $[0, T]$), the following approximation holds (see, for example, Merton 1992, Chapter 3):

$$\phi_i \sim \mathcal{N}\big(\mu \Delta t, \sigma^2 \Delta t\big) + O\big(\Delta t^{3/2}\big).$$

This, and Taylor's theorem, imply the following approximations for the recursive relations (3.3)–(3.5) of §3.1:

$$\mathrm{Var}\big[b_{i+1}(P_i(1+\phi_i)) \mid P_i\big] = b'_{i+1}(P_i)\sigma^2 P_i^2 \Delta t + O(\Delta t^2),$$

$$\mathrm{Cov}\big[\phi_i, b_{i+1}\big(P_i(1+\phi_i)\big) \mid P_i\big] = b'_{i+1}(P_i)\sigma^2 P_i \Delta t + O(\Delta t^2),$$

$$\begin{aligned}\mathrm{E}\big[b_{i+1}\big(P_i(1+\phi_i)\big) \mid P_i\big] &= b_{i+1}(P_i) + b'_{i+1}(P_{i-1})\mu P_i \Delta t \\ &\quad + b''_{i+1}(P_i)\frac{\sigma^2}{2} P_i^2 \Delta t + O(\Delta t^2),\end{aligned}$$

$$\begin{aligned}\mathrm{E}\big[c_{i+1}\big(P_i(1+\phi_i)\big) \mid P_i\big] &= c_i(P_i) + c'_{i+1}(P_{i-1})\mu P_i \Delta t \\ &\quad + c''_{i+1}(P_i)\frac{\sigma^2}{2} P_i^2 \Delta t + O(\Delta t^2).\end{aligned}$$

We can then rewrite (3.4)–(3.5) as

$$\begin{aligned}b_i(P_i) &= b_{i+1}(P_i) + b''_{i+1}(P_i)\frac{\sigma^2}{2} P_i^2 \Delta t + O(\Delta t^2),\\ c_i(P_i) &= c_{i+1}(P_i) + c'_{i+1}(P_{i-1})\mu P_i \Delta t \\ &\quad + c''_{i+1}(P_i)\frac{\sigma^2}{2} P_i^2 \Delta t + O(\Delta t^2),\end{aligned}$$

and conclude that the system (3.4)–(3.5) approximates the following system of PDEs

$$\frac{\partial b(t,P)}{\partial t} = -\frac{\sigma^2 P^2}{2}\frac{\partial^2 b(t,P)}{\partial P^2}, \tag{3.11}$$

$$\frac{\partial c(t,P)}{\partial t} = -\mu P \frac{\partial c(t,P)}{\partial P} - \frac{\sigma^2 P^2}{2}\frac{\partial^2 c(t,P)}{\partial P^2}, \tag{3.12}$$

up to $O(\Delta t)$ terms. However, (3.11) is the Black and Scholes (1973) PDE, hence we see that in the limit of continuous trading, i.e., as $N \to \infty$ and $\Delta t \to 0$ for a fixed $T \equiv N\Delta t$, the discrete-time optimal-replication strategy of Theorem 1 characterizes the Black and Scholes (1973) and Merton (1973) models.

Moreover, the equation for $c(t, P)$, (3.12), is homogeneous, hence $c(t, P) \equiv 0$, because of the boundary condition $c(T, \cdot) = 0$. This is consistent with the fact that in the Black-Scholes case it is possible to replicate the option exactly, so that the replication error vanishes in the continuous-time limit.

In particular, it can be shown that the components of the discrete-time optimal-replication strategy (2.13) converge to the following continuous-time counterparts:

$$p_i - q_i b_i \to \frac{\partial b}{\partial P}, \qquad q_i \to \frac{\mu}{\sigma^2 P},$$

hence the continuous-time limit of the optimal-replication strategy $\theta^*(\cdot)$ is given by

$$\theta^*(t, V_t, P_t) = \frac{\partial b(t, P_t)}{\partial P_t} - \frac{\mu}{\sigma^2 P_t}\left[V_t - b(t, P_t)\right]. \tag{3.13}$$

Now at time $t = 0$, and for the minimum production-cost initial wealth V_0^*, this expression reduces to

$$\theta^*(0, V_0^*, P_0) = \frac{\partial b(0, P_0)}{\partial P_0},$$

since $V_0^* = b(0, P_0)$.

Given the optimal-replication strategy (3.13), the value of the replicating portfolio V_t satisfies the stochastic differential equation

$$\begin{aligned} dV_t &= \theta^*(t, V_t, P_t)\, dP_t && (3.14)\\ &= \left(\frac{\partial b(t, P_t)}{\partial P_t} - \frac{\mu}{\sigma^2} P_t \left[V_t - b(t, P_t)\right]\right) dP_{t.} && (3.15)\end{aligned}$$

Thus, the difference between V_t and $b(t, P_t)$ is characterized by

$$\begin{aligned} d\big(V_t - b(t, P_t)\big) &\\ = \left(\frac{\partial b(t, P_t)}{\partial P_t} - \frac{\mu}{\sigma^2} P_t \left[V_t - b(t, P_t)\right]\right) dP_{t.} && (3.16)\\ - \left(\left(\frac{\partial b(t, P_t)}{\partial t} + \frac{1}{2}\frac{\partial^2 b(t, P_t)}{\partial P_t^2}\sigma^2 P_t^2\right) dt \right.&\\ \left. + \frac{\partial b(t, P_t)}{\partial P_t} dP_t\right) && (3.17)\\ = -\frac{\mu}{\sigma^2} P_t \left[V_t - b(t, P_t)\right] dP_t, && (3.18)\end{aligned}$$

subject to the initial condition

$$V_0 - b(0, P_0) = 0,$$

where we have used the fact that $b(t, P_t)$ satisfies (3.11). The solution of this stochastic differential equation is unique (see, for example, Karatzas and Shreve 1988, Theorem 2.5), and $V_t - b(t, P_t) \equiv 0$ satisfies the equation. We conclude that the value of the replicating portfolio is always equal to $b(t, P_t)$ for every realization of the stock price process, i.e.,

$$V_t = b(t, P_t),$$

for all $t \in [0, T]$, which implies that

$$\theta^*(t, V_t, P_t) = \frac{\partial b(t, P_t)}{\partial P_t}. \tag{3.19}$$

As expected, for every realization of the stock price process the optimal-replication strategy coincides with the delta-hedging strategy given by the Black-Scholes hedge ratio. However, note that the functional form of (3.13) is different from the Black-Scholes hedging formula—the optimal-replication strategy depends explicitly on the value of the replicating portfolio V_t.

3.3. Jump-Diffusion Models

In this section, we apply results of §2 to the replication and pricing of options on a stock with mixed jump-diffusion price dynamics. As before, we assume that all time intervals $t_{i+1} - t_i = \Delta t$ are regularly spaced. Following Merton (1976), we assume the following model for the stock price process:

$$P_{i+1} = P_i(1+\phi_i), \tag{3.20}$$

$$\log(1+\phi_i) = \left(\mu - \lambda k - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} z_i + \sum_{j=0}^{n_i} \log Y_j, \tag{3.21}$$

$$z_i \sim \mathcal{N}(0, 1), \tag{3.22}$$

$$k = \mathrm{E}[Y_j - 1], \tag{3.23}$$

$$\mathrm{Prob}(n_i = m) = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^m}{m!}, \tag{3.24}$$

where the jump magnitudes $\{Y_j\}$ are independently and identically distributed random variables, and jump arrivals follow a Poisson process with constant arrival rate λ.

We consider two types of jumps: jumps of deterministic magnitude, and jumps with lognormally distributed jump magnitudes. In the first case,

$$Y_j = 1 + \delta. \tag{3.25}$$

If we set $\sigma = 0$ in (3.20), this model corresponds to the continuous-time jump process considered by Cox and Ross (1976). In the second case,

$$\log Y_i \sim \mathcal{N}(0, \delta^2). \tag{3.26}$$

There are two methods of calculating the optimal-replication strategy for the mixed jump-diffusion model. One method is to begin with the solutions of the dynamic programming problem given in §2.3 and §2.4, derive a limiting system of partial differential equations, as in §3.2, and solve this system numerically, using one of the standard finite difference schemes.

The second method is to implement the solution of the dynamic programming problem directly, without the intermediate step of reducing it to a system of PDEs.

The advantage of the second method is that it treats a variety of problems in a uniform fashion, the only problem-dependent part of the approach being the specification of the stochastic process. On the other hand, the first approach yields a representation of the solution as a system of PDEs, which can often provide some information about the qualitative properties of the solution even before a numerical solution is obtained.

With these considerations in mind, we shall derive a limiting system of PDEs for the deterministic-jump-magnitude specification (3.25), and use it to find conditions on the parameters of the stochastic process that allow exact replication of the option's payoff, or, equivalently, arbitrage pricing. For the lognormal-jump-magnitude specification (3.26), we shall obtain numerical solutions directly from the dynamic programming algorithm of Theorem 1.

The Continuous-Time Limit. To derive the continuous-time limit of (3.3)–(3.5) we follow the same procedure as in §3.2, which yields the following system of PDEs:

$$\begin{aligned}\frac{\partial b(t,P)}{\partial t} &= -\lambda\big[b\big(t, P(1+\delta)\big) - b(t,P)\big] + \lambda\delta P\frac{\partial b(t,P)}{\partial P} \\ &\quad - \frac{\sigma^2 P^2}{2}\frac{\partial^2 b(t,P)}{\partial P^2} - \frac{\mu\lambda\delta}{\lambda\delta^2+\sigma^2}\bigg[\delta P\frac{\partial b(t,P)}{\partial P} \\ &\quad - \big[b\big(t,P(1+\delta)\big) - b(t,P)\big]\bigg],\end{aligned} \tag{3.27}$$

$$\begin{aligned}\frac{\partial c(t,P)}{\partial t} &= -\lambda\big[c\big(t,P(1+\delta)\big) - c(t,P)\big] \\ &\quad - (\mu - \lambda\delta)P\frac{\partial c(t,P)}{\partial P} - \frac{\sigma^2 P^2}{2}\frac{\partial^2 c(t,P)}{\partial P^2} \\ &\quad - a(t)\frac{\lambda\sigma^2}{\lambda\delta^2+\sigma^2}\bigg[\delta P\frac{\partial b(t,P)}{\partial P} - \big[b\big(t,P(1+\delta)\big) \\ &\quad - b(t,P)\big]\bigg]^2,\end{aligned} \tag{3.28}$$

$$\frac{da(t)}{dt} = \frac{\mu^2}{\lambda\delta^2+\sigma^2}a(t), \tag{3.29}$$

with boundary conditions:

$$a(T) = 1, \tag{3.30}$$

$$c(T,P) = 0, \tag{3.31}$$

$$b(T,P) = F(P). \tag{3.32}$$

We can use the boundary conditions to solve (3.29):

$$a(t) = \exp\left[\frac{\mu^2}{\lambda\delta^2+\sigma^2}(t-T)\right]. \tag{3.33}$$

The optimal-replication strategy is given by

$$\begin{aligned}\theta^*(t,V_t,P_t) &= \frac{\partial b(t,P_t)}{\partial P_t} - \frac{\mu}{(\lambda\delta^2+\sigma^2)P_t}\big[V_t - b(t,P_t)\big] \\ &\quad - \frac{\lambda\delta^2}{\lambda\delta^2+\sigma^2}\frac{\partial b(t,P_t)}{\partial P_t} + \frac{\lambda\delta}{(\lambda\delta^2+\sigma^2)P_t} \\ &\quad \cdot\big[b\big(t,P_t(1+\delta)\big) - b(t,P_t)\big].\end{aligned} \tag{3.34}$$

For exact replication to be possible, $c(t,P) \equiv 0$ must be a solution of (3.28). This implies that (3.28) is homogeneous, i.e.,

$$\frac{\lambda\sigma^2}{\lambda\delta^2+\sigma^2}\bigg\{\delta P\frac{\partial b(t,P)}{\partial P} - \big[b\big(t,P(1+\delta)\big) - b(t,P)\big]\bigg\}^2 = 0, \tag{3.35}$$

for all $b(t,P)$ satisfying (3.27), which is equivalent to

$$\lambda\delta\sigma^2 = 0. \tag{3.36}$$

Condition (3.36) is satisfied if at least one of the following is true:

- Jumps occur with zero probability;
- Jumps have zero magnitude; and
- The diffusion coefficient is equal to zero, i.e., stock price follows a pure jump process.

However, these are precisely the conditions for the arbitrage-pricing of options on mixed jump-diffusion assets, e.g., Merton (1976).

Perturbation Analysis with Small Jump Amplitudes. Consider the behavior of $b(t,P)$ and $c(t,P)$ when the jump magnitude is small, i.e., $\delta \ll 1$. In this case, the market is "almost complete" and solution of the option replication problem is obtained as a perturbation of the complete-markets solution of Black and Scholes (1973) and Merton (1973). In particular, we treat the amplitude of stock price jumps as a small parameter and look for a solution of (3.27)–(3.32) of the following form:

$$b(t,P) = b_0(t,P) + \delta b_1(t,P) + \delta^2 b_2(t,P) + \cdots, \tag{3.37}$$

$$c(t,P) = c_0(t,P) + \delta c_1(t,P) + \delta^2 c_2(t,P) + \delta^3 c_3(t,P) + \delta^4 c_4(t,P) + \cdots. \tag{3.38}$$

After substituting this expansion into (3.28)–(3.32), it is apparent that the functions $b_0(t,P)$, $b_2(t,P)$, and $c_4(t,P)$ must satisfy the following system of partial differential equations:

$$\frac{\partial b_0(t,P)}{\partial t} = -\frac{\sigma^2 P^2}{2}\frac{\partial^2 b_0(t,P)}{\partial P^2}, \tag{3.39}$$

$$\frac{\partial b_2(t,P)}{\partial t} = -\frac{\lambda P^2}{2}\frac{\partial^2 b_0(t,P)}{\partial P^2}, \tag{3.40}$$

$$\frac{\partial c_4(t,P)}{\partial t} = -\mu P\frac{\partial c_4(t,P)}{\partial P} - \frac{\sigma^2 P^2}{2}\frac{\partial^2 c_4(t,P)}{\partial P^2} - a(t)\frac{\lambda P^4}{4}\left(\frac{\partial^2 b_0(t,P)}{\partial P^2}\right)^2, \tag{3.41}$$

with boundary conditions

$$b_0(T,P) = F(P), \tag{3.42}$$

$$b_2(T,P) = 0, \tag{3.43}$$

$$c_4(T,P) = 0, \tag{3.44}$$

and

$$b_1 = c_1 = c_2 = c_3 = 0.$$

System (3.39)–(3.44) can be solved to yield

$$b(t,P) = b_0(t,P) + \frac{\lambda\delta^2}{\sigma^2}\big[b_0(t,P) - F(P)\big] + O(\delta^3), \tag{3.45}$$

where $b_0(t,P)$ is the option price in the absence of a jump component, i.e., the Black-Scholes formula in the case of put and call options. Observe that for an option with a convex payoff function $b_0(t,P) \geqslant F(P)$, which implies that $b(t,P) \geqslant b_0(t,P)$, i.e., the addition of a small jump component to geometric Brownian motion increases the price of the option. This qualitative behavior of the option price is consistent with the results in Merton (1976), which were obtained with equilibrium arguments.

The optimal-replication strategy (3.34) is given by

$$\theta^*(t,V_t,P_t) = \frac{\partial b_0(t,P_t)}{\partial P_t} + \frac{\mu}{\sigma^2 P_t}\left[b_0(t,P_t) - V_t\right] + \frac{\lambda\delta^2}{\sigma^2}\left[\frac{\partial b_0(t,P_t)}{\partial P_t} - \frac{\partial F(P_t)}{\partial P_t} + V_t - F(P_t)\right] + O(\delta^3), \tag{3.46}$$

and the corresponding replication error is

$$c(t,P) = \delta^4 c_4(t,P) + O(\delta^6) = O(\delta^4), \tag{3.47}$$

where $c_4(t,P)$ solves (3.41) and (3.44).

Equations (3.45) and (3.46) provide closed-form expressions for the replication cost and the optimal-replication strategy when the amplitude of jumps is small, i.e., when markets are almost complete, and (3.47) describes the dependence of the replication error on the jump magnitude.

3.4. Stochastic Volatility

Let stock prices follow a diffusion process with stochastic volatility as in Hull and White (1987) and Wiggins (1987):

$$dP_t = \mu P_t dt + \sigma_t P_t dW_{Pt}, \tag{3.48}$$

$$d\sigma_t = g(\sigma_t)dt + \kappa\sigma_t dW_{\sigma t}, \tag{3.49}$$

where W_{Pt} and $W_{\sigma t}$ are Brownian motions with mutual variation $dW_{Pt}dW_{\sigma t} = \rho dt$.

The Continuous-Time Solution. Although applying the results of §2 to (3.48)–(3.49) is conceptually straightforward, the algebraic manipulations are quite involved in this case. A simpler alternative to deriving a system of PDEs as the continuous-time limit of the solution in Theorem 1 is to formulate the problem in continuous time at the outset, and solve it using continuous-time stochastic control methods. This approach simplifies the calculations considerably.

Specifically, the pair of stochastic processes (P_t, σ_t) satisfies assumptions of §2.4, therefore, results of this section can be used to derive the optimal-replication strategy, the minimum production cost of optimal replication, and the replication error. In particular, the application of the results of §2.4 to (3.48)–(3.49) yields the following system of PDEs:

$$\frac{\partial a(t,\sigma)}{\partial t} = \frac{\mu^2}{\sigma^2}a(t,\sigma) - (g(\sigma)+2\rho\kappa\mu)\frac{\partial a(t,\sigma)}{\partial\sigma} + \frac{1}{a(t,\sigma)}\left(\rho\kappa\sigma\frac{\partial a(t,\sigma)}{\partial\sigma}\right)^2 - \frac{1}{2}\kappa^2\sigma^2\frac{\partial^2 a(t,\sigma)}{\partial\sigma^2}, \tag{3.50}$$

$$\frac{\partial b(t,P,\sigma)}{\partial t} = -\big(g(\sigma)-\rho\mu\kappa\big)\frac{\partial b(t,P,\sigma)}{\partial \sigma} - \frac{\kappa^2\sigma^2}{2}\frac{\partial^2 b(t,P,\sigma)}{\partial \sigma^2} - \frac{\sigma^2P^2}{2} \cdot \frac{\partial^2 b(t,P,\sigma)}{\partial P^2} - \rho\kappa\sigma^2 P \cdot \frac{\partial^2 b(t,P,\sigma)}{\partial P\partial \sigma} - (1-\rho^2)\frac{\kappa^2\sigma^2}{a(t,\sigma)} \cdot \frac{\partial a(t,\sigma)}{\partial \sigma}\frac{\partial b(t,P,\sigma)}{\partial \sigma}, \tag{3.51}$$

$$\frac{\partial c(t,P,\sigma)}{\partial t} = -g(\sigma)\frac{\partial c(t,P,\sigma)}{\partial \sigma} - \mu P\frac{\partial c(t,P,\sigma)}{\partial P} - \frac{\kappa^2\sigma^2}{2}\frac{\partial^2 c(t,P,\sigma)}{\partial \sigma^2} - \rho\kappa\sigma^2 P\frac{\partial^2 c(t,P,\sigma)}{\partial P\partial \sigma} - \frac{\sigma^2P^2}{2}\frac{\partial^2 c(t,P,\sigma)}{\partial P^2} + a(t,\sigma)\kappa^2\sigma^2(\rho^2-1) \cdot \left(\frac{\partial b(t,P,\sigma)}{\partial \sigma}\right)^2, \tag{3.52}$$

with boundary conditions

$$a(T,\sigma)=1,\ \ b(T,P,\sigma)=F(P,\sigma),\ \ c(T,P,\sigma_T)=0.$$

The optimal-replication strategy is given by

$$\theta^*(t,V_t,P_t,\sigma_t) = \frac{\partial b(t,P_t,\sigma_t)}{\partial P_t} + \frac{\rho\kappa}{P_t}\frac{\partial b(t,P_t,\sigma_t)}{\partial \sigma_t} - \frac{V_t - b(t,P_t,\sigma_t)}{a(t,\sigma_t)}\frac{\rho\kappa}{P_t}\frac{\partial a(t,\sigma_t)}{\partial \sigma_t} - \frac{\mu}{\sigma_t^2 P_t}\big[V_t - b(t,P_t,\sigma_t)\big]. \tag{3.53}$$

Exact replication is possible when the following equation is satisfied:

$$\kappa^2(\rho^2-1)=0,$$

and this corresponds to the following special cases:

- Volatility is a deterministic function of time;
- The Brownian motions driving stock prices and volatility are perfectly correlated.

Both of these conditions yield well-known special cases where arbitrage-pricing is possible (see, for example, Geske 1979 and Rubinstein 1983). If we set $\kappa = g(\sigma) = 0$, (3.51) reduces to the Black and Scholes (1973) PDE.

4. NUMERICAL ANALYSIS

The essence of the ϵ-arbitrage approach to the optimal-replication problem is the recognition that although perfect replication may not be possible in some situations, the optimal-replication strategy of Theorem 1 may come very close. How close is, of course, an empirical matter, hence, in this section, we present several numerical examples that complement the theoretical analysis of §3.

In §4.1, we describe our numerical procedure and apply it to the case of geometric Brownian motion in §4.2, a mixed jump-diffusion model with a lognormal jump magnitude in §4.3, a stochastic volatility model in §4.4, and to a path-dependent option to "sell at the high" in §4.5.

4.1. The Numerical Procedure

To implement the solution (2.17)–(2.21) of the optimal-replication problem numerically, we begin by representing the functions $a_i(P,\mathbf{Z})$, $b_i(P,\mathbf{Z})$, and $c_i(P,\mathbf{Z})$ by their values over a spatial grid $\{(P^j,\mathbf{Z}^k) : j=1,\ldots,J, k=1,\ldots,K\}$. For any given $(P,\mathbf{Z})$, values $a_i(P,\mathbf{Z})$, $b_i(P,\mathbf{Z})$, and $c_i(P,\mathbf{Z})$ are obtained from $a_i(P^j,\mathbf{Z}^k)$, $b_i(P^j,\mathbf{Z}^k)$, and $c_i(P^j,\mathbf{Z}^k)$, using a piece-wise quadratic interpolation. This procedure provides an accurate representation of $a_i(P,\mathbf{Z})$, $b_i(P,\mathbf{Z})$, and $c_i(P,\mathbf{Z})$, with a reasonably small number of sample points. The values $a_i(P^j,\mathbf{Z}^k)$, $b_i(P^j,\mathbf{Z}^k)$, and $c_i(P^j,\mathbf{Z}^k)$ are updated according to the recursive procedure (2.17)–(2.19).

We evaluate the expectations in (2.17)–(2.19) by replacing them with the corresponding integrals. For all the models considered in this paper, these integrals involve Gaussian kernels. We use Gauss-Hermite quadrature formulas (see, for example, Stroud 1971) to obtain efficient numerical approximations of these integrals.

In all cases, except for the path-dependent options, we perform numerical computations for a European put option with a unit strike price ($K=1$), i.e., $F(P_T)=\max[0,K-P_T]$, and a six-month maturity. It is apparent from (2.17)–(2.21) that for a call option with the same strike price K, the replication error $c_i(\cdot)$ is the same as that of a put option, and the replication cost $b_i(\cdot)$ satisfies the put-call parity relation. We assume 25 trading periods, defined by $t_0=0$, $t_{i+1}-t_i=\Delta t=1/50$.

4.2. Geometric Brownian Motion

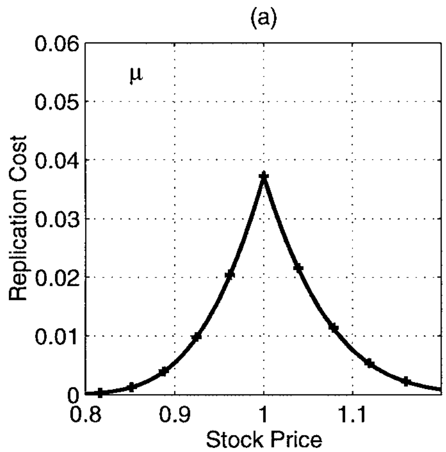
Let stock prices follow a geometric Brownian motion, which implies that returns are lognormally distributed as in (3.8)–(3.10). We set $\mu=0.07$ and $\sigma=0.13$, and to cover a range of empirically plausible parameter values, we vary each parameter by increasing and decreasing them by 25% and 50%, while holding the values of other parameter fixed. Figure 1 displays the minimum replication cost V_0^* minus the intrinsic value $F(P_0)$, for the above range of parameter values, as a function of the stock price at time 0.

Figure 1 shows that V_0^* is not sensitive to changes in μ and increases with σ. This is not surprising given that V_0^* approximates the Black-Scholes option pricing formula.

Figure 2 shows the dependence of the replication error ϵ^* on the initial stock price. Again we observe low sensitivity to the drift μ, but, as in Figure 1, the replication error tends to increase with the volatility. Also, the replication error is highest when the stock price is close to the strike price.

Another important characteristic of the replication process is the ratio of the replication error to the replication cost ϵ^*/V_0^*, which we call the *relative replication error*.

Figure 1. The difference between the replication cost and the intrinsic value of a 6-month maturity European put option, plotted as a function of the initial stock price. The stock price follows a geometric Brownian motion with parameter values $\mu = 0.07$ and $\sigma = 0.13$ corresponding to the solid line. In Panel (a), μ is varied and σ is fixed; in Panel (b), σ is varied and μ is fixed. In both cases, the variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses).



This ratio is more informative than the replication error itself, since it describes the replication error per dollar spent, as opposed to the error of replicating a single option contract.

The dependence of the relative replication error on the initial stock price is displayed in Figure 3. This figure shows that the relative replication error is an increasing function of the initial stock price, i.e., it is higher for out-of-the-money options. Also, the relative replication error decreases with volatility for out-of-the-money options. This is not surprising given that it was defined as a ratio of the replication error to the hedging cost, both of which are increasing functions of volatility. According to this definition, the dependence of the relative replication error on volatility is determined by the trade-off between increasing replication error and increasing replication cost.

4.3. Jump-Diffusion Models

In our numerical implementation of the jump-diffusion model (3.20)–(3.24) and (3.26), we restrict the number of jumps over a single time interval to be no more than three, which amounts to modifying the distribution of n_i in (3.21),

Figure 2. The replication error of a 6-month maturity European put option, plotted as a function of the initial stock price. The stock price follows a geometric Brownian motion with parameter values $\mu = 0.07$ and $\sigma = 0.13$ corresponding to the solid line. In Panel (a), μ is varied and σ is fixed; in Panel (b), σ is varied and μ is fixed. In both cases, the variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses) .

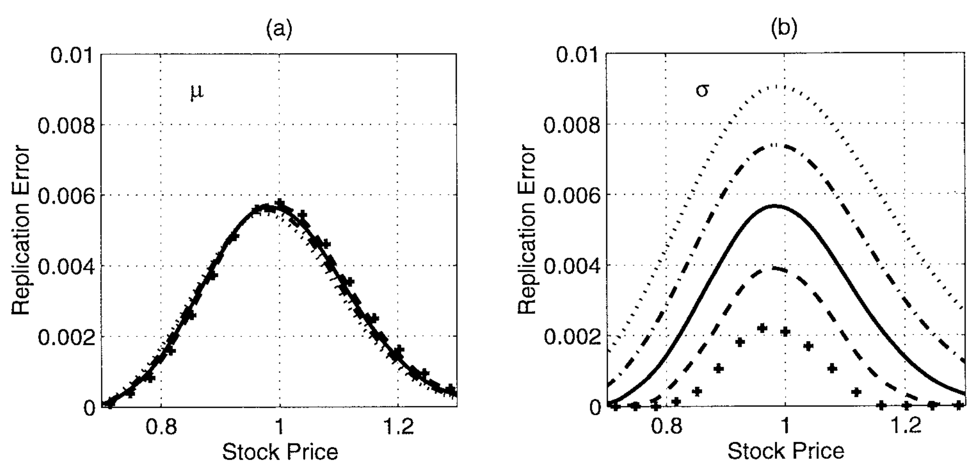


Figure 3. The relative replication error of a 6-month maturity European put option (relative to the replication cost), plotted as a function of the initial stock price. The stock price follows a geometric Brownian motion with parameter values $\mu = 0.07$ and $\sigma = 0.13$ corresponding to the solid line. In Panel (a), μ is varied and σ is fixed; in Panel (b), σ is varied and μ is fixed. In both cases, the variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses).

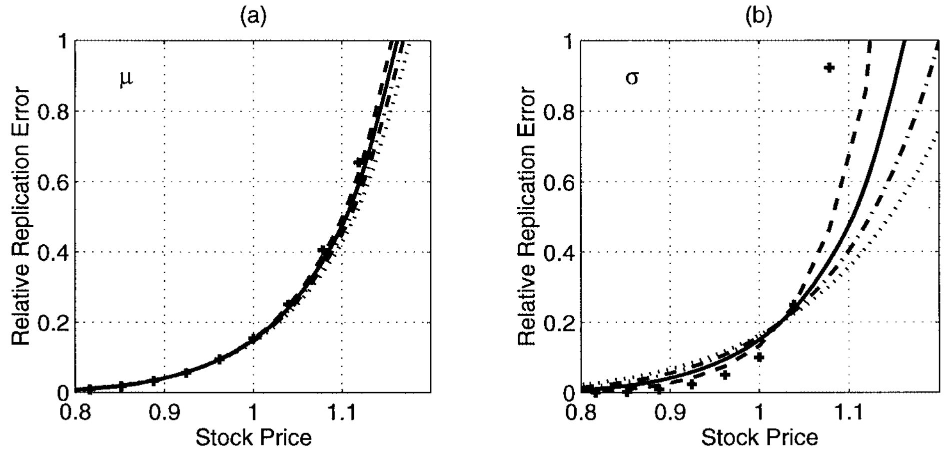


Figure 4. The difference between the replication cost and the intrinsic value of a 6-month maturity European put option, plotted as a function of the initial stock price. The stock price follows the mixed jump-diffusion process given in (3.20)–(3.23), (3.26), (4.1), and (4.2) with parameter values $\mu = 0.07$, $\sigma = 0.106$, $\lambda = 25$, and $\delta = 0.015$ corresponding to the solid line. In Panels (a)–(d), μ, σ, λ, and δ are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses).

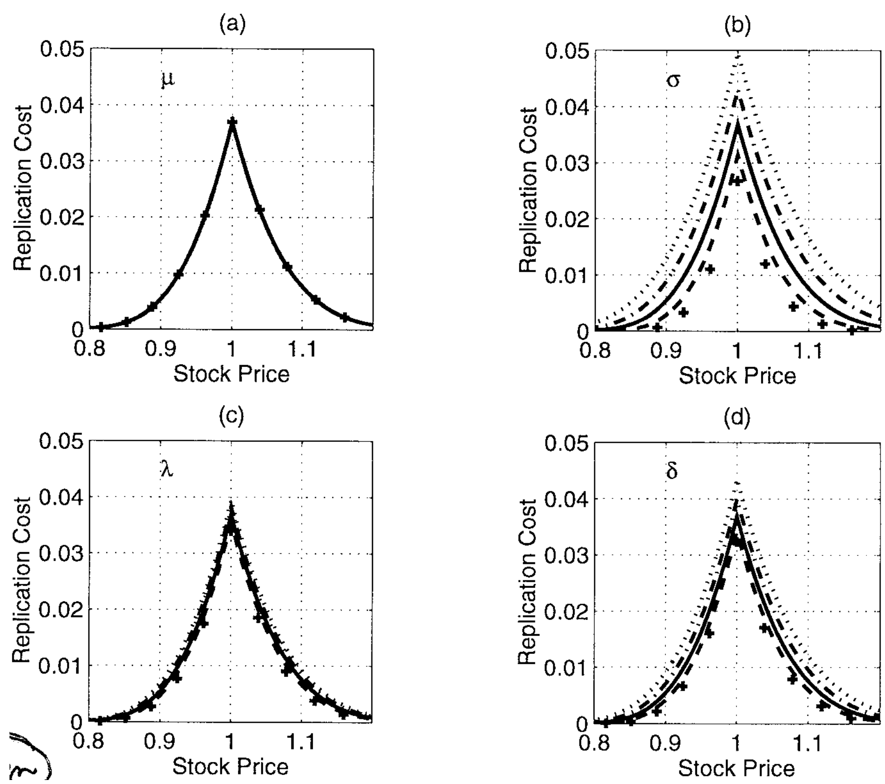
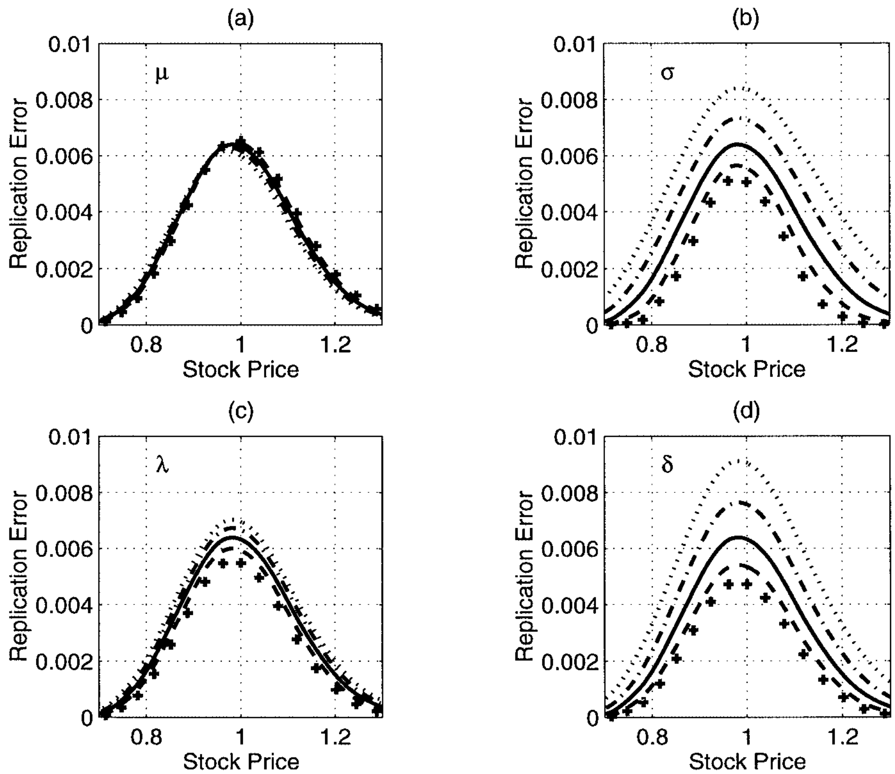


Figure 5. The replication error of a 6-month maturity European put option, plotted as a function of the initial stock price. The stock price follows the mixed jump-diffusion process given in (3.20)–(3.23), (3.26), (4.1), and (4.2) with parameter values $\mu = 0.07$, $\sigma = 0.106$, $\lambda = 25$, and $\delta = 0.015$ corresponding to the solid line. In Panels (a)–(d), μ, σ, λ, and δ are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses).



originally given by (3.24). This "truncation problem" is a necessary evil in the estimation of jump-diffusion models (see Ball and Torous 1985 for further discussion). Specifically, we replace (3.24) with

$$\text{Prob}[n_i = m] = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^m}{m!}, \qquad m = 1, 2, 3, \tag{4.1}$$

$$\text{Prob}[n_i = 0] = 1 - \sum_{m=1}^{3} \text{Prob}[n_i = m]. \tag{4.2}$$

Besides this adjustment in the distribution of returns, our numerical procedure is exactly the same as in §3.2. We start with the following parameter values:

$$\mu = 0.07, \quad \sigma = 0.106, \quad \lambda = 25, \quad \delta = 0.015.$$

Then we study sensitivity of the solution to the parameter values by increasing and decreasing them by 25% and 50%, while holding the other parameter values fixed. Our numerical results are summarized in Figures 4, 5, and 6.

Figure 4 shows that the replication cost V_0^* is not sensitive to the drift rate μ and is increasing in volatility σ, the jump intensity λ, and the standard deviation δ of the jump magnitude. It is most sensitive to σ. Figure 5 shows that the replication error ϵ^* is not sensitive to μ and increases with all other parameters, with the highest sensitivity to δ. Finally, Figure 6 shows that the relative replication error ϵ^*/V_0^* is sensitive only to σ and that it decreases as a function of σ for out-of-the-money options.

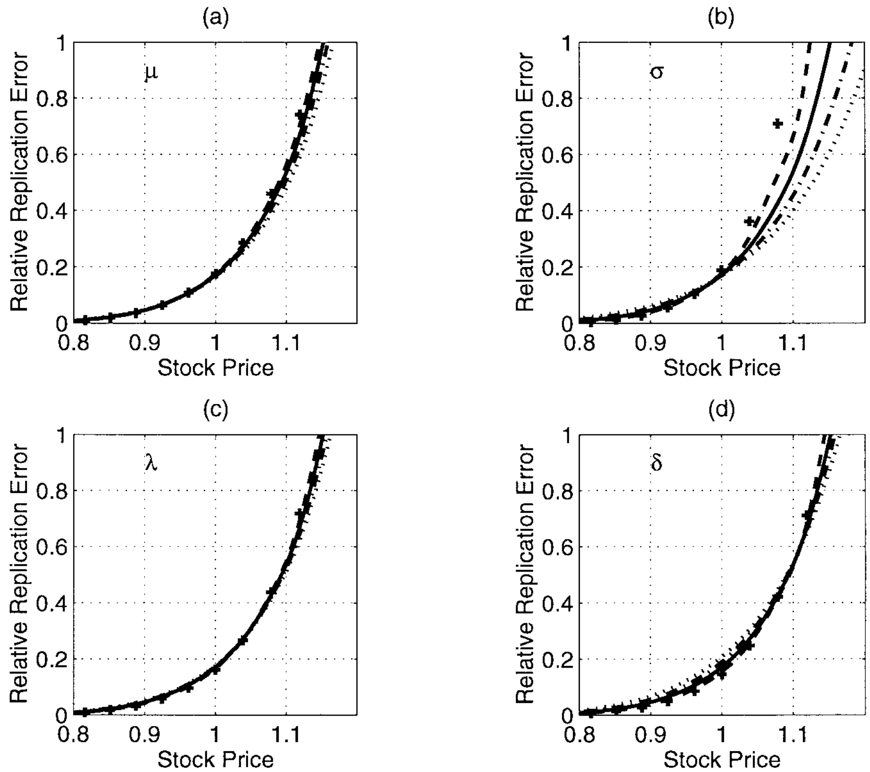
4.4. Stochastic Volatility

We begin by assuming a particular functional form for $g(\sigma)$ in (3.49):

$$g(\sigma) = -\delta\sigma(\sigma - \zeta).$$

We also assume that the Brownian motions driving the stock price and volatility are uncorrelated. Since the closed-form expressions for the transition probability density of the diffusion process with stochastic volatility are not available, we base our computations on the discrete-time approximations of this process. This is done mostly for convenience, since we can approximate the transition probability density using Monte Carlo simulations. While the discrete-time approximations lead to significantly more efficient numerical algorithms, they are also consistent with

Figure 6. The relative replication error of a 6-month maturity European put option (relative to the replication cost), plotted as a function of the initial stock price. The stock price follows the mixed jump-diffusion process (3.20)–(3.23), (3.26), (4.1), and (4.2) with parameter values $\mu = 0.07$, $\sigma = 0.106$, $\lambda = 25$, and $\delta = 0.015$ corresponding to the solid line. In Panels (a)–(d), μ, σ, λ, and δ are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses).



many estimation procedures that replace continuous-time processes with their discrete-time approximations (see, for example, Ball and Torous 1985 and Wiggins 1987).

The dynamics of stock prices and volatility are described by

$$P_{i+1} = P_i \exp\left((\mu - \sigma_i^2/2)\Delta t + \sigma_i \sqrt{\Delta t} z_{Pi}\right), \tag{4.3}$$

$$\sigma_{i+1} = \sigma_i \exp\left((-\delta(\sigma_i - \zeta) - \kappa^2/2)\Delta t + \kappa \sqrt{\Delta t} z_{\sigma i}\right), \tag{4.4}$$

where $z_{Pi}, z_{\sigma i} \sim \mathcal{N}(0, 1)$ and $\mathrm{E}[z_{Pi} z_{\sigma i}] = 0$. The parameters of the model are chosen to be

$$\mu = 0.07, \quad \zeta = 0.153, \quad \delta = 2, \quad \kappa = 0.4. \tag{4.5}$$

We also assume that at time $t = 0$ volatility σ_0 is equal to 0.13. As before, we study sensitivity of the solution to parameter values. Our findings are summarized in Figures 7, 8, and 9.

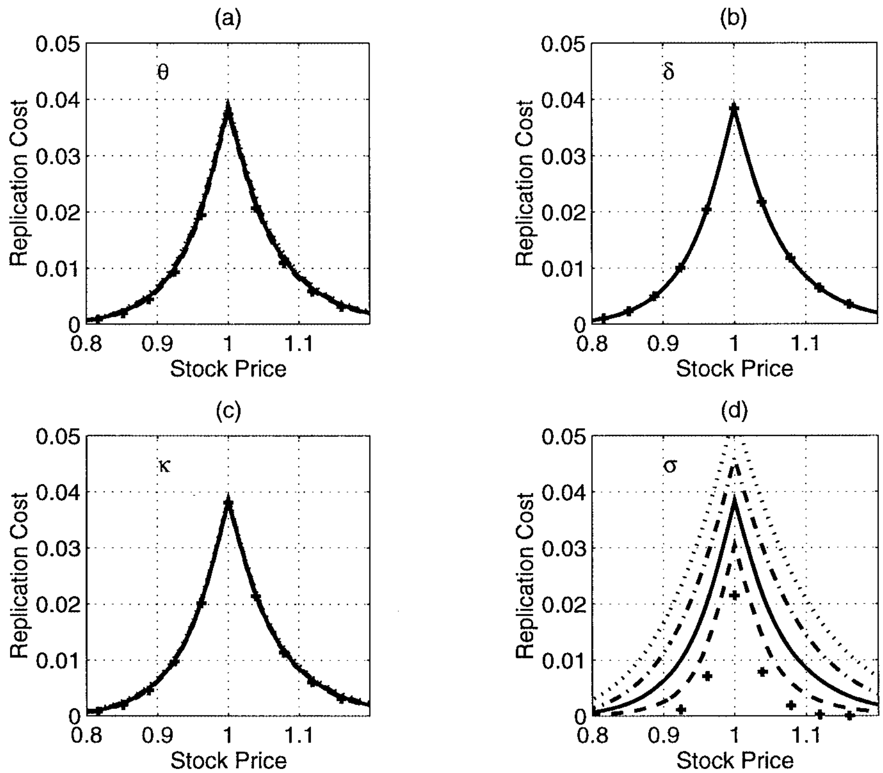
We do not display the dependence on μ in these figures, since the sensitivity to this parameter is so low. Figure 7 shows that the replication cost is sensitive only to the initial value of volatility σ_0 and, as expected, the replication cost increases with σ_0. Figure 8 shows that the replication error is sensitive to κ and σ_0 and is increasing in both of these parameters. According to Figure 9, the relative replication error is increasing in κ. It also increases in σ_0 for in-the-money options and decreases for out-of-the-money options.

In addition to its empirical relevance, the stochastic volatility model (3.48)–(3.49) also provides a clear illustration of the use of ϵ^* as a quantitative measure of dynamic market-incompleteness. Table 2 reports the results of Monte Carlo experiments, in which the optimal-replication strategy is implemented for six sets of parameter values for the stochastic volatility model, including the set that yields geometric Brownian motion.

For each set of parameter values, 1,000 independent sample paths of the stock price are simulated, each sample path containing 25 observations, and for each path, the optimal-replication strategy is implemented. The averages (over the 1,000 sample paths) of the minimum production cost V_0^*, the realized replication error $\hat{\epsilon}^*$, the initial optimal stock holdings θ_0^*, and the average optimal stock holdings $\bar{\theta}^*$ (over the 25 periods), are reported in each row. For comparison, the theoretical replication error ϵ^* is also reported.

Since stochastic volatility implies dynamically incomplete markets whereas geometric Brownian motion implies the opposite, these six sets of simulations make up a

Figure 7. The difference between the replication cost and the intrinsic value of a 6-month maturity European put option, plotted as a function of the initial stock price. The stock price follows the stochastic volatility model (4.3)–(4.4) with parameter values $\mu = 0.07$, $\zeta = 0.153$, $\delta = 2$, $\kappa = 0.4$, and $\sigma_0 = 0.13$ corresponding to the solid line. In Panels (a)–(d), ζ, δ, κ, and σ_0 are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses).



sequence of models that illustrate the fact that market completeness need not be a binary characteristic. In particular, Table 2 shows that, as the parameter values move closer to geometric Brownian motion, the average replication error ϵ^* decreases from 0.0086 to 0.0060. Moreover, the decrease between the first and second rows is considerably larger than the decrease between the second and third rows—the second and third rows imply price processes that are closer to each other in their degree of market completeness than that of the first row. Such specific rank orderings and sharp numerical comparisons are simply unavailable from standard dynamic equilibrium models that have been used to model market incompleteness.

Of course, ϵ^* is only one of many possible measures of market incompleteness—a canonical measure seems unlikely to emerge from the current literature—nevertheless it is an extremely useful measure, given the practical implications that it contains for dynamically hedging risks.

4.5. Path-dependent Options

We consider the option to "sell at the high," as described by Goldman et al. (1979), under the assumption that the stock price follows the mixed jump-diffusion process (3.20)–(3.23), (3.26), (4.1), (4.2). We define the state variable Z:

$$Z_0 = m \geqslant P_0,$$

$$Z_{i+1} = \max[Z_i, P_{i+1}].$$

According to this definition, Z_i is the running maximum of the stock price process at time t_i. The initial value of Z_i is m, i.e., we assume that at time 0 the running maximum is equal to m.

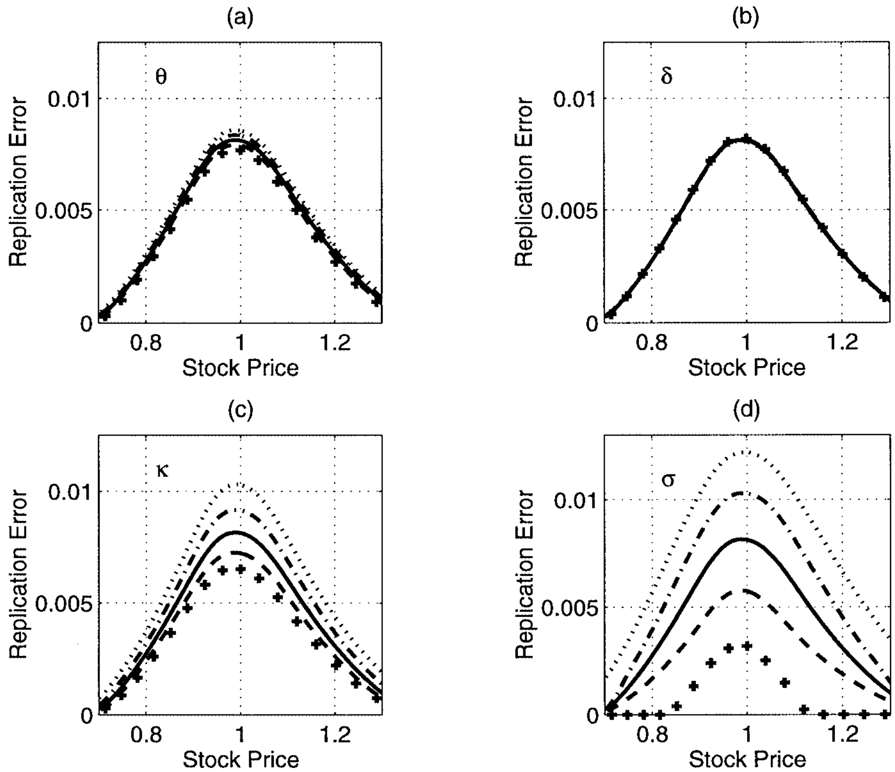
The payoff of the option is given by

$$F(P_T, Z_T) = Z_T - P_T.$$

In our numerical analysis we set $m = 1$ as a convenient normalization. Note that this convention is just a change of scale and does not lead to any loss of generality.

The parameters for the stock price process are taken to be the same as in §3.3. The sensitivity of the replication cost and replication error on the initial stock price and parameters of the stock price process are reported in Figures 10, 11, and 12.

Figure 8. The replication error of a 6-month maturity European put option, plotted as a function of the initial stock price. The stock price follows the stochastic volatility model (4.3)–(4.4) with parameter values $\mu = 0.07$, $\zeta = 0.153$, $\delta = 2$, $\kappa = 0.4$, and $\sigma_0 = 0.13$ corresponding to the solid line. In Panels (a)–(d), ζ, δ, κ, and σ_0 are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses).



The qualitative behavior of the replication cost as a function of the initial stock price is similar to that of the option price as described in Goldman et al. (1979). The difference between our model and theirs is that they assume the stock price follows a geometric Brownian motion and that continuous-time trading is allowed. Also, the running maximum of the stock price process is calculated continuously, not over a discrete set of time moments, as in our case. Figure 10 shows that the replication cost V_0 is not sensitive to the drift rate μ and is increasing in volatility σ, the jump intensity jumps λ, and the standard deviation δ of the jump magnitude. It is most sensitive to σ. These observations are consistent with the behavior of the replication error of the European put option in §3.3. According to Figure 11, the replication error ϵ^* is not sensitive to μ and is increasing in all other parameters with the highest sensitivity to δ and σ. Figure 12 shows that the relative replication error ϵ^*/V_0 is sensitive to σ and δ. It is an increasing function of δ, while the sign of the change of ϵ^*/V_0 with σ depends on the initial stock price P_0.

5. MEASURING THE DEGREE OF MARKET INCOMPLETENESS

In this section, we propose to measure the degree of market incompleteness by exploring the sensitivity of the replication error and the replication cost of a particular option contract to the specification of the stock-price dynamics. Specifically, we compare the following models: geometric Brownian motion, a mixed jump-diffusion process, and a diffusion process with stochastic volatility. The parameters of these models are calibrated to give rise to identical values of the expected instantaneous rate of return and volatility, hence we can view these three models as competing specifications of the same data-generating process.

5.1. Calibrating the Stochastic Processes

We consider a European put option with a unit strike price ($K = 1$) and a six-month maturity, i.e., $F(P_T) = \max[0, K - P_T]$. There are 25 trading periods, defined by $t_{i+1} - t_i = \Delta t = 1/50$. Since the closed-form expressions for the transition probability density of the mixed jump-diffusion process and the process with stochastic volatility are not available, we base our computations on the discrete-time approximations of these processes. The model specifications and corresponding parameter values are

1. Geometric Brownian Motion. Returns on the stock are lognormal, given by (3.8)–(3.10). We use the following parameter values:

$$\mu = 0.07, \quad \sigma = 0.13. \tag{5.1}$$

Figure 9. The relative replication error of a 6-month maturity European put option (relative to the replication cost), plotted as a function of the initial stock price. The stock price follows the stochastic volatility model (4.3)–(4.4) with parameter values $\mu = 0.07$, $\zeta = 0.153$, $\delta = 2$, $\kappa = 0.4$, and $\sigma_0 = 0.13$ corresponding to the solid line. In Panels (a)–(d), ζ, δ, κ, and σ_0 are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses).

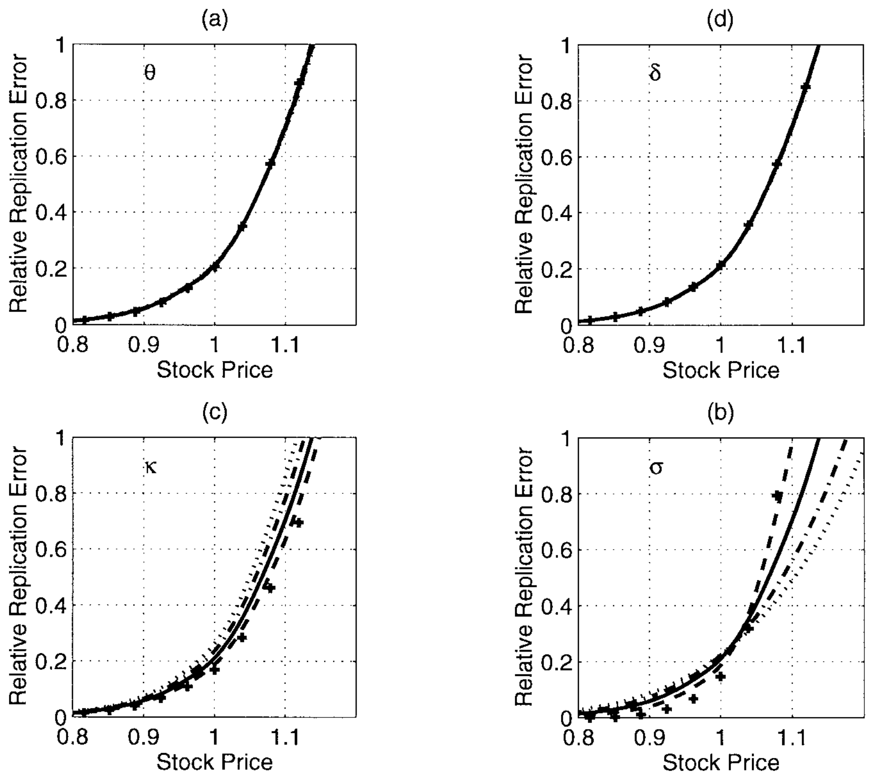
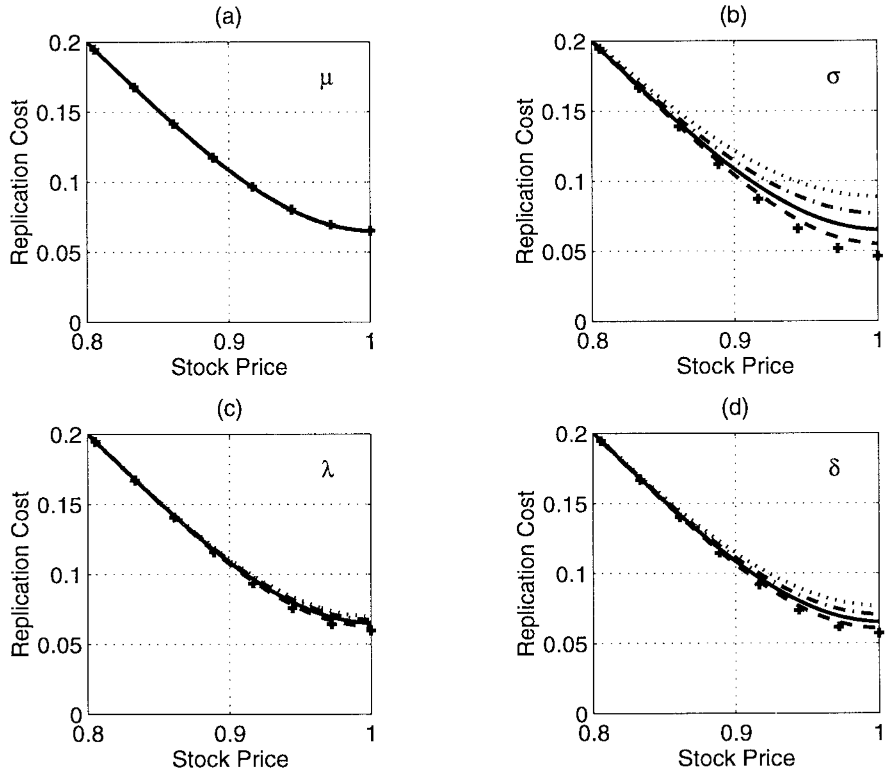


Table 2. Monte Carlo simulation of the optimal-replication strategy θ^* for replicating a 6-month at-the-money European put-option.

Model				Performance of Optimal Replication Strategy				
σ_0	ζ	δ	κ	V_0^*	ϵ^*	$\hat{\epsilon}^*$	θ_0^*	$\bar{\theta}^*$
0.13	0.153	2.00	0.400	0.0374	0.0084	0.0086	−0.474	−0.44
0.13	0.137	1.50	0.200	0.0367	0.0067	0.0068	−0.475	−0.43
0.13	0.133	1.00	0.100	0.0366	0.0062	0.0063	−0.475	−0.42
0.13	0.131	0.50	0.050	0.0366	0.0060	0.0059	−0.475	−0.41
0.13	0.130	0.25	0.025	0.0365	0.0060	0.0060	−0.475	−0.43
0.13	0.130	0.00	0.000	0.0365	0.0060	0.0062	−0.475	−0.42

*For six sets of parameter values of the Stochastic Volatility Model (4.3)–(4.4), including the set of parameter values that yields a geometric Brownian motion (last row). For each set of parameter values, 1,000 independent sample paths were simulated, each path containing 25 periods, and $P_0 = 1$.

Figure 10. The replication cost of a 6-month maturity European option to "sell at the high," plotted as a function of the initial stock price. The stock price follows the mixed jump-diffusion process (3.20)–(3.23), (3.26), (4.1), and (4.2) with parameter values $m = 1$, $\mu = 0.07$, $\sigma = 0.106$, $\lambda = 25$, and $\delta = 0.015$ corresponding to the solid line. In Panels (a)–(d), μ, σ, λ, and δ are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses).



2. Mixed Jump-Diffusion. The distribution of returns on the stock is given by (3.20)–(3.23), (3.26), (4.1), and (4.2). We use the following parameter values:

$$\mu = 0.07, \quad \sigma = 0.106, \quad \lambda = 25, \quad \delta = 0.015. \tag{5.2}$$

3. Diffusion with Stochastic Volatility. Stock-price and volatility dynamics are given by (4.3)–(4.4). We assume that at time $t = 0$, volatility σ_0 is equal to 0.13, and the other parameter values are

$$\mu = 0.07, \quad \zeta = 0.153, \quad \delta = 2, \quad \kappa = 0.4. \tag{5.3}$$

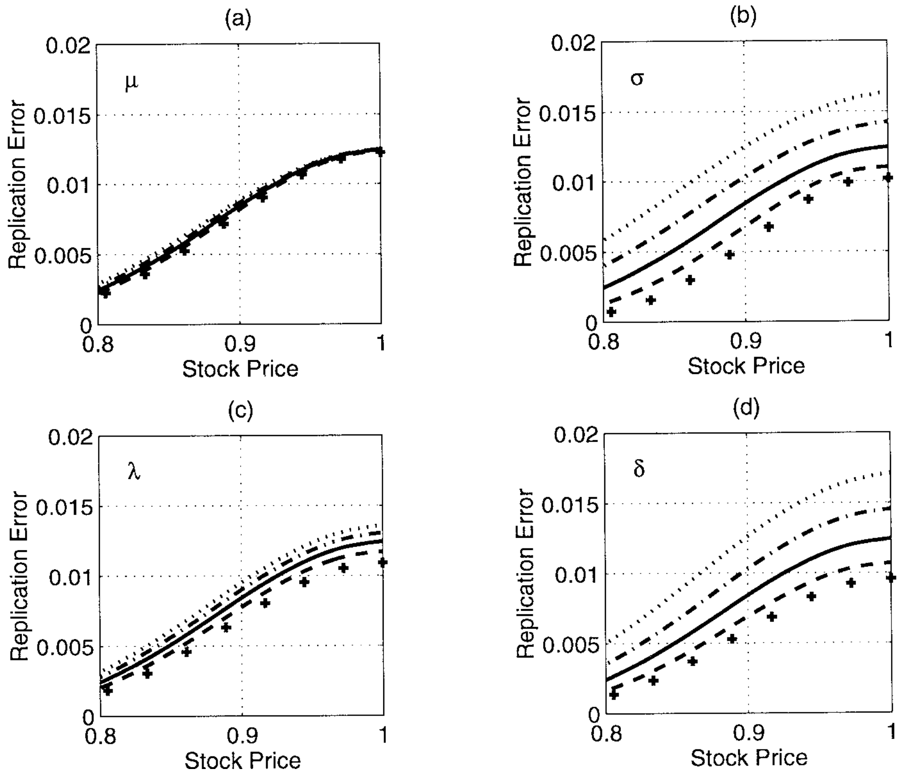
5.2. Numerical Results

Figures 13, 14, and 15 and Table 3 summarize our numerical results. Figure 13 presents the replication cost V_0^* minus the intrinsic value $F(P_0)$ for the three models as a function of the stock price at time $t = 0$. The hedging costs for the first two models are practically identical, while the stochastic volatility model can give rise to a significantly higher hedging cost for a deep-out-of-money option. Figure 14 and Table 3 show the dependence of the replication error ϵ^* on the initial stock price.

All three models exhibit qualitatively similar behavior: The replication error is highest when close to the strike price. For our choice of parameter values, the replication error is highest for the stochastic volatility model and lowest for geometric Brownian motion. However, this need not hold in general. As we demonstrate in §3.3, the replication error of the mixed jump-diffusion process depends critically on δ and λ in (3.24, 3.26), thus, by varying these parameters, one can reverse the order of the curves in Figure 14 without changing the annualized volatility of the mixed jump-diffusion process.

The dependence of the relative replication error on the initial stock price is captured in Figure 15. As in Figure 13, the relative replication errors for the first two models are practically identical, while the stochastic volatility model can exhibit considerably higher errors. Also, while the relative replication error can be significant, particularly for an out-of-money option, the variation across the models is not as significant as one would expect. When continuous-time trading is allowed, the replication error for the geometric Brownian motion model is zero, while the other two models give rise to strictly positive replication errors. This is an implication of the fact that the first model describes a dynamically complete market, while the other two corre-

Figure 11. The replication error of a 6-month maturity European option to "sell at the high," plotted as a function of the initial stock price. The stock price follows the mixed jump-diffusion process (3.20)–(3.23), (3.26), (4.1), and (4.2) with parameter values $m = 1$, $\mu = 0.07$, $\sigma = 0.106$, $\lambda = 25$, and $\delta = 0.015$ corresponding to the solid line. In Panels (a)–(d), μ, σ, λ, and δ are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses).



spond to markets that are dynamically incomplete (due to the absence of a sufficient number of traded instruments).

Nevertheless, as Figure 15 illustrates, the transition from continuous- to discrete-time trading can smear the differences between these models, leading to relative replication errors of comparable magnitude. This shows that the inability to trade continuously is just as important a source of market incompleteness as the absence of traded instruments.

6. CONCLUSION

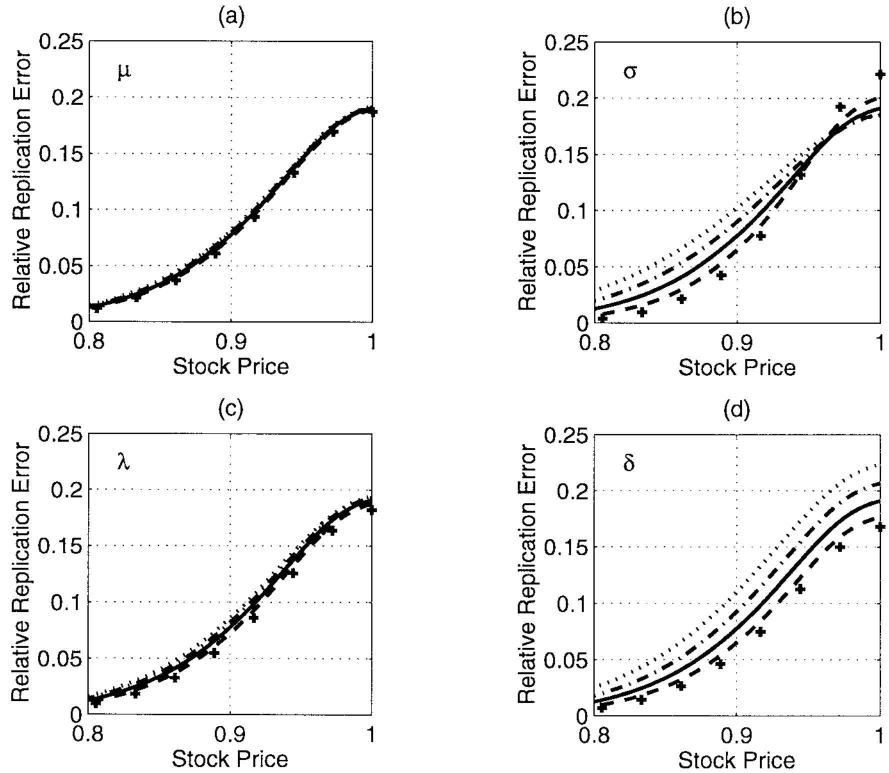
We have proposed a method for replicating derivative securities in dynamically incomplete markets. Using stochastic dynamic programming, we construct a self-financing dynamic portfolio strategy that best approximates an arbitrary payoff function in a mean-squared sense. When markets are dynamically complete, as in the Black and Scholes (1973) and Merton (1973) models, our optimal-replication strategy coincides with the delta-hedging strategies of such arbitrage-based models. Moreover, we provide an explicit algorithm for computing such strategies, which can be a formidable challenge despite market completeness, e.g., path-dependent derivatives such as "look-back" options.

When markets are not dynamically complete, as in the case of options on assets with stochastic volatility or with jump components, our approach yields the minimum production cost of a self-financing portfolio strategy with a terminal value that comes as close as possible—in mean-squared error—to the option's payoff. This is the essence of the ϵ-arbitrage approach to synthetically replicating a derivative security.

We also argue that the replication error of the optimal-replication strategy can be used as a quantitative measure for the degree of market incompleteness. Despite the difficulties in making welfare comparisons between markets with different types of incompleteness (see, for example, Duffie 1987; Duffie and Shafer 1985, 1986; Hart 1974), the minimum replication error of an ϵ-arbitrage strategy does provide one practical metric by which market completeness can be judged. After all, if it is possible to replicate the payoff of a derivative security to within some small error ϵ, the market for that security may be considered complete for all practical purposes even if ϵ is not zero.

Of course, this is only one of many possible measures of market completeness and we make no claims of generality here. Instead, we hope to have shown that Merton's (1973) seminal idea of dynamic replication has far broader

Figure 12. The relative replication error of a 6-month maturity European option to "sell at the high" (relative to the replication cost), plotted as a function of the initial stock price. The stock price follows the mixed jump-diffusion process (3.20)–(3.23), (3.26), (4.1), and (4.2) with parameter values $m = 1$, $\mu = 0.07$, $\sigma = 0.106$, $\lambda = 25$, and $\delta = 0.015$ corresponding to the solid line. In Panels (a)–(d), μ, σ, λ, and δ are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.25 (dashed-dotted line), 1.5 (dots), 0.75 (dashed line), and 0.5 (pluses).



implications than the dynamically complete-markets setting in which it was originally developed. We plan to explore other implications in future research.

APPENDIX

The proofs of Theorems 1 and 2 are conceptually straightforward but notationally quite cumbersome. Therefore, we present only a brief sketch of the proofs here—interested readers can contact the authors for the more detailed mathematical appendix.

A.1. Proof of Theorem 1

The proof of Theorem 1 follows from dynamic programming. For $i = N$, (2.14)–(2.16) are clearly true, given (2.10). We now show that (2.17)–(2.21) describe the solution of the optimization problem in (2.9). First, as we observed in §2.3, the functions $a_i(\cdot,\cdot)$ are positive. Together with (2.3) this implies that

$$\mathrm{E}[J_{i+1}(V_i + \theta_i(P_{i+1} - P_i), P_{i+1}, \mathbf{Z}_{i+1}) \mid V_i, P_i, \mathbf{Z}_i]$$

is a convex function of θ_i. Therefore, we can use the first-order condition to solve the optimization problem in (2.11):

$$\frac{d}{d\theta_i}\mathrm{E}[J_{i+1}(V_i + \theta_i(P_{i+1} - P_i), P_{i+1}, \mathbf{Z}_{i+1}) \mid V_i, P_i, \mathbf{Z}_i] = 0, \tag{A.1}$$

where $J_{i+1}(\cdot,\cdot,\cdot)$ is given by (2.12). Equation (A.1) is a linear equation in θ_i and it is straightforward to check that its solution, $\theta^*(i, V_i, P_i, \mathbf{Z}_i)$, is given by (2.13), (2.17), and (2.18). We now substitute (2.13) into (2.3) and use (2.11) to calculate

$$J_i(V_i, P_i, \mathbf{Z}_i) = \mathrm{E}\big[J_{i+1}(V_i + \theta^*(i, V_i, P_i, \mathbf{Z}_i) \cdot (P_{i+1} - P_i), P_{i+1}, \mathbf{Z}_{i+1}) \mid V_i, P_i, \mathbf{Z}_i\big]. \tag{A.2}$$

Equations (2.19)–(2.21) are obtained by rearranging terms in (A.2). □

A.2. Proof of Theorem 2

The more tedious algebraic manipulations of this proof were carried out using the symbolic algebra program *Maple*. Therefore, we shall outline the main ideas of the proof without reporting all of the details.

Figure 13. The difference between the replication cost and the intrinsic value of a 6-month maturity European put option, plotted as a function of the initial stock price. Several processes for the stock price are plotted: geometric Brownian motion (3.8)–(3.10) (solid line); the mixed jump-diffusion model (3.20)–(3.23), (3.26), (4.1), and (4.2) (dashed line); and the stochastic volatility model (4.3)–(4.4) (dashed-dotted line). The parameter values are given by (5.1), (5.2), and (5.3).

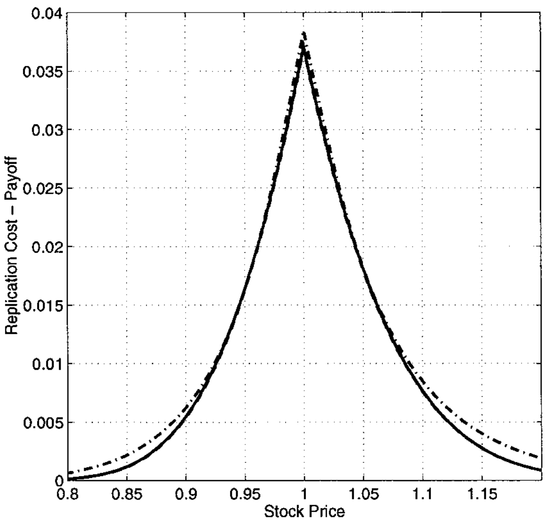


Figure 14. The replication error of a 6-month maturity European put option, plotted as a function of the initial stock price. Several processes for the stock price are plotted: geometric Brownian motion (3.8)–(3.10) (solid line); the mixed jump-diffusion model (3.20)–(3.23), (3.26), (4.1), and (4.2) (dashed line); and the stochastic volatility model (4.3)–(4.4) (dashed-dotted line). The parameter values are given by (5.1), (5.2), and (5.3).

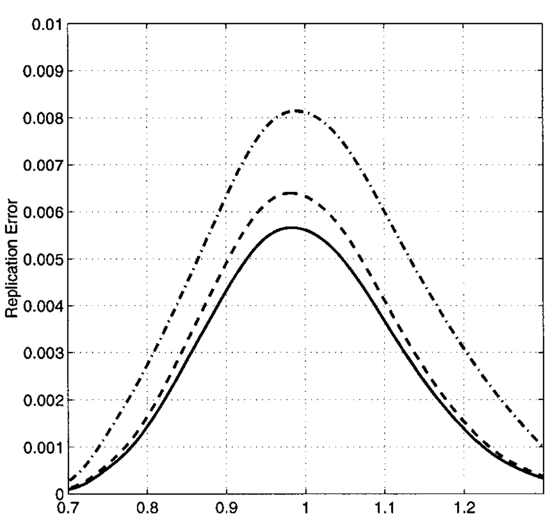
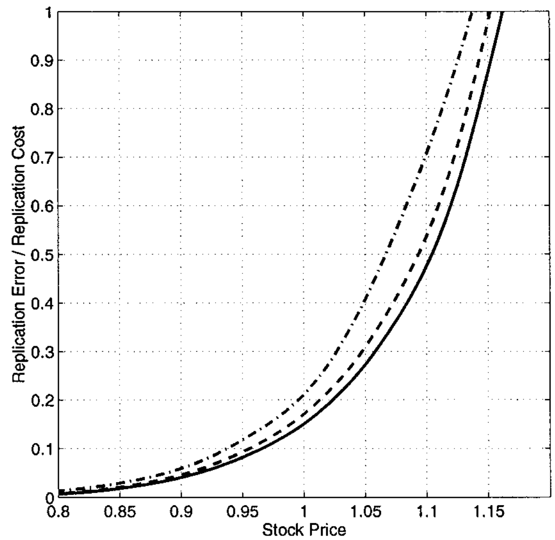


Figure 15. The relative replication error of a 6-month maturity European put option (relative to the replication cost), plotted as a function of the initial stock price. Several processes for the stock price are plotted: geometric Brownian motion (3.8)–(3.10) (solid line); the mixed jump-diffusion model (3.20)–(3.23), (3.26), (4.1), and (4.2) (dashed line); and the stochastic volatility model (4.3)–(4.4) (dashed-dotted line). The parameter values are given by (5.1), (5.2), and (5.3).



The cost-to-go function $J(t, V_t, P_t, \mathbf{Z}_t)$ satisfies the dynamic programming equation

$$\frac{\partial J}{\partial t} + \min_{\theta_t} \left\{ \left[\sum_{j=0}^{J} \mu_j Z_j \frac{\partial}{\partial Z_j} + \frac{1}{2} \sum_{i,j=0}^{J} \sigma_i \sigma_j Z_i Z_j \rho_{ij} \frac{\partial^2}{\partial Z_i \partial Z_j} + \theta_t \mu_0 Z_0 \frac{\partial}{\partial W} + \frac{1}{2} (\theta_t \sigma_0 Z_0)^2 \frac{\partial^2}{\partial W^2} + \theta_t \sum_{j=0}^{J} \sigma_0 \sigma_j \rho_{0j} Z_0 Z_j \frac{\partial^2}{\partial W \partial Z_j} \right] J \right\} = 0 \tag{A.3}$$

with boundary condition:

$$J(T, V_T, P_T, \mathbf{Z}_T) = [V_T - F(P_T, \mathbf{Z}_T)]^2, \tag{A.4}$$

where some of the functional dependencies were omitted to simplify the notation.

We must now check that the function $J(t, V_t, P_t, \mathbf{Z}_t)$, given by (2.26), (2.27)–(2.30), and the optimal control (2.31), satisfies (A.3)–(A.4). Boundary conditions (2.30) immediately imply (A.4). Next we substitute (2.26) into (A.3). It is easy to check, using Equation (2.27), that function $a(\cdot)$ is positive. Therefore, the first-order condition is sufficient for the minimum in (A.3). This condition is a linear equation in θ_t, which is solved by (2.31).

Table 3. Comparison of replication costs and errors of the optimal-replication strategy for replicating a 6-month European put option under competing specifications of price dynamics.

Price Dynamics	Initial Stock Price P_0 0.90	0.95	1.00	1.05	1.10
	Replication Cost Minus Intrinsic Value $(V_0^* - F(P_0))$				
Geometric Brownian Motion	0.0054	0.0161	0.0365	0.0176	0.0074
Jump/Diffusion	0.0053	0.0159	0.0364	0.0175	0.0074
Stochastic Volatility	0.0060	0.0168	0.0374	0.0185	0.0082
	Replication Error (ϵ^*)				
Geometric Brownian Motion	0.0046	0.0058	0.0060	0.0052	0.0038
Jump/Diffusion	0.0051	0.0066	0.0068	0.0059	0.0043
Stochastic Volatility	0.0061	0.0079	0.0084	0.0074	0.0056
	Relative Replication Error (ϵ^*/V_0^*)				
Geometric Brownian Motion	0.043	0.088	0.164	0.292	0.509
Jump/Diffusion	0.049	0.100	0.187	0.335	0.585
Stochastic Volatility	0.058	0.119	0.226	0.400	0.679

*Geometric Brownian motion (3.8)–(3.10); the mixed jump-diffusion model (3.20)–(3.23), (3.26), (4.1), and (4.2); and the stochastic volatility model (4.3)–(4.4). The parameter values are given by (5.1), (5.2), and (5.3).

It is now straightforward to verify that, whenever functions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ satisfy (2.27)–(2.29), (A.3) is satisfied as well.

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