

Amoeboid: A Partisan Combinatorial Game of Chance

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Rules

Set up

This game can be played on any graph (or n -dimensional board). For simplicity we will consider a $n \times n$ Board. The game starts with two amoebae, of size 1, in diametrically opposite corners of the board; each player controls one amoeba.

Definitions

1. Amoeba:

It's a piece that has position, composition, size and owner.

(a) Position:

The place on the board (or node on a graph) occupied by this piece.

(b) Composition

The amoeba is composed of two parts represented by numbers a and b . Each number represents each player's control over the piece. It can also be thought as a set of a unit amoebae with $a = 1$, $b = 0$ and b unit amoebae with $a = 0$, $b = 1$.

(c) Size

The sum of the two numbers in the composition.

(d) Owner.

The player whose control number over the piece is bigger. If both numbers are equal the piece becomes neutral and has no owner.

Game play

The players A and B alternate taking turns. Each turn consist of the following two elements:

1. Dice roll The player rolls a six sided die, then picks one of their amoebae and increases his control number on it by the number on the die (subsequently the size increases by the same number).
2. Move

The player chooses a subset of unit amoebae from the set of a controlled amoeba and move them (as a new single amoeba) to an adjacent space; the adjacent space has to be:

 - (a) Empty: In this case we say the player moved
 - (b) Occupied by an amoeba of smaller or equal size (of moving amoeba). This results in a single amoeba whose control numbers are summed: In this case we say the player eats an amoeba.

End Game

The game ends when one does not own any of the pieces on the board. If a player owns any amoeba at the end of the game he wins; otherwise it's a tie.

Analysis

Advantage:

In this section we consider two variables of the game that will give us a good idea of the development of the game and will be crucial for starting to grab a strategy.

1. Advantage 1 (simple total difference) The two players are rolling dice alternately, let σ_A and σ_B be the sum of the rolls for player A and B respectively in a given moment (If after three turns A has rolled 1,2,3 and B has rolled 4,5,6, $\sigma_A = 6$ and $\sigma_B = 15$). Also we define $\delta = \sigma_A - \sigma_B$.

Definition: We say player A has *advantage 1* over player B if $\delta > 0$, similarly we say B has *advantage 1* if $\delta < 0$.

Claim : Player A can only win if he has advantage 1 and player B can only win if he has advantage 1.

Proof: If a player wins (Let's say A), the other player does not control any amoebae thus in any single amoeba the control of player A is greater than or equal to the size of B. In addition since A won, he controls at least one Amoeba. Then adding all the sizes of amoebae, size of A's control is greater than the size of B's control and this is exactly $\sigma_A > \sigma_B$ or $\delta > 0$

Theorem (Symmetry of difference): The expected value of δ is always 0.

Proof: the players A and B start and play with the same conditions thus by symmetry in the game $P(\delta = k) = P(\delta = -k)$ and $\sum_k P(\delta = k)k = 0$ which implies that the expected value is 0.

Conjecture (Unboundedness of difference): Let $P_M(n)$ be the probability of δ being bigger than M , before n turns. Then $\lim_{n \rightarrow \infty} P_M(n) = 1$ for any integer M .

Justification: We have run computer modeling that supports this claim.

2. Advantage 2 (Total size difference): The two players control some amoebae, let Σ_A and Σ_B be the sum of the sizes of amoebae controlled by player A and B respectively in a given moment. Also we define $\Delta = \Sigma_A - \Sigma_B$.

Definition: We say player A has *advantage 2* over player B if $\Delta > 0$, similarly we say B has *Advantage 2* if $\Delta < 0$.

Claim : Player A can only win if he has advantage 2 and player B can only win if he has advantage 2.

Proof: If a player wins (Let's say A), the other player does not control any amoebae thus $\Sigma_B = 0$, $\Sigma_A > 0$ and this is $\Sigma_A > \Sigma_B$ or $\Delta > 0$

For two players rolling dice alternately, let σ_A and σ_B be the sum of the rolls for player A and B respectively in a given moment. Then we have the following conjectures:

Opening strategies:

In the following we analyze the opening strategies where no amoebae has been eaten or split. We consider different cases depending on the difference of size of initial amoebae, δ will be this difference in size.

Case I: When the two amoeba are not adjacent and the $\delta < 3.5$.

We claim that the optimal strategy here is to move to one of the two adjacent positions not adjacent to the enemy amoeba. We will first describe why splitting is not an optimal strategy. Splitting in this case guarantees that at least one of your amoebae is smaller than the opponent's amoeba and the δ between each of your amoebae and the opponent's amoeba increases in favor of the opponent. This gives you no advantage while weakening your position, so we can rule out splitting. The only strategies left are moving to one of the three adjacent squares. One of these squares is adjacent to the opponent's amoeba and moving to this square is also not optimal. The expected value of the opponents roll is 3.5, so the opponent's amoeba is likely to become bigger than your amoeba. Therefore the opponent is likely to gain Advantage 1 and 2, so moving adjacent to the opponent would constitute a loss. On the other hand, should the opponent not roll enough to win, his amoeba can split and subsequently avoid immediate loss with a chance of a comeback. Moving to this square means likely loss or unlikely survival, but no immediate guaranteed victory. Therefore, the best strategy here is to stall the game until you gain in Advantage 1 and 2 by increasing the δ in your favor to a larger amount. This is done by moving your amoeba to one of the two adjacent squares not adjacent to the enemy amoeba repeatedly. For example at the start of the game, this strategy would

look like $\left[\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & B \end{array} \right]$ and $\left[\begin{array}{c|c|c} 0 & A & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & B & 0 \end{array} \right]$ alternating.

Case II: When two amoebae are not adjacent and $\delta = 4, 5$ or 6 .

- If $\delta = 4$ moving to the center has at least a possibility of $\frac{1}{2}$ of not winning (if B rolls 4 he will probably choose to tie because he will roll again and get an advantage). So the best choice is to play as in case 1.
- If $\delta = 5$ moving to the center has at least a possibility of $\frac{1}{3}$ of not winning (same as before, B will choose to tie). This is not an extremely good situation but might be very advantageous if the game is advanced (i.e. the two amoebae have a significant size), if this is the case the cases where B rolls 1,2,3 or 4 will result on A eating half of B's size giving him a huge advantage. The decision of moving to the center will depend on how advanced the game is and on the previous dice rolls. This is a complicated case which would have to be analyzed case by case. Here is one extreme case:

Suppose the size of both amoebae is 1000, and in the last 10 turns A and B have only roll 5's and 6's and a single 3. Each amoeba is in a corner and A rolls a 5. In this case the probability of B rolling 5 or 6 is extremely low, so A shouldn't worry about losing. More over if B fails to roll a 6 or 5 (let's say a 4) A will eat at least 500 in size making the amoebae (1006,502) and (0,502). (for clarification the tuples given represent the control of each player over an amoeba) Then B will never be able to eat A until he gets 1006 more in size, this will take him more than 200 turns. Thus making possibility of A losing small, so A should move to the middle.

- If $\delta = 6$, A is never going to lose by moving to the center but B will chose to tie if he rolls a 6. In all the other cases, with probability $\frac{5}{6}$, A will take at least half of B's size, getting a valuable advantage.

Case III: When the two amoeba are not adjacent and the $\delta > 6$ in your favor.

We claim that the optimal strategy here is to move your amoeba to the center. Splitting here has the chance to get rid of your advantage and serves no real purpose. In similar fashion to case 1, splitting can be ruled out. We can show that by moving to the center with an Advantage 2 in your favor, the opponent cannot win. After your amoeba moves to the center, your opponent's amoeba grows but not by more than 6. Therefore in order to avoid losing, the opponent's best move is to split in half. Your next optimal play depends on the new Advantage 2. If the new Advantage 2 is greater than 6, the optimal play is to split your amoeba enough to make one of the opponent's amoeba neutral while keeping control of the center. Again your opponent's amoeba must split in half to avoid losing and it cannot eat the neutral piece which is twice its size. While Δ remains larger than 6, the optimal strategy is to repeat this process of neutralizing one of the opponent's split amoeba. This repeats until either there are no squares left unoccupied in which case you win, or the Δ is no longer greater than 6 in your favor. When your Advantage 2 is no longer greater than

6 (either after the first turn or several after moving to the center), the optimal play is to eat one of the opponent's split amoebae. From here, the Δ may continue to decrease from your advantage in which case the optimal move is to move to an opposite square from your opponent. This would constitute a kind of "reset" wherein similar to the beginning of the game the Advantage 1 is small and two players each control one amoeba on opposite sides of the board. The point of having made these plays is that your controlled amoeba now has some of the opponent's amoeba that it ate. So while the Advantage 1 may go in your opponent's favor, your Advantage 2 may still be larger and your opponent cannot eat your amoeba to win.

Case IV: When two amoebae are not adjacent and $\delta > 44$.

We claim that, if amoeba from player A is bigger, player A can win the game in finitely many turns (A max of 8).

1. A moves to the center were he is adjacent to the other amoeba. B's only option is to split his amoeba, obviously in half. Example:

$$\text{A moves to the center } \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 60A & 0 \\ \hline 0 & 0 & 16B \end{array} \right] \text{ B rolls a 6 and splits } \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 60A & 11B \\ \hline 0 & 0 & 11B \end{array} \right]$$

2. A eats the bigger of the two amoebae with an amoeba just as big. Again B can't eat anything without loosing, so he will have to split again in half. Example:

$$\text{A rolls a 1 and eats } \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 50A & 9B \\ \hline 0 & 0 & 11a, 11B \end{array} \right] \text{ B rolls 6 and splits } \left[\begin{array}{c|c|c} 0 & 0 & 8B \\ \hline 0 & 50A & 9B \\ \hline 0 & 0 & 11a, 11B \end{array} \right]$$

3. A repeats step (2) as long as B is splitting, and there is more than one square empty. Examples:

$$\begin{aligned} \text{A rolls 1 and eats } & \left[\begin{array}{c|c|c} 0 & 0 & 8B \\ \hline 0 & 42A & 9A, 9B \\ \hline 0 & 0 & 11a, 11B \end{array} \right] \text{ B rolls 6 and splits } & \left[\begin{array}{c|c|c} 0 & 7B & 7B \\ \hline 0 & 42A & 9A, 9B \\ \hline 0 & 0 & 11a, 11B \end{array} \right] \\ \text{A rolls 1 and eats } & \left[\begin{array}{c|c|c} 0 & 7B & 7A, 7B \\ \hline 0 & 36A & 9A, 9B \\ \hline 0 & 0 & 11a, 11B \end{array} \right] \text{ B rolls 6 and splits } & \left[\begin{array}{c|c|c} 6b & 7B & 7A, 7B \\ \hline 0 & 36A & 9A, 9B \\ \hline 0 & 0 & 11a, 11B \end{array} \right] \\ \text{A rolls 1 and eats } & \left[\begin{array}{c|c|c} 6B & 7A, 7B & 7A, 7B \\ \hline 0 & 30A & 9A, 9B \\ \hline 0 & 0 & 11a, 11B \end{array} \right] \text{ B rolls 6 and splits } & \left[\begin{array}{c|c|c} 6B & 7A, 7B & 7A, 7B \\ \hline 6B & 30A & 9A, 9B \\ \hline 0 & 0 & 11a, 11B \end{array} \right] \\ \text{A rolls 1 and eats } & \left[\begin{array}{c|c|c} 6B & 7A, 7B & 7A, 7B \\ \hline 6B & 25A & 9A, 9B \\ \hline 0 & 0 & 11a, 11B \end{array} \right] \text{ B rolls 6 and splits } & \left[\begin{array}{c|c|c} 6A, 6B & 7A, 7B & 7A, 7B \\ \hline 6B & 25A & 9A, 9B \\ \hline 6B & 0 & 11a, 11B \end{array} \right] \end{aligned}$$

4. When only one square is empty A will eat B's amoeba with a number 1 higher (gaining control of the amoeba).

B will play same as before. Examples:

A rolls 1 and eats $\left[\begin{array}{c|c|c} 6A, 6B & 7A, 7B & 7A, 7B \\ \hline 7A, 6B & 19 & 9A, 9B \\ \hline 6B & 0 & 11a, 11B \end{array} \right]$ B rolls 6 and splits

$\left[\begin{array}{c|c|c} 6A, 6B & 7A, 7B & 7A, 7B \\ \hline 7A, 6B & 19 & 9A, 9B \\ \hline 6B & 6B & 11a, 11B \end{array} \right]$

A rolls 1 and eats $\left[\begin{array}{c|c|c} 6A, 6B & 7A, 7B & 7A, 7B \\ \hline 7A, 6B & 13 & 9A, 9B \\ \hline 7A, 6B & 6B & 11a, 11B \end{array} \right]$

5. When all the squares are taken, B can't split any further. He can't eat any amoeba because the adjacent amoebae are bigger than him (at least 13 if he is 6 and so forth). In any case A has an amoeba of bigger size and can eat him winning control of the new amoeba, therefore winning the game. Example:

B doesn't have where to go even if rolling a 6 $\left[\begin{array}{c|c|c} 6A, 6B & 7A, 7B & 7A, 7B \\ \hline 7A, 6B & 13 & 9A, 9B \\ \hline 7A, 6B & 12B & 11a, 11B \end{array} \right]$

A eats and wins $\left[\begin{array}{c|c|c} 6A, 12B & 7A, 7B & 7A, 7B \\ \hline 7A, 6B & 13 & 9A, 9B \\ \hline 7A, 6B & 12A, 12B & 11a, 11B \end{array} \right]$

Implementation

Board Game:

As a board game, the best way we found to play this game was with a board and coins. For the board, a chess board is suitable as well as anything else that is gridded. As for the amoebae, we found that either using two sets of different colored uniform Lego pieces or pennies and dimes work best. The two different kinds of pieces represent a players control and eating and splitting is preformed by stacking or unstacking the pieces. A player's control can easily be kept track of by looking at the proportion of the players pieces in the amoeba. The limitations of the board version is that the amoeba can get unwieldy for large sizes (imagine 40 stacked pennies and dimes).

Pen and Paper with Example Game:

We have found classic pen and paper to be the best way to play. We still need a die, but amoebae can easily be kept track as tuples of two integers representing

each player's control and the sum of these integers is the amoeba's size. For the board, an N by N grid keeps track of the positions. If the game has not been made clear yet, we end this paper with an example pen and paper game that we played.

Game Start: $\left[\begin{array}{c|c|c} (1,0) & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & (0,1) \end{array} \right]$ Player 1 rolls a 3 and moves to the right,

Player 2 rolls a 2 and moves to the left. $\left[\begin{array}{c|c|c} 0 & (4,0) & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & (0,3) & 0 \end{array} \right]$ Player 1 rolls a 3 and moves to the right again, Player 2 rolls a 1 and moves to the left again.

$\left[\begin{array}{c|c|c} 0 & 0 & (7,0) \\ \hline 0 & 0 & 0 \\ \hline (0,4) & 0 & 0 \end{array} \right]$ Player 1 rolls a 2 and makes a risky move to the center,

Player 2 rolls a 4 and splits in half upwards. $\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline (0,4) & (9,0) & 0 \\ \hline (0,4) & 0 & 0 \end{array} \right]$ Player 1 rolls a 5 and eats the amoeba to his left, Player 2 rolls a 2 and splits his amoeba in half

to the right. $\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline (14,4) & 0 & 0 \\ \hline (0,3) & (0,3) & 0 \end{array} \right]$ Player 1 rolls a 4 and moves to the center,

Player 2 rolls a 4 and moves that piece to the right. $\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & (18,4) & 0 \\ \hline (0,3) & 0 & (0,7) \end{array} \right]$

Player 1 rolls a 6 and neutralizes the bottom right amoeba, Player 2 rolls a 4 and splits upwards. $\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline (0,4) & (16,3) & 0 \\ \hline (0,3) & 0 & (8,8) \end{array} \right]$ Player 1 rolls a 1 and neutralizes the amoeba to its left, Player 2 rolls a 6 and splits to the right.

$\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline (5,5) & (12,2) & 0 \\ \hline (0,5) & (0,4) & (8,8) \end{array} \right]$ Player 1 rolls a 5, splits and eats the piece under it.

Player 2 rolls a 4 and eats player 1's center amoeba. $\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline (5,5) & (6,11) & 0 \\ \hline 0 & (11,4) & (8,8) \end{array} \right]$

Player 1 rolls a 5 and eats player 2's center amoeba winning the game.