QUANTITATIVE FINANCE VOLUME 1 (2001) 45–72
INSTITUTE OF PHYSICS PUBLISHING

RESEARCH PAPER
quant.iop.org

Asset allocation and derivatives

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Received 14 November 2000

Abstract

The fact that derivative securities are equivalent to specific dynamic trading strategies in complete markets suggests the possibility of constructing buy-and-hold portfolios of options that mimic certain dynamic investment policies, e.g. asset-allocation rules. We explore this possibility by solving the following problem: given an optimal dynamic investment policy, find a set of options at the start of the investment horizon which will come closest to the optimal dynamic investment policy. We solve this problem for several combinations of preferences, return dynamics and optimality criteria, and show that under certain conditions, a portfolio consisting of just a few options is an excellent substitute for considerably more complex dynamic investment policies.

1. Introduction

It is now well known that under certain conditions, complex financial instruments such as options and other derivative securities can be replicated by sophisticated dynamic trading strategies involving simpler securities such as stocks and bonds. This 'delta-hedging' strategy—for which Robert Merton and Myron Scholes shared the Nobel Memorial Prize in Economics in 1998—is largely responsible for the multitrillion-dollar derivatives industry and is now part of the standard toolkit of every derivatives dealer in the world.

The essence of delta-hedging is the ability to actively manage a portfolio continuously through time, and to do so in a 'self-financing' manner, i.e. no cash inflows or outflows after the initial investment, so that the portfolio's value tracks the value of the derivative security without error at each point in time, until the maturity date of the derivative. If such a portfolio strategy were possible, then the cost of implementing it must equal the price of the derivative, otherwise an arbitrage opportunity would exist. Black and Scholes (1973) and Merton (1973) used this argument to deduce the celebrated Black–Scholes option-pricing formula, but an even more significant outcome of their research was the insight that there exists a correspondence between dynamic trading strategies over a period of time and complex securities at a single point in time.

In this paper, we consider the reverse implications of this correspondence by constructing an optimal portfolio of complex securities at a single point in time to mimic the properties of a dynamic trading strategy over a period of time. Specifically, we focus on dynamic investment policies, i.e. asset-allocation rules, that arise from standard dynamic optimization problems in which an investor maximizes the expected utility of his end-of-period wealth, and we pose the following problem: given an investor's optimal dynamic investment policy for two assets, stocks and bonds, construct a 'buy-and-hold' portfolio—a portfolio that involves no trading once it is established—of stocks, bonds and options at the start of the investment horizon that will come closest to the optimal dynamic policy. By defining 'closest' in three distinct ways—expected utility, mean-squared error of terminal wealth and utility-weighted mean-squared error of terminal wealth—we propose three sets of numerical algorithms for solving this problem in general, and characterize specific solutions for several sets of preferences (constant relative risk-aversion, constant absolute risk-aversion) and return dynamics (geometric Brownian motion, mean-reverting processes).

The optimal buy-and-hold problem is an interesting one for several reasons. First, it is widely acknowledged that the continuous-time framework in which most of modern finance has been developed is an approximation to reality—it is currently impossible to trade continuously, and even if it were possible, market frictions would render continuous trading infinitely costly. Consequently, any practical implementation

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1469-7688/01/010045+28$30.00 PII: S1469-7688(01)18977-3

of continuous-time asset-allocation policies invariably requires some discretization in which the investor's portfolio is rebalanced only a finite number of times, typically at equally spaced time intervals, with the number of intervals chosen so that the discrete asset-allocation policy 'approximates' the optimal continuous-time policy in some metric. However, Merton's (1973) insight suggests that it may be possible to approximate a continuous-time trading strategy in a different manner, i.e. by including a few well-chosen options in the portfolio at the outset and trading considerably less frequently. In particular, Merton (1995) observes that derivatives can be an effective substitute for dynamic open-market operations of central banks seeking to engage in interest-rate stabilization policies. Therefore, in the presence of transactions costs, derivative securities may be an efficient way to implement optimal dynamic investment policies[2]. Indeed, we find that under certain conditions, a buy-and-hold portfolio consisting of just a few options is an excellent substitute for considerably more complex dynamic investment policies.

Second, the approximation errors between the optimal dynamic policy and the buy-and-hold policy will reveal the importance of dynamic trading, the 'completeness' of financial markets, and the ability of investors to achieve certain financial goals in a cost-effective manner[3]. In particular, the conditions that guarantee dynamic completeness are non-trivial restrictions on market structure and price dynamics (see, for example, Duffie and Huang (1985)), hence there are situations in which exact replication is impossible. These instances of *market incompleteness* are often attributable to institutional rigidities and market frictions—transactions costs, periodic market closures and discreteness in trading opportunities and prices—and while the pricing of derivative securities can still be accomplished in some cases via equilibrium arguments[4], this still leaves open the question of how expensive it is to achieve certain financial objectives, or how close one can come to those objectives for a given budget?

Finally, the optimal buy-and-hold portfolio can be used to develop a measure of the risks associated with the corresponding dynamic investment policy that the buy-and-hold portfolio is designed to replicate. While there is general agreement in the financial community regarding the proper measurement of risk in a static context—the market beta from the Capital Asset Pricing Model—there is no consensus regarding the proper measurement of risk for dynamic investment strategies. Market betas are notoriously unreliable in a multiperiod setting[5], and other measures such as the Sharpe ratio, the Sortino ratio and maximum drawdown have been used to capture different risk exposures of dynamic investment strategies. By developing a correspondence between a dynamic investment strategy and a buy-and-hold portfolio, it may be possible to construct a more comprehensive set of risk measures for the dynamic strategy through the characteristics of the buy-and-hold portfolio and the approximation error.

In section 2 we provide a brief review of the strands of the asset allocation and derivatives pricing literature that are most relevant to our problem. We describe the buy-and-hold alternative to the standard asset-allocation problem in section 3 and propose three methods for solving it: maximization of expected utility, minimization of mean-squared error and a hybrid of the two (minimization of utility-weighted mean-squared error). While the first approach is the most direct, it is also the most computationally intensive. The latter two approaches are simpler to implement, however, they do not maximize expected utility and as a result, the portfolios that they generate may be suboptimal. These issues are addressed in more detail in sections 4 and 5 where we implement the three methods for geometric Brownian motion, the Ornstein–Uhlenbeck process, and a bivariate linear diffusion process with a stochastic mean-reverting drift. Extensions, qualifications and other aspects of the optimal buy-and-hold portfolio are discussed in section 6, and we conclude in section 7.

2. Literature review

The literature on asset allocation is vast and addresses a broad set of issues, many that are beyond the scope of this paper's main focus[6]. Most studies that consider derivatives in the context of asset allocation use option-pricing methods to gauge the economic value of market-timing skills, e.g. Merton (1981), Henriksson and Merton (1981), and Evnine and Henriksson (1987). Carr *et al* (2000) solve the asset-allocation problem in an economy where derivatives are required to complete the market. Carr and Madan (2000) consider a single-period model where agents are permitted to trade the stock, bond and European options with a continuum of strikes. Because of the inability to trade dynamically, options constitute a new asset class and the impact of beliefs and preferences on the agent's positions in the three asset classes is studied. In a general equilibrium framework, they derive conditions for mutual-fund separation where some of the separating funds are composed of derivative securities. None of these papers explores the possibility of substituting a simple buy-and-hold portfolio for a dynamic investment policy.

Three other strands of the literature are relevant to our paper: Merton's (1995) functional approach to understanding

[2] Taxes can be viewed as another type of transactions cost and the optimal buy-and-hold portfolio offers several additional advantages over the optimal dynamic investment policy for taxable investors.

[3] Financial markets are said to be 'complete' (in the Arrow–Debreu sense) if it is possible to construct a portfolio of securities at a point in time which guarantees a specific payoff in a specific state of nature at some future date. The notion of 'dynamic completeness' is the natural extension of this idea to dynamic trading strategies. See Harrison and Kreps (1979) and Duffie and Huang (1985) for a more detailed discussion.

[4] Examples of continuous-time incomplete-markets models include Breeden (1979), Duffie and Shafer (1985, 1986), Föllmer and Sonderman (1986), Duffie (1987) and He and Pearson (1991). Examples of discrete-time incomplete-markets models include Scheinkman and Weiss (1986), Aiyagari and Gertler (1991), Heaton and Lucas (1992, 1996), Weil (1992), Telmer (1993), Aiyagari (1994), Lucas (1994) and He and Modest (1995). Other aspects of pricing and hedging in incomplete markets have been considered by Magill and Quinzii (1996), Kallsen (1999), Kramkov and Schachermeyer (1999), Bertsimas *et al* (2000b), Goll and Rueschendorf (2000) and Schäl (2000).

[5] See, for example, the short-put strategy described in Lo (2000).

[6] See Sharpe (1987), Arnott and Fabozzi (1992) and Bodie *et al* (1999) for more detailed expositions of asset allocation.

the dynamics of financial innovation[7], the literature on dynamic portfolio choice with transactions costs, and the literature on option replication.

Among the many examples contained in Merton (1995) illustrating the importance of function in determining institutional structure is the example of the German government's issuance in 1990 of ten-year Schuldschein bonds with put-option provisions. Merton (1995) observes that the put provisions have the same effect as an interest-rate stabilization policy in which the government repurchases bonds when bond prices fall and sells bonds when bond prices rise. More importantly, Merton (1995) writes that '... the put bonds function as the equivalent of a dynamic, 'open market', trading operation without any need for actual transactions'. This automatic stabilization policy is a 'proof of concept' for the possibility of substituting a buy-and-hold portfolio for a particular dynamic investment strategy, and the optimal buy-and-hold portfolio of section 3 may be viewed as a generalization of Merton's automatic stabilization policy to the asset-allocation problem.

Magill and Constantinides (1976) were among the first to point out that in the presence of transactions costs, trading occurs only at discrete points in time. More recent studies by Davis and Norman (1990), Aiyagari and Gertler (1991), Heaton and Lucas (1992, 1996) and He and Modest (1995) have contributed to the growing consensus that trading costs have a significant impact on investment performance and, therefore, investor behaviour. Despite the recent popularity of internet-based day-trading, it is now widely accepted that buy-and-hold strategies such as indexation are difficult to beat—transactions costs and management fees can quickly dissipate the value-added of many dynamic asset-allocation strategies.

The option-replication literature is relevant to our paper primarily because of the correspondence between a complex security and a dynamic trading strategy in simpler securities, an insight which gave rise to this literature. The classic references are Black and Scholes (1973), Merton (1973), Cox and Ross (1976), Harrison and Kreps (1979), Duffie and Huang (1985), and Huang (1985a, b). More recently, several studies have considered the option-replication problem directly, in some cases using mean-squared error as the objective function to be minimized[8], and in other cases with transactions costs[9]. In the latter set of studies, the existence of transactions costs induces discrete trading intervals, and the optimal replication problem is solved for some special cases, e.g. call and put options on stocks with geometric Brownian motion or constant-elasticity-of-variance price dynamics, or for more general derivative securities under vector-Markov price processes.

We take these somewhat disparate literatures as our starting point. Merton's (1995) automatic stabilization policy illustrates the possibility of substituting a static buy-and-hold portfolio for a specific dynamic trading strategy, i.e. an interest-rate stabilization policy. The fact that trading is costly implies that continuous asset-allocation is not feasible, and that alternatives to frequent trading are important to investors. The technology for replicating options is clearly well established, and a natural generalization of that technology is to construct portfolios of options that replicate more general dynamic trading strategies. We begin developing this generalization in the next section.

[7] See also, Bodie and Merton (1995) and Merton (1997).

[8] See, for example, Duffie and Jackson (1990), Schweizer (1992, 1995, 1996), Schäl (1994), Delbaen and Schachermeyer (1996) and Bertsimas *et al* (2000a).

[9] See Leland (1985), Hodges and Neuberger (1989), Bensaid *et al* (1992), Boyle and Vorst (1992), Davis *et al* (1993), Edirisinghe *et al* (1993), Henrotte (1993), Avellaneda and Paras (1994), Neuberger (1994), Whalley and Wilmott (1994), Grannan and Swindle (1996) and Toft (1996).

3. The optimal buy-and-hold portfolio

The asset-allocation problem has become one of the classic problems of modern finance, thanks to Samuelson's (1969) and Merton's (1969) pioneering studies over three decades ago. The simplest formulation—one without intermediate consumption—consists of an investor's objective to maximize the expected utility $\mathrm{E}[U(W_T)]$ of end-of-period wealth W_T by allocating his wealth W_t between two assets, a risky security (the 'stock') and a riskless security (the 'bond'), over some investment horizon $[0, T]$. The bond is assumed to yield a riskless instantaneous return of $r\,\mathrm{d}t$ and with an initial market price of \$1, the bond price at any date t is simply $\exp(rt)$. The stock price is denoted by P_t and is typically assumed to satisfy an Itô stochastic differential equation:

$$\mathrm{d}P_t = \mu(P_t, t)\, P_t\, \mathrm{d}t + \sigma(P_t, t)\, P_t\, \mathrm{d}B_t \tag{3.1}$$

where B_t is standard Brownian motion and $\mu(P_t, t)$ and $\sigma(P_t, t)$ satisfy certain regularity conditions that ensure the existence of a solution to (3.1). The standard asset-allocation problem is then:

$$\mathrm{Max}_{\{\omega_t\}} \mathrm{E}[U(W_T)] \tag{3.2}$$

subject to

$$\mathrm{d}W_t = [r + \omega_t(\mu - r)]W_t\, \mathrm{d}t + \omega_t W_t \sigma\, \mathrm{d}B_t \tag{3.3}$$

where ω_t is the fraction of the investor's portfolio invested in the stock at time t and (3.3) is the budget constraint that wealth W_t must satisfy at all times $t \in [0, T]$.[10]

Denote by $\{\omega_t^*\}$ the optimal dynamic investment policy, i.e. the solution to (3.2) and (3.3), and let W_T^* denote the end-of-period wealth generated by the optimal policy. The question we wish to answer in this paper is: how close can we come to this optimal policy with a buy-and-hold portfolio of stocks, bonds and options? We measure closeness in three ways: a direct approach in which we maximize the expected utility of the buy-and-hold portfolio, and two indirect approaches in which we minimize the mean-squared error and weighted mean-squared error between W_T^* and the end-of-period wealth of the buy-and-hold portfolio. These three approaches are described in sections 3.1–3.3, respectively.

3.1. Maximizing expected utility

Our reformulation of the standard asset-allocation problem (3.2) and (3.3) contains only two modifications: (1) we allow the investor to include up to n European call options in his

[10] See Merton (1992, chapter 5) for details.

portfolio at date 0 which expire at date T;[11] and (2) we do not allow the investor to trade after setting up his initial portfolio of stocks, bonds and options. Specifically, denote by D_i the date-T payoff of a European call option with strike price equal to k_i, hence:

$$D_i = (P_T - k_i)^+. \tag{3.4}$$

Then the 'buy-and-hold' asset-allocation problem for the investor is given by:

$$\text{Max}_{\{a,b,c_i,k_i\}} \text{E}[U(V_T)] \tag{3.5}$$

subject to

$$V_T \equiv a \exp(rT) + b\, P_T + c_1 D_1 + c_2 D_2 + \cdots + c_n D_n \tag{3.6}$$
$$W_0 = \exp(-rT)\, \text{E}^Q[V_T] \tag{3.7}$$

where a and b denote the investor's position in bonds and stock, and $c_1, \ldots, c_n$ the number of options with strike prices $k_1, \ldots, k_n$, respectively. Note that we use V_T instead of W_T to denote the investor's end-of-period wealth to emphasize the distinction between this case and the standard asset-allocation problem in which stocks and bonds are the only assets considered and intermediate trading is allowed.

The budget constraint is given by (3.7), where $\text{E}^Q[\cdot]$ is the expectation operator under the equivalent martingale measure Q.[12] This constraint is highly nonlinear in the option strikes $\{k_i\}$, creating significant computational challenges for any optimizer. Moreover, for certain utility functions, it is necessary to impose solvency constraints to avoid bankruptcy, and such constraints add to the computational complexity of the problem.

For these reasons, our approach for solving (3.5)–(3.7) consists of two steps. In the first step, we assume that the strike prices $\{k_i\}$ are fixed, in which case (3.5)–(3.7) reduces to maximizing a concave objective function subject to linear constraints. Such a problem has a unique global optimum that is generally quite easy to find. This is done by discretizing the distribution of P_T and solving the Karush–Kuhn–Tucker conditions which, in this case, are sufficient for an optimal solution[13]. We will refer to this problem—where the strikes $\{k_i\}$ are fixed—as the 'subproblem'.

The second step involves determining the best set of strikes. We propose to solve this problem by specifying in advance a large number, $N \gg n$, of possible strikes where the N strikes are chosen to be representative of the distribution of P_T. We then solve the subproblem for each of the $\binom{N}{n}$ possible combinations of options and select the best combination.

In selecting the set of N strikes, we must ensure that their range spans a significant portion of the support of P_T. Therefore, the distribution of P_T must be taken into account in specifying the strikes. Given a distribution for P_T, we select an interval of its support and choose N points—spaced either evenly (for simplicity) or according to the probability mass of the distribution of P_T (for efficiency)—so that approximately 4 to 6 standard deviations of P_T are contained within the interval.

In solving each subproblem, we discretize the distribution of P_T. This yields a straightforward nonlinear optimization problem with a concave objective function and linear constraints, which can be solved relatively quickly.

One subtlety arises for CRRA utility: the function is not defined for negative wealth. In such cases, the following $n+2$ solvency constraints must be imposed along with the budget constraint to ensure non-negative wealth:

$$\begin{aligned}
0 &\leqslant a\exp(rT) \\
0 &\leqslant a\exp(rT) + bk_1 \\
0 &\leqslant a\exp(rT) + (b + c_1)k_2 - c_1k_1 \\
&\vdots \\
0 &\leqslant a\exp(rT) + (b + c_1 + \cdots + c_{n-1})k_n \\
&\quad -(c_1k_1 + \cdots + c_{n-1}k_{n-1}) \\
0 &\leqslant b + c_1 + \cdots + c_n \\
0 &\leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n.
\end{aligned} \tag{3.8}$$

3.2. Minimizing mean-squared error

In situations where the computational demands of the buy-and-hold asset-allocation problem of section 3.1 are too great, a less demanding alternative is to use mean-squared error as the metric for measuring the closeness of the end-of-period wealth V_T of the buy-and-hold portfolio of stocks, bonds, and options with the end-of-period wealth W_T^* of the optimal portfolio. In addition, for dynamic investment policies that are not derived from maximization of expected utility, e.g. dollar-cost averaging, a mean-squared-error objective function may be appropriate. In this case, the buy-and-hold portfolio problem becomes:

$$\text{Min}_{\{a,b,c_i,k_i\}} \text{E}[(W_T^* - V_T)^2] \tag{3.9}$$

subject to

$$V_T \equiv a \exp(rT) + b\, P_T + c_1 D_1 + c_2 D_2 + \cdots + c_n D_n \tag{3.10}$$
$$W_0 = \exp(-rT)\, \text{E}^Q[V_T] \tag{3.11}$$

If W_T^* depends only on the terminal stock price P_T and not on any of its path $\{P_t\}$—as is the case when $\{P_t\}$ follows a geometric Brownian motion and W_T^* is the end-of-period wealth from an optimization of an investor's expected utility—it can be shown that V_T can be made arbitrarily close to W_T^* in mean-square as the number of options n in the buy-and-hold portfolio increases without bound. If we do not impose any additional constraints beyond the budget constraint (such as the solvency constraints (3.8) of section 3.1), the corresponding subproblems for (3.9)–(3.11) can be solved very quickly, and the first-order conditions, which are necessary and sufficient, merely amount to solving a series of linear equations.

Specifically, the subproblem associated with (3.9)–(3.11) consists of selecting portfolio weights for stocks, bonds and options to minimize the mean-squared error between W_T^* and V_T, holding fixed the strike prices $\{k_i\}$ of the n options available

[11] Without loss of generality, we focus exclusively on call options for expositional simplicity. Parallel results for put options can be easily derived via put-call parity (see, for example, Cox and Rubinstein (1985)).

[12] Note that specifying Q yields pricing formulae for all the options contained in our optimal buy-and-hold portfolio since $\exp(-rT)\text{E}^Q[D_i]$ is the date-0 price of option i. Therefore, option-pricing formulae are implicit in (3.7). For example, it is easy to verify that under geometric Brownian motion, $\exp(-rT)\text{E}^Q[D_i]$ reduces to the celebrated Black–Scholes formula.

[13] See, for example, Bertsekas (1999).

to the investor. It is clear from (3.9)–(3.11) that for fixed strike prices, the objective function is convex so the first-order conditions are sufficient to characterize an optimal solution. These conditions may be written as

$$\begin{bmatrix} \exp(rT) & \mathrm{E}[P_T] & \mathrm{E}[D_1] & \cdots & \mathrm{E}[D_n] & \exp(-rT) \\ \exp(rT)\mathrm{E}[P_T] & \mathrm{E}\left[P_T^2\right] & \mathrm{E}[D_1 P_T] & \cdots & \mathrm{E}[D_n P_T] & P_0 \\ \exp(rT)\mathrm{E}[D_1] & \mathrm{E}[P_T D_1] & \mathrm{E}\left[D_1^2\right] & \cdots & \mathrm{E}[D_n D_1] & \exp(-rT)\mathrm{E}^Q[D_1] \\ \exp(rT)\mathrm{E}[D_2] & \mathrm{E}[P_T D_2] & \mathrm{E}[D_1 D_2] & \cdots & \mathrm{E}[D_n D_2] & \exp(-rT)\mathrm{E}^Q[D_2] \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \exp(rT)\mathrm{E}[D_n] & \mathrm{E}[P_T D_n] & \mathrm{E}[D_1 D_n] & \cdots & \mathrm{E}\left[D_n^2\right] & \mathrm{E}\left[D_n^2\right] \\ \exp(rT) & \exp(rT)P_0 & \mathrm{E}^Q[D_1] & \cdots & \mathrm{E}^Q[D_n] & 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c_1 \\ c_2 \\ \vdots \\ c_n \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathrm{E}\left[W_T^*\right] \\ \mathrm{E}\left[W_T^* P_T\right] \\ \mathrm{E}\left[W_T^* D_1\right] \\ \mathrm{E}\left[W_T^* D_2\right] \\ \vdots \\ \mathrm{E}\left[W_T^* D_n\right] \\ \exp(rt)W_0 \end{bmatrix} \tag{3.12}$$

or, in matrix notation:

$$\boldsymbol{\Sigma}\,\boldsymbol{\eta} = \boldsymbol{\epsilon} \tag{3.13}$$

where λ is the Lagrange multiplier corresponding to the budget equation.

Inverting (3.13) to compute

$$\hat{\boldsymbol{\eta}} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\epsilon} \tag{3.14}$$

and then substituting $\hat{\boldsymbol{\eta}} \equiv [\hat{a}\ \hat{b}\ \hat{c}_1\ \cdots\ \hat{c}_n \hat{\lambda}]'$ into the objective function (3.9) yields the optimal value for a given subproblem. Repeating this procedure for all $\binom{N}{n}$ subproblems and selecting the best of these solutions gives an approximate solution to (3.9)–(3.11).

However, for some utility functions, it is necessary to impose the solvency constraints (3.8), in which case the solution to the subproblem cannot be simplified according to (3.14).

3.3. Minimizing weighted mean-squared error

A third alternative to the two approaches outlined in sections 3.1 and 3.2 is to maximize expected utility but where we substitute an approximation for the utility function. This yields a *weighted* mean-squared-error objective function where the weighting function is the second derivative of the utility function evaluated at the optimal end-of-period wealth W_T^*. This is a hybrid of the two approaches proposed above that provides important economic motivation for mean-squared error, and approximates the direct approach of maximizing expected utility described in section 3.1.

Specifically, consider the subproblem of section 3.1 in which we maximize expected utility holding fixed the strike prices $\{k_i\}$:

$$\mathrm{Max}_{\{a,b,c_i\}}\mathrm{E}[U(V_T)]$$

subject to the budget (3.7) and solvency constraints (3.8). Take a Taylor expansion of $U(W_T^* \pm \lambda(W_T^* - V_T))$ about the global optimal W_T^*:

$$\begin{aligned} \mathrm{E}[U(W_T^* \pm \lambda(W_T^* - V_T))] \\ &\approx \mathrm{E}[U(W_T^*)] \pm \lambda \mathrm{E}[(W_T^* - V_T)U'(W_T^*)] \\ &\quad + \frac{\lambda^2}{2}\mathrm{E}[(W_T^* - V_T)^2 U''(W_T^*)]. \end{aligned} \tag{3.15}$$

If V_T were 'budget feasible', by which we mean that $\exp(-rT)\mathrm{E}^Q[V_T] = W_0$, and V_T were sufficiently close to W_T^*, then this implies that $\pm\lambda(W_T^* - V_T)$ is a feasible direction of travel from W_T^*. For sufficiently small λ, (3.15) implies that

$$\mathrm{E}[(W_T^* - V_T)U''(W_T^*)] = 0$$

under certain regularity conditions. Therefore, maximizing $\mathrm{E}[U(V_T)]$ should be equivalent to maximizing

$$\tfrac{1}{2}\mathrm{E}[(W_T^* - V_T)^2\, U''(W_T^*)] \tag{3.16}$$

for V_T sufficiently close to W_T^*. This gives rise to a third approach to the buy-and-hold asset-allocation problem, one that involves approximating W_T^* in mean-square rather than explicitly maximizing expected utility:

$$\mathrm{Min}_{\{a,b,c_i,k_i\}}\mathrm{E}[-U''(W_T^*)(W_T^* - V_T)^2] \tag{3.17}$$

subject to

$$V_T \equiv a\ \exp(rT) + b\,P_T + c_1 D_1 + c_2 D_2 + \cdots + c_n D_n \tag{3.18}$$

$$W_0 = \exp(-rT)\,\mathrm{E}^Q[V_T]. \tag{3.19}$$

For CRRA utility, we still need to impose solvency constraints, but even with such constraints we can solve the subproblem much more quickly in the weighted mean-squared error case than in the maximization of expected utility proposed in section 3.1. Indeed, the computational challenges for the weighted mean-squared error approach are comparable to the mean-squared error approach of section 3.2.

A potential difficulty with the utility-weighted mean-squared-error approach is that some of the expectations in (3.17) may not be defined. Even when the expectations are defined, it is possible that some of them are very difficult to compute when they are ill-conditioned, i.e. 'close' to being undefined. In such cases the approach either will not work or will be very difficult to implement. This typically occurs for low values of relative risk aversion. Fortunately, it is precisely for low values of risk aversion that a direct maximization of expected utility works best. The reason is that the discretization of the support of P_T leads to approximation errors that can be extreme for high values of risk aversion. In particular, the discretized distribution has finite support, hence the optimal buy-and-hold strategy obtained with this distribution may perform poorly outside this finite support. The power-law specification of CRRA preferences will magnify small approximation errors of this type when the risk-aversion parameter is large.

Therefore, the maximization of expected utility and the minimization of utility-weighted mean-squared-error complement each other. As we will see in section 5, when both approaches work well, they result in almost identical portfolios and certainty equivalents. Therefore, in the numerical examples of section 5, we will maximize expected utility for low values of relative risk aversion and minimize utility-weighted mean-squared-error for higher values when computing the utility-optimal buy-and-hold portfolios.

4. Three leading cases

To derive the optimal buy-and-hold portfolios according to the three criteria of section 3, we require a few auxiliary results that depend on the specific utility function of the investor and the stochastic process for stock prices. In this section, we derive these results for CRRA and CARA utility under three leading cases for the stock-price process: geometric Brownian motion (section 4.1), the trending Ornstein–Uhlenbeck process (section 4.2), and a bivariate linear diffusion process with a stochastic mean-reverting drift (section 4.3).

In the case of geometric Brownian motion, the required results are straightforward—we are able to characterize W_T^* explicitly for both CRRA and CARA preferences, and all three approaches to the optimal buy-and-hold portfolio can be readily implemented. However, for the other two stochastic processes, the optimal dynamic asset-allocation strategies are path dependent, which implies that no buy-and-hold portfolio of stocks, bonds and European call options can ever achieve the same certainty equivalents as the optimal dynamic strategies. In such situations, we propose an alternative to W_T^* as a target for the optimal buy-and-hold portfolio, and derive this alternative explicitly in sections 4.2 and 4.3.

4.1. Geometric Brownian motion

In the case of geometric Brownian motion, the stock price P_t satisfies the following stochastic differential equation (SDE):

$$\mathrm{d}P_t = \mu P_t\,\mathrm{d}t + \sigma P_t\,\mathrm{d}B_t \tag{4.1}$$

where B_t is a standard Brownian motion. Recall that the standard asset-allocation problem in the absence of derivatives is given by (3.2) and (3.3):

$$\mathrm{Max}_{\{\omega_t\}}\mathrm{E}[U(W_T)]$$

subject to the budget equation

$$\mathrm{d}W_t = [r + \omega_t(\mu - r)]W_t\,\mathrm{d}t + \omega_t W_t \sigma\,\mathrm{d}B_t$$

where ω_t is the fraction of the investor's portfolio invested in the stock at time t (see Merton (1969, 1971) for a more detailed exposition). For concreteness, we consider two specific utility functions: constant absolute risk-aversion (CARA) and constant relative risk-aversion (CRRA) utility. These are well-known utility functions for which there are closed-form solutions to the standard asset-allocation problem. In particular, for CRRA utility, we have:

$$U(W_T) = \frac{W_T^\gamma}{\gamma} \tag{4.2}$$

$$W_T^* = W_0 \exp\left(rT - \frac{\xi^2 T\,(2\gamma - 1)}{2\,(1-\gamma)^2} + \frac{\xi B_T}{(1-\gamma)}\right) \tag{4.3}$$

$$\omega_t^* = \frac{\mu - r}{(1-\gamma)\,\sigma^2} \tag{4.4}$$

and for CARA utility,

$$U(W_T) = -\frac{\exp(-\gamma W_T)}{\gamma} \tag{4.5}$$

$$W_T^* = \frac{\gamma W_0 \exp(rT) + \xi^2 T + \xi B_T}{\gamma}, \qquad \xi \equiv \frac{\mu - r}{\sigma} \tag{4.6}$$

$$\omega_t^* = \frac{\exp(-r(T-t))\xi}{\gamma \sigma W_t}. \tag{4.7}$$

Given these closed-form solutions, we can make explicit comparisons of the optimal buy-and-hold portfolio of stocks, bonds and options with the standard optimal asset-allocation strategies for the two utility functions.

4.2. The Ornstein–Uhlenbeck process

If stock prices are predictable to some degree, the asset-allocation problem becomes considerably more challenging since the optimal investment strategy is path-dependent. This implies that of W_T^* is also path-dependent and very difficult to compute explicitly, hence the mean-squared-error approaches of sections 3.2 and 3.3 are not feasible. However, in certain cases, it is possible to derive an upper bound on the certainty equivalent of the optimal buy-and-hold portfolio of stocks, bonds and options, which provides some indication of the benefits of options in replicating dynamic investment strategies. We present such an upper bound in this section for the case where log-prices $X_t \equiv \log P_t$ follow a trending Ornstein–Uhlenbeck process[14]:

$$\mathrm{d}X_t = [-\delta\,(X_t - \mu t - X_0) + \mu]\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \qquad \delta > 0. \tag{4.8}$$

which has the solution:

$$X_t = X_0 + \mu t + \sigma \exp(-\delta t) \int_0^t \exp(\delta s)\,\mathrm{d}B_s. \tag{4.9}$$

The solution to the standard asset-allocation problem (3.2) and (3.3) in this case is characterized by the following Hamilton–Jacobi–Bellman equation:

$$\begin{aligned} 0 = \mathrm{Max}_{\omega_t}\Big\{ & J_t + W_t J_W\big(r + \omega_t[-\delta(X_t - \mu t - X_0) + \mu \\ & + \tfrac{1}{2}\sigma^2 - r]\big) + J_X(-\delta(X_t - \mu t - X_0) + \mu) \\ & + \tfrac{1}{2}\omega_t^2\sigma^2 W_t^2 J_{WW} + \tfrac{1}{2}\sigma^2 J_{XX} + \sigma^2\omega_t W_t J_{XW}\Big\} \end{aligned} \tag{4.10}$$

where

$$J(W_t, X_t, t) \equiv \mathrm{Max}_{\omega_t}\mathrm{E}_t[U(W_T)]. \tag{4.11}$$

The solutions to (4.10) for CRRA and CARA utility are given in the appendix.

Because W_T^* is path-dependent in this case, even if we allow the number of options n in the buy-and-hold portfolio to increase without bound, the certainty equivalent of the buy-and-hold portfolio will never approach the certainty equivalent of W_T^*. However, an upper bound on the certainty equivalent of any buy-and-hold portfolio can be derived by allowing the investor to purchase an unlimited number of options at all

[14] See Lo and Wang (1995) for a more detailed exposition of its properties. We also derive results for the standard Ornstein–Uhlenbeck process (without trend), which are included in the appendix.

possible strike prices. The certainty equivalent of the end-of-period wealth in this case, which we denote by V_T^∞, is clearly an upper bound for any buy-and-hold portfolio containing a finite number n of options.

To derive V_T^∞, we require the conditional state-price density of the terminal stock price P_T, defined as:

$$\pi_T^b \equiv \mathrm{E}\left[\pi_T | P_T = b\right] \tag{4.12}$$

where π_T is the unconditional state-price density of the terminal stock price[15]. The economic interpretation of π_T^b is the price per unit probability of 1 unit of wealth at time T in the event that $P_T = b$. By definition, π_T^b is given by:

$$\pi_T^b = \mathrm{E}\left[\pi_T | P_T = b\right] = \frac{\mathrm{E}\left[\pi_T 1_{\{P_T=b\}}\right]}{\mathrm{E}\left[1_{\{P_T=b\}}\right]}. \tag{4.13}$$

The numerator of (4.13) is computed by applying Girsanov's theorem and noting that the Radon–Nikodym derivative $\mathrm{d}Q/\mathrm{d}F$ of the equivalent martingale measure Q with respect to the true probability measure F is equal to $\exp(rT)\pi_T$. Under Q, the stock price at time T is given by

$$P_T^Q = \exp\left(Z_T\right) \equiv P_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma \widehat{B_T}\right). \tag{4.14}$$

where $\widehat{B_T}$ is a standard Brownian motion under Q. Under the true probability measure, F, recall that the stock price at time T is given by

$$P_T = \exp\left(X_T\right) \equiv \exp\left(X_0 + \mu T + \sigma \mathrm{e}^{-\delta T} \int_0^T \mathrm{e}^{\delta s} \mathrm{d}B_s\right). \tag{4.15}$$

With this in mind, we can write (4.13) as

$$\pi_T^b = \frac{\exp(-rT) f_{P_T}^Q(b)}{f_{P_T}(b)} \tag{4.16}$$

where f_{P_T} and $f_{P_T}^Q$ denote the log-normal density functions of P_T under F and Q respectively. Simplifying (4.16) yields:

$$\pi_T^b = \left(\frac{\sigma_x}{\sigma_z}\right) \exp\left(-rT - \frac{1}{2}\left[\left(\frac{\log b - \mu_z}{\sigma_z}\right)^2 - \left(\frac{\log b - \mu_x}{\sigma_x}\right)^2\right]\right) \tag{4.17}$$

where

$$\mu_x = X_0 + \mu T, \qquad \sigma_x^2 = \frac{\sigma^2}{2\delta}\left(1 - \exp(-2\delta T)\right)$$
$$\mu_z = X_0 + \left(r - \frac{\sigma^2}{2}\right) T, \qquad \sigma_z^2 = \sigma^2 T. \tag{4.18}$$

Using π_T^b as the state-price density process, we can derive the optimal buy-and-hold portfolio in which options of all possible strikes may be included. Using the approach proposed in Cox and Huang (1989) for the case of CRRA utility, the problem reduces to:

$$\max \mathrm{E}\left[\frac{(V_T)^\gamma}{\gamma}\right] \qquad \text{subject to} \qquad \mathrm{E}\left[\pi_T^b V_T\right] = W_0 \tag{4.19}$$

which has the solution:

$$V_T^\infty = \frac{W_0 \left(\pi_T^b\right)^{\frac{1}{\gamma-1}}}{\mathrm{E}\left[\left(\pi_T^b\right)^{\frac{\gamma}{\gamma-1}}\right]} \tag{4.20}$$

where

$$\mathrm{E}\left[\left(\pi_T^b\right)^{\frac{\gamma}{\gamma-1}}\right] = \frac{\sigma_o}{\sigma_x}\left(\frac{\sigma_x}{\sigma_z}\right)^{\frac{\gamma}{\gamma-1}} \exp\left(\frac{-rT\gamma}{\gamma-1} + \frac{\gamma\left[\mu_x - \mu_z\right]^2}{2\left[\gamma-1\right]\left[\gamma\sigma_x^2 - \sigma_z^2\right]}\right)$$

and

$$\sigma_o^2 = \frac{\sigma_x^2\sigma_z^2(\gamma - 1)}{\left(\gamma\sigma_x^2 - \sigma_z^2\right)}. \tag{4.21}$$

This, in turn, implies:

$$U^\infty \equiv \mathrm{E}\left[\frac{\left(V_T^\infty\right)^\gamma}{\gamma}\right] = \frac{W_0^\gamma}{\gamma} \mathrm{E}\left[\left(\pi_T^b\right)^{\frac{\gamma}{\gamma-1}}\right]^{1-\gamma}$$
$$\mathrm{CE}(V_T^\infty) = \left(\gamma U^\infty\right)^{\frac{1}{\gamma}}$$

where $\mathrm{CE}(\cdot)$ denotes the certainty equivalent operator.

The case of CARA utility can also be handled in a similar manner.

Having solved for the optimal buy-and-hold portfolio and its certainty equivalent in the infinite options case, we can now compare this upper bound to the optimal buy-and-hold portfolios with a finite number of options. We use the same method as in the geometric Brownian motion case (see section 4.1), hence we omit the details.

4.3. A bivariate linear diffusion process

We now turn to a third set of price dynamics for P_t, one in which there are two sources of uncertainty, implying that markets are incomplete. Nevertheless, we are still able to compute optimal buy-and-hold portfolios of stocks, bonds and options, and can also derive the upper bound to the buy-and-hold certainty equivalents as in section 4.2. Specifically, let $X_t \equiv \log P_t$ satisfy the following bivariate linear diffusion process:

$$\mathrm{d}X_t = \left(\mu_t - \frac{\sigma_1^2}{2}\right)\mathrm{d}t + \sigma_1 \mathrm{d}B_{1t} \tag{4.22}$$
$$\mathrm{d}\mu_t = \kappa\left(\theta - \mu_t\right)\mathrm{d}t + \sigma_2 \mathrm{d}B_{2t} \tag{4.23}$$

where B_{1t} and B_{2t} are two standard Brownian motions with instantaneous correlation coefficient ρ. Kim and Omberg (1993, 1996) derive the optimal value function for the standard asset-allocation problem with these price dynamics for an investor with CARA utility. Despite the fact that markets are incomplete, it is clear that options can be replicated using

[15] See Duffie (1996) for a more detailed exposition of state-price densities.

trading strategies in only the stock and the bond[16], hence options can be priced by arbitrage in this case. Therefore, we can perform the same analysis for these dynamics as we did for geometric Brownian motion in section 4.1 and the Ornstein–Uhlenbeck process in section 4.2.

To derive V_T^∞ for the bivariate diffusion (4.22) and (4.23), we perform a similar set of calculations as in section 4.2. We begin by solving (4.22) and observing that P_T is log-normally distributed with parameters:

$$\mu_X = X_0 + (\theta - \frac{\sigma_1^2}{2})T + \frac{\theta - \mu_o}{\kappa}(\exp(-\kappa T) - 1) \tag{4.24}$$

$$\sigma_x^2 = \sigma_1^2 T + \frac{2\sigma_1\sigma_2\rho}{\kappa}\left[T + \frac{\exp(-\kappa T)}{\kappa} - \frac{1}{\kappa}\right] + \frac{\sigma_2^2}{\kappa^3}\left[T\kappa - \frac{3}{2} + 2\exp(-\kappa T) - \frac{\exp(-2\kappa T)}{2}\right]. \tag{4.25}$$

The conditional state-price density then follows in the same manner as (4.17):

$$\pi_T^b = \left(\frac{\sigma_x}{\sigma_z}\right)\exp\left(-rT - \frac{1}{2}\left[\left(\frac{\log b - \mu_z}{\sigma_z}\right)^2 - \left(\frac{\log b - \mu_x}{\sigma_x}\right)^2\right]\right) \tag{4.26}$$

where

$$\mu_z = X_0 + \left(r - \frac{\sigma^2}{2}\right)T, \qquad \sigma_z^2 = \sigma^2 T \tag{4.27}$$

With the conditional state-price density in hand, V_T^∞ and its certainty equivalent are readily derived.

5. Numerical results

To illustrate the practical relevance of our optimal buy-and-hold portfolio, we provide numerical results in this section for CRRA preferences under each of the three stochastic processes of section 4 using the nonlinear programming solver LOQO and the algebraic mathematical programming language AMPL[17]. Before turning to those results, we begin with a simple example to motivate our analysis. Let stock prices follow geometric Brownian motion (4.1) and set

$$\begin{aligned} U(W_T) &= \frac{W_T^\gamma}{\gamma}, & \gamma &= -4, \\ W_0 &= \$100\,000, & T &= 20 \text{ years}, \\ P_0 &= \$50, & r &= 0.05, \\ \mu &= 0.15, & \sigma &= 0.20 \end{aligned}$$

which implies a relative risk-aversion coefficient of 5, a portfolio weight ω_t^* of 50% for the stock in the optimal dynamic asset-allocation policy (4.4), and a certainty equivalent of \$448 169 for W_T^*. Now consider the problem of constructing an optimal buy-and-hold portfolio containing stocks, bonds, and a maximum of two options, assuming that there are only four possible options to choose from, with the following strikes:

$$k_1 = \$176, \quad k_2 = \$976, \quad k_3 = \$1775, \quad k_4 = \$2575.$$

For the approach outlined in section 3.1, we maximize the expected utility:

$$\mathrm{Max}_{\{a,b,c_i,k_i\}} \mathrm{E}[U(V_T)]$$

subject to

$$V_T \equiv a\,\exp(rT) + b\,P_T + c_1 D_1 + c_2 D_2$$
$$W_0 = \exp(-rT)\,\mathrm{E}^Q[V_T]$$

and the corresponding solvency constraints. We discretize the support of P_T using a grid of 4000 points, chosen in such a way that the weight associated with each point in the objective function is equal to 1/4000. A direct optimization then yields the following certainty equivalents for subproblems of the optimal buy-and-hold problem for the various combinations of strikes:

Options Used:	CE(V_T^*):
1 and 2	\$447 307
1 and 3	\$447 137
1 and 4	\$447 067
2 and 3	\$437 971
2 and 4	\$437 850
3 and 4	\$436 506

(5.1)

From (5.1), it is apparent that the optimal buy-and-hold strategy is to use options with strikes $k_1 = 176$ and $k_2 = 976$, and the optimal portfolio positions are:

$$a^* = \$36\,097, \quad b^* = 1521, \quad c_1^* = -907, \quad c_2^* = -353. \tag{5.2}$$

With only two options, the optimal buy-and-hold portfolio yields an estimated certainty equivalent of \$447 307[18], which is 99.8% of the certainty equivalent of the optimal dynamic asset-allocation strategy, a strategy that requires continuous trading over a 20-year period!

Note that the portfolio weights implied by the positions (5.2) are 36.1% in bonds, 76.1% in stocks and −12.2% in options. The optimal buy-and-hold portfolio consists of shorting options 1 and 2, and investing the proceeds—approximately \$12 100—in stocks and bonds along with the initial wealth of \$100 000.

Alternatively, we can minimize the mean-squared error between V_T and W_T^* according to section 3.2:

$$\mathrm{Min}_{\{a,b,c_i,k_i\}} \mathrm{E}[(W_T^* - V_T)^2]$$

subject to

$$V_T \equiv a\,\exp(rT) + b\,P_T + c_1 D_1 + c_2 D_2$$
$$W_0 = \exp(-rT)\,\mathrm{E}^Q[V_T]$$

[16] For further discussion, see Lo and Wang (1995).

[17] AMPL is described in Fourer *et al* (1999). Information on LOQO can be obtained from http://www.princeton.edu/ loqo/.

[18] The estimation error is due to the discretization of the distribution of P_T. Once we obtain the strategy (5.2), we can compute the certainty equivalent exactly, and in this case, it is \$446 034, which is 99.5% of the certainty equivalent of the optimal dynamic asset-allocation strategy.

and also subject to the solvency constraints (3.8). The root-mean-squared-error (RMSE) (as a percentage of $\mathrm{E}[W_T^*]$) of each of the subproblems is given by:

Options Used:	RMSE (%):
1 and 2	6.27
1 and 3	4.73
1 and 4	5.69
2 and 3	6.47
2 and 4	5.75
3 and 4	9.95

Under the mean-squared-error criterion, the optimal buy-and-hold portfolio consists of a different set of options than under the expected-utility criterion—in this case, options 1 and 3—and the optimal positions are:

$$a^* = \$20\,928, \quad b^* = 1980, \quad c_1^* = -1508, \quad c_2^* = -291. \tag{5.3}$$

With such a buy-and-hold portfolio, the root-mean-squared-error is 4.73% of the expected value of W_T^*, and the certainty equivalent of this portfolio is \$436 034, which is 97.3% of the certainty equivalent of the optimal dynamic asset-allocation strategy. Despite the fact that (5.3) is only an indirect method of approximating W_T^*, the certainty equivalent is almost identical to that of the optimal dynamic strategy. The portfolio weights corresponding to (5.3) are 20.9% in bonds, 99.0% in stocks and −19.9% in options.

Finally, if we minimize the weighted mean-squared-error according to section 3.3,

$$\mathrm{Min}_{\{a,b,c_i,k_i\}} \mathrm{E}[-U''(W_T^*)(W_T^* - V_T)^2]$$

subject to

$$V_T \equiv a\ \exp(rT) + b\,P_T + c_1 D_1 + c_2 D_2$$
$$W_0 = \exp(-rT)\,\mathrm{E}^Q[V_T]$$

and the solvency constraints (3.8), we obtain the following weighted RMSEs for the various subproblems:

Options Used:	Weighted RMSE:
1 and 2	0.738
1 and 3	0.764
1 and 4	0.777
2 and 3	1.830
2 and 4	1.839
3 and 4	2.013

which yields an optimal buy-and-hold portfolio containing options 1 and 2 and positions:

$$a^* = \$35\,321, \quad b^* = 1523, \quad c_1^* = -930, \quad c_2^* = -349. \tag{5.4}$$

Although the weighted RMSE of the optimal buy-and-hold portfolio, 0.738, is somewhat difficult to interpret, the certainty of equivalent of the portfolio is \$445 967 which is 99.5% of the certainty equivalent of the optimal dynamic asset-allocation strategy. With portfolio weights of 35.3% in bonds, 76.2% in stocks and −11.5% in options, the minimum utility-weighted mean-squared-error approach yields an almost-identical solution to the maximum expected-utility approach (recall that the portfolio weights of the latter are 36.1% in bonds, 76.1% in stocks and −12.2% in options). Therefore, the hybrid approach provides an excellent approximation to the maximization of expected utility.

In sections 5.1–5.3, we perform more computationally intense optimizations for the three stochastic processes of section 4 under CRRA preferences using the three approaches described in section 3: maximizing expected utility, and minimizing mean-squared error and weighted mean-squared error. In particular, for each stochastic process, we compute two optimal buy-and-hold portfolios for each of six different values of the relative risk aversion coefficient (RRA = 1, 2, 5, 10, 15, 20): a utility-optimal buy-and-hold portfolio obtained by either direct maximization of expected utility or minimization of utility-weighted mean-squared error (as in sections 3.1 and 3.3, respectively), and a mean-square-optimal buy-and-hold portfolio (as in section 3.2). For each stochastic process and each value of the relative risk-aversion coefficient, we consider $N = 45$ possible strike prices and up to $n = 3$ options for the utility-optimal buy-and-hold portfolios and up to $n = 5$ options for the mean-square-optimal buy-and-hold portfolios. This yields up to $\binom{45}{3} = 14\,190$ and $\binom{45}{5} = 1221\,759$ subproblems for each of the two optimizations, respectively.

The strikes are selected in the following way. Letting μ_x and σ_x denote the mean and variance of $X_T \equiv \log P_T$, we partition the interval $[\,\exp(\mu_x - 3\sigma_x)\,,\ \exp(\mu_x + 3\sigma_x)\,]$ into 45 evenly spaced points which we denote by $s_1 \equiv \exp(\mu_x - 3\sigma_x), \ldots, s_{45} \equiv \exp(\mu_x + 3\sigma_x)$. We then use these points as our strikes, $k_i = s_i$, $i = 1, \ldots, 45$. Such a procedure for choosing the set of strikes $\{k_i\}$ is simple to implement, however, more sophisticated methods can be employed to improve the performance of the overall optimization process.

To facilitate comparisons across different optimal buy-and-hold portfolios we use one set of 45 strikes for each of the three stochastic processes considered in sections 5.1–5.3, i.e. for each stochastic process, we construct one set of 45 strikes and keep these fixed as we vary the values of relative risk aversion and the number of options n in the buy-and-hold portfolio. This is clearly suboptimal—for example, when $n = 1$, we can optimize the buy-and-hold portfolio over several thousand possible strike prices very quickly—but holding the strikes fixed allows us to gauge the impact of other parameters such as the risk-aversion coefficient and the number of options on the objective function being optimized. In practical applications, the set of possible strikes should be optimized for each specification of the buy-and-hold problem; in our limited experience, simple heuristics for optimizing the set of strikes can lead to substantial improvements in overall performance.

For each of the three cases considered in sections 5.1–5.3, we maintain the following set of assumptions:

$$\begin{aligned} &U(W_T) = \frac{W_T^\gamma}{\gamma} \\ &\gamma = 0, -1, -4, -9, -14, -19 \\ &W_0 = \$100\,000 \\ &T = 20 \text{ years} \\ &P_0 = \$50 \\ &r = 0.05 \\ &\mathrm{E}[\log(P_t/P_{t-1})] = 0.15 \\ &\mathrm{Var}[\log(P_t/P_{t-1})] = 0.20^2 \end{aligned} \tag{5.5}$$

where the values of γ correspond to relative risk-aversion coefficients of 1, 2, 5, 10, 15, and 20, respectively.

5.1. Geometric Brownian motion

For geometric Brownian motion (4.1), we set the parameters (μ, σ) to match the mean and variance of continuously compounded returns specified in (5.5). Based on our algorithm for constructing the set of strike prices from the distribution of $\log P_T$, we select the strikes for our n options from among the following 45 possibilities (in dollars):

69	401	733	1 066	1 398
1 731	2 063	2 396	2 728	3 061
3 393	3 725	4 058	4 390	4 723
5 055	5 388	5 720	6 052	6 385
6 717	7 050	7 382	7 715	8 047
8 379	8 712	9 044	9 377	9 709
10 042	10 374	10 706	11 039	11 371
11 704	12 036	12 369	12 701	13 033
13 366	13 698	14 031	14 363	14 696

Utility-optimal buy-and-hold portfolios. Table 1 reports the utility-optimal buy-and-hold portfolios for various levels of risk aversion and, for each risk-aversion parameter, for the number of options n varying from 0 to 3. For example, the first panel of table 1 contains results for the log-utility case ($\gamma = 0$, or RRA = 1). This is a very low level of risk aversion—by most empirical and experimental accounts, an unrealistically low level—and implies that the investor's objective is to maximize the expected geometric average rate of return of his portfolio. Examples of investors with such preferences are proprietary traders and hedge-fund managers. The results for the RRA = 1 panel were obtained by maximizing expected utility directly using a discretized distribution for P_T (see section 3.1). The results for the remaining five panels of table 1 were obtained by minimizing the utility-weighted mean-squared error (see section 3.3).

The first row of table 1's first panel corresponds to the optimal buy-and-hold portfolio with no options ($n = 0$)—for log-utility, the optimal portfolio is to put 100% of the investor's wealth into the stock[19]. Not surprisingly, the certainty equivalent of such a strategy is only 20.2% of the certainty equivalent of the optimal dynamic strategy $\mathrm{CE}(W_T^*)$.

[19] In fact, in the absence of solvency constraints, the optimal portfolio weight for the stock would be much greater than 100%, i.e. for CRRA preferences, the solvency constraints are binding.

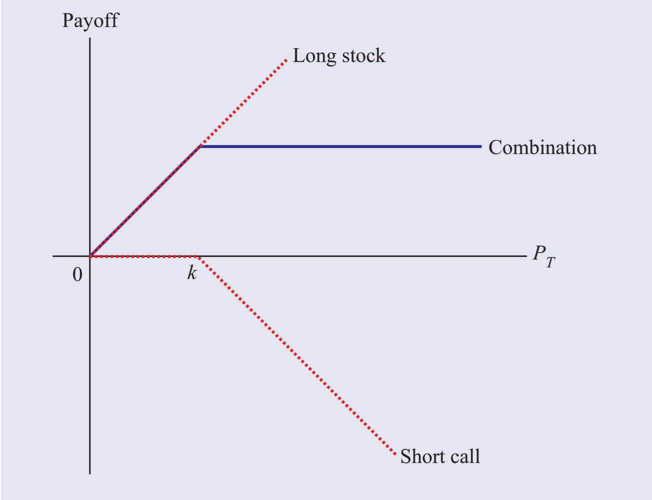

Figure 1. Payoff diagram of hedged position (long stock and short call).

By not allowing the investor to trade at all over the 20-year period, and without access to any options, the investor's welfare is reduced by approximately 80%. As the number of options is increased, his welfare increases so that for $n = 3$ options the certainty equivalent of the optimal buy-and-hold is 92.2% of $\mathrm{CE}(W_T^*)$.

For log utility, it is interesting to note that the RMSE is approximately 3 650% even for $n = 3$ and despite the fact that the certainty equivalent of the optimal buy-and-hold portfolio is close to that of the optimal dynamic investment policy. This, and the very slow rate at which the RMSE decreases as we increase n from 0 to 3, suggests that it may be possible to obtain an excellent approximation to the optimal dynamic strategy—in terms of expected utility—without being able to approximate W_T^* very well in mean-square.

Note that within each relative risk-aversion panel of table 1, the RMSEs decrease monotonically as the number of options n increases from 0 to 3. This, of course, need not be the case since we are maximizing expected utility, not minimizing RMSE. In fact, it is quite possible for the RMSE to increase as we increase n. However, the fact that they do decrease monotonically suggests that there is some correlation between smaller RMSE and a more preferred buy-and-hold portfolio. Of course, as n becomes arbitrarily large, the RMSE must converge to 0.

Perhaps the most interesting feature of table 1 is how the results fall naturally into two distinct groups. The first group consists of the first two panels, corresponding to investors who are not very risk averse (relative risk-aversion coefficients of 1 and 2, respectively) and who, in the standard dynamic asset-allocation framework, would optimally hold more than 100% of their wealth in the risky asset. In a buy-and-hold portfolio without options ($n = 0$), these investors are bound by the solvency constraints (3.8), making it difficult for them to approximate $\mathrm{CE}(W_T^*)$ very well (the certainty equivalents $\mathrm{CE}(V_T^*)$ of the optimal buy-and-hold portfolios are only 20.2% of $\mathrm{CE}(W_T^*)$ for the log-utility investor and 81.9% for the investor with RRA = 2). Options are of particular benefit to these investors, who purchase call options so that they can

Table 1. Utility-optimal buy-and-hold portfolios of stocks, bonds and n European call options for CRRA utility under geometric Brownian motion stock-price dynamics with parameters (μ, σ) calibrated to match the following moments: $E[\log(P_t/P_{t-1})] = 0.15$, $\text{Var}[\log(P_t/P_{t-1})] = 0.04$. Other calibrated parameters include: riskless rate $r = 5\%$, initial stock price $P_0 = \$50$, initial wealth $W_0 = \$100\,000$, and time period $T = 20$ years. 'RRA' denotes the coefficient of relative risk aversion, '$CE(W_T^*)$' denotes the certainty equivalent of the optimal dynamic stock/bond policy, and '$CE(V_T^*)$' denotes the certainty equivalent of the optimal buy-and-hold portfolio, reported as a percentage of $CE(W_T^*)$.

					Option positions in optimal portfolio with n options		
n	Options (%)	Stock (%)	$CE(V_T^*)$ (%)	RMSE (%)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)
		$CE(W_T^*) = \$9\,948\,433$		RRA = 1 (Log utility)			
0	0.0	100.0	20.2	3 659.6			
1	60.4	39.6	68.7	3 653.4	54 653		
					733		
2	80.0	20.0	87.7	3 642.4	13 371	143 901	
					401	1 398	
3	99.3	0.7	92.2	3 642.4	786	12 518	144 890
					69	401	1 398
		$CE(W_T^*) = \$1\,644\,465$		RRA = 2			
0	0.0	100.0	81.9	206.2			
1	63.3	36.7	94.5	188.6	2 214		
					69		
2	59.1	40.9	99.2	146.2	1 647	3 040	
					69	401	
3	59.3	40.7	99.4	97.4	1 661	2 795	4 120
					69	401	1 398
		$CE(W_T^*) = \$558\,453$		RRA = 5			
0	0.0	62.0	97.3	143.5			
1	−46.3	131.9	99.1	103.8	−1 620		
					69		
2	−36.9	120.4	99.8	35.9	−1 207	−602	
					69	401	
3	−37.0	120.5	99.8	14.8	−1 215	−573	−197
					69	401	1 398
		$CE(W_T^*) = \$389\,619$		RRA = 10			
0	0.0	26.5	96.6	154.9			
1	−47.1	102.0	98.9	104.5	−1 647		
					69		
2	−37.5	90.0	99.7	25.4	−1 258	−387	
					69	401	
3	−37.6	90.1	99.7	5.9	−1 262	−377	−91
					69	401	1 731

increase their risk exposure[20]. They do not invest in bonds at all, but divide their wealth between stocks and options. As the number of options allowed increases, the fraction of wealth devoted to options in the optimal buy-and-hold portfolio for the log-utility investor also increases, from 60.4% for $n = 1$ to 99.3% for $n = 3$. For a relative risk-aversion coefficient of 2, the proportion of the optimal buy-and-hold portfolio devoted to options declines slightly as n increases, apparently stabilizing at approximately 59% for $n = 3$.

[20] Call options are generally more risky than the underlying stock on which they are based. See, for example, Cox and Rubinstein (1985).

The second group consists of the remaining four panels, which correspond to investors who, in the standard dynamic asset-allocation framework, would optimally hold less than 100% of their wealth in the risky asset. For these investors a buy-and-hold portfolio with no options has a certainty equivalent that is approximately 97% of $CE(W_T^*)$. It is remarkable that a well-chosen buy-and-hold portfolio in the stock and the bond can do so well over a 20-year horizon.

When just 1 or 2 options are added to the buy-and-hold portfolio in these cases, the certainty equivalents $CE(V_T^*)$ of the optimal portfolios increase to approximately 99.7% of

Table 1. Continued.

n	Options (%)	Stock (%)	CE(V_T^*) (%)	RMSE (%)	Option positions in optimal portfolio with n options: Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)
		CE(W_T^*) = $345 561		RRA = 15			
0	0.0	16.2	97.2	124.0			
1	−37.1	76.6	99.1	82.3	−1 297		
					69		
2	−29.6	67.1	99.8	17.6	−1 000	−262	
					69	401	
3	−29.7	67.2	99.8	4.2	−1 003	−256	−50
					69	401	1 731
		CE(W_T^*) = $325 437		RRA = 20			
0	0.0	11.6	97.7	101.0			
1	−30.0	60.7	99.3	66.5	−1 048		
					69		
2	−24.0	53.0	99.8	13.3	−812	−196	
					69	401	
3	−24.0	53.1	99.8	3.2	−814	−192	−34
					69	401	1 731

CE(W_T^*). In contrast to the first two panels, investors with higher risk-aversion parameters are net sellers of call options, forgoing some of the upside gain in order to limit losses on the downside. The value of these option positions ranges from 24% to 37% of their initial wealth. The optimal buy-and-hold portfolios invest the option premia, together with the initial wealth of $100 000, in stocks and bonds.

The combination of a short position in a call option and a long position in the underlying stock is often called a 'hedged position' since the gains (losses) of one security offset to some degree the losses (gains) of the other. Figure 1 provides an example of such a hedged position: a long position in one share of stock and a short position in a call option on that stock with strike price k. The combination yields a payoff that has limited upside—beyond k, the payoff is constant at k—which a sufficiently risk-averse investor might find attractive, since he receives cash now in exchange for an uncertain upside.

For risk-aversion coefficients greater than or equal to 5, table 1 shows that the optimal buy-and-hold portfolios all include hedged positions in which part of the upside potential in the stock is relinquished in exchange for option premia that are invested in stocks and bonds. For a relative risk-aversion coefficient of 10, the optimal buy-and-hold portfolio with 3 options consists of a -37.6% investment in options, 90.1% in the stock, and 47.5% in bonds. Since this portfolio yields an excellent approximation to the optimal dynamic investment strategy (it has a certainty equivalent CE(V_T^*) of 99.7%), we can be fairly confident that these rather unorthodox positions do, in fact, accurately represent the investor's preferences. Indeed, by graphing the payoff diagram of this optimal buy-and-hold portfolio along the lines of figure 1, we can obtain a visual representation of the investor's dynamic risk exposures at a single point in time.

A common characteristic in all of the panels of table 1 is the optimal strike prices of the options in the buy-and-hold portfolio. Despite the fact that the possible strikes range from $69 to $14 696, the highest strike selected by the optimization algorithm is $1731. Under geometric Brownian motion, the expected stock price 20 years into the future is:

$$\mathrm{E}_0[P_T] = P_0 \exp(\mu T) = \$50 \times \exp(0.17 \times 20) = \$1498.$$

Therefore, almost all of the options selected by the optimal buy-and-hold portfolio are in-the-money relative to the expected terminal price $\mathrm{E}_0[P_T]$, which characterizes another aspect of the investor's risk profile.

Also, the fact that among the 45 possible strikes, only 5 are employed in the optimal buy-and-hold portfolios over the range of relative risk-aversion coefficients from 1 to 20 suggests the possibility of standardizing a small number of 'canonical' long-dated options that will appeal to a broad set of investors.

Mean-square-optimal buy-and-hold portfolios. Table 2 reports the mean-square-optimal buy-and-hold portfolios for various levels of risk aversion and, for each risk-aversion parameter, for the number options n varying from 0 to 5. We use a larger number of options in this case to illustrate the fact that even with a larger number of options, a mean-square-optimal portfolio need not come close in certainty equivalence to the optimal dynamic investment policy.

The first row of table 2's first panel corresponds to the optimal buy-and-hold portfolio with no options ($n = 0$), which is identical to the first row of table 1's first panel. As the number of options n is increased, the investor's welfare increases, so that for $n = 5$, the certainty equivalent of the optimal

Table 2. Mean-square-optimal buy-and-hold portfolios of stocks, bonds and n European call options for CRRA utility under geometric Brownian motion stock-price dynamics with parameters (μ, σ) calibrated to match the following moments: $\mathrm{E}[\log(P_t/P_{t-1})] = 0.15$, $\mathrm{Var}[\log(P_t/P_{t-1})] = 0.04$. Other calibrated parameters include: riskless rate $r = 5\%$, initial stock price $P_0 = \$50$, initial wealth $W_0 = \$100\,000$, and time period $T = 20$ years. 'RRA' denotes the coefficient of relative risk aversion, '$\mathrm{CE}(W_T^*)$' denotes the certainty equivalent of the optimal dynamic stock/bond policy, and '$\mathrm{CE}(V_T^*)$' denotes the certainty equivalent of the optimal buy-and-hold portfolio, reported as a percentage of $\mathrm{CE}(W_T^*)$.

					Option positions in optimal portfolio with n options				
n	Options (%)	Stock (%)	$\mathrm{CE}(V_T^*)$ (%)	RMSE (%)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)
				$\mathrm{CE}(W_T^*) = \$9\,948\,433$	RRA = 1 (Log utility)				
0	0.0	100.0	20.2	3 659.6					
1	0.2	99.8	20.4	2 889.6	29.0×10^{-6} 14 696				
2	62.9	−4.0	10.2	2 886.7	331 561 1 398	28.3×10^{-6} 14 696			
3	2.8	97.2	23.1	2 870.6	2 431 277 5 388	-68.2×10^{-6} 14 363	94.7×10^{-6} 14 696		
4	8.0	92.0	28.1	2 869.5	943 747 3 393	2 987 657 8 712	-85.9×10^{-6} 14 363	111.0×10^{-6} 14 696	
5	15.5	84.5	34.9	2 869.3	465 463 2 396	1 415 411 6 052	2 917 259 10 374	-92.0×10^{-6} 14 363	116.2×10^{-6} 14 696
				$\mathrm{CE}(W_T^*) = \$1\,644\,465$	RRA = 2				
0	0.0	100.0	81.9	206.2					
1	0.8	99.2	83.4	57.2	14 846 2 063				
2	3.9	96.1	86.6	26.6	9 233 1 066	15 764 9 044			
3	7.6	92.4	89.3	15.8	6 873 733	8 120 4 390	15 063 14 696		
4	19.6	68.4	89.4	13.8	4 908 401	4 815 2 063	6 273 5 388	14 315 14 696	
5	17.5	82.5	94.0	13.3	4 342 401	3 797 1 731	3 753 3 725	4 430 6 717	13 501 14 696
				$\mathrm{CE}(W_T^*) = \$558\,453$	RRA = 5				
0	0.0	24.4	84.7	28.3					
1	−0.1	42.5	94.2	7.8	−543 1 398				
2	−0.6	50.8	96.6	3.6	−554 733	−225 4 058			
3	−2.5	61.8	98.4	2.2	−624 401	−246 1 731	−159 6 385		
4	−2.3	60.5	98.2	1.4	−569 401	−218 1 398	−134 3 725	−110 10 374	
5	−2.1	59.0	98.1	1.0	−500 401	−191 1 066	−134 2 396	−102 5 388	−91 13 698

buy-and-hold strategy is 34.9% of $\mathrm{CE}(W_T^*)$. Although this is a considerable improvement over the $n = 0$ case, it is still quite far below the optimal dynamic strategy's certainty equivalent. This is not unexpected in light of the fact that we are minimizing mean-squared-error, not maximizing expected utility. As n increases beyond 5, this approximation will improve eventually, but the optimization process becomes considerably more challenging for larger n. For example, the $n = 15$ case involves $\binom{45}{15} = 344\,867\,425\,584$ subproblems, and if each subproblem requires 0.01 seconds to solve, the overall optimization would take approximately 109.4 years to complete.

Unlike table 1, in table 2 the certainty equivalents of the optimal buy-and-hold portfolio, $\mathrm{CE}(V_T^*)$, do not increase monotonically with the number of options n. For example, in the case of log utility (RRA = 1), $\mathrm{CE}(V_T^*)$ is 20.4% of $\mathrm{CE}(W_T^*)$ for $n = 1$ option, but declines to 10.2% for $n = 2$ options. This underscores the fact that we are minimizing

Table 2. Continued.

n	Options (%)	Stock (%)	CE(V_T^*) (%)	RMSE (%)	Option positions in optimal portfolio with n options: Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)
			CE(W_T^*) = $389 619	RRA = 10					
0	0.0	6.7	86.4	27.0					
1	−0.1	17.7	94.8	6.4	−297				
					1 066				
2	−1.8	30.0	98.0	3.0	−450	−107			
					401	2 396			
3	−1.5	27.9	97.7	1.7	−375	−101	−52		
					401	1 398	4 723		
4	−1.4	27.3	97.6	1.3	−343	−93	−54	−32	
					401	1 066	2 728	7 715	
5	−68.0	139.6	96.5	0.9	−2345	−236	−100	−54	−32
					69	401	1 066	2 728	7 715
			CE(W_T^*) = $345 561	RRA = 15					
0	0.0	3.7	89.6	21.5					
1	−0.3	14.0	97.0	5.0	−248				
					733				
2	−1.2	19.2	98.3	2.2	−306	−60			
					401	2 396			
3	−1.0	18.0	98.1	1.3	−260	−60	−27		
					401	1 398	4 723		
4	−47.6	96.4	98.5	1.0	−1 637	−189	−62	−27	
					69	401	1 398	4 723	
5	−51.2	102.2	98.0	0.7	−1 767	−163	−56	−29	−19
					69	401	1 066	2 396	6 385
			CE(W_T^*) = $325 437	RRA = 20					
0	0.0	2.5	91.7	17.5					
1	−0.2	10.0	97.5	3.9	−180				
					733				
2	−0.9	14.1	98.5	1.7	−230	−41			
					401	2 396			
3	−0.8	13.2	98.4	1.1	−198	−40	−18		
					401	1 398	4 390		
4	−37.8	75.7	98.9	0.8	−1 304	−142	−42	−18	
					69	401	1 398	4 390	
5	−40.7	80.2	98.5	0.5	−1 405	−122	−40	−20	−12
					69	401	1 066	2 396	6 385

mean-squared-error in the optimal buy-and-hold portfolios of table 2, not maximizing expected utility. In fact, it is possible for a buy-and-hold portfolio to exhibit a small RMSE and a small certainty equivalent at the same time[21]. Therefore, while RMSE must decline monotonically with n, the certainty equivalents need not. Of course, as the number of options n increases without bound, CE(V_T^*) will approach CE(W_T^*) eventually, even if not monotonically.

The option positions in the optimal buy-and-hold portfolios provide additional insight into the differences between maximizing expected utility and minimizing mean-squared-error in constructing the optimal buy-and-hold portfolio. As n increases from 0 to 1 in the first panel of table 2, the optimal buy-and-hold portfolio changes from 100% stocks to 99.8% stocks and 0.2% options, with a huge position (29.0 million) in the option with strike price $14 696. Given a current stock price of $50, this option is obviously deeply out-of-the-money, hence its price is extremely close to zero, so close that 29.0 million options amount to only 0.2% of the investor's initial portfolio. Moreover, recall that these are 20-year options, hence a strike price of $14 696 should be compared not only with the current stock price but with the expected stock price at maturity, P_T. Recall that under geometric Brownian motion, the expected stock price 20 years

[21] This typically occurs when the buy-and-hold strategy results in a final wealth V_T^* that is close to zero over some interval of P_T.

into the future is \$1498. Therefore, even taking into account the expected appreciation in the stock over the next 20 years, the strikes are still extraordinarily high.

The $n = 2$ case differs dramatically from the $n = 1$ case. When given the opportunity to include 2 options in the buy-and-hold portfolio, the optimal weights become 62.9% in options, −4.0% in the stock, and the remaining 41.1% in bonds. The optimal buy-and-hold portfolio involves shorting \$4000 of the stock and putting the proceeds, as well as the original \$100 000, into bonds and options. The options component consists of two positions: 331 561 options with a strike of \$1398, and 28.3 million options with a strike of \$14 696. The latter position is similar to that of the $n = 1$ case, and accounts for a relatively small part of the portfolio. The majority of the 69.2% allocated to options is due to the former position in options of a much lower strike price. The lower strike price implies a higher option price, hence the cost of 331 561 of these options dwarfs the cost of 28.3 million of the higher-strike options. While this buy-and-hold portfolio is indeed optimal from a mean-squared-error criterion, the certainty equivalent reported in table 2 shows that the investor's welfare has actually declined by half, as compared to the $n = 1$ case. Moreover, the RMSE declines only slightly, suggesting that we treat this case cautiously and with a certain degree of skepticism.

As the investor's risk-aversion parameter increases, table 2 shows that the optimal buy-and-hold portfolio performs considerably better in terms of certainty equivalence, in most cases attaining 90% or more of the certainty equivalent of the optimal dynamic strategy. For risk-aversion coefficients greater than 2, the RMSE of the buy-and-hold portfolio is less than 5% with only one or two options. The intuition for this pattern follows from the fact that investors with higher risk aversion invest a smaller proportion of their wealth in the stock market, hence their final wealth W_T^* has lower variance which makes it easier to approximate W_T^* with a buy-and-hold strategy.

The option positions in optimal buy-and-hold portfolios are also different for higher levels of risk aversion, consisting of fewer options and at lower strike prices. To see why, observe that for risk-aversion coefficients of 5 and greater, the optimal buy-and-hold portfolios with no options ($n = 0$) consist largely of bonds (75.6% in bonds for RRA = 5, 95.3% for RRA = 10, 96.3% for RRA = 15, and 97.5% for RRA = 20). When options are allowed in the buy-and-hold portfolios, additional risk-reduction possibilities become feasible and the optimization algorithm takes advantage of such opportunities. In particular, for risk-aversion levels of 5 and greater, the option positions are generally negative—the optimal buy-and-hold portfolios consist of selling options and investing the proceeds as well as the original \$100 000 initial wealth in stocks and bonds. For example, the third panel of table 2 shows that with a risk-aversion coefficient of 5, the optimal buy-and-hold portfolio with 5 options is 59.0% in stocks, 43.1% in bonds and −2.1% in options, with short positions in all 5 options, and where the optimal strikes range from \$401 to \$13 698. These results correspond well with those of table 1, in which the optimal buy-and-hold portfolios of investors with higher risk-aversion coefficients contained hedged positions (long positions in the stock and short positions in options).

5.2. The Ornstein–Uhlenbeck process

To calibrate the parameters of the trending Ornstein–Uhlenbeck process (4.8), we observe that the moments of the stationary distribution of $\{P_t\}$ are given by:

$$\mathrm{E}[\log(P_t/P_{t-1})] = \mu$$

$$\mathrm{Var}[\log(P_t/P_{t-1})] = \frac{\sigma^2}{\delta}(1-\exp(-\delta))$$

$$\mathrm{Corr}[\log(P_t/P_{t-1}), \log(P_{t-1}/P_{t-2})] = -\frac{1}{2}(1-\exp(-\delta)).$$

Therefore, using the parameters in (5.5) and setting the first-order autocorrelation coefficient equal to −0.05 uniquely calibrates the parameter vector (μ, σ, δ). The distribution of $\log P_T$ implied by these parameters yields the following 45 possible strikes (in dollars) from which we select our n options in the optimal buy-and-hold portfolio:

265	346	426	506	587
667	748	828	908	989
1 069	1 150	1 230	1 310	1 391
1 471	1 552	1 632	1 712	1 793
1 873	1 954	2 034	2 114	2 195
2 275	2 356	2 436	2 517	2 597
2 677	2 758	2 838	2 919	2 999
3 079	3 160	3 240	3 321	3 401
3 481	3 562	3 642	3 723	3 803

Note that the distribution of possible strikes lies in a much narrower range in this case than in the geometric Brownian motion case of section 5.1: 265 to 3723 for the trending Ornstein–Uhlenbeck process versus 69 to 14 363 for geometric Brownian motion. This is an implication of the mean-reverting nature of the trending Ornstein–Uhlenbeck process, a stochastic process in which log-prices are stationary about a deterministic trend, in contrast to geometric Brownian motion in which log-prices are difference-stationary. In the former case, the variance of the log-price process is bounded as the horizon increases without bound, whereas in the latter case, the variance is proportional to the horizon, implying a wider range of strikes.

Recall from section 4.2 that because the optimal dynamic asset-allocation strategy is path-dependent under (4.8), the certainty equivalent of V_T^* will not approach the certainty equivalent of W_T^* as the number of options n in the buy-and-hold portfolio increases without bound. Indeed, there is an upper bound for $\mathrm{CE}(V_T^*)$, which is the certainty equivalent of the optimal buy-and-hold portfolio with an infinite number of options, $\mathrm{CE}(V_T^\infty)$, and for path-dependent dynamic portfolio strategies, $\mathrm{CE}(V_T^\infty)$ is strictly less than $\mathrm{CE}(W_T^*)$. In the case of the trending Ornstein–Uhlenbeck process (4.8) and CRRA preferences, we have an explicit expression for V_T^∞ (see section 4.2), hence we can construct a mean-square optimal buy-and-hold portfolio where the benchmark is V_T^∞, not W_T^*.

Table 3. Utility-optimal buy-and-hold portfolios of stocks, bonds, and n European call options for CRRA utility under a trending Ornstein–Uhlenbeck stock-price process where the parameters (σ, μ, δ) have been calibrated to match the following moments: $\mathrm{E}[\log(P_t/P_{t-1})] = 0.15$, $\mathrm{Var}[\log(P_t/P_{t-1})] = 0.04$, $\mathrm{Corr}[\log(P_t/P_{t-1}), \log(P_{t-1}/P_{t-2})] = -0.05$. Other calibrated parameters include: riskless rate $r = 5\%$, initial stock price $P_0 = \$1$, initial wealth $W_0 = \$100\,000$, and time period $T = 20$ years. 'RRA' denotes the coefficient of relative risk aversion, '$\mathrm{CE}(W_T^*)$' denotes the certainty equivalent of the optimal dynamic stock/bond policy, '$\mathrm{CE}(V_T^\infty)$' denotes the certainty equivalent of the optimal buy-and-hold portfolio with a continuum of options, and '$\mathrm{CE}(V_T^*)$' denotes the certainty equivalent of the optimal buy-and-hold portfolio with a finite number n of options, reported as a percentage of $\mathrm{CE}(V_T^\infty)$.

					Option positions in optimal portfolio with n options		
n	Options (%)	Stock (%)	$\mathrm{CE}(V_T^*)$ (%)	RMSE (%)	Quantity Strike (\$)	Quantity Strike (\$)	Quantity Strike (\$)
	$\mathrm{CE}(W_T^*) = \$13\,162\,500$		$\mathrm{CE}(V_T^\infty) = \$12\,417\,350$		RRA = 1	(Log utility)	
0	0.0	100.0	16.2	115.7			
1	96.6	3.4	89.1	42.3	24 955		
					426		
2	98.8	1.2	96.2	40.3	6 251	30 824	
					346	587	
3	98.8	1.2	97.8	12.6	6047	33 494	−39 564
					346	587	1793
	$\mathrm{CE}(W_T^*) = \$6166\,222$		$\mathrm{CE}(V_T^\infty) = \$5814\,196$		RRA = 2		
0	0.0	100.0	31.3	87.5			
1	83.9	16.1	89.0	29.1	10 204		
					265		
2	83.0	17.0	90.5	34.2	7 822	4 810	
					265	426	
3	83.0	17.0	91.6	6.7	8 283	6 623	−15 246
					265	506	1 391
	$\mathrm{CE}(W_T^*) = \$2011\,701$		$\mathrm{CE}(V_T^\infty) = \$1874\,790$		RRA = 5		
0	0.0	100.0	72.2	30.8			
1	16.7	83.3	79.8	56.6	2 036		
					265		
2	19.5	80.5	82.7	26.9	5 777	−5 062	
					265	346	
3	18.9	81.1	83.2	13.9	5 304	−4 202	−2 385
					265	346	989
	$\mathrm{CE}(W_T^*) = \$957\,797$		$\mathrm{CE}(V_T^\infty) = \$900\,296$		RRA = 10		
0	0.0	100.0	91.9	117.3			
1	−6.6	106.6	96.8	11.2	−1 712		
					426		
2	−7.0	107.0	97.1	3.8	−1 047	−951	
					346	748	
3	−6.8	106.8	97.1	2.5	−958	−641	−525
					346	587	1 150

Utility-optimal buy-and-hold portfolios. Table 3 summarizes the utility-optimal buy-and-hold portfolios for the same combination of risk-aversion parameters and number of options n as in the geometric Brownian motion case of table 1. The results for the panels with RRA $= 1, 2, 5$ were obtained by maximizing expected utility directly using a discretized distribution for P_T (see section 3.1), and the results for the remaining three panels of table 3 were obtained by minimizing the utility-weighted mean-squared error (see section 3.3).

Note that for each level of risk aversion, the certainty equivalent $\mathrm{CE}(W_T^*)$ of the optimal dynamic strategy is considerably larger than that of the geometric Brownian motion case. The presence of predictability can be exploited by the investor and in doing so, his expected utility is increased dramatically, e.g. from a certainty equivalent of \$9948 433 in the geometric Brownian motion case to \$13 162 500 in the Ornstein–Uhlenbeck case for log-utility. A more direct measure of the economic value of predictability can be obtained by considering the difference between the certainty equivalents of the optimal dynamic strategy and those of the optimal buy-and-hold portfolio with an infinite number of options. For a log-utility investor, this difference is \$745 150

Table 3. Continued.

n	Options (%)	Stock (%)	CE(V_T^*) (%)	RMSE (%)	Option positions in optimal portfolio with n options: Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)
		CE(W_T^*) = $681 834		CE(V_T^∞) = $647 654	RRA = 15		
0	0.0	75.8	87.9	151.9			
1	−15.9	108.7	95.3	10.3	−1 937		
					265		
2	−15.5	108.1	95.2	2.0	−1 869	−285	
					265	1 150	
3	−16.9	110.2	95.7	3.9	−2 844	1 435	−689
					265	346	587
		CE(W_T^*) = $560 880		CE(V_T^∞) = $537 074	RRA = 20		
0	0.0	54.1	87.9	138.6			
1	−13.9	89.0	93.2	5.1	−1 686		
					265		
2	−15.5	91.7	93.7	11.6	−2 385	740	
					265	346	
3	−15.7	92.0	93.8	4.0	−2 868	1 753	−643
					265	346	506

or 5.6% of CE(W_T^*), a significant amount. As the level of risk aversion increases, this difference declines in absolute terms—less wealth is allocated to the risky asset, hence predictability has less of an impact—but is relatively stable as a percentage of CE(W_T^*), fluctuating between 4% and 6%.

The most interesting feature of table 3 is that the certainty equivalents of the buy-and-hold portfolios do not approach CE(V_T^∞) as quickly as the certainty equivalents of table 1. This is most easily seen in the third panel (RRA = 5) in which the certainty equivalent of the optimal buy-and-hold portfolio with 3 options is only 83.2% of CE(V_T^∞). However, as we remarked earlier, the data for this panel were computed by maximizing expected utility through a discretization of the distribution of P_T using a grid of 4000 points. Because of the relatively high value of RRA, any interval in the support of P_T where $W_T(P_T)$ is close to 0 will result in a large negative contribution to the certainty equivalent. We can address this issue by using a finer grid, but only at the expense of computational complexity[22].

Another interesting feature of table 3 is that there is no investment in the bond in any of the buy-and-hold portfolios in the first four panels (RRA = 1, 2, 5, 10). While this is not unexpected for low levels of risk aversion—such investors seek higher expected returns by the nature of their risk preferences—it is quite surprising for investors with RRA = 10. The intuition for this result comes from the fact that stock returns are predictable in this case, hence there is greater value to be gained from investing in stocks for each level of risk aversion. Alternatively, the predictability in stock returns make stocks less risky, *ceteris paribus*, hence even a risk-averse investor will hold a larger fraction of his wealth in stocks in this case.

As in tables 1 and 2, the optimal buy-and-hold portfolios for less risk-averse investors (RRA = 1, 2, 5) are net positive in options, ranging from 98.8% when RRA = 1 to 18.9% when RRA = 5, for $n = 3$. However, unlike the geometric Brownian motion case, the optimal buy-and-hold portfolios do contain short positions in some options, even for these lower levels of risk aversion. For example, when RRA = 2 and $n = 3$, the optimal buy-and-hold portfolio consists of long positions in the $265-strike and $506-strike options, but a short position of 15 246 options in the $1391-strike option. For higher levels of risk aversion, the situation is reversed: the optimal buy-and-hold portfolios are net negative in options, but they do contain long positions in certain options. For example, when RRA = 20 and $n = 3$, the optimal buy-and-hold portfolio consists of short positions in the $265-strike and $506-strike options, but a long position of 1753 options in the $346-strike option.

These long and short positions underscore the complexity of an investor's ideal risk exposures, and may provide a useful benchmark for comparing different dynamic investment policies at a single point in time. In particular, it may be possible to re-interpret these option positions as classic spread trades, e.g. bull/bear and butterfly spreads, or combinations, e.g. strips, straps, straddles and strangles[23]. By doing so, we may be able to gain insight into the implicit bets that a particular dynamic asset-allocation strategy contains, and

[22] Since these numerical results are mainly for illustrative purposes, we have not endeavoured to optimize them within each panel. Instead, to ensure comparability across risk-aversion parameters and other specifications, we have attempted to hold fixed as many aspects of the optimization process as possible.

[23] For this purpose, it may be useful to convert some of the call-option positions into their put-option equivalents using the put-call parity relation (see, for example, Cox and Rubinstein (1985)).

Table 4. Mean-square-optimal buy-and-hold portfolios of stocks, bonds and n European call options for CRRA utility under a trending Ornstein–Uhlenbeck stock-price process with parameters (σ, μ, δ) calibrated to match the following moments: $\mathrm{E}[\log(P_t/P_{t-1})] = 0.15$, $\mathrm{Var}[\log(P_t/P_{t-1})] = 0.04$, $\mathrm{Corr}[\log(P_t/P_{t-1}), \log(P_{t-1}/P_{t-2})] = -0.05$. Other calibrated parameters include: riskless rate $r = 5\%$, initial stock price $P_0 = \$50$, initial wealth $W_0 = \$100\,000$, and time period $T = 20$ years. 'RRA' denotes the coefficient of relative risk aversion, '$\mathrm{CE}(W_T^*)$' denotes the certainty equivalent of the optimal dynamic stock/bond policy, '$\mathrm{CE}(V_T^\infty)$' denotes the certainty equivalent of the optimal buy-and-hold portfolio with a continuum of options, and '$\mathrm{CE}(V_T^*)$' denotes the certainty equivalent of the optimal buy-and-hold portfolio with a finite number n of options, reported as a percentage of $\mathrm{CE}(V_T^\infty)$.

					Option positions in optimal portfolio with n options				
n	Options (%)	Stock (%)	$\mathrm{CE}(V_T^*)$ (%)	RMSE (%)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)
		$\mathrm{CE}(W_T^*) = \$13\,162\,500$			$\mathrm{CE}(V_T^\infty) = \$12\,417\,350$		RRA = 1 (Log utility)		
0	0.0	100.0	16.2	115.7					
1	91.7	8.3	87.4	37.5	32 697				
					506				
2	86.8	13.2	81.3	9.3	44 428	−44 691			
					587	1712			
3	87.0	13.0	81.5	6.3	44 694	−55 819	10 865		
					587	1 793	3 803		
4	88.7	−3.4	56.7	4.5	45 736	−33 938	−29 561	17 831	
					587	1 632	2 195	3 803	
5	90.2	9.8	56.7	2.7	24 169	24 603	−32 245	−33 388	16 665
					506	748	1 552	2 114	3 803
		$\mathrm{CE}(W_T^*) = \$6166\,222$			$\mathrm{CE}(V_T^\infty) = \$5814\,196$		RRA = 2		
0	0.0	100.0	31.3	87.5					
1	87.2	12.8	88.5	28.9	10 599				
					265				
2	75.4	24.6	79.8	5.0	14 198	−14 689			
					346	1 471			
3	82.7	−9.0	24.3	3.6	15 686	−6 392	−9 114		
					346	1 150	1 632		
4	79.8	7.4	64.4	2.3	15 136	−8 530	−9 152	2 399	
					346	1 230	1 873	3 803	
5	82.1	−5.9	37.4	1.6	15 582	−5 987	−6 895	−6 095	3 515
					346	1 150	1 552	2 195	3 803
		$\mathrm{CE}(W_T^*) = \$2011\,701$			$\mathrm{CE}(V_T^\infty) = \$1874\,790$		RRA = 5		
0	0.0	100.0	72.2	30.8					
1	−0.2	100.2	72.4	26.8	−2 005				
					1 793				
2	17.1	82.9	81.4	6.3	2 421	−3 647			
					265	908			
3	20.6	79.4	81.5	2.8	3 139	−3 022	−1 705		
					265	667	1 391		
4	26.4	73.6	80.5	1.8	5 151	−3 730	−1 731	−1 162	
					265	426	908	1 552	
5	29.0	71.0	79.9	1.4	7 665	−5 731	−1 481	−1 063	−810
					265	346	748	1 150	1 632

develop a standard lexicon for comparing those bets across investment policies.

Mean-square-optimal buy-and-hold portfolios. Table 4 summarizes the mean-square-optimal buy-and-hold portfolios for the same combination of strikes, risk-aversion parameters, and number of options n as in table 3. Table 4 shows that the RMSE of the optimal buy-and-hold portfolio declines rapidly. With only one or two options, the optimal buy-and-hold portfolio is typically within 5% of the upper bound $\mathrm{CE}(V_T^\infty)$. For example, in the case where relative risk-aversion is 2, the RMSE of the optimal buy-and-hold portfolio with no options is 87.5%; with 1 option, the RMSE declines to 28.9%; and with 2 options, the RMSE is 5.0%. With 5 options, the RMSE is less than 2.0% for all but the lowest level of risk aversion (RRA = 1, for which the RMSE is 2.7%). But as

Table 4. Continued.

n	Options (%)	Stock (%)	CE(V_T^*) (%)	RMSE (%)	Quantity Strike (\$)	Quantity Strike (\$)	Quantity Strike (\$)	Quantity Strike (\$)	Quantity Strike (\$)
					Option positions in optimal portfolio with n options				
			CE(W_T^*) = \$957 797		CE(V_T^∞) = \$900 296		RRA = 10		
0	0.0	39.0	67.9	28.4					
1	−1.7	74.7	86.5	4.5	−1 377				
					748				
2	−4.0	92.4	93.0	1.8	−1 367	−464			
					506	1 230			
3	−5.2	98.1	94.8	1.0	−1 200	−485	−277		
					426	828	1 471		
4	−7.4	107.4	97.2	0.7	−1 166	−479	−288	−215	
					346	667	989	1 552	
5	−7.1	106.2	96.9	0.6	−1 061	−418	−278	−209	−158
					346	587	828	1 150	1 632
			CE(W_T^*) = \$681 834		CE(V_T^∞) = \$647 654		RRA = 15		
0	0.0	22.0	70.3	25.4					
1	−2.1	53.9	87.2	3.7	−1 003				
					587				
2	−4.0	64.4	90.7	1.4	−1 008	−259			
					426	1 069			
3	−6.1	72.9	93.0	0.8	−1 028	−283	−146		
					346	748	1 391		
4	−5.9	71.8	92.8	0.5	−974	−233	−136	−93	
					346	667	989	1 552	
5	−9.6	84.4	95.4	0.4	−1 037	−290	−168	−106	−86
					265	506	748	1 069	1 552
			CE(W_T^*) = \$560 880		CE(V_T^∞) = \$537 074		RRA = 20		
0	0.0	15.0	73.7	22.6					
1	−1.5	37.6	86.8	3.0	−708				
					587				
2	−3.0	46.4	89.9	1.2	−760	−155			
					426	1 069			
3	−4.7	53.6	91.8	0.6	−808	−177	−86		
					346	748	1 391		
4	−8.5	66.4	94.4	0.4	−971	−191	−98	−67	
					265	587	908	1 471	
5	−7.8	63.9	94.1	0.3	−866	−193	−105	−64	−50
					265	506	748	1 069	1 552

in table 2, the certainty equivalents of the optimal buy-and-hold portfolio do not increase monotonically as the number of options increases, since we are optimizing mean-squared-error, not expected utility. For example, in the second panel (RRA = 2) the certainty equivalent drops precipitously from 64.4% to 37.4% of the upper bound CE(V_T^∞) as the number of options increases from 4 to 5. However, for higher levels of risk aversion, the certainty equivalents do tend to increase with the number of options in the portfolio (and are guaranteed to converge to the upper bound CE(V_T^∞) as n increases without bound).

As risk aversion increases, the optimal buy-and-hold portfolios behave in a similar manner to those in table 2: options are used to hedge long positions in the stock. For risk-aversion levels of 10 or greater, all options positions are negative.

5.3. A bivariate linear diffusion process

We calibrate the parameters $(\kappa, \theta, \sigma_1, \sigma_2, \rho)$ of the bivariate linear diffusion (4.22) and (4.23) using the following values:

$$\begin{aligned} &E[\log(P_t/P_{t-1})] = 0.15 \\ &\mathrm{Var}[\log(P_t/P_{t-1})] = 0.04 \\ &\mathrm{Var}[\mu_t] = 0.025^2 \\ &\mathrm{Corr}[\mu_t, \mu_{t-1}] = 0.05 \\ &\rho = 0. \end{aligned} \tag{5.6}$$

The first two moments are calibrated with the same values as those in the geometric Brownian motion and trending Ornstein–Uhlenbeck cases. The value for the variance of μ_t implies a standard deviation of 250 basis points for the conditional mean μ_t, and we assume that μ_t is only slightly autocorrelated over time, and not correlated at all with the Brownian motion driving prices. This calibration implies the following 45 possible strikes (in dollars) from which we select our n options in the optimal buy-and-hold portfolio:

68	403	737	1 071	1 405
1 739	2 073	2 407	2 742	3 076
3 410	3 744	4 078	4 412	4 746
5 081	5 415	5 749	6 083	6 417
6 751	7 085	7 420	7 754	8 088
8 422	8 756	9 090	9 425	9 759
10 093	10 427	10 761	11 095	11 429
11 764	12 098	12 432	12 766	13 100
13 434	13 768	14 103	14 437	14 771

Note the similarity between the range of these strikes and that of geometric Brownian motion in section 5.1. This suggests that the economic properties of the bivariate linear diffusion process are close to those of geometric Brownian motion, which will be borne out by the optimal buy-and-hold portfolios described below.

As in the case of the trending Ornstein–Uhlenbeck process, under (4.22) and (4.23) the optimal dynamic asset-allocation strategy is path-dependent. Therefore, we shall again use the upper bound V_T^∞ as the benchmark in our mean-square-optimal buy-and-hold portfolio, and compare its certainty equivalent $\mathrm{CE}(V_T^*)$ to $\mathrm{CE}(V_T^\infty)$.

Utility-optimal buy-and-hold portfolios. Table 5 reports the optimal buy-and-hold portfolios under (4.22) and (4.23) for CRRA preferences with the same risk aversion levels as in tables 1–4. The results of the first two panels of table 5 were computed by maximizing expected utility according to section 3.1 and the results of the remaining panels were computed by minimizing utility-weighted mean-squared-error according to section 3.3.

Table 5 contains certain features in common with tables 1 and 3, but also exhibits some important differences. As in table 3, the certainty equivalents of V_T^∞ are lower than their counterparts for W_T^*, but in table 5 the gap declines monotonically as risk aversion increases. For log-utility, $\mathrm{CE}(V_T^\infty)$ is 15.5% less than $\mathrm{CE}(W_T^*)$, but this difference is only 7.5% when RRA = 2, 3.1% when RRA = 5, and 0.8% when RRA = 20. In contrast, the gap between $\mathrm{CE}(W_T^*)$ and $\mathrm{CE}(V_T^\infty)$ in table 3 is still 4.2% when RRA = 20. This underscores the fact that the predictability of the bivariate linear diffusion is of a different form from that of the trending Ornstein–Uhlenbeck process.

Indeed, there are striking similarities between tables 5 and 1, another indication that the terminal stock price P_T and option prices corresponding to the two stochastic processes—as we have calibrated them—have much in common. However, note that the certainty equivalents in table 1 are relative to $\mathrm{CE}(W_T^*)$, not $\mathrm{CE}(V_T^\infty)$. Nevertheless, even the values of $\mathrm{CE}(V_T^\infty)$ in table 5 are extremely close to the values of $\mathrm{CE}(W_T^*)$ in table 1. This close correspondence suggests that for all practical purposes, the bivariate process (4.22) and (4.23) offers the same buy-and-hold investment opportunities to the investor as geometric Brownian motion.

Mean-square-optimal buy-and-hold portfolios. Table 6 reports the mean-square-optimal buy-and-hold portfolios under (4.22) and (4.23) for CRRA preferences with the same risk aversion levels as in table 5. These results match those in table 2 quite closely. Specifically, as in table 2, the optimal buy-and-hold portfolio is a particularly poor approximation to both W_T^* and V_T^∞ in the log-utility case, with RMSE's greater than 3500%, certainty equivalents $\mathrm{CE}(V_T^*)$ no greater than 35% of $\mathrm{CE}(V_T^\infty)$, and large swings in portfolio weights as n is changed from 1 to 2 and from 2 to 3. For higher levels of risk aversion, the optimal buy-and-hold portfolios in table 6 are remarkably close to those in table 2 in terms of portfolio weights, option positions, and certainty equivalents, providing further confirmation that the bivariate linear diffusion, calibrated according to (5.6), shares many of the same economic properties as geometric Brownian motion.

6. Discussion

For expositional purposes, we have made a number of simplifying assumptions, many of which can be relaxed at the expense of notational and computational complexity. In section 6.1, we consider some practical issues regarding the implementation of the optimal buy-and-hold portfolio. We discuss the advantages of using more complex derivative securities in section 6.2, and in section 6.3 we consider extending our analysis to other preferences and price processes. Finally, in section 6.4 we argue that the gap between $\mathrm{CE}(W_T^*)$ and $\mathrm{CE}(V_T^\infty)$ is a useful measure of the economic value of predictability, and discuss the role of taxes and transactions costs in interpreting the gap.

6.1. Practical considerations

An obvious prerequisite to any practical implementation of the optimal buy-and-hold portfolio proposed in section 3 is the existence of options with the appropriate maturity T and strike prices $\{k_i^*\}$. These two issues—time-to-maturity and the set of available strikes—are related, since a longer time-to-maturity generally implies a greater dispersion for the optimal strikes (to accommodate the greater dispersion in the terminal stock-price distribution). For horizons less than one year, there are relatively liquid options on the S&P 500 and other indexes, usually with a reasonable number of strikes above and below the spot price, hence the possibility of replacing certain dynamic investment strategies with an optimal buy-and-hold portfolio is plausible. However, for longer maturities such as the 20-year horizons proposed in the numerical examples of section 5, exchange-traded options do not exist.

This might seem to be a serious impediment to implementing the optimal buy-and-hold strategy for realistic investment horizons. However, we think there is hope for several reasons. First, longer-maturity index options are

Table 5. Utility-optimal buy-and-hold portfolios of stocks, bonds and n European call options for CRRA utility under a bivariate linear diffusion stock-price process with parameters $(\sigma_1, \sigma_2, \rho, \kappa, \theta)$ of the steady-state distribution calibrated to match the following moments: $E[\log(P_t/P_{t-1})] = 0.15$, $\text{Var}[\log(P_t/P_{t-1})] = 0.04$, $\text{Var}[\mu_t] = 0.025^2$, $\text{Corr}[\mu_t, \mu_{t-1}] = 0.05$, and $\rho = 0$. Other calibrated parameters include: riskless rate $r = 5\%$, initial stock price $P_0 = \$50$, initial wealth $W_0 = \$100\,000$, and time period $T = 20$ years. 'RRA' denotes the coefficient of relative risk aversion, '$\text{CE}(W_T^*)$' denotes the certainty equivalent of the optimal dynamic stock/bond policy, '$\text{CE}(V_T^\infty)$' denotes the certainty equivalent of the optimal buy-and-hold portfolio with a continuum of options, and '$\text{CE}(V_T^*)$' denotes the certainty equivalent of the optimal buy-and-hold portfolio with a finite number n of options, reported as a percentage of $\text{CE}(V_T^\infty)$.

					Option positions in optimal portfolio with n options		
n	Options (%)	Stock (%)	$\text{CE}(V_T^*)$ (%)	RMSE (%)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)
	$\text{CE}(W_T^*) = \$11\,861\,394$		$\text{CE}(V_T^\infty) = \$10\,142\,498$		RRA = 1	(Log utility)	
0	0.0	100.0	19.8	4 346.6			
1	60.3	39.7	68.2	4 341.5	56 231		
					737		
2	79.9	20.1	87.5	4 332.2	13 577	151 097	
					403	1 405	
3	99.3	0.7	91.9	4 332.2	786	12 725	152 096
					68	403	1 405
	$\text{CE}(W_T^*) = \$1778\,906$		$\text{CE}(V_T^\infty) = \$1\,645\,135$		RRA = 2		
0	0.0	100.0	81.8	214.7			
1	64.6	35.4	94.4	197.4	2 258		
					68		
2	58.0	42.0	99.2	155.4	1 600	3 159	
					68	403	
3	58.2	41.8	99.4	106.3	1 624	2 811	4 379
					68	403	1 405
	$\text{CE}(W_T^*) = \$575\,004$		$\text{CE}(V_T^\infty) = \$557\,315$		RRA = 5		
0	0.0	61.8	97.4	141.8			
1	−45.9	131.2	99.1	103.0	−1 604		
					68		
2	−36.7	119.8	99.8	35.4	−1 201	−595	
					68	403	
3	−36.8	120.0	99.8	14.8	−1 207	−568	−192
					68	403	1 405
	$\text{CE}(W_T^*) = \$395\,205$		$\text{CE}(V_T^\infty) = \$389\,080$		RRA = 10		
0	0.0	26.5	96.7	154.7			
1	−46.8	101.5	98.9	105.1	−1 635		
					68		
2	−37.3	89.5	99.7	25.3	−1 250	−386	
					68	403	
3	−37.4	89.6	99.7	6.1	−1 254	−376	−90
					68	403	1 739

always available through custom OTC derivatives contracts, although this is admittedly a very expensive alternative. Second, the scarcity of longer-maturity contracts is a reflection of existing demand—if optimal buy-and-hold portfolios become popular, this will create new demand for such contracts, leading to increased supply. Recent legislative debate regarding the privatization of the US social security system suggests the possibility of a huge increase in demand for such products and services. Third, insurance companies now provide various policies that have similar features to long-dated options, e.g. annuities with call and put features, contingent life-insurance policies etc, hence they may be a natural supplier of optimal buy-and-hold portfolios. And finally, an imperfect alternative to long-dated options is a carefully managed sequence of shorter-term options, and it may be possible to derive a dynamic trading strategy consisting of a sequence of overlapping options contracts that will yield the same investment profile as the optimal buy-and-hold strategy[24]. A dynamic trading strategy seems contrary to our

[24] See Bertsimas *et al* (2000b) for an example of how such a strategy might be derived.

Table 5. Continued.

n	Options (%)	Stock (%)	$\mathrm{CE}(V_T^*)$ (%)	RMSE (%)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)
					Option positions in optimal portfolio with n options		
		$\mathrm{CE}(W_T^*) = \$348\,828$		$\mathrm{CE}(V_T^\infty) = \$345\,214$	RRA = 15		
0	0.0	16.2	97.3	124.2			
1	−36.9	76.2	99.1	82.9	−1 288		
					68		
2	−29.5	66.8	99.8	17.6	−994	−261	
					68	403	
3	−29.5	66.8	99.8	4.3	−996	−255	−49
					68	403	1 739
		$\mathrm{CE}(W_T^*) = \$327\,732$		$\mathrm{CE}(V_T^\infty) = \$325\,182$	RRA = 20		
0	0.0	11.6	97.7	101.2			
1	−29.8	60.3	99.3	67.1	−1 041		
					68		
2	−23.8	52.8	99.8	13.3	−807	−196	
					68	403	
3	−23.9	52.8	99.8	3.3	−809	−191	−33
					68	403	1 739

motivation for constructing a buy-and-hold alternative to the standard optimal dynamic asset-allocation policy. However, the inclusion of a few well-chosen short-maturity options from time to time in an otherwise passive buy-and-hold portfolio might be a very cost-effective and efficient alternative to the optimal dynamic policy, and we are investigating this possibility in our current research program.

Another issue that arises in the practical implementation of the optimal buy-and-hold strategy is computational challenges associated with the optimization procedure. As discussed in section 5, there are limits to the number of subproblems that can be handled in a reasonable amount of time, which imposes limits on the number of possible strikes that can be considered, as well as the number of options n in the buy-and-hold portfolio. In our numerical examples, we have made no attempt to optimize our algorithm for numerical and computational efficiency, preferring instead to maintain consistency across examples to facilitate comparisons. For example, when solving for the optimal buy-and-hold portfolio with $n = 1$ option, there was no need to limit ourselves to just 45 possible strikes. In fact, this problem can be solved very efficiently even if we were to consider several thousand possible strikes. In addition, by selecting the range of strikes as a function of the relative risk-aversion parameter, it is possible to obtain considerably better results than those of tables 1–6[25]. Therefore, the numerical results of section 5 should be taken as illustrative only, and not necessarily indicative of the best possible performance of the optimal buy-and-hold portfolios.

[25] Specifically, having selected the N possible strikes, we solve the $\binom{N}{n}$ subproblems as described in section 5.1. Once this is completed, we use the strikes $\{\hat{k}_i\}$ from the subproblem with the best optimum to select another set of N possible strikes. This new set of N strikes spans a smaller interval than the original set, but still contains $\{\hat{k}_i\}$. We then solve another $\binom{N}{n}$ subproblems and select the best optimum as our solution, and denote the corresponding strikes as $\{k_i^*\}$. This two-stage procedure for determining the set of possible strikes often yields significant improvements in the objective function.

6.2. Other derivative securities

For simplicity, we have used only European call options in our buy-and-hold strategies. A natural extension is to include more complex derivatives, perhaps with path dependences such as knock-out or average-rate options. This extension may be especially relevant in the presence of predictability, since in such cases we cannot attain $\mathrm{CE}(W_T^*)$ with a buy-and-hold strategy even if we include an infinite number of European options. In fact, the specific form of predictability may suggest a class of derivatives that are particularly suitable. For example, in the case of the trending Ornstein–Uhlenbeck process (4.8), it seems reasonable to conjecture that derivatives whose payoffs depend on

$$\int_0^T h(|X_t - X_0 - \mu t|)\,\mathrm{d}t \tag{6.1}$$

for some function $h(\cdot)$ would be most useful for approximating W_T^* in a buy-and-hold portfolio. This should be true more generally for other mean-reverting stock-price processes. On the other hand, if the stock-price process displays some type of persistence or 'momentum', a different class of derivatives might be more appropriate.

6.3. Other preferences and price processes

Although we have confined much of our analysis in sections 4 and 5 to the special cases of CRRA and CARA preferences under three specific price processes, we wish to emphasize that

Table 6. Mean-square-optimal buy-and-hold portfolios of stocks, bonds, and n European call options for CRRA utility under a bivariate linear diffusion stock-price process with parameters $(\sigma_1, \sigma_2, \rho, \kappa, \theta)$ of the steady-state distribution calibrated to match the following moments: $\mathrm{E}[\log(P_t/P_{t-1})]=0.15$, $\mathrm{Var}[\log(P_t/P_{t-1})]=0.04$, $\mathrm{Var}[\mu_t]=0.025^2$, $\mathrm{Corr}[\mu_t, \mu_{t-1}]=0.05$, and $\rho=0$. Other calibrated parameters include: riskless rate $r=5\%$, initial stock price $P_0 = \$50$, initial wealth $W_0=\$100\,000$, and time period $T=20$ years. 'RRA' denotes the coefficient of relative risk aversion, '$\mathrm{CE}(W_T^*)$' denotes the certainty equivalent of the optimal dynamic stock/bond policy, '$\mathrm{CE}(V_T^\infty)$' denotes the certainty equivalent of the optimal buy-and-hold portfolio with a continuum of options, and '$\mathrm{CE}(V_T^*)$' denotes the certainty equivalent of the optimal buy-and-hold portfolio with a finite number n of options, reported as a percentage of $\mathrm{CE}(V_T^\infty)$.

					Option positions in optimal portfolio with n options				
n	Options (%)	Stock (%)	$\mathrm{CE}(V_T^*)$ (%)	RMSE (%)	Quantity Strike (\$)	Quantity Strike (\$)	Quantity Strike (\$)	Quantity Strike (\$)	Quantity Strike (\$)
			$\mathrm{CE}(W_T^*) = \$11\,861\,394$		$\mathrm{CE}(V_T^\infty) = \$10\,142\,498$		RRA = 1 (Log utility)		
0	0.0	100.0	19.8	4 346.6					
1	0.2	99.8	20.0	3 579.8	35.36×10^{-6}				
					14 771				
2	100.0	0.0	0.0	3 578.5	247 608	34.8×10^{-6}			
					1 071	14 771			
3	2.8	97.2	22.7	3 564.7	2 695 502	-75.6×10^{-6}	108.2×10^{-6}		
					5 415	14 437	14 771		
4	8.1	91.9	27.7	3 563.8	1 023 256	3412 389	-96.3×10^{-6}	127.2×10^{-6}	
					3 410	8 756	14 437	14 771	
5	15.6	84.4	34.5	3 563.7	497 357	1 586 801	3 353 435	-103.3×10^{-6}	133.3×10^{-6}
					2 407	6 083	10 427	14 437	14 771
			$\mathrm{CE}(W_T^*) = \$1778\,906$		$\mathrm{CE}(V_T^\infty) = \$1645\,135$		RRA = 2		
0	0.0	100.0	81.8	214.7					
1	0.5	99.5	82.9	62.2	16 875				
					2 407				
2	2.0	98.0	85.0	29.3	11 060	18 392			
					1 405	10 761			
3	7.6	92.4	89.3	17.9	7 084	8 847	17 361		
					737	4 412	14 771		
4	19.8	66.3	88.7	16.0	5 044	5 127	6 858	16 539	
					403	2 073	5 415	14 771	
5	17.5	82.5	94.0	15.5	4 417	4 021	4 093	4 828	15 662
					403	1 739	3 744	6 751	14 771
			$\mathrm{CE}(W_T^*) = \$575\,004$		$\mathrm{CE}(V_T^\infty) = \$557\,315$		RRA = 5		
0	0.0	24.6	85.0	27.9					
1	−0.1	42.5	94.3	7.7	−536				
					1 405				
2	−0.6	50.6	96.6	3.5	−546	−223			
					737	4 078			
3	−2.4	61.5	98.4	2.1	−615	−243	−158		
					403	1 739	6 417		
4	−2.2	60.2	98.3	1.3	−561	−215	−133	−109	
					403	1 405	3 744	10 427	
5	−2.0	58.8	98.1	1.0	−493	−189	−132	−101	−91
					403	1 071	2 407	5 415	13 768

the optimal buy-and-hold portfolio can be derived for many other preferences and price processes. For example, the class of hyperbolic absolute risk-aversion (HARA) preferences can be accommodated, as well as any price process for which the conditional state-price density can be computed. Even more general preferences and price processes are allowable at the expense of computational complexity. For example, for price processes that do not admit closed-form expressions for the conditional state-price densities, these can be estimated non-parametrically as in Aït-Sahalia and Lo (1998).

6.4. The predictability gap

As we have seen in sections 4.2 and 4.3, in the presence of predictability in the stock-price process, buy-and-hold portfolios of stocks, bonds and European call options cannot

Table 6. Continued.

					Option positions in optimal portfolio with n options				
n	Options (%)	Stock (%)	$\text{CE}(V_T^*)$ (%)	RMSE (%)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)	Quantity Strike ($)
			$\text{CE}(W_T^*) = \$395\,204$		$\text{CE}(V_T^\infty) = \$389\,080$		RRA = 10		
0	0.0	6.7	86.5	26.9					
1	−0.1	17.7	94.9	6.4	−295				
					1 071				
2	−1.7	29.8	98.0	3.0	−446	−107			
					403	2 407			
3	−1.5	27.8	97.7	1.7	−371	−101	−53		
					403	1 405	4 746		
4	−1.4	27.2	97.6	1.3	−340	−93	−54	−32	
					403	1 071	2 742	7 754	
5	−68.4	140.1	96.4	0.9	−2 357	−235	−100	−53	−33
					68	403	1 071	2 742	7 754
			$\text{CE}(W_T^*) = \$348,828$		$\text{CE}(V_T^\infty) = \$345\,214$		RRA = 15		
0	0.0	3.7	89.7	21.4					
1	−0.3	13.9	97.1	5.0	−246				
					737				
2	−1.2	19.1	98.3	2.2	−304	−60			
					403	2 407			
3	−1.0	17.9	98.1	1.3	−257	−59	−27		
					403	1 405	4 746		
4	−47.8	96.7	98.5	1.0	−1 644	−188	−62	−27	
					68	403	1 405	4 746	
5	−51.5	102.6	97.9	0.7	−1 776	−162	−56	−29	−19
					68	403	1 071	2 407	6 417
			$\text{CE}(W_T^*) = \$327\,732$		$\text{CE}(V_T^\infty) = \$325\,182$		RRA = 20		
0	0.0	2.5	91.8	17.5					
1	−0.2	9.9	97.6	3.9	−179				
					737				
2	−0.9	14.0	98.6	1.7	−228	−41			
					403	2 407			
3	−0.8	13.2	98.4	1.1	−197	−40	−18		
					403	1 405	4 412		
4	−38.2	76.2	98.9	0.8	−1 316	−140	−44	−18	
					68	403	1 405	4 746	
5	−40.9	80.5	98.5	0.5	−1 411	−121	−40	−20	−12
					68	403	1 071	2 407	6 417

approximate W_T^* arbitrarily well, even as the number of options increases without bound. We use the term 'predictability gap' to denote the difference between $\text{CE}(W_T^*)$ and $\text{CE}(V_T^\infty)$, which depends on the investor's preferences as well as the parameters of the stock-price process.

The natural question to ask is how significant is this predictability gap? Given that the end-of-period wealth W_T^* of the optimal dynamic asset-allocation policy is generally unattainable due to transaction costs and other market frictions, $\text{CE}(W_T^*)$ can be viewed as a theoretical upper bound on how well an investor can do. On the other hand, if V_T^∞ is well approximated by an optimal buy-and-hold portfolio V_T^* with just a few options, it is more likely to be attainable in practice given that only a few trades are required to establish the portfolio and there are few costs to bear thereafter. Therefore, if the predictability gap is small, the buy-and-hold portfolio may well be optimal even in the presence of predictable stock returns. To investigate this possibility, we must consider the impact of transaction costs on $\text{CE}(W_T^*)$.

Most of the studies in the transaction costs literature ignore predictability, assuming independently and identically distributed (IID) returns instead[26]. Such studies may underestimate the impact of transaction costs because the

[26] For example, Magill and Constantinides (1976), Constantinides (1986), Davis and Norman (1990), Dumas and Luciano (1991) and Gennotte and Jung (1992) all assume IID return-generating processes.

presence of predictability provides another motive for trade, i.e. time-varying investment opportunity sets. Therefore, we might expect transaction costs—as a percentage of initial wealth W_0—to be higher if stock returns are predictable.

Balduzzi and Lynch (1999) do consider transaction costs in the case of predictability, computing the impact on an investor's expected utility when transaction costs exist but are ignored by the investor. They do not report the difference between the certainty equivalent of the optimal asset-allocation policy in an economy without transaction costs and the certainty equivalent of the optimal policy in an economy with transaction costs (though their framework should allow them to do so easily). They do mention, however, that '... the presence of transaction costs ... decreases the utility cost of ignoring predictability'. This suggests that $\mathrm{CE}(W_T^*)$ might be reduced significantly, reducing the predictability gap and providing more compelling motivation for the optimal buy-and-hold portfolio.

An even more compelling motivation for the optimal buy-and-hold portfolio is the presence of taxes. For taxable investors, $\mathrm{CE}(W_T^*)$ is reduced by the present value of the sequence of capital gains taxes that are generated by an optimal dynamic asset-allocation strategy. In contrast, all of the capital gains taxes are deferred until date T in a buy-and-hold portfolio. Therefore, the economic value of predictability is likely to be even lower for taxable investors, and the optimal buy-and-hold portfolio that much more attractive.

7. Conclusion

In this paper, we compare optimal buy-and-hold portfolios of stocks, bonds and options to the standard optimal dynamic asset-allocation policies involving only stocks and bonds. Under certain conditions, buy-and-hold portfolios are excellent approximations—in terms of certainty equivalence and mean-squared-error of end-of-period wealth—to their dynamic counterparts, suggesting that in those cases, dynamic trading strategies may be 'automated' by simple buy-and-hold portfolios with just a few options. Cases where the approximation breaks down are also of interest, since such situations highlight the importance of dynamic trading opportunities.

There are a number of extensions of this research that may be worth pursuing. The most obvious is to perform similar analyses for other stochastic processes and preferences, those that are more consistent with the empirical evidence. The main challenge in this case is, of course, tractability and computational complexity.

A more important extension is to consider approximating other dynamic investment strategies with buy-and-hold portfolios of derivatives. Although we focus on optimal dynamic asset-allocation strategies in this paper, there is no reason to confine our attention to such a narrow class of strategies. For example, deriving optimal buy-and-hold strategies to approximate dollar-cost averaging strategies or other popular dynamic investment strategies—strategies that need not be based on expected utility maximization—might be of considerably broader interest.

Finally, the composition of the optimal buy-and-hold portfolio provides an interesting summary of the risk exposures of the optimal dynamic asset-allocation policy that the buy-and-hold portfolio approximates. By examining the payoff structure of the optimal buy-and-hold portfolio, and its sensitivities to various market factors and economic shocks, we can develop insights into the risks of dynamic investment policies using measures computed at a single point in time. We hope to explore these and other extensions in the near future.

Acknowledgments

This research was partially supported by the MIT Laboratory for Financial Engineering and the National Science Foundation (grant no SBR–9709976). We thank Michael Dempster, Doyne Farmer, Leonid Kogan, Jiang Wang and participants at the Boston University Derivative Securities Conference and the MathSoft Conference on Statistical Modeling and Computation in Finance for valuable discussions and comments.

Appendix A

In this appendix, we derive the optimal value function $J(\cdot)$ from the Hamilton–Jacobi–Bellman equation (4.10) for the trending and standard Ornstein–Uhlenbeck processes.

Appendix A.1. Trending Ornstein–Uhlenbeck value function

We present here the solution to the Hamilton–Jacobi–Bellman equation (4.10) of section 4.2. Recall that this equation is given by:

$$\begin{aligned} 0 = \mathrm{Max}_{\omega_t} \Big\{ & J_t + W_t J_W \big(r + \omega_t[-\delta(X_t - \mu t - X_0) \\ & + \mu + \tfrac{1}{2}\sigma^2 - r]\big) + J_X(-\delta(X_t - \mu t - X_0) + \mu) \\ & + \tfrac{1}{2}\omega_t^2 \sigma^2 W_t^2 J_{WW} + \tfrac{1}{2}\sigma^2 J_{XX} + \sigma^2 \omega_t W_t J_{XW} \Big\}. \end{aligned} \tag{A.1}$$

Solving for ω_t and substituting back into (A.1) yields the following partial differential equation (PDE):

$$\begin{aligned} 0 = & J_t J_{WW} + [-\delta(X_t - \mu t - X_0) + \mu] J_X J_{WW} \\ & + r W J_W J_{WW} + \frac{\sigma^2}{2} J_{XX} J_{WW} - \frac{\sigma^2}{2} J_{XW}^2 \\ & - \left(-\delta(X_t - \mu t - X_0) + \mu + \frac{\sigma^2}{2} - r\right) J_W J_{XW} \\ & - \frac{\left(-\delta(X_t - \mu t - X_0) + \mu + \frac{\sigma^2}{2} - r\right)^2}{2\sigma^2} J_W^2 \end{aligned} \tag{A.2}$$

subject to $J(W, X, T) = U(W)$. We solve this PDE by conjecturing that

$$J(W, X, t) = U(W \exp[r(T-t)]) \exp(\alpha(t) + \beta(t)X + \zeta(t)X^2)$$

where $\alpha(T) = \beta(T) = \zeta(T) = 0$. Therefore solving the PDE reduces to solving three ordinary differential equations. We then solve these differential equations for $\alpha(t)$, $\beta(t)$ and $\zeta(t)$.

CRRA utility

For an investor with the CRRA utility function, $U(W) = W^{\gamma}/\gamma$, it is only possible to solve explicitly for $\beta(t)$ and $\zeta(t)$. Solving for $\alpha(t)$ required evaluating a number of definite integrals for which there did not seem to be analytic solutions. These integrals are easy to solve numerically, however, and it is therefore possible to find a very good numerical solution to the value function, $J(W, X, t)$. We present here the solutions for $\beta(t)$ and $\zeta(t)$. Let

$$\begin{aligned} a &= 1+\sqrt{1-\gamma}, & b &= 1-\sqrt{1-\gamma}, \\ q &= \tfrac{-2\delta}{\sqrt{1-\gamma}}, & H &= \tfrac{-\gamma\delta^2\mu}{\sigma^2\sqrt{1-\gamma}} \\ I &= \tfrac{(2r-\sigma^2)\gamma\delta}{2\sigma^2}, & J &= \tfrac{\gamma\delta(2r-2\mu-\sigma^2)}{2\sigma^2\sqrt{1-\gamma}}, \\ K &= \tfrac{-\gamma\delta\mu}{\sigma^2}. \end{aligned} \tag{A.3}$$

Then $\beta(t)$ and $\zeta(t)$ are given by:

$$\begin{aligned} \beta(t) = \frac{\sqrt{1-\gamma}}{\delta(a-b\exp[q(T-t)])}\Bigg[&(Ht+K+I+J) \\ &-(Ht-K+J-I)\exp(q(T-t)) \\ &-2(K+I)\exp\left(\frac{q(T-t)}{2}\right)\Bigg] \end{aligned} \tag{A.4}$$

$$\zeta(t) = \frac{\gamma\delta}{2\sigma^2}\left[\frac{1-\exp(q(T-t))}{a-b\exp(q(T-t))}\right]. \tag{A.5}$$

CARA utility

For the CARA utility function, $U(W) = -\exp(-\gamma W)/\gamma$, $\alpha(t)$, $\beta(t)$, $\zeta(t)$ are:

$$\alpha(t) = \frac{\Gamma_1\left(t^3-T^3\right)}{3} + \frac{\Gamma_2\left(t^2-T^2\right)}{2} + \Gamma_3(t-T) \tag{A.6}$$

$$\beta(t) = \Delta_1\left(t^2-T^2\right) + \Delta_2(T-t) \tag{A.7}$$

$$\zeta(t) = \frac{\delta^2(t-T)}{2\sigma^2} \tag{A.8}$$

where

$$\Delta_1 = \frac{\delta^2}{4} - \frac{r\delta^2}{2\sigma^2} - \frac{\delta^2\mu}{2\sigma^2} \tag{A.9}$$

$$\Delta_2 = \frac{\delta}{2\sigma^2}\left(\delta\sigma^2 T - 2r\delta T + 2\mu + \sigma^2 - 2r\right) \tag{A.10}$$

$$\Gamma_1 = \frac{\delta^2\mu^2}{2\sigma^2} + \left(\frac{\sigma^2}{2} - r\right)\Delta_1 \tag{A.11}$$

$$\Gamma_2 = \frac{\delta\mu}{\sigma^2}\left(\mu - r + \frac{\sigma^2}{2}\right) - \left(\frac{\sigma^2}{2} - r\right)\Delta_2 - \frac{\delta^2}{2} \tag{A.12}$$

$$\begin{aligned} \Gamma_3 = &\frac{\left(\mu - r + \frac{\sigma^2}{2}\right)^2}{2\sigma^2} + \Delta_1 T^2\left(r - \frac{\sigma^2}{2}\right) \\ &- \Delta_2 T\left(r - \frac{\sigma^2}{2}\right) + \frac{\delta^2 T}{2}. \end{aligned} \tag{A.13}$$

Appendix A.2. Non-trending Ornstein–Uhlenbeck value function

Recall that $X_t \equiv \log P_t$ and let X_t satisfy the following stochastic differential equation:

$$\mathrm{d}X_t = -\delta(X_t - \alpha)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t \tag{A.14}$$

where α and δ are both positive. The solution to (A.14) is given by:

$$X_t = \alpha + \exp(-\delta t)[X_0 - \alpha] + \sigma\exp(-\delta t)\int_0^t \exp(\delta s)\,\mathrm{d}B_s \tag{A.15}$$

and the corresponding Hamilton–Jacobi–Bellman equation is given by

$$\begin{aligned} 0 = \mathrm{Max}_\omega \Bigg[& J_t + WJ_W\left((1-\omega)r - \delta\omega(\log P - \alpha) + \frac{\omega\sigma^2}{2}\right) \\ & + PJ_P\left(\frac{\sigma^2}{2} - \gamma(\log P - \alpha)\right) + \frac{1}{2}W^2\sigma^2 J_{WW}\omega^2 \\ & + \frac{1}{2}P^2\sigma^2 J_{PP} + WP\sigma^2\omega J_{PW}\Bigg]. \end{aligned} \tag{A.16}$$

We solve this PDE by conjecturing that the value function is of the form:

$$J(W, X, t) = U(W\exp[r(T-t)])\exp(\alpha(t)+\beta(t)X+\zeta(t)X^2) \tag{A.17}$$

where $\alpha(T) = \beta(T) = \zeta(T) = 0$.

CRRA utility

For $U(W) = W^{\gamma}/\gamma$, we have the following system of ODEs:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\gamma}{\gamma-1}\left[\frac{\sigma^2}{2\delta}\beta^2 + \frac{\Delta_3}{2}\beta + \frac{\Delta_2}{2\sigma^2} - \sigma^2\zeta\right] \tag{A.18}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \left(2\sigma^2\zeta - \delta\right)\frac{\beta}{\gamma-1} + \frac{\gamma}{\gamma-1}\left[\Delta_3\zeta + \frac{\Delta_1}{2\sigma^2}\right] \tag{A.19}$$

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \frac{2\sigma^2}{\gamma-1}\zeta^2 - \frac{2\delta}{\gamma-1}\zeta + \frac{\delta^2\gamma}{2\sigma^2(\gamma-1)} \tag{A.20}$$

where

$$\Delta_1 = 2r\delta - \sigma^2\delta - 2\alpha\delta^2 \tag{A.21}$$

$$\Delta_2 = \sigma^4/4 + \alpha^2\delta^2 + r^2 - r\sigma^2 + \sigma^2\delta\alpha - 2r\delta\alpha \tag{A.22}$$

$$\Delta_3 = \sigma^2 - 2r + \frac{2\alpha\delta}{\gamma} \tag{A.23}$$

Then

$$\begin{aligned} \beta(t) = &\left(2(d_1+d_2) - d_1\left(\mathrm{e}^{s(T-t)} + \mathrm{e}^{-s(T-t)}\right) - d_2\left(a\mathrm{e}^{s(T-t)}\right.\right. \\ &\left.\left. + b\mathrm{e}^{-s(T-t)}\right)\right)\left(s\left(a\mathrm{e}^{s(T-t)} - b\mathrm{e}^{-s(T-t)}\right)\right)^{-1} \end{aligned} \tag{A.24}$$

$$\zeta(t) = \frac{\delta\gamma}{2\sigma^2}\left[\frac{1-\mathrm{e}^{q(T-t)}}{a-b\mathrm{e}^{q(T-t)}}\right] \tag{A.25}$$

where

$$\begin{aligned} d_1 &= \frac{\gamma^2\Delta_3\delta}{2(\gamma-1)\sigma^2}, & d_2 &= \frac{\gamma\Delta_1}{2(\gamma-1)\sigma^2} \\ a &= 1+\sqrt{1-\gamma}, & b &= 1-\sqrt{1-\gamma} \\ s &= \frac{\delta(a-\gamma)}{(1-\gamma)a}, & q &= \frac{-2\delta}{\sqrt{1-\gamma}} \end{aligned} \tag{A.26}$$

To define $\alpha(t)$, let:

$$\begin{aligned} f_1 &= \frac{\sigma^2}{2(\gamma-1)}, & f_2 &= \frac{\gamma\Delta_3}{2(\gamma-1)} \\ f_3 &= \frac{\gamma\Delta_2}{2(\gamma-1)\sigma^2}, & \rho_1 &= \frac{2(d_1+d_2)}{s} \\ \rho_2 &= \frac{-(d_1+ad_2)}{s}, & \rho_3 &= \frac{-(d_1+bd_2)}{s} \end{aligned} \tag{A.27}$$

and

$$I_1 = \left[2as\left(ae^{2s(T-t)} - b\right)\right]^{-1} \quad (A.28)$$

$$I_2 = \frac{-1}{2s}\left[\frac{1}{a^2}\log\left(\frac{a - be^{-2s(T-t)}}{be^{-2s(T-t)}}\right) - \frac{1}{a\left(a - be^{-2s(T-t)}\right)}\right] \quad (A.29)$$

$$I_3 = \frac{-1}{2s}\left[\frac{1}{b^2}\log\left(\frac{ae^{2s(T-t)}}{ae^{2s(T-t)} - b}\right) - \frac{1}{b\left(ae^{2s(T-t)} - b\right)}\right] \quad (A.30)$$

$$I_4 = \frac{1}{2s}\left[\frac{e^{s(T-t)}}{b\left(ae^{2s(T-t)} - b\right)} + \frac{1}{b\sqrt{-ab}}\tan^{-1}\left(\sqrt{\frac{-a}{b}}e^{s(T-t)}\right)\right] \quad (A.31)$$

$$I_5 = \frac{1}{2s}\left[\frac{e^{s(T-t)}}{a\left(ae^{2s(T-t)} - b\right)} - \frac{1}{a\sqrt{-ab}}\tan^{-1}\left(\sqrt{\frac{-a}{b}}e^{s(T-t)}\right)\right] \quad (A.32)$$

$$I_7 = \frac{-1}{s\sqrt{-ab}}\tan^{-1}\left(\sqrt{\frac{-a}{b}}e^{s(T-t)}\right) \quad (A.33)$$

$$I_6 = I_1 \quad (A.34)$$

$$I_8 = \frac{-1}{2as}\log\left(ae^{2s(T-t)} - b\right) \quad (A.35)$$

$$I_9 = \frac{-1}{2bs}\log\left(a - be^{-2s(T-t)}\right) \quad (A.36)$$

$$I_{10} = \frac{\delta\gamma}{2\sigma^2}\left[\frac{1}{aq}\log\left(ae^{-q(T-t)} - b\right) - \frac{1}{bq}\log\left(a - be^{q(T-t)}\right)\right] \quad (A.37)$$

Then

$$\alpha(t) = f_1\left[\rho_1^2 I_1 + \rho_2^2 I_2 + \rho_3^2 I_3 + 2\rho_1\rho_2 I_4 + 2\rho_1\rho_3 I_5 + 2\rho_2\rho_3 I_6\right] + f_2\left[\rho_1 I_7 + \rho_2 I_8 + \rho_3 I_9\right] - \sigma^2 I_{10} + f_3 t + G \quad (A.38)$$

where G is the constant defined by the condition $\alpha(T) = 0$.

CARA utility

The solution of (A.16) for CARA utility, $U(W) = -\exp(-\gamma W)/\gamma$ is given by:

$$\alpha(t) = \frac{\delta^2}{6\sigma^2}\left(\frac{\sigma^2}{2} - r\right)^2 (t-T)^3 + \left[\frac{\Delta_1}{4\sigma^2}\left(\frac{\sigma^2}{2} - r\right) - \frac{\delta^2}{4}\right](T-t)^2 - \frac{\Delta_2}{2\sigma^2}(T-t) \quad (A.39)$$

$$\beta(t) = \frac{\delta^2}{2}\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)(t-T)^2 - \frac{\Delta_1}{2\sigma^2}(T-t) \quad (A.40)$$

$$\zeta(t) = \frac{\delta^2}{2\sigma^2}(t-T) \quad (A.41)$$

where the solution is of the form:

$$J(W, X, t) = U\left(W\exp[r(T-t)]\right)\exp(\alpha(t) + \beta(t)X + \zeta(t)X^2). \quad (A.42)$$

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