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A Convergent Frequency Estimator [1]

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Abstract

Online identification of sinusoidal components is an important problem that occurs in active noise control, vibration suppression, on line health monitoring, and radar, sonar, and seismic applications. We adopt a new approach to this identification problem which consists of the utilization of the underlying nonlinearlity and an algorithm that is based on the nonlinear parameterization. The algorithm is shown to result in global convergence in the presence of two unknown frequencies. Extensions to n unknown frequencies for $n \geq 2$ that have unknown amplitudes are also discussed.

1 Introduction

This paper focuses on the problem of online identifying of sinusoidal components, namely the amplitude and frequency, of a signal that is a combination of many sinusoids. Specifically, given a signal $y(u)$ defined as

$$y(u) = \sum_{i=1}^{N} a_i \cos(\omega_i u) \qquad (1)$$

where N is the number of sinusoidal components, u is the known input variable, a_i and ω_i are the amplitude and frequency of the i^{th} component respectively, the goal is to design a globally stable online identification algorithm to identify all a_i's and ω_i's. Online identification of sinusoidal components plays a significant role in many engineering applications. An example is active noise and vibration control. For any machine with rotating components, the resulting noises or vibrations are often modeled as periodic signal. Examples of this class of applications include noises in turboprop aircraft [1], in helicopters [2], in HVAC systems [3]. Health monitoring is another application of sinusoidal identification in which the interest involves online monitoring the performance of a complex system such as the Space Shuttle Main Engines by continuously detecting structural degradations that potentially lead the catastrophic failures [4]. Furthermore, sinusoidal identification is also essential in radar, sonar, and seismic applications [5]. In this paper, we take a first step towards such sinusoidal identification by making direct use of the structure of the sinusoidal function in (1), rather than transform (1) into a dynamic system where ω_i are transformed into linear parameters of a differential equation as in [6, 5].

The rest of the paper is organized as follows. In Section 2, a new identification algorithm for the case when $y(u)$ consists of two unknown frequencies is proposed. In Section 3, we give several useful Lemmas. In Section 4, we prove the global convergence of the algorithm proposed in Section 2.

2 Statement of the Problem

This section considers the sinusoidal identification problem with y having the form

$$y(t) = g(w, u(t)) = \cos(\omega_1 u(t)) + \cos(\omega_2 u(t)) \qquad (2)$$

where ω_1 and ω_2 are the unknown frequencies to be identified, and $\omega = [\omega_1,\ \omega_2]^T$. $y(t)$ and $u(t)$ are assumed to be measured at each instant t.
The following assumptions are made regarding (2): (i) u switches between u_1 and u_2; (ii) $\overline{\omega} > \omega_1 \geq \omega_2 \geq \underline{\omega}$, and $\underline{\omega}, \overline{\omega}$ are known. It follows that for $u_1 \leq \pi/(2\overline{\omega})$, $\cos(\omega u)$ is concave on $[\underline{\omega}, \overline{\omega}]$.

The following identification algorithm is proposed to identify ω as $\widehat{\omega} = [\widehat{\omega}_1, \widehat{\omega}_2]^T$:

$$\begin{aligned} &\hat{y}(t) = g(\widehat{\omega}, u(t)) = \textstyle\sum_{i=1}^{2} \cos(\widehat{\omega}_i u(t)) \\ &\dot{\widehat{\omega}}_i = -\tilde{y} \nabla_{\widehat{\omega}_i} \hat{y}, \qquad \tilde{y} = \hat{y} - y. \end{aligned} \qquad (3)$$

Defining $L = \{(\omega_1, \omega_2), (\omega_2, \omega_1)\}$, the goal is to establish the convergence of $\widehat{\omega}$ to L.

3 Preliminaries

The following definitions are useful for proving the main theorem. We define

$$\Omega = \{\widehat{\omega} \mid \widehat{\omega}_{min} \leq \widehat{\omega}_i \leq \widehat{\omega}_{max},\ \ i = 1, 2\}.$$

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and

$$f(\widehat{\omega},u) = g(\widehat{\omega},u) - g(\omega,u).$$

Without loss of generality, we shall assume that $\widehat{\omega} \in \Omega$. Let M_1 and M_2 represent the curves $\tilde{y} = 0$ when $u = u_1$ and $u = u_2$, respectively, in the $\widehat{\omega}$ space. That is,

$$\begin{aligned} M_1 &= \{\widehat{\omega} \mid f(\widehat{\omega},u_1) = 0\} \\ M_2 &= \{\widehat{\omega} \mid f(\widehat{\omega},u_2) = 0\} \end{aligned} \tag{4}$$

and are illustrated in Figure 1.

The following properties of M_1 and M_2 can be derived:

Lemma 3.1 M_1 and M_2 are monotonically decreasing concave functions.

Lemma 3.2 $M_1 \bigcap M_2 = L$.

Proofs of Lemmas can be found in [7].

4 Convergence Results

We now state the main result of the paper. In what follows, we define

$$E = \{\widehat{\omega} \mid \widehat{\omega} \in \Omega \text{ and } \widehat{\omega}_1 = \widehat{\omega}_2\}. \tag{5}$$

Theorem 1 *Under assumptions (i)-(ii), the above proposed identification algorithm is stable and*
(a) if $\widehat{\omega}(t_0) \in E$, then $\widehat{\omega}(t) \in E$ $\forall t \geq t_0$
(b) if $\widehat{\omega}(t_0) \in \Omega \setminus E$, then $\widehat{\omega}(t) \to L$ as $t \to \infty$.

Proof.
(a) If $\widehat{\omega}(t) \in E$, $\nabla_{\widehat{\omega}_1} \cos(\widehat{\omega}_1 u) = \nabla_{\widehat{\omega}_2} \cos(\widehat{\omega}_2 u)$, which implies that $\widehat{\omega}_1(t) = \widehat{\omega}_2(t) \qquad \forall t \geq t_0$.

(b) We define two scalar functions V_{u_1} and V_{u_2} as

$$V_{u_1}(\widehat{\omega}) = \frac{1}{2}(f(\widehat{\omega},u_1))^2, \quad V_{u_2}(\widehat{\omega}) = \frac{1}{2}(f(\widehat{\omega},u_2))^2.$$

Let

$$F(\widehat{\omega}) = (-\nabla_{\widehat{\omega}} V_{u_1}(\widehat{\omega}))^T(-\nabla_{\widehat{\omega}} V_{u_2}(\widehat{\omega})), \tag{6}$$

where $\nabla_{\widehat{\omega}} V_{u_i}(\widehat{\omega})$ denote the gradient of V_{u_i} with respect to $\widehat{\omega}$, and is given for $i = 1,2$ by

$$-\nabla_{\widehat{\omega}} V_{u_i}(\widehat{\omega}) = -f(\widehat{\omega},u_i)\nabla_{\widehat{\omega}} g(\widehat{\omega},u_i) \tag{7}$$

We divide Ω into two regions:

$$\begin{aligned} \Omega_1 &= \{\widehat{\omega} \mid \widehat{\omega} \in \Omega\,,\ F(\widehat{\omega}) > 0\} \\ \Omega_2 &= \{\widehat{\omega} \mid \widehat{\omega} \in \Omega\,,\ F(\widehat{\omega}) \leq 0\}. \end{aligned} \tag{8}$$

We prove Theorem 1 using the following three steps.

Step 1: For any t_0, if $\widehat{\omega}(t_0) \in \Omega_1$, then, either

(1a) $\exists$ finite value $\bar{t}$, such that $\widehat{\omega}(t_0 + \bar{t}) \notin \Omega_1$, or

(1b) $\widehat{\omega}(t)$ converges to L as $t \to \infty$.

Step 2: For any t_0, if $\widehat{\omega}(t_0) \in \Omega_2$, then $\widehat{\omega}(t) \in \Omega_2 \ \ \forall t \geq t_0$.
Step 3: For any t_0, if $\widehat{\omega}(t) \in \Omega_2 \backslash E$, then $\widehat{\omega}(t)$ converges to L as $t \to \infty$. •

4.1 Proof of Step 1

Proof: We will now establish step 1 by contradiction. Assume that both (1a) and (1b) do not hold, that is, suppose for any $\widehat{\omega}(t) \in \Omega_1$ and $t \geq t_0$, $\widehat{\omega}(t)$ does not converge to L as $t \to \infty$.

Define sets T_{u_1} and T_{u_2} such that

$$\begin{aligned} T_{u_1} &= \{t \mid u(t) = u_1,\ t \geq t_0\} \\ T_{u_2} &= \{t \mid u(t) = u_2,\ t \geq t_0\} \end{aligned} \tag{9}$$

From the definition of V_{u_1} and V_{u_2} and from (3), it follows that

$$\dot{\widehat{\omega}}(t) = \begin{cases} -\nabla_{\widehat{\omega}} V_{u_1}(\widehat{\omega}(t)) & \forall\, t \in T_{u_1} \\ -\nabla_{\widehat{\omega}} V_{u_2}(\widehat{\omega}(t)) & \forall\, t \in T_{u_2} \end{cases} \tag{10}$$

The time derivative of $V_{u_1}(\widehat{\omega}(t))$ and $V_{u_2}(\widehat{\omega}(t))$ can be calculated as:

$$\begin{aligned} \dot{V}_{u_1}(\widehat{\omega}(t)) &= (\nabla_{\widehat{\omega}} V_{u_1}(\widehat{\omega}))\dot{\widehat{\omega}}(t) \\ \dot{V}_{u_2}(\widehat{\omega}(t)) &= (\nabla_{\widehat{\omega}} V_{u_2}(\widehat{\omega}))\dot{\widehat{\omega}}(t) \end{aligned} \tag{11}$$

Suppose $t \in T_{u_1}$, then from (10) and (11), we have

$$\dot{V}_{u_1}(\widehat{\omega}(t)) = (\nabla_{\widehat{\omega}} V_{u_1}(\widehat{\omega}(t)))^T \cdot (-\nabla_{\omega} V_{u_1}(\widehat{\omega}(t))). \tag{12}$$

It should be noted that since $\dot{V}_{u_1}(\widehat{\omega}(t))$ is negative, from the definition of Ω_1, it follows that

$$\nabla_{\widehat{\omega}} V_{u_1}(\widehat{\omega}) \neq 0. \tag{13}$$

From (12) and (13), it follows that

$$\dot{V}_{u_1}(\widehat{\omega}(t)) < 0. \qquad \forall\, t \in T_{u_1} \text{ and } \widehat{\omega}(t) \in \Omega_1. \tag{14}$$

Suppose $t \in T_{u_2}$ and $\widehat{\omega}(t) \in \Omega_1$, then from (10) and (11), we have

$$\dot{V}_{u_1}(\widehat{\omega}(t)) = (\nabla_{\widehat{\omega}} V_{u_1} \widehat{\omega}(t))^T \cdot (-\nabla_{\omega} V_{u_2}(\widehat{\omega}(t))) \tag{15}$$

From the definition of Ω_1 and (15), we can conclude that

$$\dot{V}_{u_1}(\widehat{\omega}(t)) < 0 \qquad \forall\, t \in T_{u_2} \text{ and } \widehat{\omega}(t) \in \Omega_1. \tag{16}$$

Therefore, from (14) and (16) we conclude that

$$\dot{V}_{u_1}(\widehat{\omega}(t)) < 0 \qquad \forall\, \widehat{\omega}(t) \in \Omega_1 \text{ and } t \geq t_0.$$

It can be proved in a similar manner that

$$\dot{V}_{u_2}(\widehat{\omega}(t)) < 0 \qquad \forall\, \widehat{\omega}(t) \in \Omega_1 \text{ and } t \geq t_0.$$

Since we have assumed that *(1a)* and *(1b)* do not hold, it follows that indeed $\widehat{\omega}(t) \in \Omega_1$ for all $t \geq t_0$. Therefore $\forall t \geq t_0$, $\dot{V}_{u_1}(\widehat{\omega}(t)) < 0$, and hence $V_{u_1}(\widehat{\omega}(t))$ monotonically decreases. Since $V_{u_1}(\widehat{\omega}(t))$ is continuous and has a lower bound (of zero), it reaches a limit point. The same is true for $V_{u_2}(\widehat{\omega}(t))$ as well. Suppose

$$\Omega_l = \{\widehat{\omega}_l \mid \lim_{t\to\infty} \widehat{\omega}(t) = \widehat{\omega}_l\},$$

it follows that (i) $\lim_{t\to\infty} \widehat{\omega}(t) = \widehat{\omega}_l$, (ii)$\lim_{t\to\infty} V_{u_i}(\widehat{\omega}(t)) = V_{u_i}(\widehat{\omega}_l)$ and (iii) $\lim_{t\to\infty} \dot{V}_{u_i}(\widehat{\omega}(t)) = 0$ for $i = 1, 2$. It also follows that

$$\lim_{t\to\infty} \dot{V}_{u_i}(\widehat{\omega}_l) = 0 \implies -\nabla_{\widehat{\omega}} V_{u_i}(\widehat{\omega}_l) = 0 \qquad i = 1, 2. \tag{17}$$

We note that

$$\sin(\widehat{\omega} u) > 0 \qquad \widehat{\omega} \in \Omega, \text{ for } i = 1, 2, \tag{18}$$

and therefore the elements of $\nabla_{\widehat{\omega}} g(\widehat{\omega}, u)$ are negative. Hence, (17) and (18) imply that

$$f(\widehat{\omega}_l, u_i) = 0, \ \forall \widehat{\omega}_l \in \Omega_l \qquad i = 1, 2. \tag{19}$$

From (19) and the definition of M_1 and M_2 in (4), it follows that

$$\widehat{\omega}_l \subset M_{u_1} \bigcap M_{u_2} \tag{20}$$

From (20) and *lemma 3.2*, it follows that $\omega_l \subset L$. We have shown that if $\widehat{\omega}(t) \in \Omega_l$, $\widehat{\omega}(t) \to L$ as $t \to \infty$. This contradicts our earlier assumption that *(1a)* and *(1b)* in step 1 do not hold. Therefore Step 1 is proved.

4.2 Proof of Step 2

Before processing to the proof, we include the following definitions.

$$\begin{aligned} \Omega_2^b &= \{\widehat{\omega} \mid \widehat{\omega} \in \Omega\,,\, F(\widehat{\omega}) = 0\} \\ \Omega_2^i &= \{\widehat{\omega} \mid \widehat{\omega} \in \Omega\,,\, F(\widehat{\omega}) < 0\} \end{aligned} \tag{21}$$

where $F(\widehat{\omega})$ is defined in (6).
The following properties of Ω_2^b and Ω_2^i follow directly from the above definitions:

(1) $\Omega_2 = \Omega_2^b \bigcup \Omega_2^i$

(2) $\Omega_2^b = M_1 \bigcup M_2$.

For any t_1 such that $\widehat{\omega}(t_1) \in \Omega_2^i$, since $F(\widehat{\omega})$ is a continuous function of $\widehat{\omega}$, only two possibilities arise.

(a) $\widehat{\omega}(t) \in \Omega_2^i$, $\quad \forall t \geq t_1$; or

(b) $\widehat{\omega}(t_2) \in \Omega_2^b$ for a finite $t_2 \geq t_1$

If (a) holds, proof of step 2 is complete. If (b) holds, it implies that $\widehat{\omega}(t_2) \in \Omega_2^b$. Without loss of generality, we assume that $\widehat{\omega}(t_2) \in M_1$. We note that

$$V_{u_1}(\widehat{\omega}(t_2)) = 0, \qquad \text{if } \widehat{\omega}(t_2) \in M_1.$$

and

$$\dot{V}_{u_1}(\widehat{\omega}(t)) < 0, \qquad \text{if } \widehat{\omega}(t) \in \Omega_1 \text{ and } \forall t \geq t_2.$$

Therefore $\widehat{\omega}(t)$ cannot enter Ω_1 starting from any $\widehat{\omega}(t_2) \in M_1$. It can be proved in a similar manner that $\widehat{\omega}(t)$ cannot enter Ω_1 starting from any $\widehat{\omega}(t_2) \in M_2$. That is, if $\widehat{\omega}(t_2) \in \Omega_2^b$, then $\widehat{\omega}(t)$ must enter Ω_2^i or stay in Ω_2^b for all $t \geq t_2$. This proves Step 2.

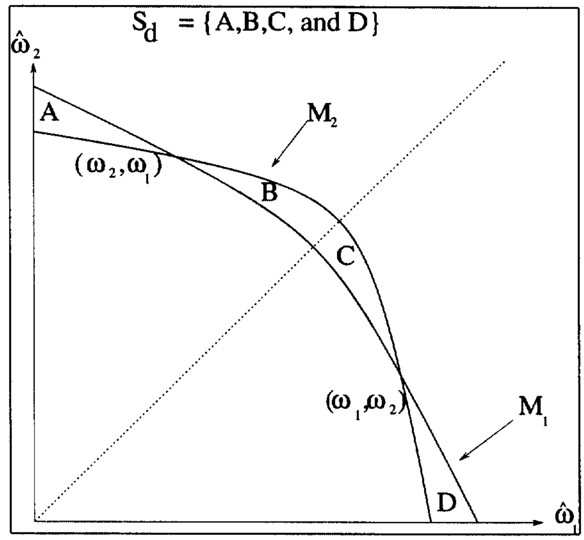

Figure 1: M_1 and M_2 curves, *A*,*B*,*C* and *D* Regions.

4.3 Proof of Step 3

We begin with $\widehat{\omega}(t_0)$ such that $\widehat{\omega}(t_0) \in \Omega_2 \setminus E$. The goal is to show that $\widehat{\omega}(t) \to L$ as $t \to \infty$. The following lemmas are useful in the proof of Step 3.
Lemma 4.3.1 $\Omega_2 \setminus E = A \bigcup B \bigcup C \bigcup D$ where

$$\begin{aligned} A &= \{\widehat{\omega} \mid f(\widehat{\omega}, u_1) \leq 0, f(\widehat{\omega}, u_2) \geq 0,\ \widehat{\omega}_2 > \widehat{\omega}_1\} \\ B &= \{\widehat{\omega} \mid f(\widehat{\omega}, u_1) \geq 0, f(\widehat{\omega}, u_2) \leq 0, \widehat{\omega}_2 > \widehat{\omega}_1\} \\ C &= \{\widehat{\omega} \mid f(\widehat{\omega}, u_1) \leq 0, f(\widehat{\omega}, u_2) \geq 0, \widehat{\omega}_2 < \widehat{\omega}_1\} \\ D &= \{\widehat{\omega} \mid f(\widehat{\omega}, u_1) \geq 0, f(\widehat{\omega}, u_2) \leq 0, \widehat{\omega}_2 < \widehat{\omega}_1\} \end{aligned} \tag{22}$$

are shown in figure 2.
Proof of Lemma 4.3.1 From (6) and (7), it follows that

$$F(\widehat{\omega}) \leq 0 \implies f(\widehat{\omega}, u_1) f(\widehat{\omega}, u_2) \leq 0$$

Therefore, we can write Ω_2 as

$$\begin{aligned} \Omega_2 = & \{\omega \mid f(\widehat{\omega}, u_1) \geq 0 \text{ and } f(\widehat{\omega}, u_2) \leq 0\} \\ & \bigcup \{\omega \mid f(\widehat{\omega}, u_1) \leq 0 \text{ and } f(\widehat{\omega}, u_2) \geq 0\}. \end{aligned} \tag{23}$$

For (23) and (5), it can be seen that

$$\Omega_2 \setminus E = A \bigcup B \bigcup C \bigcup D \qquad \bullet$$

Lemma 4.3.2 If (i) $0 < \omega_2 < \omega_1$, (ii) $0 < u_2 < u_1$ and (ii) $\omega_1 u_1 \leq \frac{\pi}{2}$, then $Q(\omega_1, \omega_2, u_1, u_2) > 0$ where

$$Q = \sin(w_2 u_1)\sin(\omega_1 u_2) - \sin(\omega_2 u_2)\sin(\omega_1 u_1).$$

We now prove *Step 3* in three substeps.
Substep 1: For any $\widehat{\omega}(t_0)$, if $u(t) = u_i$, $\forall t \geq t_0$, then the set of all limit points $\widehat{\omega}_l$ in Ω_l belong to M_i, for $i = 1, 2$.

Proof of Substep 1: If $\widehat{\omega}(t_0) \in \Omega_2$, we note that if $u(t) = u_i, \ \forall t \geq t_0$, then

$$\lim_{t\to\infty} \dot{V}_{u_i}(\widehat{\omega}(t)) = (\nabla_{\widehat{\omega}} V_{u_i}(\widehat{\omega}(t))^T(-\nabla_{\widehat{\omega}} V_{u_i}(\widehat{\omega}(t)).$$

Therefore $V(u_i(\widehat{\omega}(t))$ monotonically decreases. As in step 1, we can show that

$$\lim_{t\to\infty} V_{u_i}(\widehat{\omega}(t)) = V_{u_i}(\widehat{\omega}_l) \text{ and } \lim_{t\to\infty} \dot{V}_{u_i}(\widehat{\omega}(t)) = 0.$$

It follows that

$$\dot{V}_{u_i}(\widehat{\omega}_l) = 0. \quad (24)$$

From (24) and (12), it follows

$$\nabla_{\widehat{\omega}} V_{u_i}(\widehat{\omega}_l) = 0 \quad (25)$$

Because the elements of $\nabla_{\widehat{\omega}} g(\widehat{\omega}, u)$ are negative, it follows from (25) that

$$f(\widehat{\omega}_l, u_i) = 0.$$

This proves that $\widehat{\omega}_l \in M_i$. •

Substep 2: If $\widehat{\omega}(t_0) \in A$, then $\widehat{\omega}(t)$ converges to (ω_2, ω_1) as $t \to \infty$.

Proof of Substep 2 If $\widehat{\omega}_l = \lim_{t\to\infty} \widehat{\omega}(t)$, then *Substep 1* implies that $\widehat{\omega}_l \in M_2$ for any $\widehat{\omega}(t_0) \in \Omega_2$ and in particular, for any $\widehat{\omega}(t_0) \in A \bigcap M_1$ if $u(t) = u_2 \ \ \forall\, t \geq t_0$. Let $N_i(t)$ denote the solution of $\widehat{\omega}(t)$ in (3) if $u(t) = u_j \ \ \forall\, t \geq t_0$, with $\widehat{\omega}(t_0) \in A \bigcap M_i$, $i = 1, 2, j = 2, 1$.

Define the sets L_1 and L_2 as

$$\begin{array}{rcl} L_1(v) &=& \{\hat{v} \mid \hat{v} = N_1(t), \ v = \widehat{\omega}(t_0) \in A \bigcap M_1\}, \\ L_2(v) &=& \{\hat{v} \mid \hat{v} = N_2(t), \ v = \widehat{\omega}(t_0) \in A \bigcap M_2\}. \end{array} \quad (26)$$

The following sublemmas summarize properties of L_1 and L_2.

Sublemma 1 If $v_1 \neq v_2$, then $L_i(v_1) \cap L_i(v_2) = \emptyset$ for $i = 1, 2$.

Proof of Sublemma 1: We will establish this by contradiction. Let $i = 1$. If $v_3 \in L_1(v_1) \cap L_1(v_2)$, it means that the curves $L_1(v_1)$ and $L_1(v_2)$ either (1)intersect at v_3, or (2)they approach the same limit point v_3. If case (1) holds, it implies that there exists a $t \in T_{u_2}$ with $\widehat{\omega}(t) = v_3$ such that the velocity field has two distinct values determined by $L_1(v_1)$ and $L_1(v_2)$ respectively. This contradicts the fact that $\dot{\widehat{\omega}}(t)$ is unique at each $t \in T_{u_2}$. Using a similar argument, we can establish a contradiction for case (2) as well. We prove $L_1(v_1) \cap L_1(v_2) = \emptyset$ if $v_1 \neq v_2$. The proof for $i = 2$ follows similarly. •

Sublemma 2 $\forall v \in (A \bigcap M_1)$, $L_1(v) \in A$

Proof of Sublemma 2: If $v \in M_1 \bigcap E$, from (a) of *Theorem 1*, we note that $L_1(v) = E \bigcap \Omega_2$ with the limit point being $(M_2 \bigcap E)$.

Now that we know $L_1((\omega_2, \omega_1)) = (\omega_2, \omega_1)$, From *Sublemma 1*, it means $\forall v \in (A \bigcap M_1)$, $L_1(v) \in A$. •

For any $\widehat{\omega} \in A$, we can find $v \in (A \bigcap M_1)$ which satisfy $\widehat{\omega} \in L_1(v)$. We define

$$v = G_1(\widehat{\omega}), \text{ if } \widehat{\omega} \in L_1(v)$$

where $v \in A \bigcap M_1$. Similarly, we define

$$v = G_2(\omega), \text{ if } \widehat{\omega} \in L_2(v)$$

where $v \in A \bigcap M_2$.

For any $\widehat{\omega} \in M_i$, we define $D_i(\omega)$ as the distance between (ω_2, ω_1) and $\widehat{\omega}$ along the curve M_i, $i = 1, 2$.

We define two scalar functions W_1 and W_2 of $\widehat{\omega} \in A$ as

$$\begin{array}{rcl} W_1(\widehat{\omega}) &=& D_1(G_1(\widehat{\omega})) \\ W_2(\widehat{\omega}) &=& D_2(G_2(\widehat{\omega})) \end{array} \quad (27)$$

We now evaluate the orientation of the trajectories with respect to a specific $\omega(t) \in A$.

Suppose that at $t = t_0$, $\widehat{\omega}(t_0) \in A$ with $v_0 = G_1(\widehat{\omega}(t_0))$. If $u(t) = u_2$ for all $t \geq t_0$, then

$$W_1(\widehat{\omega}(t)) = W_1(\widehat{\omega}(t_0))$$

which implies that

$$\dot{W}_1(\widehat{\omega}(t)) = 0,$$

if $u(t) = u_2 \ \forall t \geq t_0$. That is

$$\dot{W}_1(\widehat{\omega}) = 0, \qquad \text{if } t \in T_{u_2}$$

We also note that the vector field $\dot{\widehat{\omega}}(t)$ is along the curve $L_1(G_1(\widehat{\omega}))$ if $t \in T_{u_2}$ which simply follows from the definition of (26). This direction is given by

$$f(\widehat{\omega}, u_2)[-u_2 \sin(\hat{\omega_1} u_2) \quad - u_2 \sin(\hat{\omega_2} u_2)]^T \quad (28)$$

From (28), we can compute that the unit vector normal to $L_1(G_1(\widehat{\omega}(t))$ at $\widehat{\omega}$ in a direction towards $(\omega_2, \ \omega_1)$ is given by

$$e_1(\widehat{\omega}) = a[u_2 \sin(\widehat{\omega}_2 u_2) \quad - u_2 \sin(\widehat{\omega}_1 u_2)]^T$$

where a is a positive scalar such that $||e_1(\widehat{\omega})|| = 1$.

If on the other hand $t \in T_{u_1}$, it follows from (10) that the vector field is along the direction

$$\dot{\widehat{\omega}}(t) = f(\widehat{\omega}, u_1)[-u_1 \sin(\hat{\omega_1} u_1) \quad - u_1 \sin(\hat{\omega_2} u_1)]^T$$

We denote $DL_1(\widehat{\omega})$ as the inner product of $e_1(\widehat{\omega})$ and $\dot{\widehat{\omega}}(t)$ at $t \in T_{u_1}$. If $DL_1(\widehat{\omega}(t)) = 0$, $\widehat{\omega}(t)$ will stay in $L_1(G_1(\widehat{\omega}(t)))$ which implies that $\dot{W}_1(\widehat{\omega}(t)) = 0$. If $DL_1(\widehat{\omega}(t)) > 0$, then $\widehat{\omega}(t)$ will leave $L_1(G_1(\widehat{\omega}(t)))$. Since the unit vector is pointed towards $(\omega_2, \ \omega_1)$,$W_1(\widehat{\omega}(t')) < W_1(\widehat{\omega}(t))$ for any $t' > t$. In summary,

$$\dot{W}_1(\widehat{\omega}(t)) \begin{cases} = 0 & \text{if } t \in T_{u_2} \\ = 0 & \text{if } DL_1 = 0 \text{ and } t \in T_{u_1} \\ < 0 & \text{if } DL_1 > 0 \text{ and } t \in T_{u_1} \end{cases} \quad (29)$$

In a similar manner, the inner product DL_2 can be defined for $t \in T_{u_2}$ and it can be shown that

$$\dot{W}_2(\widehat{\omega}(t)) \begin{cases} = 0 & \text{if } t \in T_{u_1} \\ = 0 & \text{if } DL_2 = 0 \text{ and } t \in T_{u_2} \\ < 0 & \text{if } DL_2 > 0 \text{ and } t \in T_{u_2} \end{cases} \quad (30)$$

(29) and (30) suffice to prove substep 2 as shown below: By definition of A, it follows that

$$\widehat{\omega}_2 > \widehat{\omega}_1, \; f(\widehat{\omega}, u_1) \leq 0, \; f(\widehat{\omega}, u_2) \geq 0. \qquad (31)$$

Since $u_1 > u_2$, the conditions of *Lemma 4.3.2* are satisfied for $\widehat{\omega}_1 = \omega_2$ and $\widehat{\omega}_2 = \omega_1$, it follows that

$$Q(\widehat{\omega}_2, \widehat{\omega}_1, u_1, u_2) > 0 \qquad \forall \widehat{\omega} \in A. \qquad (32)$$

It can be shown that

$$DL_1(\widehat{\omega}) = -a u_1 u_2 f(\widehat{\omega}, u_1) Q(\widehat{\omega}_2, \widehat{\omega}_1, u_1, u_2),$$

and hence

$$DL_i \geq 0 \qquad \forall \widehat{\omega} \in A \qquad i = 1, 2.$$

and hence

$$\dot{W}_i(\widehat{\omega}(t)) \leq 0, \qquad \forall \widehat{\omega}(t) \in A \qquad i = 1, 2.$$

Hence $W_1(\widehat{\omega})$ does not increase, since $W_i(\widehat{\omega})$ is continuous and has a lower bound of zero, it follows that $\lim_{t\to\infty} \widehat{\omega}(t) = \widehat{\omega}_l$, $\lim_{t\to\infty} W_i(\widehat{\omega}(t)) = W_i(\widehat{\omega}_A)$, and $\lim_{t\to\infty} \dot{W}_i(\widehat{\omega}(t)) = 0$. From (29) and (30) it also follows that

$$DL_i(\widehat{\omega}_l) = 0, \qquad i = 1, 2.$$

Because we know that a,b,u_1,u_2 are all positive and from equations (31) and (32), it follows that

$$f(\widehat{\omega}_l, u_1) = f(\widehat{\omega}_2, u_2) = 0$$

at $\omega = \omega_l$ which in turn implies, from *Theorem 3.2*, that

$$\widehat{\omega}_l \in L.$$

Because $\widehat{\omega}_l \in A$, it follows that $\forall \widehat{\omega}(t_0) \in A$, $\widehat{\omega}(t)$ converges to (ω_2, ω_1) as $t \to \infty$, proves *Substep 2*. •

Substep 3: (i) $\forall \widehat{\omega}(t_0) \in B$, $\widehat{\omega}(t)$ converges to (ω_2, ω_1) as $t \to \infty$;

(ii) $\forall \widehat{\omega}(t_0) \in C \bigcup D$, $\widehat{\omega}(t)$ converges to (ω_1, ω_2) as $t \to \infty$.

The proof of *Substep 3* follows in a similar manner of *Substep 2*.

In summary, if $\widehat{\omega}(t_0) \in \Omega_2 \setminus E$, then $\widehat{\omega}(t)$ converges to L as $t \to \infty$, which proves *Theorem 1* •

Numerical results from simulating the above algorithm are plotted in Figure 2 for four trajectories (1,2,4,5) with different initial conditions. Each trajectory, independent on its initial condition, converges to the true solution.

The same approach as above can be used for the system in (1) when a_i and ω_i are unknown for $i = 1, 2$ [7]. The number of values between which u switches increases with the number of unknowns. Extensions to the case when $i > 2$ are currently under investigation.

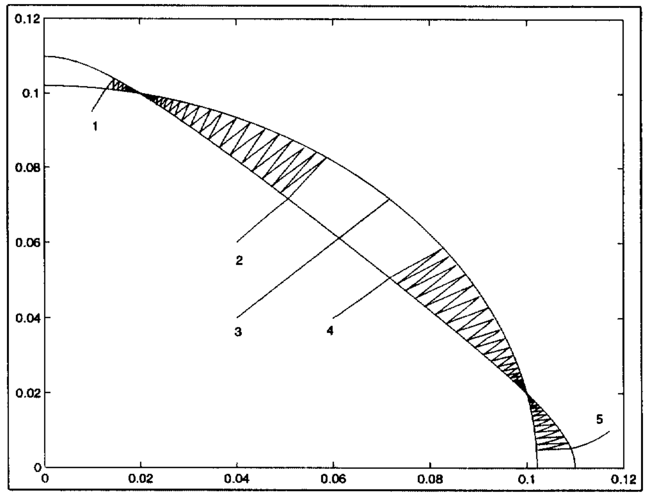

Figure 2: Time evolution of $\widehat{\omega}_2$ vs. $\widehat{\omega}_1$ with initial conditions 1, 2, 3, 4, and 5. $\omega_1 = .02, \omega_2 = .1, u_1 = 25, u_2 = 5$.

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