Water Towers

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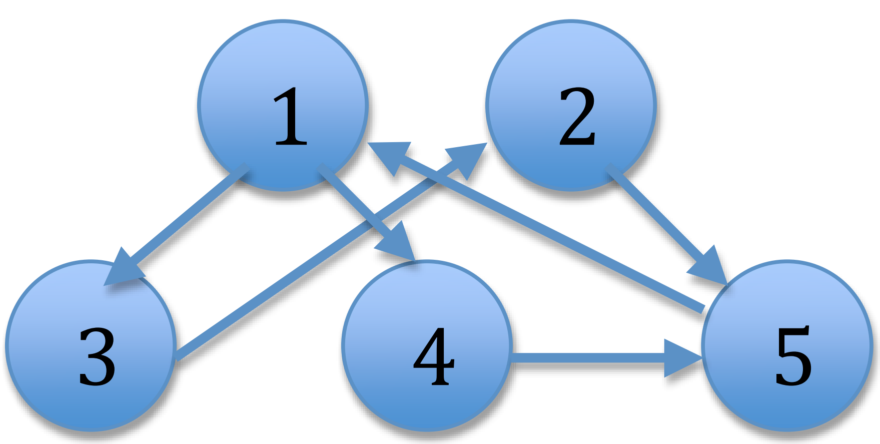
Description: Water Towers is a game based upon the principles of equilibrium. The set up consists of towers full of water and chutes that connect the towers together. The ultimate goal of each player is to be the one who creates a state of equilibrium in the system and can be played by any number of people individually or as part of teams. Upon each turn, each player or team has a variety of options. They can either build a tower, build a chute, change the direction of water flow, increase the rate of water flow in the same direction by one, or decrease the rate of water flow in the same direction by one. This game is the most fun when played with actual materials, but the construction of such a game is inhibiting considering the intricacy of maintaining constant water flow and monitoring a state of equilibrium.

Objective: The ultimate goal of the game is to be the person who creates equilibrium in the system. This means that at the end of that person's turn, the net flow in and out of every water tower is equal to zero. All water towers must also be connected by chutes, so that the system is closed.

Mathematical Model: In order to play this game, it is most effective to create a model. This model consists of nodes connected by edges. It is commonly referred to as a graph. Graphs of this kind can display a function in the form of f(x) but to play this game, it is more effective to create an incidence matrix with n nodes connected by m edges. In this game, m >n due to the nature of how equilibrium is achieved. If a matrix has more matrix has more nodes than edges, n>m, than the graph will be a tree with no closed loops. Although this is a fine intragame state, the final state must be closed. As a general mathematical principle, the maximum number of chutes will be $m = \frac{n(n-1)}{2}$, while the minimum will be $m = n - 1$, assuming that the graph is connected. In order to model the system, each column will represent a water tower and every row will represent a chute. Obviously, your matrix will be static and continually changing as the game progresses, which will involve imperfect anticipation of future action by your competitors. In order to model flow, one must consider that positive flow in one direction can only occur when negative flow occurs in the opposite direction. Conceptually, consider a log flume with water flowing in one direction. Traveling in the opposite direction of flow is not a neutral process, it involves an input of energy, and therefore must be modeled as being negative flow. To give an example of how a system would be modeled, consider the following diagram where numbers represent towers and the arrows represent chutes. For simplicity, we will also assume that the flow rate between towers is just one.



This would be modeled in matrix form as:

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix} = A$$

With the graph modeled in this form, it is very easy to see that rows are dependent when the edges form a closed loop system, while independent rows form a tree. Because of this, it is necessary for our matrix to be singular in order to have a solution to our equilibrium goal. From here we want to examine possible methods for solution, and the repercussions behind those methods.

For any graph matrix, A, we want the potential difference in water flow across each tower to be equal to zero. This is most easily accomplished by multiplying your graph matrix by another matrix called x in the form of m rows and 1 column.

Using the example from before:

$$Ax = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_3 - x_1 \\ x_4 - x_1 \\ x_2 - x_3 \\ x_5 - x_4 \\ x_5 - x_2 \end{pmatrix}$$

The resultant matrix is the matrix of potential differences of flow across each water tower, while x_1, x_2 ,x_3, x_4, and x_5 represent the potential flow at each water tower. In order to solve for the equilibrium state, we want to find the solution where $Ax = 0$.

This is the same as finding the nullspace for A, which is where all four potentials are equal. If a nullspace does not exist, than we know that the potential difference of water flow across the towers is not equal to zero, and a state of equilibrium will not be reached. Simply having a vector in the nullspace other than the trivial 0 vector is not enough for the system to be in equilibrium though. The second half of analysis involves summing every row to create one row. If this new row is a row of zeros, than it is assured that the net flow through every water tower is equal to zero. This concept can be seen very easily because each column is the sum of flow through each tower.

Strategy: The main strategy in this game is assuring that you are able to develop a mathematical model that accurately depicts what is occurring in the game. By doing this, you can continually assure that your move will not allow your opponent to win on the next turn. The true test is who is able to analyze the most complicated model mentally. In the face of uncertainty, there are always neutral moves, like building another tower, so this automatically removes the potential for a concrete strategy.