Adaptive Posicast Controller for Time-delay Systems with Relative Degree $n^* \leqslant 2$ ⋆

Yildiray Yildiz [a], Anuradha Annaswamy [a],
Ilya V. Kolmanovsky [b], Diana Yanakiev [b]

[a] *Active Adaptive Control Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

[b] *Research and Innovation Center, Ford Motor Company 2101 Village Road, Dearborn, MI 48121, USA*

Abstract

In this paper, we present an Adaptive Posicast Controller that deals with parametric uncertainties in linear systems with delays. It is assumed that the plant has no right half plane zeros and the delay is known. The adaptive controller is based on the Smith Predictor and Finite-Spectrum Assignment with time-varying parameters adjusted online. A novel Lyapunov-Krasovskii functional is used to show semi-global stability of the closed-loop error equations. The controller is applied to engine fuel-to-air ratio control. The implementation results show that the Adaptive Posicast Controller dramatically improves the closed loop performance when compared to the case with the existing baseline controller.

Key words: Adaptive control; Time-delay systems; Model matching.

1 Introduction

A time-delay system can be defined as one where there is a time-interval from the application of a control signal to any observable change in the measured variable [1]. Time-delays are ubiquitous in dynamical systems, present

⋆ This paper was not presented at any IFAC meeting. Corresponding author Yildiray Yildiz. Tel. +1 650 6044382.

Email addresses: yildiray.yildiz@nasa.gov (Yildiray Yildiz), aanna@mit.edu (Anuradha Annaswamy), ikolmano@ford.com (Ilya V. Kolmanovsky), dyanakie@ford.com (Diana Yanakiev).

as computational delays, input delays, measurement delays and transportation/convection lags to name a few. Often, higher order dynamical systems can be modeled as a low order system together with a time-delay. Examples of time-delay systems can be found in a wide range of applications including chemical, biological, mechanical, physiological and electrical systems (see [2], [3]). Detailed surveys of time-delay systems can be found in [4] and [5].

The stabilization of systems involving time-delays is a difficult problem since the existence of a delay may induce instability or bad performance for the closed loop system. In many controller designs, the delay is neglected and stability and robustness margins are given with respect to delay. The same approach can also be found in some adaptive control designs [6]. However, in general, these approaches may produce small delay margins.

One of the most popular methods for controlling systems with time-delays is the Smith Predictor (SP), proposed by Otto Smith in the 1950s [7]. The main idea in this approach is predicting the future output of the plant, using a plant model, and using this prediction to cancel the effect of the time-delay [8–12]. This future prediction inspired the name for the "posi-cast", which stands for "positively casting", or "forecasting" [13]. The SP method, however, is not suitable for unstable systems due to the possibility of unstable pole-zero cancellations. In [14], a method based on finite spectrum assignment (FSA) was introduced by Manitius and Olbrot that allows control of unstable systems with time-delay, where the main idea was using finite time integrals to prevent unstable pole zero cancellations (see [15–17] for other variations of FAS methods). A combination of the FSA together with a pole-placement based control design was proposed in [18], and an adaptive version of the same was proposed in [19] and deployed the augmented error approach proposed in [20]. The approach in [19] guarantees a stable adaptive system for any linear time-invariant plant with parametric uncertainties, stable zeros and a known order, time-delay, and upper-bound on the relative degree.

In [21], the control of time-delay systems whose relative degree does not exceed two was addressed. It was shown that a simple adaptive law, similar to that in the time-delay free case [20], can be used to lead to stability. In [21], as in [18–20], the plant poles are restricted to multiplicity one. In addition, the underlying Lyapunov function arguments deploy structures that depend on finite-dimensional state-space that may lead to additional restrictions on the locations of the plant poles and zeros.

In this paper, we provide an adaptive control design for plants with a relative degree less than or equal to two with unknown parameters and a known time-delay. No restrictions on the pole locations are imposed, and the zeros are assumed to be stable. Unlike the control structure in [18–21], we derive a controller using predictions of future states, which gives a better insight into its design. A novel Lyapunov-Krasovskii functional that only consists of partial states of the overall infinite dimensional system is used to establish stability. This is carried out at the expense of a slightly more complex set of induction based arguments. The overall control design is shown to remain simple, despite its ability to stabilize a larger class of plants with time-delays. As in [22], it can be shown that this controller can be extended to stabilize plants with arbitrary relative degree.

The simplicity of the underlying control design together with its adaptive feature has led to a very successful implementation in several full-scale experimental studies [23–25] in automotive applications that pertained different problems in powertrain systems with large time-delays. A brief description of the results obtained is included in this paper for the sake of completeness.

The organization of the paper is as follows: In section 2 and 3, we explain the APC design for time-delay systems when the plant is first order and when the plant is higher order but the state variables are accessible, respectively. In section 4, we give the APC design for systems with relative degree smaller than or equal to two, where we have only input-output measurements. In this section, we first give a simple model matching controller design for time-delay systems and later we introduce the APC building upon this model matching controller. We give the stability proof by induction, employing Lyapunov-Krasovskii functionals. In section 5, we present the results of experiments where we applied the APC to fuel-to-air ratio control problem, using a test vehicle.

2 First-Order Plant

We begin with a simple problem, where the plant is given by

$$\dot{x}(t) = ax(t) + u(t) \tag{1}$$

for which a control input of the form

$$u(t) = \theta^* x(t) + r(t) \qquad \theta^* = a_{\mathrm{m}} - a, \quad a_{\mathrm{m}} < 0 \tag{2}$$

ensures stable tracking for any a. One can provide a more formal guarantee of such a tracking by choosing a reference model of the form

$$\dot{x_{\mathrm{m}}}(t) = a_{\mathrm{m}} x_{\mathrm{m}}(t) + r(t) \tag{3}$$

which leads to error dynamics of the form

$$\dot{e}(t) = a_{\mathrm{m}} e(t) \qquad e(t) = x(t) - x_{\mathrm{m}}(t) \tag{4}$$

for which it can be simply shown that $V(t) = \frac{1}{2}e^2(t)$ is a Lyapunov function, with $\dot{V} < 0$, leading to exponential stability.

2.1 The Posicast Controller

We now introduce a time-delay in (1) so that

$$\dot{x}(t) = ax(t) + u(t - \tau) \tag{5}$$

where the goal is to stabilize the plant and track the output of a stable reference model. The results of [7] and [14] inspire us to establish the following: A posicast controller that "positively" forecasts the output is chosen as

$$u(t) = \theta^* x(t + \tau) + r(t) \tag{6}$$

which in turn leads to a closed-loop system of the form

$$\dot{x}(t) = a_{\mathrm{m}} x(t) + r(t - \tau), \tag{7}$$

an obviously stable plant. The non-causal controller in (6) can be shown to be indeed causal with a clever algebraic manipulation established in [14]. This is enabled by observing that the plant equation in (5) can be written in an integral form as

$$x(t + \tau) = e^{a\tau} x(t) + \int_{-\tau}^{0} e^{-a\eta} u(t + \eta) d\eta, \tag{8}$$

leading to a causal controller determined by (6) together with (8). The above observation also leads us to a PosiCast Lyapunov function for the closed-loop system given by (5)-(6), for the case when $r(t) = 0$, given by

$$V(t) = \frac{1}{2} x^2(t + \tau). \tag{9}$$

It can be shown from (8) and some algebra that

$$\dot{V}(t) = a_{\mathrm{m}} x^2(t + \tau). \tag{10}$$

2.2 The Adaptive Posicast Controller

We now proceed to the case when a is unknown. Using customary adaptive control procedures [20], suppose we choose a control input of the form

$$u(t) = \theta x(t+\tau) + r(t) \tag{11}$$

it leads to a closed-loop system of the form

$$\begin{aligned}\dot{x}(t) = {} & a_{\mathrm{m}} x(t) + \tilde{\theta}(t-\tau)\left[e^{a\tau}x(t-\tau) + \int_{-\tau}^{0} e^{-a\eta}u(t+\eta-\tau)d\eta\right] \\ & + r(t-\tau)\end{aligned} \tag{12}$$

where $\tilde{\theta}(t) = \theta(t) - \theta^*$. While indeed this suggests that a reference model can be chosen in the form

$$\dot{x}_{\mathrm{m}}(t) = a_{\mathrm{m}} x_{\mathrm{m}}(t) + r(t-\tau), \tag{13}$$

it can be seen that it poses a difficulty, since the underlying error model can be derived using (12) and (13) as

$$\dot{e}(t) = a_{\mathrm{m}} e(t) + \tilde{\theta}(t-\tau)\left[e^{a\tau}x(t-\tau) + \int_{-\tau}^{0} e^{-a\eta}u(t+\eta-\tau)d\eta\right]. \tag{14}$$

Equation (14) is however not in a form that lends itself to a Lyapunov function since the term inside the brackets includes the unknown parameter a. We therefore choose a different control input that is still motivated by the non-adaptive controller given in (6).

Equations (6) and (8) imply that the Posicast control input is essentially of the form

$$u(t) = \theta_{\mathrm{x}}^* x(t) + \int_{-\tau}^{0} \lambda^*(\eta)u(t+\eta)d\eta + r(t) \tag{15}$$

where

$$\theta_{\mathrm{x}}^* = \theta^* e^{a\eta}, \qquad \lambda^*(t) = \theta^* e^{a\eta}$$

Therefore, a choice of a control input of the form

$$u(t) = \theta_{\mathrm{x}}(t)x(t) + \int_{-\tau}^{0} \lambda(t,\eta)u(t+\eta)d\eta + r(t) \tag{16}$$

leads to a closed-loop system of the form

$$\dot{x}(t) = ax(t) + \theta^* \left[e^{a\tau} x(t-\tau) + \int_{-\tau}^{0} e^{-a\eta} u(t-\tau+\eta) d\eta \right]$$
$$+\tilde{\theta}_{\mathrm{x}}(t-\tau)x(t-\tau) + \int_{-\tau}^{0} \tilde{\lambda}(t-\tau,\eta)u(t-\tau+\eta)d\eta + r(t-\tau) \tag{17}$$

where

$$\tilde{\theta}_{\mathrm{x}}(t) = \theta_{\mathrm{x}}(t) - \theta_{\mathrm{x}}^*, \qquad \tilde{\lambda}(t,\eta) = \lambda(t,\eta) - \lambda^*.$$

From (8), it follows that (17) can be written as

$$\dot{x}(t) = a_{\mathrm{m}} x(t) + \tilde{\theta}_{\mathrm{x}}(t-\tau)x(t-\tau) + \int_{-\tau}^{0} \tilde{\lambda}(t-\tau,\eta)u(t-\tau+\eta)d\eta$$
$$+ r(t-\tau) \tag{18}$$

As a result, defining $e(t) = x(t) - x_{\mathrm{m}}(t)$, (18) and (13) imply that the underlying error model is of the form

$$\dot{e}(t) = a_{\mathrm{m}} e(t) + \tilde{\theta}_{\mathrm{x}}(t-\tau)x(t-\tau) + \int_{-\tau}^{0} \tilde{\lambda}(t-\tau,\eta)u(t-\tau+\eta)d\eta. \tag{19}$$

This error model is discussed in Section 4, where a Lyapunov-Krasovskii functional leads to semi-global stability in τ. Here, we give the stability result and leave the proof to Section 4, where the general case, n^{th} order system, is investigated.

Theorem 1 Given initial conditions $\tilde{\theta}_{\mathrm{x}}(0)$, $x(\xi)$, $\tilde{\lambda}(\xi,\eta)$ for $\xi \in [-\tau, 0]$ and $u(\zeta)$ for $\zeta \in [-2\tau, 0]$, there exists a τ^* such that for all $\tau \in [0, \tau^*]$, the plant in (1), controller in (16), and adaptive laws given by

$$\dot{\theta}_{\mathrm{x}}(t) = -\gamma_1 e(t) x(t-\tau)$$
$$\frac{\partial \lambda}{\partial t}(t,\eta) = -\gamma_\lambda e(t) u(t+\eta-\tau) \tag{20}$$

have bounded solutions for all $t \geqslant 0$.

3 State variables accessible

The plant considered here is of the form

$$\dot{x}(t) = Ax(t) + bku(t-\tau) \tag{21}$$

where A and k are an unknown matrix and a scalar, respectively, (A, b) is controllable, and b is a known vector. We choose a reference model of the form

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}}x(t) + br(t-\tau) \tag{22}$$

where A_{m} is a suitable Hurwitz matrix. Taking a cue from Eq. (16) in the previous section, we choose a control input of the form

$$u(t) = \theta_{\mathrm{x}}^T(t)x(t) + \int_{-\tau}^{0} \lambda(t,\eta)u(t+\eta)d\eta + \theta_{\mathrm{r}}(t)r(t) \tag{23}$$

and adaptive laws of the form

$$\begin{aligned} \dot{\theta}_{\mathrm{x}}(t) &= -\gamma_1 e^T Pbx(t-\tau) \\ \dot{\lambda}(t,\eta) &= -\gamma_\lambda e^T Pbu(t-\tau+\eta) \\ \dot{\theta}_{\mathrm{r}}(t) &= -\gamma_{\mathrm{r}} e^T(t)Pbr(t-\tau) \end{aligned} \tag{24}$$

We show below that the closed-loop system specified by (21)-(24) leads to semi-global boundedness in τ. The desired parameters for $\theta_{\mathrm{x}}(t)$, $\lambda(t,\eta)$ and $\theta_{\mathrm{r}}(t)$ are defined as

$$\begin{aligned} \theta_{\mathrm{x}}^* &= e^{A^T\tau}\theta^* \\ \lambda^*(\eta) &= \theta^{*T}e^{A\tau}bk \qquad (25) \\ \theta_{\mathrm{r}}^*k &= 1 \qquad (26) \end{aligned}$$

where

$$A + bk\theta^{*T} = A_{\mathrm{m}}. \tag{27}$$

This in turn, after several algebraic manipulations, leads to an error equation of the form

$$\begin{aligned} \dot{e}(t) = A_{\mathrm{m}}e(t) + bk[\tilde{\theta}_{\mathrm{x}}(t-\tau)x(t-\tau) + \int_{-\tau}^{0} \tilde{\lambda}(t-\tau,\eta)u(t-\tau+\eta)d\eta \\ +\tilde{\theta}_{\mathrm{r}}(t-\tau)r(t-\tau)] \end{aligned} \tag{28}$$

As in Section 2, using a Lyapunov Krasovskii functional, (24) and (28) can be shown to have semi-globally bounded solutions. The underlying Theorem is stated below:

Theorem 2 Given initial conditions $\tilde{\theta}_{\mathrm{x}}(0)$, $\tilde{\theta}_{\mathrm{r}}(0)$, $x(\xi)$, $\tilde{\lambda}(\xi, \eta)$ for $\xi \in [-\tau, 0]$ and $u(\zeta)$ for $\zeta \in [-2\tau, 0]$, there exists a τ^* such that for all $\tau \in [0, \tau^*]$, the plant in (21), controller in (23), and adaptive laws given in (24) have bounded solutions for all $t \geqslant 0$.

4 Adaptive Posicast Control in the presence of output measurements with $n^* \leqslant 2$

In this section, the focus is on higher order time-delay systems with relative degree, n^*, smaller than two.

4.1 Exact model matching for delayed systems

Consider the plant with the time-delay τ whose input-output description is given as

$$y(t) = W_{\mathrm{p}}(s)u(t-\tau), \qquad W_{\mathrm{p}}(s) = k_{\mathrm{p}}\frac{Z_{\mathrm{p}}(s)}{R_{\mathrm{p(s)}}} \tag{29}$$

where $Z_{\mathrm{p}}(s)$ and $R_{\mathrm{p}}(s)$ are monic coprime polynomials with order m and n and $n^* = n - m > 0$ is defined as the relative order of the finite dimensional part of the plant. It is also assumed that $Z_{\mathrm{p}}(s)$ is Hurwitz and k_{p} is a constant gain parameter. The reference input-output description is given by

$$y_{\mathrm{m}}(t) = W_{\mathrm{m}}(s)r(t-\tau), \qquad W_{\mathrm{m}}(s) = k_{\mathrm{m}}\frac{Z_{\mathrm{m}}(s)}{R_{\mathrm{m}}(s)} \tag{30}$$

where $Z_{\mathrm{m}}(s)$ and $R_{\mathrm{m}}(s)$ are monic Hurwitz polynomials of degrees m_{m} and n_{m} respectively, and k_{m} is a constant gain parameter. Further, it is assumed that $n_{\mathrm{m}} - m_{\mathrm{m}} \geqslant n - m$.

The model matching problem is to determine a bounded control input $u(t)$ to the plant such that the closed loop transfer transfer function of the plant together with the controller, from $r(t)$ to $y_{\mathrm{p}}(t)$, matches the reference model transfer function.

Consider the following state space representation of the plant dynamics (29), together with two "signal generators" formed by a controllable pair Λ, l:

$$\dot{x}_{\mathrm{p}}(t) = A_{\mathrm{p}} x_{\mathrm{p}}(t) + b_{\mathrm{p}} u(t-\tau), \;\; y(t) = h_{\mathrm{p}}^T x_{\mathrm{p}}(t) \tag{31}$$
$$\dot{\omega}_1(t) = \Lambda \omega_1(t) + l u(t-\tau) \tag{32}$$
$$\dot{\omega}_2(t) = \Lambda \omega_2(t) + l y(t) \tag{33}$$

where, $\Lambda \in \Re^{nxn}$ and $l \in \Re^n$. Defining $\bar{x}(t) \triangleq x(t+\tau)$, $\bar{\omega}_i(t) \triangleq \omega_i(t+\tau)$, $i = 1, 2$, $\bar{y}(t) \triangleq y(t+\tau)$, (31)-(33) can be rewritten in the following form:

$$\dot{\bar{x}}_{\mathrm{p}}(t) = A_{\mathrm{p}} \bar{x}_{\mathrm{p}}(t) + b_{\mathrm{p}} u(t), \;\; \bar{y}(t) = h_{\mathrm{p}}^T \bar{x}_{\mathrm{p}}(t) \tag{34}$$
$$\dot{\bar{\omega}}_1(t) = \Lambda \bar{\omega}_1(t) + l u(t) \tag{35}$$
$$\dot{\bar{\omega}}_2(t) = \Lambda \bar{\omega}_2(t) + l \bar{y}(t) \tag{36}$$

Under our assumptions, it follows [20] that there exists $\beta_1^*, \beta_2^* \in R^n$ and $k^* \in R$ such that the control law

$$u(t) = \beta_1^{*T} \bar{\omega}_1(t) + \beta_2^{*T} \bar{\omega}_2(t) + k^* r(t) \tag{37}$$

satisfies the exact model matching condition.

$$\frac{\bar{y}(t)}{r(t)} = k_{\mathrm{m}} \frac{Z_{\mathrm{m}}(s)}{R_{\mathrm{m}}(s)}. \tag{38}$$

Note that the model matching is achieved between the rational parts of the closed loop transfer function and of the reference model. On the other hand, it follows directly from (38) that

$$\frac{y(t)}{r(t)} = k_{\mathrm{m}} \frac{Z_{\mathrm{m}}(s)}{R_{\mathrm{m}}(s)} e^{-\tau s} \tag{39}$$

which shows that the exact model matching condition is also satisfied between the total closed loop and the reference model transfer functions. Since $\bar{\omega}_2(t)$ requires the output measurement at time $t+\tau$, the control law given in (37) is non-causal. We refer to this closed loop system with the non-causal controller as "Controller NC" and it is depicted in Fig. 1.

It is also well known [20] that with zero initial conditions, there exists $c, d \in R^n$ such that $\bar{y}(t) = c^T \bar{\omega}_1(t) + d^T \bar{\omega}_2(t)$, which implies

$$y(t) = c^T \omega_1(t) + d^T \omega_2(t). \tag{40}$$

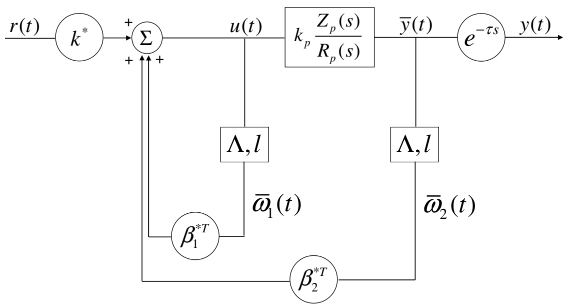

Fig. 1. Exact model matching, Controller NC.

Substituting (40) into (33) and rewriting (32) and (33) together in vector-matrix form, we get

$$\begin{pmatrix} \dot{\omega}_1(t) \\ \dot{\omega}_2(t) \end{pmatrix} = A \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} + bu(t-\tau)$$

$$\text{where, } A = \begin{bmatrix} \Lambda & 0 \\ lc^T & \Lambda + ld^T \end{bmatrix} \text{ and } b = \begin{bmatrix} l \\ 0 \end{bmatrix}. \tag{41}$$

From (41) we get

$$\begin{pmatrix} \bar{\omega}_1(t) \\ \bar{\omega}_2(t) \end{pmatrix} = e^{A\tau} \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} + \int_{-\tau}^{0} e^{-A\sigma} bu(t+\sigma)\, \mathrm{d}\sigma. \tag{42}$$

Substituting (42) into (37) we obtain that

$$\begin{aligned} u(t) = &\left\{ \begin{bmatrix} \beta_1^{*T} & \beta_2^{*T} \end{bmatrix} e^{A\tau} \right\} \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} \\ &+ \int_{-\tau}^{0} \left\{ \begin{bmatrix} \beta_1^{*T} & \beta_2^{*T} \end{bmatrix} e^{-A\sigma} b \right\} u(t+\sigma)\, \mathrm{d}\sigma \\ &+ k^* r(t), \end{aligned} \tag{43}$$

which can equivalently be expressed as

$$\begin{aligned} u(t) = &\, \alpha_1^{*T} \omega_1(t) + \alpha_2^{*T} \omega_2(t) + \int_{-\tau}^{0} \phi^*(\sigma) u(t+\sigma) \mathrm{d}\sigma \\ &+ k^* r(t). \end{aligned} \tag{44}$$

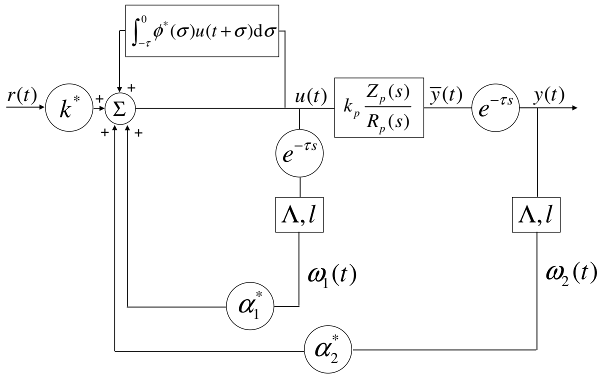

Fig. 2. Exact model matching, Controller C.

The controller (44) is causal and we refer to it as "Controller C". Defining $\beta^* = [\beta_1^{*T} \ \ \beta_2^{*T}]$ and $\alpha^* = [\alpha_1^{*T} \ \ \alpha_2^{*T}]$, we get the following relationships between the controller parameters of Controller NC and Controller C

$$\alpha^* = \beta^* e^{A\tau} \tag{45}$$

$$\phi^*(\sigma) = \beta^* e^{-A\sigma} b. \tag{46}$$

The block diagram of Controller C is shown in Fig. 2.

In summary, an exact model matching controller, Controller C, can be designed for the time-delay system (29) using a well known procedure for the delay free systems [20]. Note that the equivalence between Controller NC and Controller C shows that the latter relies on predicting the future output, $\bar{y}(t)$ to achieve exact model matching. This can be viewed as an equivalence result for higher order systems similar to that shown in Section 2 between (11) and (15).

Note that the model matching controller (44) was already introduced in [18] and [14]. The main contribution here is that we derive this controller by using the delay free part of the system dynamics, which considerably simplifies the design. In addition, instead of proposing a controller structure and proving that it is a model matching controller, we *derive* the controller using future state prediction which gives an insight about the controller structure. Finally, this derivation removes the assumption that the plant has poles with multiplicity one, which was assumed in the earlier papers. A similar result is given in [26] where it is stated that state prediction is a fundamental concept for time-delay systems.

4.2 Adaptive Controller

The goal of this section is to design a model reference adaptive controller, motivated by the equivalence of the Controller NC and Controller C, for the plant described in (29) with unknown coefficients for the polynomials $Z_{\mathrm{p}}(s)$ and $R_{\mathrm{p}}(s)$ and with unknown, positive high frequency gain k_{p}.

4.2.1 Design for $n^* = 1$

In this section, we assume that the relative degree n^* of the plant is one. The first step of the adaptive controller design is to design a fixed controller which satisfies the exact model matching with the reference model assuming that the plant parameters are known. This step was already completed in Section 4.1 resulting in the controller given in (44). In the second step, the controller parameters $\alpha_1^*, \alpha_2^*, \phi^*(\sigma)$ and k^* are replaced by time-varying parameters $\alpha_1(t), \alpha_2(t), \phi(t,\sigma)$ and $k(t)$ and the goal is to determine the laws by which they are adjusted so as to result in a stable system.

We define the following parameters:

$$
\begin{aligned}
&\tilde{k}(t) \triangleq k(t) - k^*, && \tilde{\alpha}_1(t) \triangleq \alpha_1(t) - \alpha_1^*,\\
&\tilde{\alpha}_2(t) \triangleq \alpha_2(t) - \alpha_2^*, && \tilde{\phi}(t,\sigma) \triangleq \phi(t,\sigma) - \phi^*,\\
&\alpha(t) \triangleq \left[\alpha_1^T(t)\ \ \alpha_2^T(t)\right]^T, && \beta(t) \triangleq \left[\beta_1^T(t)\ \ \beta_2^T(t)\right]^T,\\
&\Omega(t) \triangleq \left[\omega_1^T(t)\ \ \omega_2^T(t)\ \ r(t)\right]^T, && \omega(t) \triangleq \left[\omega_1^T(t)\ \ \omega_2^T(t)\right]^T,\\
&\theta(t) \triangleq \left[\alpha_1^T(t)\ \ \alpha_2^T(t)\ \ k(t)\right]^T, && \tilde{\theta}(t) \triangleq \theta(t) - \theta^*,\\
&\theta^* \triangleq \left[(\alpha_1^*)^T\ \ (\alpha_2^*)^T\ \ k^*\right]
\end{aligned}
$$

Consider a controller

$$
\begin{aligned}
u(t) &= \alpha^T(t)\omega(t) + \int_{-\tau}^{0} \phi(t,\sigma)u(t+\sigma)\mathrm{d}\sigma + k(t)r(t)\\
&= \alpha_1^{*T}\omega_1(t) + \alpha_2^{*T}\omega_2(t) + \int_{-\tau}^{0} \phi^*(\sigma)u(t+\sigma)\mathrm{d}\sigma\\
&\quad + k^*r(t) + \tilde{\alpha}_1^T(t)\omega_1(t) + \tilde{\alpha}_2^T(t)\omega_2(t)\\
&\quad + \int_{-\tau}^{0} \tilde{\phi}(t,\sigma)u(t+\sigma)\mathrm{d}\sigma + \tilde{k}(t)r(t).
\end{aligned} \tag{47}
$$

Using the equivalence between (37) and (44), we can rewrite (47) as

$$
\begin{aligned}
u(t) &= \beta_1^{*T}\bar{\omega}_1(t) + \beta_2^{*T}\bar{\omega}_2(t) + k^*r(t) + \tilde{\alpha}_1^T(t)\omega_1(t)\\
&\quad + \tilde{\alpha}_2^T(t)\omega_2(t) + \int_{-\tau}^{0} \tilde{\phi}(t,\sigma)u(t+\sigma)\mathrm{d}\sigma\\
&\quad + \tilde{k}(t)r(t).
\end{aligned} \tag{48}
$$

The differential equations, describing the plant (31) together with the controller (48) can then be represented as

$$
\begin{aligned}
\dot{X}_{\mathrm{p}}(t) &= A_{\mathrm{m}} X_{\mathrm{p}}(t) + b_{\mathrm{m}}[\tilde{\theta}^T(t-\tau)\Omega(t-\tau) \\
&\quad + \int_{-\tau}^{0} \tilde{\phi}(t-\tau,\sigma)u(t-\tau+\sigma)\mathrm{d}\sigma + k^* r(t-\tau)], \\
y_{\mathrm{p}}(t) &= h_{\mathrm{m}}^T X_{\mathrm{p}}(t)
\end{aligned}
\tag{49}
$$

where

$$
\begin{aligned}
A_{\mathrm{m}} &\triangleq \begin{bmatrix} A_{\mathrm{p}} & b_{\mathrm{p}}\beta_1^{*T} & b_{\mathrm{p}}\beta_2^{*T} \\ 0 & \Lambda + l\beta_1^{*T} & l\beta_2^{*T} \\ lh_{\mathrm{p}}^T & 0 & \Lambda \end{bmatrix}, \ b_{\mathrm{m}} \triangleq \begin{bmatrix} b_{\mathrm{p}} \\ l \\ 0 \end{bmatrix}, \\
X_{\mathrm{p}}(t) &\triangleq \left[x_{\mathrm{p}}^T(t) \ \omega_1^T(t) \ \omega_2^T(t)\right]^T, \ h_{\mathrm{m}}^T \triangleq \left[h_{\mathrm{p}}^T \ 0 \ 0\right], \\
y_{\mathrm{p}} &= y.
\end{aligned}
\tag{50}
$$

We showed in Section 4.1 that when the parameter errors are equal to zero, the closed loop transfer function is identical to that of the reference model. Therefore, the reference model can be described by the $(3n)th$ order differential equation

$$
\dot{X}_{\mathrm{m}}(t) = A_{\mathrm{m}} X_{\mathrm{m}}(t) + b_{\mathrm{m}} k^* r(t-\tau), \ y_{\mathrm{m}}(t) = h_{\mathrm{m}}^T X_{\mathrm{m}}(t) \tag{51}
$$

where,

$$
\begin{aligned}
X_{\mathrm{m}}(t) &\triangleq \left[x_{\mathrm{p}}^{*T} \ \omega_1^{*T} \ \omega_2^{*T}\right]^T, \\
h_{\mathrm{m}}^T \left(sI - A_{\mathrm{m}}\right)^{-1} b_{\mathrm{m}} k^* &= k_{\mathrm{m}} \frac{Z_{\mathrm{m}}}{R_{\mathrm{m}}}.
\end{aligned}
\tag{52}
$$

Note that $x_{\mathrm{p}}^*(t), \omega_1^*(t)$ and $\omega_2^*(t)$ can be considered as the signals in the reference model corresponding to $x_{\mathrm{p}}(t), \omega_1(t)$ and $\omega_2(t)$ in the closed loop system. Therefore, subtracting (51) from (49), we get an error equation for the overall system as

$$
\begin{aligned}
\dot{e}(t) &= A_{\mathrm{m}} e(t) + b_{\mathrm{m}}[\tilde{\theta}^T(t-\tau)\Omega(t-\tau) \\
&\quad + \int_{-\tau}^{0} \tilde{\phi}(t-\tau,\sigma)u(t-\tau+\sigma)\mathrm{d}\sigma],
\end{aligned}
$$

$$e_1(t) = h_{\mathrm{m}}^T e(t). \tag{53}$$

where $e(t) = X_{\mathrm{p}} - X_{\mathrm{m}}$ and $e_1(t) = y_{\mathrm{p}}(t) - y_{\mathrm{m}}(t)$.

The adaptive control law which guarantees convergence of the tracking error to zero and boundedness of all signals has the following form:

$$\begin{aligned}
\dot{\tilde{\alpha}}(t) &= -\gamma_\alpha e_1(t)\omega(t-\tau) \\
\dot{\tilde{\phi}}(t,\sigma) &= -\gamma_\phi e_1(t) u(t-\tau+\sigma), -\tau \leqslant \sigma \leqslant 0
\end{aligned} \tag{54}$$

where, γ_α and γ_ϕ are positive, real adaptation gains.

4.2.2 Design for $n^* = 2$

When the relative degree n^* is equal to unity, it is easy to define a strictly positive real reference model $W_{\mathrm{m}}(s)$. When $n^* = 2$, an addition of an input u_{a} to u as

$$\begin{aligned}
u_{\mathrm{a}} &= \dot{\theta}^T \Omega' + \int_{-\tau}^{0} \dot{\phi}(t,\sigma) u'(t+\sigma) \mathrm{d}\sigma, \\
\dot{\Omega}' &= -aI\Omega' + \Omega, \\
\dot{u}' &= -au' + u, \quad a > 0
\end{aligned} \tag{55}$$

can be used to derive yet another error equation of the form

$$e(t) = W_{\mathrm{m}}(s)(s+a)e^{-s\tau}\left[\tilde{\theta}^T(t)\Omega(t) + \int_{-\tau}^{0} \tilde{\phi}(t,\sigma)u(t+\sigma)\mathrm{d}\sigma\right] \tag{56}$$

where $a > 0$ is chosen such that $(s+a)W_{\mathrm{m}}(s)$ is strictly positive real. Therefore it suffices to consider the stability of (53). The results can then be extended to the case when $n^* = 2$ by making use of the additional input u_{a}.

4.3 Stability Analysis

The following theorem confirms the desirable properties of the adaptive control law (54).

Theorem 2 Given initial conditions $x_{\mathrm{p}}(0)$, $u(\eta)$, $\eta \in [t_0 - 2\tau, t_0]$, $\tilde{\alpha}(\xi), \tilde{\phi}(\xi)$, $\xi \in [t_0 - \tau, t_0]$, $\exists \tau^*$ s.t. $\forall \tau \in [0, \tau^*]$, the plant (29), the controller (47), and the adaptive laws given by (54) have bounded solutions $\forall t \geqslant 0$ and $lim_{t\to\infty} e_1(t) \to 0$.

We first state and prove the following lemma.

Lemma 1 Suppose a variable $u(t)$ is of the form

$$u(t) = f(t) + \int_{-\tau}^{0} \phi(t,\sigma)u(t+\sigma)\mathrm{d}\sigma \tag{57}$$

where $u, f : [t_0 - \tau, \infty) \to \Re$, $\phi : [t_0, \infty)\mathrm{x}[-\tau, 0] \to \Re$ and constants $t_i^{'}$, $\bar{f}, c_0, c_1 \in \Re^{+}$ exist such that $\|f(t)\| \leqslant \bar{f}$,

$$\int_{-\tau}^{0} \phi^2(t,\sigma)\mathrm{d}\sigma \leqslant c_0^2, \tag{58}$$

and

$$\int_{-\tau}^{0} u^2(t+\sigma)d\sigma = c_1^2, \qquad \forall t \geqslant t_i^{'}. \tag{59}$$

Then,

$$\left|u(t_j^{'})\right| \leqslant 2(\bar{f} + c_1 c_0)e^{c_0^2(t_j^{'} - t_i^{'})}, \qquad \forall t_j^{'} \geqslant t_i^{'}. \tag{60}$$

Proof of Lemma 1 Since (57) is an implicit integral equation, we derive inequality (60) by considering a sequence u_n, and let u be the limit of this sequence as $n \to \infty$ [27].

Define

$$\begin{aligned} u_0(t_j^{'}) &\triangleq u(t_j^{'}), \qquad t_j^{'} < t_i^{'} \\ &\triangleq 0, \qquad t_j^{'} \geqslant t_i^{'} \end{aligned} \tag{61}$$

$$\begin{aligned} u_{n+1}(t_j^{'}) &\triangleq u(t_j^{'}), \qquad t_j^{'} < t_i^{'}, \\ &\triangleq \int_{-\tau}^{0} \phi(t_j^{'},\sigma)u_n(t_j^{'}+\sigma)\mathrm{d}\sigma + f(t_j^{'}), \qquad t_j^{'} \geqslant t_i^{'}, \\ &\quad n = 0, 1..\infty \end{aligned} \tag{62}$$

For $t_j^{'} \geqslant t_i^{'}$ and $n = 1$, we have that

$$
\begin{aligned}
&\left|u_1(t_j^{'}) - u_0(t_j^{'})\right| = |f(t_j^{'}) + \int_{-\tau}^{0} \phi(t_j^{'},\sigma) u_0(t_j^{'}+\sigma)\mathrm{d}\sigma| \\
&\leqslant \bar{f} + \left(\int_{-\tau}^{0} \left(\phi(t_j^{'},\sigma)\right)^2 \mathrm{d}\sigma\right)^{1/2} \left(\int_{-\tau}^{0} \left(u(t_i^{'}+\sigma)\right)^2 \mathrm{d}\sigma\right)^{1/2}
\end{aligned} \tag{63}
$$

since the last parenthesis on the right hand side is being calculated over the time interval $[t_i^{'} - \tau, t_i^{'}]$. Using (58) and (59), (63) can be rewritten as

$$
\left|u_1(t_j^{'}) - u_0(t_j^{'})\right| \leqslant \bar{f} + c_0 c_1 \tag{64}
$$

For $t_j^{'} \geqslant t_i^{'}$ and $n = 2$ we have that

$$
\begin{aligned}
&\left|u_2(t_j^{'}) - u_1(t_j^{'})\right| = \left|\int_{-\tau}^{0} \phi(t_j^{'},\sigma) \left(u_1(t_j^{'}+\sigma) - u_0(t_j^{'}+\sigma)\right) \mathrm{d}\sigma\right| \\
&\leqslant \left(\int_{-\tau}^{0} \phi^2(t_j^{'},\sigma)\right)^{1/2} \left(\int_{-\tau}^{0} \left|u_1(t_j^{'}+\sigma) - u_0(t_j^{'}+\sigma)\right|^2 \mathrm{d}\sigma\right)^{1/2}
\end{aligned} \tag{65}
$$

Using inequality (58) and a change of variables $\zeta = t + \sigma$, (65) becomes

$$
\begin{aligned}
&\left|u_2(t_j^{'}) - u_1(t_j^{'})\right| \leqslant c_0 \left(\int_{t_j^{'}-\tau}^{t_j^{'}} \left|u_1(\zeta) - u_0(\zeta)\right|^2 \mathrm{d}\zeta\right)^{1/2} \\
&= c_0 \left(\int_{t_j^{'}-\tau}^{t_i^{'}} \left|u_1(\zeta) - u_0(\zeta)\right|^2 \mathrm{d}\zeta + \int_{t_i^{'}}^{t_j^{'}} \left|u_1(\zeta) - u_0(\zeta)\right|^2 \mathrm{d}\zeta\right)^{1/2}
\end{aligned} \tag{66}
$$

We know from (61) and (62) that $u_1(\zeta) - u_0(\zeta) = 0$ for $\zeta < t_i^{'}$. Therefore, (66) can be further simplified as

$$
\left|u_2(t_j^{'}) - u_1(t_j^{'})\right| \leqslant c_0 \left(\int_{t_i^{'}}^{t_j^{'}} \left|u_1(\zeta) - u_0(\zeta)\right|^2 \mathrm{d}\zeta\right)^{1/2} \tag{67}
$$

Substituting (64) into (67) we obtain that

$$
\left|u_2(t_j^{'}) - u_1(t_j{'})\right| \leqslant \left(\bar{f} + c_1 c_0\right) c_0 \sqrt{(t_j^{'} - t_i^{'})} \tag{68}
$$

Continuing this procedure iteratively, we obtain that

$$
\left|u_{n+1}(t_j^{'}) - u_n(t_j^{'})\right| \leqslant \left(\bar{f} + c_1 c_0\right) \left(\frac{\left(c_0^2(t_j^{'} - t_i^{'})\right)^n}{n!}\right)^{1/2}. \tag{69}
$$

Note that the term $(t_j^{'} - t_i^{'})^n/n!$ in (69) is obtained due to successive integrations of $(t_j^{'} - t_i^{'})$. It can be shown using the ratio test [28] that the series $\sum_{n=1}^{\infty} \left(\bar{f} + c_0 c_1\right) \left(\frac{\left(c_0^2(t-t_i^{'})\right)^n}{n!}\right)^{1/2}$ converges. This in turn implies that if S is defined as

$$S = \sum_{n=1}^{\infty} (u_n - u_{n-1}) \tag{70}$$

it can be shown that

$$S \leqslant 2(\bar{f} + c_1 c_0) e^{c_0^2 (t_j^{'} - t_i^{'})}. \tag{71}$$

Please see Appendix A for a proof of (71).

Defining $u \triangleq \lim_{n\to\infty} u_n$, from (70) and (71), we obtain that

$$\left|u(t_j^{'}) - u_0(t_j^{'})\right| \leqslant 2(\bar{f} + c_1 c_0) e^{c_0^2 (t_j^{'} - t_i^{'})} \tag{72}$$

This implies that,

$$\left|u(t_j^{'})\right| \leqslant 2(\bar{f} + c_1 c_0) e^{c_0^2 (t_j^{'} - t_i^{'})} \qquad t_j^{'} \geqslant t_i^{'} \tag{73}$$

and this completes the proof. Next, we describe the proof of Theorem 2.

Proof of Theorem 2 The proof is provided using the method of induction. Let the statement S be given by

$$S : \|X_{\mathrm{p}}(\xi)\| \leqslant I_0, \ \ |u(\xi)| \leqslant U(I_0) \ \ \forall \xi \in [t_0, t_0 + k\tau) \tag{74}$$

where $U(\cdot)$ is an analytic, bounded function of its arguments.

We note that $X_{\mathrm{p}}(\xi)$ and $u(\xi)$ is bounded for $\xi \in [t_0 - 2\tau, t_0)$. Using this fact, we complete the proof by showing that

I. S is true for $k = 1$.
II. If S is true for k, then it is true for $k + 1$

I and II allow us to conclude that all signals are bounded for all $t \geqslant t_0$. Finally, from Barbalat's Lemma, convergence of the error e to zero follows.

4.3.1 I. S is true for $k=1$.

The proof of I is given in four steps, each of which starts with a brief summary.

Step 1. *Some bounds on the signals are assumed in the time interval $[t_0 - 2\tau)$ and the negative semi-definiteness of the Lyapunov functional time derivative in $[t_0, t_0+\tau)$ is shown, which yields the boundedness of the signals in $[t_0, t_0+\tau)$. In addition, using Lemma 1, an upper bound for the control signal $u(t)$ in $[t_0, t_0+\tau)$ is given.*

Consider a Lyapunov Functional

$$\begin{aligned} V(t) = &e(t)^T P e(t) + \tilde{\theta}(t)^T \tilde{\theta}(t) + \int_{-\tau}^{0} \tilde{\phi}(t,\sigma)^2 \mathrm{d}\sigma \\ &+ \int_{-\tau}^{0} \int_{t+\nu}^{t} \dot{\tilde{\theta}}(\xi)^T \dot{\tilde{\theta}}(\xi) \mathrm{d}\xi \mathrm{d}\nu \\ &+ \int_{-\tau}^{0} \int_{t+\nu}^{t} \int_{-\tau}^{0} \left(\dot{\tilde{\phi}}(\xi,\sigma)\right)^2 \mathrm{d}\sigma \mathrm{d}\xi \mathrm{d}\nu \end{aligned} \tag{75}$$

where $P > 0$. The error model (53) and the Lyapunov Functional (75) has been discussed in [21]. After algebraic manipulations, upper bound on the Lypaunov Function derivative can be computed as follows:

$$\begin{aligned} \dot{V}(t) \leqslant &-e(t)^T [Q - 2\tau(\|\omega(t-\tau)\|^2 \\ &+ \int_{-\tau}^{0} \|u(t-\tau+\sigma)\|^2 \mathrm{d}\sigma) h_{\mathrm{m}} h_{\mathrm{m}}^T] e(t) \end{aligned} \tag{76}$$

where $Q > 0$ satisfies $A_{\mathrm{m}}^T P + P A_{\mathrm{m}} = -Q$. For the non-positiveness of $\dot{V}(t)$, we need to satisfy

$$Q - 2\tau \left(\|\omega(t-\tau)\|^2 + \int_{-\tau}^{0} \|u(t-\tau+\sigma)\|^2 \mathrm{d}\sigma \right) h_{\mathrm{m}} h_{\mathrm{m}}^T \geqslant 0 \tag{77}$$

Since ω and u are dependent variables, condition (77) may not be easy to check. Note however that the bound on $\dot{V}(t)$ is given by some bounds on ω defined at $t-\tau$ and on u defined on the whole interval $[t-2\tau, t-\tau]$. It is shown below that this condition can be replaced by bounds on signals ω and u over the time interval $[t_0-\tau, t_0]$ and $[t_0-2\tau, t_0]$, respectively.

Suppose that

$$\sup_{\xi \in [t_0 - \tau, t_0)} \|\omega(\xi)\|^2 \leqslant \gamma_1 \tag{78}$$

$$\sup_{\xi \in [t_0 - 2\tau, t_0)} \|u(\xi)\|^2 \leqslant \gamma_2 \tag{79}$$

for some $\gamma_1 > 0$, $\gamma_2 > 0$ and a $\tau_1 > 0$ is such that

$$2\tau_1 (\gamma_1 + \gamma_2) h_{\mathrm{m}} h_{\mathrm{m}}^T < Q \tag{80}$$

Then, the following inequality is satisfied:

$$Q - 2\tau \left(\|\omega(\xi - \tau)\|^2 + \int_{-\tau}^{0} \|u(\xi - \tau + \sigma)\|^2 \, \mathrm{d}\sigma \right) h_{\mathrm{m}} h_{\mathrm{m}}^T > 0,$$
$$\forall \xi \in [t_0, t_0 + \tau), \forall \tau \in [0, \tau_1]. \tag{81}$$

It follows that $V(t)$ is non-increasing for $t \in [t_0, t_0 + \tau)$. Thus, we have

$$\|X_{\mathrm{p}}(\xi)\| \leqslant \sqrt{\frac{V(t_0)}{\lambda_{min}(P)}} + \|X_{\mathrm{m}}(\xi)\| \tag{82}$$

and hence,

$$\|\omega(\xi)\| \leqslant \sqrt{\frac{V(t_0)}{\lambda_{min}(P)}} + \|X_{\mathrm{m}}(\xi)\|, \forall \xi \in [t_0, t_0 + \tau). \tag{83}$$

The inequality in (83) is due to the fact that ω is a part of the state vector X_{p}. We also have the following inequalities as a result of non-increasing Lyapunov functional:

$$\left\|\tilde{\theta}(\xi)\right\|^2 \leqslant V(t_0), \tag{84}$$

$$\left\|\int_{-\tau}^{0} \tilde{\phi}(\xi, \sigma)^2 \mathrm{d}\sigma\right\| \leqslant V(t_0). \tag{85}$$

To simplify the notation, we define

$$I_0 \triangleq \max \left(\sqrt{\frac{V(t_0)}{\lambda_{min}(P)}} + \|X_{\mathrm{m}}\|, V(t_0), \sqrt{V(t_0)} \right). \tag{86}$$

An upper bound on the control signal $u(t)$ for $t \in [t_0, t_0 + \tau)$ can be derived by using Lemma 1. In particular, setting $t_i^{'} = t_0$, $t_j^{'} = t_0 + \tau$, $c_0^2 = V(t_0)$ and using (47), (79), (84), (85) and (86), we obtain that

$$|u(\xi)| \leqslant 2\left(\bar{f} + \left(\int_{-\tau}^{0} (u(t_0+\sigma))^2 \, \mathrm{d}\sigma\right)^{1/2} I_0\right) e^{I_0 \tau}$$
$$\forall\, \xi \in [t_0, t_0+\tau)\,, \tag{87}$$

where $\bar{f}$ depends only on I_0. For simplicity, we will define $g(\gamma_2, I_0, \tau) \triangleq 2\left(\bar{f} + \gamma_2 I_0 \sqrt{\tau}\right) e^{I_0\tau}$ and rewrite (87) as

$$|u(\xi)| \leqslant g(\gamma_2, I_0, \tau), \forall \xi \in [t_0, t_0+\tau)\,, \tag{88}$$

Step 2. *A delay value is found that leads to a non-increasing $\dot{V}$ over $[t_0, t_0 + 2\tau]$, which in turn shows that X_{p} is bounded over the same interval.* Consider a delay value of $\tau_2 > 0$ that satisfies

$$2\tau_2 \left(I_0^2 + (\max(\gamma_2, g(\gamma_2, I_0, \tau_2)))^2 \tau_2\right) h_{\mathrm{m}} h_{\mathrm{m}}^T < Q. \tag{89}$$

For $\bar{\tau}_2 = \min(\tau_1, \tau_2)$, (77) is satisfied in the interval $[t_0, t_0 + 2\tau)$, for all $\tau \in [0, \bar{\tau}_2]$. Therefore, we obtain that

$$\|X_{\mathrm{p}}(\xi)\| \leqslant I_0, \forall \xi \in [t_0, t_0+2\tau)\,. \tag{90}$$

Step 3. *It is shown that the bound on the control signal u over the time interval $[t_0, t_0+\tau)$ depends only on $A_{\mathrm{p}}, b_{\mathrm{p}}, T, I_0$ and τ. The proof given in this section is similar to the one given in [29], Lemma 5.*

Let $\Psi_{\mathrm{u}} = \{t |\, |u(t)| = \sup_{\sigma \leqslant t} |u(\sigma)|\}$.

Let $[t_i - \tau, t_i - \tau + T] \subset \Psi_{\mathrm{u}} \subset [t_0, t_0 + \tau)$. Defining $z(t) = u(t-\tau)$, we can solve (31) as

$$x_{\mathrm{p}}(t_i + T) = x_{\mathrm{p}}(t_i) e^{A_{\mathrm{p}} T} + \int_{t_i}^{t_i+T} e^{A_{\mathrm{p}}(T+t_i-t)} b_{\mathrm{p}} z(t) \mathrm{d}t \tag{91}$$

Positive constants c_6 and c_7 exists such that $||e^{A_{\mathrm{p}}T}|| \leqslant c_6$ and $|| \int_0^T e^{A_{\mathrm{p}}\sigma} b_{\mathrm{p}} \mathrm{d}\sigma|| \geqslant c_7$ and

$$\|x_{\mathrm{p}}(t_i+T)\| \geqslant |c_7 |z(t_i)| - c_6 \, \|x_{\mathrm{p}}(t_i)\| | \tag{92}$$

Note that we obtain (92) by selecting T such that the terms of the vector $e^{A_{\mathrm{p}}\sigma} b_{\mathrm{p}}$ does not change sign for $\sigma \in [0, T]$. In addition, note that once T is selected properly, the inequality (92) is satisfied for any time interval $[t_j, t_j + T]$ as soon as $[t_j - \tau, t_j + T - \tau] \subset \Psi_{\mathrm{u}} \subset [t_0, t_0 + \tau)$ since the constants c_6 and c_7 are determined only by the size of T.

From (92) we obtain that

$$\|x_{\mathrm{p}}(t_i + T)\| + \|x_{\mathrm{p}}(t_i)\| \, c_6 \geqslant c_7 \, |z(t_i)| \tag{93}$$

or

$$|u(t_i - \tau)| \leqslant \frac{c_6}{c_7} \|x_{\mathrm{p}}(t_i)\| + \frac{1}{c_7} \|x_{\mathrm{p}}(t_i + T)\| \tag{94}$$

Since x_{p} is a part of the state vector X_{p}, $\|x_{\mathrm{p}}(t)\| \leqslant \|X_{\mathrm{p}}(t)\|$, for any t. In addition, we find using (90) that $\|x_{\mathrm{p}}(t)\| \leqslant I_0$ in the time interval $[t_0, t_0 + 2\tau)$. Therefore, (94) can be simplified as

$$|u(t_i - \tau)| \leqslant \frac{(1 + c_6)}{c_7} I_0 \tag{95}$$

Using Lemma 1 by setting $t_i^{'} = t_i - \tau$, $t_j^{'} = t_i - \tau + T$ and $c_0^2 = V(t_0)$, using (86) and noting that $\left(\int_{-\tau}^{0} (u(t_i - \tau + \sigma))^2 \, \mathrm{d}\sigma\right)^{1/2} = c_1$, we obtain that

$$|u(t_i + T - \tau)| \leqslant 2 \left(\bar{f} + I_0 \sqrt{\int_{-\tau}^{0} (u(t_i - \tau + \sigma))^2 \, \mathrm{d}\sigma} \right) e^{I_0 T}. \tag{96}$$

Since $|u(t)| = \sup_{\sigma \leqslant t} |u(\sigma)|$ over $t \in [t_i - \tau, t_i + T - \tau]$, (96) can be simplified as

$$|u(t_i + T - \tau)| \leqslant 2 \left(\bar{f} + u(t_i - \tau) I_0 \sqrt{\tau} \right) e^{I_0 T} \tag{97}$$

Using (95), (97) can be simplified as

$$|u(t_i + T - \tau)| \leqslant 2 \left(\bar{f} + \frac{\sqrt{\tau}(1 + c_6)}{c_7} I_0^2 \right) e^{c_0 T} \tag{98}$$

Since $|u(t)| = \sup_{\sigma \leqslant t} |u(\sigma)|$ over $t \in [t_i - \tau, t_i - \tau + T]$ and t_i is any arbitrary time instant in $\Phi_{\mathrm{u}} \cap [t_0, t_0 + \tau)$, it follows that $\forall t \in \Psi_{\mathrm{u}} \cap [t_0, t_0 + \tau)$,

$$|u(t)| \leqslant 2 \left(\bar{f} + \frac{\sqrt{\tau}(1 + c_6)}{c_7} I_0^2 \right) e^{I_0 T}. \tag{99}$$

When $t \notin \Psi_{\mathrm{u}}$, the inequality (99) is strengthened further. Hence, it follows that the inequality holds for the whole interval $[t_0, t_0 + \tau)$. Note that the right hand side of the inequality is a function of $A_{\mathrm{p}}, b_{\mathrm{p}}, T$ and I_0. For simplicity, we are going to drop the dependence on $A_{\rm p}, b_{\rm p}$ and T and represents the inequality as in the following, which simplifies the notation:

$$|u(t)| \leqslant U(I_0),\ t \in [t_0, t_0 + \tau) \tag{100}$$

where U is a continuous function.

Step 4. *Finally, a uniform upper bound τ^* is given for the time-delay τ such that I is satisfied.*

Let the delay value τ_3 satisfy the following inequality:

$$2\tau_3[I_0^2 + (\max(U(I_0), g(U(I_0), I_0, \tau_3)))^2\, \tau_3] h_{\rm m} h_{\rm m}^T < Q. \tag{101}$$

For $\tau^* = \min(\bar{\tau}_2, \tau_3)$, the following inequalities hold:

$$\|X_{\rm p}(\xi)\| \leqslant I_0,\ \ |u(\xi)| \leqslant U(I_0)\ \ \forall \xi \in [t_0, t_0 + \tau)\ \ \forall \tau \in [0, \tau^*] \tag{102}$$

This completes the proof of I.

Note that, in Steps 1-3, I is proved for a delay value $\tau \in [0, \bar{\tau}_2]$. Finally in Step 4, a new upper bound, τ^*, is given for the allowable time-delay τ. The need for the introduction of this new upper bound will be clear in the next section, where we prove II.

4.3.2 II. If S is true for k, then it is true for $k + 1$

Assume that

$$\|X_{\rm p}(\xi)\| \leqslant I_0, \quad |u(\xi)| \leqslant U(I_0) \quad \xi \in [t_0, t_0 + k\tau]. \tag{103}$$

Then, using (101) we conclude that the Lyapunov function is non-increasing in the time interval $[t_0 + k\tau, t_0 + (k+1)\tau]$ since $\tau \leqslant \tau^* \leqslant \tau_3$. This means that $\|X_{\rm p}(t)\| \leqslant I_0$, and, using Lemma 1, $|u(t)| \leqslant g(U(I_0), I_0, \tau)$ for $t \in [t_0 + k\tau, t_0 + (k+1)\tau]$. Again, using (101) and the fact that $\tau \leqslant \tau^* \leqslant \tau_3$, we conclude that the Lyapunov function is non-increasing in the time interval $[t_0+(k+1)\tau, t_0+(k+2)\tau]$. Hence, $\|X_{\rm p}(t)\| \leqslant I_0$ for $t \in [t_0+(k+1)\tau, t_0+(k+2)\tau]$. But this means that $|u(t)| \leqslant U(I_0)$ for $t \in [t_0 + k\tau, t_0 + (k+1)\tau]$ using the same procedure as in Step 3. Therefore, we have that

$$\|X_{\rm p}(\xi)\| \leqslant I_0, \quad |u(\xi)| \leqslant U(I_0) \quad \xi \in [t_0, t_0 + (k+1)\tau] \tag{104}$$

This completes the proof of II.

Therefore, we proved the boundedness of the signals in the system, using induction. The second derivative of the Lyapunov functional includes these signals as well as their derivatives which can be shown to be bounded using the adaptation laws (54). Hence, the requirements of the Barbalat's Lemma are satisfied, which completes the stability proof. Note that the region of stability depends on the time-delay and therefore the stability is semi-global.

Remark 1 We make a few comments about the development of the overall control architecture. The first step in this architecture is the design of a fixed, non-adaptive controller. This was carried out by beginning with a control design (Figure 1) quite similar to an SP-controller. The resulting controller is rational but non-causal, and therefore is transformed into a causal, infinite-dimensional controller (44). The second step of this architecture is the development of adaptation laws for time-varying parameters which estimate the unknown control parameters for the case when the plant has unknown parameters. This is carried out in a similar manner to that proposed in [21].

Remark 2 A Lyapunov-Krasovskii functional (75) is used to begin the stability proof, which is similar to the procedure in [21]. Where this paper differs from [21] is that this functional, in our case, does not depend on the infinite dimensional state introduced by the control input history of one delay interval. The structure of the control input is then exploited to show the boundedness of the all state variables. Growth rate conditions, similar to those used in [20], are used in this regard. This procedure helps us remove the assumption of plant poles having multiplicity one.

A few remarks about the nature of the proof are in order.

Remark 3 Induction is the tool that is used for the stability proof. In the first part, we showed that all system states are bounded in the first delay interval given that the delay value is bounded by τ^*. This is achieved by using the Lyapunov-Krasovskii functional (75). Note that in general, Lyapunov-Krasovskii functionals include the infinite dimensional state introduced by one delay interval time history of the control signal to satisfy the positivity requirement. However, (75) does not include this state and therefore additional tools were needed to be used to show the boundedness of the control signal. In particular, we used signal growth rate conditions [20], which simply gives rules that defines the relationship between the states of an LTI system and its outputs. Using this relationship, we showed in Lemma 1 that we can put an upper bound on the control signal at time t in terms of system state values during the time interval $[t - T, t]$, for any T. This lemma is used in Part I as well as in Part II of the stability proof, where we showed that given all the states are bounded for the k^{th} interval, the states for the next interval are also bounded.

5 Experimental Results

Here we briefly present experimental results that are obtained from the application of the APC to fuel-to-air ratio (FAR) control problem of gasoline engines. The details can be found in [25].

In FAR control, the objective is to maintain the in-cylinder fuel-to-air ratio at a prescribed set point, determined primarily by the state of the Three-Way Catalyst (TWC), so that the pollutants in the exhaust are removed with the highest efficiency. The main components of the FAR dynamics are the wall wetting dynamics, gas mixing dynamics, sensor dynamics and the time-delay. The time-delay in the system comprises two basic components [30]: the time it takes from the fuel injection calculation to exhaust gas exiting the cylinders and the time it takes for the exhaust gases to reach the UEGO sensor location. System dynamics with the time-delay can be closely approximated with a first order transfer function and a time-delay in series [31]:

$$G(s) = \frac{1}{\tau_m s + 1} e^{-\tau s} \tag{105}$$

where, τ_m is the system time constant, which may be uncertain due to changing operating conditions, and τ is the total time-delay. The controller must handle the time-delay and the uncertainty in the time constant. In addition, the disturbances due to canister vapor purging and due to inaccuracies in air charge estimation and wall wetting compensation, must be rejected.

The purpose of our FAR control experiments was to compare the performances of the APC and the baseline controller, while emulating canister vapor purge disturbance rejection tests. These experiments were conducted with the test vehicle idling at different speeds.

The test started with the engine speed at 700 rpm. At 300 sec, the engine speed increased to 1000 rpm and at 600 sec it decreased back to 700 rpm. Every 20 sec the fuel injector gains were changed to emulate the purge disturbance. Overall, the performance of the APC was 70 percent better, in terms of integral error which influences catalyst oxygen storage and emissions, than the baseline controller during the test which lasted 15 minutes. Figure 3 shows a time window from the test where the engine speed was 700 rpm. The APC performs considerably better than the baseline controller as its features enable it to better account for the delay and achieve faster response. The performance measure for these tests are the integral of the tracking error.

Figure 4 shows how the equivalence ratio changes during the same test but now the engine speed is 1000 rpm. Again, the performance of the APC is

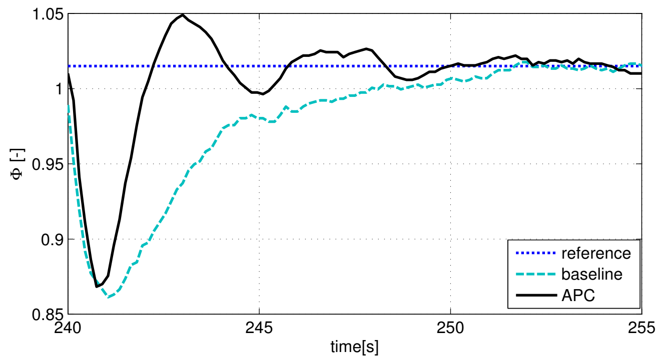

Fig. 3. Comparison of baseline controller with APC for purge disturbance rejection at 700 rpm.

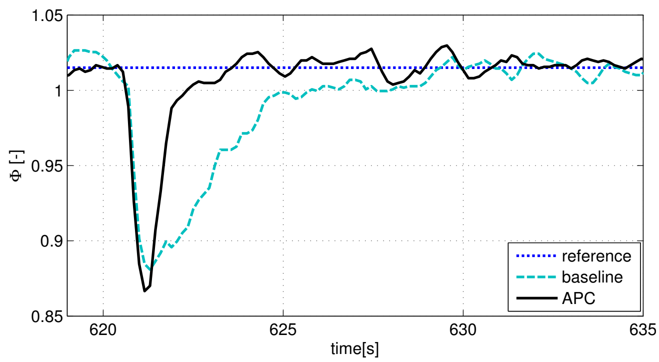

Fig. 4. Comparison of baseline controller with APC for purge disturbance rejection at 1000 rpm, with $c = 1$.

better than that of the baseline controller.

Remark 4 Another experimental application we have recently treated using these techniques is the idle speed control. In idle speed control, where the goal is to regulate the engine speed at a certain set point, we have a similar transfer function with a time-delay in series. In this control problem, the time-delay is the time it takes from the throttle opening to the torque generation in the power cycle. Successful implementation of the APC for the idle speed control of gasoline engines can be found in [23], where the APC outperforms the baseline controller existing in the test vehicle.

6 Summary

In this paper, we proposed an Adaptive Posicast Controller (APC) for time-delay systems. The controller is based on the Smith Predictor and finite-spectrum assignment controller and is modified so as to accommodate parametric uncertainties in the plant dynamics. We first proposed a model matching controller for time-delay systems with known time-delay, where, a non-causal controller is designed first, for the rational part of the plant dynamics. Then, by using the known delay value, the future prediction of the system states are calculated and used to show that the non-causal controller is equivalent to a causal controller. Using this equivalence, we derived the closed loop error dynamics for the APC and showed the stability of the overall system by defining a suitable Lyapunov-Krasovskii functional.

The simplicity of the APC makes it a good candidate for industrial applications where adaptation is needed. To verify this, experimental results that show the success of the APC in an important automotive control problem were presented. These implementation results show that the APC can considerably outperform the existing controller in the test vehicles.

Acknowledgements

This work was supported through the Ford-MIT Alliance Initiative. The authors would like to acknowledge Dr. Davor Hrovat of Ford Motor Company for his support and encouragement during this project.

References

[1] Q.-G. Wang, T. H. Lee, and K. K. Tan, *Finite Spectrum Assignment for Time-Delay Systems*, ser. Lecture Notes in Control and Information Sciences, M. Thoma, Ed. New York: Springer-Verlag, 1999, vol. 239.

[2] V. B. Kolmanovsky and A. Myshkis, *Introduction to the theory and applications of functional differential equations.* Dordrecht: Kluwer Academy, 1999.

[3] S. I. Niculescu, *Delay effects on stability.* Berlin: Springer., 2001.

[4] J.-P. Richard, "Time-delay systems:an overview of some recent advances and open problems," *Automatica*, vol. 39, pp. 1667–1694, 2003.

[5] K. Gu and S.-I. Niculescu, "Survey on recent results in the stability and control of time-delay systems," *Journal of Dynamic Systems, Measurement and Control*, vol. 125, no. 2, pp. 158–165, 2003.

[6] J. Jang, "Adaptive control design with guaranteed margins for nonlinear plants," Ph.D. dissertation, Massachusetts Institute of Technology, 2008.

[7] O. J. Smith, "A controller to overcome dead time," *ISA Journal*, vol. 6, 1959.

[8] K. J. Astrom, C. C. Hang, and B. C. Lim, "A new Smith predictor for controlling a process with an integrator and long dead-time," *IEEE Transactions on Automatic Control*, vol. 39, no. 2, pp. 343–345, Feb. 1994.

[9] S. Majhi and D. Atherton, "A new smith predictor and controller for unstable and integrating processes with time delay," in *Proc. Conference on decision and control*, Tampa, FL, Dec. 1998.

[10] M. Fliess, R. Marquez, and H. Mounier, "PID-like regulators for a class of linear delay systems," in *Proc. Europ. Control Conf.*, Porto, Portugal, Sep. 2001.

[11] Z. J. Palmor, "Time-delay compensation-Smith predictor and its modifications," in *The control handbook*, W. Levine, Ed. Boca Raton, FL, USA: CRSC Press, 1996, pp. 224–237.

[12] O. J. M. Smith, *Feedback Control Systems.* New York: McGraw-Hill, Inc., 1976.

[13] S. Park, D. Wee, and G. A. F. Annaswamy, A. M., "Adaptive low-order posicast control of a combustor test-rig model," in *Proc. of Conference on Decision and Control*, Dec. 2002, pp. 3698–3703.

[14] A. Z. Manitius and A. W. Olbrot, "Finite spectrum assignement problem for systems with delays," *IEEE Transactions on Automatic Control*, vol. 24, no. 4, 1979.

[15] K. Watanabe, E. Nobuyama, T. Kitamori, and M. Ito, "A new algorithm for finite spectrum assignment of single-inputsystems with time delay," *IEEE Transactions on Automatic Control*, vol. 37, no. 9, pp. 1377–1383, Sep. 1992.

[16] H. Glusing-Luerben, "A behavioral approach to delay-differential systems," *SIAM Journal on Control and Optimization*, vol. 35, no. 2, pp. 480–499, 1997a.

[17] J. J. Loiseau and D. Brethe, "An efective algorithm for finite spectrum assignment of single-input systems with delays," *Mathematics and Computers in Simulation,*, vol. 45, no. 3-4, pp. 339–348, 1998.

[18] K. Ichikawa, "Frequency-domain pole assignement and exact model-matching for delay systems," *International Journal of Control*, vol. 41, pp. 1015–1024, 1985.

[19] R. Ortega and R. Lozano, "Globally stable adaptive controller for systems with delay," *International Journal of Control*, vol. 47, no. 1, pp. 17–23, 1988.

[20] K. S. Narendra and A. M. Annaswamy, *Stable adaptive systems.* New York: Dover Publications, 2005.

[21] S.-I. Niculescu and A. M. Annaswamy, "An adaptive smith-controller for time-delay systems with relative degree $n \leqslant 2$," *Systems and Control Letters*, vol. 49, pp. 347–358, 2003.

[22] S. N. A. P. D. S. Evesque, A. M. Annaswamy, "Adaptive control of a class of time-delay systems," vol. 125, no. 2, pp. 186–194, 2003.

[23] Y. Yildiz, A. Annaswamy, D. Yanakiev, and I. Kolmanovsky, "Adaptive idle speed control for internal combustion engines," in *Proc. Amer. Control Conf.*, New York City, July 2007, pp. 3700–3705.

[24] ——, "Adaptive air fuel ratio control for internal combustion engines," in *Proc. Amer. Control Conf.*, Seattle, Washington, June 2008, pp. 2058–2063.

[25] ——, "Automotive powertrain control problems involving time delay: An adaptive control approach," in *Proc. of ASME Dynamic Systems and Control Conference*, Ann Arbor, Michigan, Oct. 2008.

[26] L. Mirkin and N. Raskin, "Every stabilizing dead-time controller has an observer-predictor-based structure," *Automatica*, vol. 39, no. 10, pp. 1747–1754, 2003.

[27] L. Debnath and P. Mikusinski, *Introduction to Hilbert Spaces with Applications*, 3rd ed. Elsevier, 2005.

[28] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., A. A. Arthur and S. L. Langman, Eds. New York: McGraw-Hill, Inc., 1976.

[29] K. S. Narendra, A. M. Annaswamy, and R. P. Singh, "A general approach to the stability analysis of adaptive systems," *International Journal of Control*, vol. 41, no. 1, pp. 193–216, 1985.

[30] Z. F., K. Grigoriadis, M. Franchek, and I. Makki, "Linear parameter varying lean burn air-fuel ratio control for a spark ignition engine," *Journal of Dynamic Systems, Measurement and Control*, vol. 129, pp. 404–414, 2007.

[31] L. Guzzella and C. H. Onder, *Introduction to Modeling and Control Internal Combustion Engine Systems*, 1st ed. Springer, 2004.

A The Bound on the Series S

Defining $a \triangleq c_0^2(t - t_i')$ and $S' \triangleq \sum_{n=1}^{\infty} \left(\frac{a^n}{n!}\right)^{1/2}$, we obtain that

$$S = \left(\bar{f} + c_0 c_1\right) S' \tag{A.1}$$

Summing up the odd terms of S' we have that

$$\begin{aligned} S_o' &= \sqrt{a} + \sqrt{\frac{a^3}{3!}} + \sqrt{\frac{a^5}{5!}} + \ldots \\ &= \sqrt{a} + a\sqrt{\frac{a}{3!}} + \frac{a^2}{2!}\sqrt{\frac{a}{30}} + \ldots \end{aligned} \tag{A.2}$$

If $a < 1$, then we obtain that

$$S_o' \leqslant 1 + a + \frac{a^2}{2!} + \ldots = e^a \tag{A.3}$$

If $a > 1$, then we obtain that

$$\begin{aligned} S_o^{'} &\leqslant a + \frac{a^2}{2!}\frac{1}{\sqrt{1.5}} + \frac{a^3}{3!}\frac{1}{\sqrt{10/3}} + ... \\ &\leqslant 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + ... \\ &= e^a. \end{aligned} \tag{A.4}$$

Summing up the even terms, we have

$$\begin{aligned} S_e^{'} &= \frac{a}{\sqrt{2!}} + \frac{a^2}{\sqrt{4!}} + \frac{a^3}{\sqrt{6!}} + ... \\ &\leqslant 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + ... \\ &= e^a \end{aligned} \tag{A.5}$$

Using (A.1), (A.3), (A.4) and (A.5) we obtain that

$$\begin{aligned} S &= \left(\bar{f} + c_0 c_1\right)\left(S_o^{'} + S_e^{'}\right) \\ &\leqslant 2\left(\bar{f} + c_0 c_1\right) e^a. \end{aligned} \tag{A.6}$$

Finally, substituting the definition of a in (A.6) we have

$$S \leqslant 2\left(\bar{f} + c_0 c_1\right) e^{c_0^2 (t - t_i^{'})}. \tag{A.7}$$

This completes the proof.