

## Grundy's Game - Periodic?

We are going to try and look at an open problem in the field of combinatorial game theory each week. This week we look at Grundy's Game.

### Rules of Play

Start with a heap of  $n$  coins. Players take turns splitting a heap into unequal piles. Play continues until there are only heaps of size 0, 1, and 2 left, meaning no more piles can be split into unequal piles.

### Nim values

The following begins to list the Nim-values of Grundy's game heaps of size  $n$ :

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12
$SG(n)$	0	0	0	1	0	2	1	0	2	1	0	2	1

The sequence continues 3,2,1,3,2,4,3,0, . . . . As you can see, the sequence appears to start getting periodic with period 3. In fact, values for  $n$  up to  $2^{35}$  have been calculated, but it is still not known whether this sequence is periodic or not. (You can even look it up in the "Online Encyclopedia of Integer Sequences... who knew such a site existed?").

It is conjectured in *Winning Ways* that this sequence eventually does become periodic. We discuss in class their reasons for thinking this.

## More Impartial Games

Try the following games, most of which can be analyzed as being exactly equivalent to games of Nim or sums of games of Nim.

### 21 Takeaway

Start with a pile of 21 coins. Each player alternates taking anywhere from one to three coins away from the pile. The player to take the last coin wins.

## Turning Turtles

A horizontal line of  $n$  coins is laid out randomly with some coins showing heads and some tails. A move consists of turning over one of the coins from heads to tails, and in addition, if desired, turning over one other coin to the left of it (from heads to tails or tails to heads). For example consider the sequence of  $n = 13$  coins:

T	H	T	T	H	T	T	T	H	H	T	H	T
1	2	3	4	5	6	7	8	9	10	11	12	13

One possible move in this position is to turn the coin in place 9 from heads to tails, and also the coin in place 4 from tails to heads.

The player to turn the last coin from heads to tails wins.

## Nimble

Nimble is played on a game board consisting of a line of squares labeled: 0, 1, 2, 3, . . . . A finite number of coins is placed on the squares with possibly more than one coin on a single square. A move consists of taking one of the coins and moving it to any square to the left, possibly moving over some of the coins, and possibly onto a square already containing one or more coins. The players alternate moves and the game ends when all coins are on the square labeled 0. The last player to move wins.

## Silver Dollar Game

From John Conway's "On Numbers and Games": This game is played on a semi-infinite strip of squares, with a finite number of coins, no one of which is a Silver Dollar. Each coin is placed on a separate square, and the legal move is to move some coin leftwards (i.e. towards the finite end of the strip), not passing over any other coin, onto any unoccupied square. The game ends when some player has no legal move, because the coins are in a traffic jam at the end of the strip.

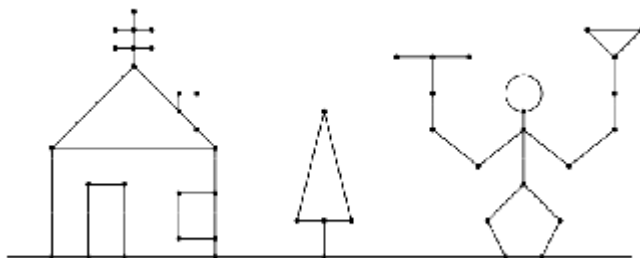
## 2D Nim

This game is played exactly like Nimble, except on a 2-dimensional checkerboard. Start with 4 coins placed anywhere on the board. On each turn a player moves a coin either any number of squares downward or any number of squares left. The player to move the last coin to the bottom left corner square wins.

## Green Hackenbush

Two players take turns cutting edges on a connected rooted graph or a collection of connected rooted graphs. For our purposes, there will only be finitely many edges on the board, we will associate all the roots with "the ground" and we will call the edges "branches". When a player cuts a branch, the branch disappears along with any branches that are no longer connected to the ground. The player who cuts the last branch wins.

The following is an example game of hackenbush:



## Sprouts

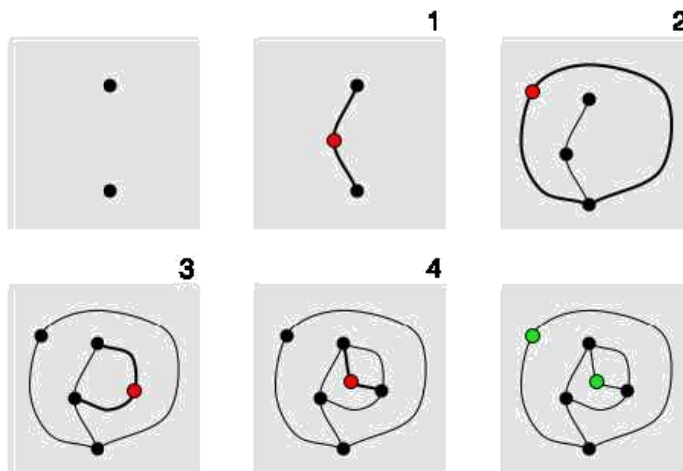
(This game is not so easily related back to Nim, and has not even been completely solved, but nevertheless has some interesting theory behind it so we present it here.)

In the game of sprouts two players start with some number of dots and alternate making moves. A move consists of connecting two dots (called spots)

with a curve and marking a new dot anywhere on the curve. The segments of curves connecting two dots are called edges. subject to the following rules:

1. The curves do not intersect (other curves or themselves).
2. No more than three edges emanate from any one spot.
3. A curve may connect a spot to itself.

The player who draws the last curve wins. Below is an example game:



## Northcott's Game

A position in Northcott's game is a checkerboard with one black and one white checker on each row. "White" moves the white checkers and "Black" moves the black checkers. A checker may move any number of squares along its row, but may not jump over or onto the other checker. Players move alternately and the last to move wins.

## Solutions to Some of the Games

These hints will help you solve the games once you know the secret to Nim.

## Northcott's Game

The the number of spaces between the two tokens on each row are the sizes of the Nim heaps. This is the same as Poker Nim. If your opponent increases the number of spaces between two tokens, just decrease it on your next move. Else, play the game of Nim and make the Nim-sum of the number of spaces between the tokens on each row be 0.

## 21 Takeaway

Always make the size of the heap  $0 \pmod 4$  (so 20, 16, 12, etc.)

In general, if players can take at most  $k$  coins away, always try to make the size of the pile  $0 \pmod k + 1$ .

## Turning Turtles

This game is essentially the same as Nim. If the turtles are numbered 1, ...,  $n$  from left to right, then a Turning Turtles position where the coins with heads showing are the ones with numbers  $a, b, \dots, z$  is equivalent to a Nim position with heaps of size  $a, b, \dots, z$ . The correspondence between moves is as follows:

1. A Nim move which removes a heap of size  $a$  corresponds to flipping over the coin numbered  $a$  from heads to tails.
2. A Nim move which decreases a heap from size  $a$  to size  $b$ , where there was no heap of size  $b$  already on the board, corresponds to flipping over the coin numbered  $a$  from heads to tails, and flipping over the coin numbered  $b$  from tails to heads.
3. A Nim move which decreases a heap from size  $a$  to size  $b$ , where there was a heap of size  $b$  already on the board, corresponds to flipping over the coin numbered  $a$  from heads to tails, and flipping over the coin numbered  $b$  from heads to tails.

In the last case, the Nim position arrived at, as compared to the Turning Turtles position arrived at, has an extra pair of heaps of size  $b$ . However, the presence of a pair of equal heaps, or any number of pairs of equal heaps, never makes any difference to the outcome of a Nim game. This is because whoever

had the win before the pairs of heaps were added can continue to play this strategy on the heaps originally present, and when the other player plays on one of the pairs of equal heaps, he can make the same play on the heap equal to the heap the other player moved on. In this way he is guaranteed the last move in the pairs of equal heaps, and as he also has the last move on the heaps originally present, he has the last move in the overall game.

## Nimble

A coin on square  $n$  is the same as a nim-heap of size  $n$ .

## Silver Dollar Nim

This one is kind of tricky. Starting from the rightmost coin, count the number of squares in alternate spaces between the coins, and let these numbers be the sizes of Nim-heaps.

## 2D Nim

This is the sum of 2 Nimble (Nim) games: one in the vertical direction and one in the horizontal direction. So a square at  $(x,y)$  is a heap of size  $x$  in the horizontal game and size  $y$  in the vertical game. By the Sprague-Grundy theorem, you can calculate the SG value of the  $x$  and  $y$  games as  $SG(x) + SG(y)$ , where  $SG(x)$  is just the num sum of all the  $x$  coordinates.

## Green Hackenbush

(Much more about this game next week)

When played only with “bamboo stalks” (see pictures) each stalk with  $n$  segments is the same as a nim heap of size  $n$ .

When played with a forest of “trees” (**Colon Principle**): When branches come together at a vertex, one may replace the branches by a non-branching stalk of length equal to their nim sum.

For general graphs: We fuse two neighboring vertices by bringing them together into a single vertex and bending the edge joining them into a loop.

Each loop can now be seen as a nim-heap of size 1.

## Sprouts

The analysis of this game hasn't actually been solved for arbitrarily large cases. What is known is below:

Suppose that a game starts with  $n$  spots and lasts for exactly  $m$  moves.

Each spot starts with three lives (opportunities to connect a line) and each move reduces the total number of lives in the game by one (two lives are lost at the ends of the line, but the new spot has one life). So at the end of the game there are  $3n - m$  remaining lives. Each surviving spot has only one life (otherwise there would be another move joining that spot to itself), so there are exactly  $3n - m$  survivors. There must be at least one survivor, namely the spot added in the final move. So  $3n - m \geq 1$ ; hence a game can last no more than  $3n - 1$  moves.

At the end of the game each survivor has exactly two dead neighbors, in a technical sense of "neighbor"; see the diagram below.



No dead spot can be the neighbor of two different survivors, for otherwise there would be a move joining the survivors. All other dead spots (not neighbors of a survivor) are called pharisees (from the Hebrew for "separated ones"). Suppose there are  $p$  pharisees. Then

$$n + m = 3n - m + 2(3n - m) + p$$

since initial spots + moves = total spots at end of game = survivors +

neighbors + pharisees. Rearranging gives:

$$m = 2n + p/4$$

So a game lasts for at least  $2n$  moves, and the number of pharisees is divisible by 4.

Real games seem to turn into a battle over whether the number of moves will be  $m$  or  $m+1$  with other possibilities being quite unlikely. One player tries to create enclosed regions containing survivors (thus reducing the total number of moves that will be played) and the other tries to create pharisees (thus increasing the number of moves that will be played).