

ADAPTIVE COMBUSTION INSTABILITY CONTROL WITH SATURATION

S. Evesque*, A.M. Annaswamy†, A.P. Dowling‡

Abstract

The control of a class of combustion systems, susceptible to damage from self-excited combustion oscillations, is considered. An adaptive stable controller, called Self-Tuning Regulator (STR), has recently been developed, which meets the apparently contradictory challenge of relying as little as possible on a particular combustion model while providing some guarantee that the controller will cause no harm. The controller injects some fuel unsteadily into the burning region, thereby altering the heat release, in response to an input signal detecting the oscillation. This paper focuses on an extension of the STR design, when, due to stringent emission requirements and to the danger of flame extension, the amount of fuel used for control is limited in amplitude. A Lyapunov stability analysis is used to prove the stability of the modified STR when the saturation constraint is imposed. The practical implementation of the modified STR remains straightforward, and simulation results, based on the nonlinear premixed flame model developed by Dowling, show that in the presence of a saturation constraint, the self-excited oscillations are damped more rapidly with the modified STR than with the original STR.

1 Introduction

Self-excited combustion oscillations arise from a coupling between unsteady combustion and acoustic waves, and can cause structural damage to many combustion systems, such as gas-fired power stations and aircraft engines. Active control provides a way of

extending their stable operating range by interrupting the damaging thermo-acoustic interaction. As in most full-scale demonstrations of active combustion control,^{9,13,17} the active controller considered injects some fuel unsteadily into the burning region, thereby altering the heat release rate, in response to an input signal detecting the oscillation.

To be useful in practice, the active controller needs to be effective across a large range of operating conditions. An efficient approach is to use an adaptive controller in which the control parameters are continuously updated as the engine conditions change.

A family of adaptive controllers, based on the Least Mean Square (LMS) algorithm, has been proposed to control combustion instabilities.^{3,6,11} Though very attractive due to its algorithmic simplicity and the fact that no model of the system to be controlled is required (this is a *model-independent* adaptive controller), the LMS controller presents a major drawback: no theoretical guarantee on the long-term stability can be provided.

On the contrary, *model-based* adaptive controllers have been developed¹ which provide strong guarantees that the controller will cause no harm to the combustion system, a stringent requirement in particular for aeroengines. However, such model-based controllers require a detailed knowledge of the combustion system, a major disadvantage for full industrial systems for which models are very crude.

Therefore, the fundamental challenge in the adaptive control design is to rely as little as possible on a model of a particular combustor, but at the same time provide some guarantee that the controller will not go unstable and cause any harm. For this purpose, Evesque et al⁷ point out that some simple and general features of self-excited combustion systems are available, that can be exploited in the control design in order to guarantee its long term stability. This approach fundamentally differs from the LMS controller, in which the unstable combustion system to be controlled is considered as an unknown black box. How-

*Engineering Department, University of Cambridge, Cambridge CB2 1PZ, UK. Current affiliation : Research Associate, Lehrstuhl fuer Thermodynamik, Technische Universitaet Muenchen, Boltzmannstrasse 15, 85748 Garching, Germany

†Principal Research Scientist, Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139-4307

‡Professor, Engineering Department, University of Cambridge, Cambridge CB2 1PZ, UK

ever, the general features used in this control design remain independent of a particular combustion system, which ensures the applicability of the adaptive controller, called Self-Tuning Regulator (STR), on a wide class of combustion systems. In particular, a very attractive feature of the STR is that it can safely stabilise combustion systems having significant time delays due to convection, mixing, etc. The STR is very easy to implement in practice, and promising results have been obtained numerically and experimentally.^{5,7}

In this paper, a further challenge in the STR design is considered: due to stringent emission specifications and the danger of flame extinction, severe constraints on the magnitude of the fuel used for control are imposed in practice. The STR design proposed by Evesque et al⁷ is modified in this paper to ensure the long term stability of the system when an amplitude saturation on the amount of fuel used for control is imposed. The paper is divided as follows: section 1 briefly recalls the STR design without saturation constraint. Section 2 describes the modified STR design which takes into account the saturation constraint. A detailed proof for stability is provided. In section 3 are given some simulation results based on a nonlinear premixed flame model⁴ in which the modified STR design has been implemented.

2 STR design without amplitude saturation

A wide class of combustion systems, including LPP combustors and aeroengines, can be modelled as a combustion section embedded within a network of pipes, as shown in figure 1. The underlying dynamics in such a system can be represented as a coupled system, with the forward loop representing the acoustics and the feedback loop representing the flame dynamics (see figure 2). Below we briefly describe this combustion system and its control loop. Further details can be found in Evesque et al.⁷ A transfer function $G(s)$ describes the generation of unsteady velocity $u_1(t)$ at the flame, due to the unsteady heat released by the flame $Q(t)$.^{*} A second transfer function $H(s)$ is introduced to describe the combustion response $Q(t)$ to incoming flow disturbances $u_1(t)$. The eigenfrequencies of the self-excited combustion system satisfy

$$1 - G(s)H(s) = 0. \quad (1)$$

A fuel injector that delivers a controlled heat release $Q_c(t)$ is chosen as an actuator, and is driven by a volt-

age V_c . A pressure transducer is chosen as the sensor which measures the unsteady pressure P_{ref} at an axial location x_{ref} . $W_{ac}(s)$ represents the dynamics of the fuel injector.

An analytical expression of the open-loop transfer function $W(s)$ from the controller output voltage V_c to the pressure P_{ref} has been derived,^{5,7} and is of the form

$$W(s) = \frac{P_{ref}(s)}{V_c(s)} = W_0(s)e^{-s\tau_{tot}}, \quad (2)$$

where

$$\tau_{tot} = \tau_{det} + \tau_{ac} \quad (3)$$

is the total time delay which occurs between the generation of the control signal V_c and the pressure measurement P_{ref} , τ_{ac} is the mixing and transport delay between injection and burning, τ_{det} is the detection time delay due to the pressure measurement location, and

$$W_0(s) = \frac{F(s)G(s)W_{ac}(s)}{1 - G(s)H(s)}. \quad (4)$$

We make four general and non-restrictive assumptions on the class of combustion models shown in figure 1:

(I) The amplitude of the reflected pressure wave at a combustor boundary is smaller than the amplitude of the incoming wave, but may include some time delay. Simple duct terminations like open and choked ends trivially satisfy this condition, provided appropriate energy loss mechanisms are included. Evesque et al⁷ showed that this is also satisfied by reflection from general pipework configurations with negligible mean flow.

(II) The flame is stable when there is no driving velocity u_1 , which means that the poles of the flame transfer function $H(s)$ are ‘stable’ (ie, are in the half plane $Real(s) < 0$ and so lead to eigenmodes with negative growth rate).

(III) The flame response has a limited bandwidth, ie $H(s) \rightarrow 0$ when $s \rightarrow \infty$.

(IV) The actuator dynamics $W_{ac}(s)$ have no unstable zeros. This is true for the case of a simple valve modulating the fuel supply.

^{*}The Laplace variable $s = i\omega$ is used throughout.

The assumptions (II) and (III) on the structure of the flame transfer function $H(s)$ fit many flame models given in the literature, including premixed flames^{4,8} and LPP systems.^{10,12,15,16}

Under these four assumptions, it was shown that the open-loop plant $W_0(s)$ (given in equation (4)) has only stable zeros and a positive gain at high frequencies. Further, a rational approximation of $W_0(s)$, whose order n can be very high for a good accuracy at high frequencies, has a relative degree n^* equal to that of the actuator dynamics $W_{ac}(s)$, ie usually 1 or 2.

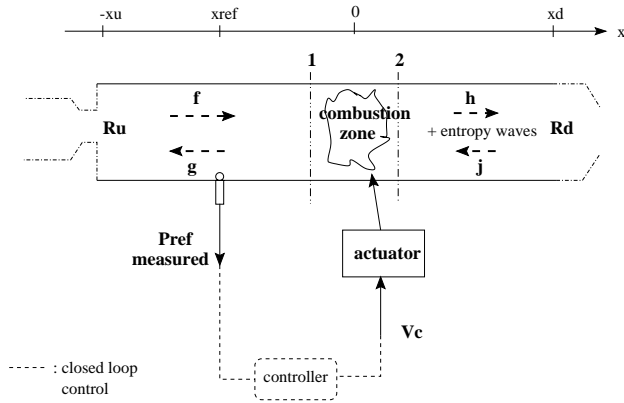


Figure 1: A general class of self-excited combustion systems with actuation.

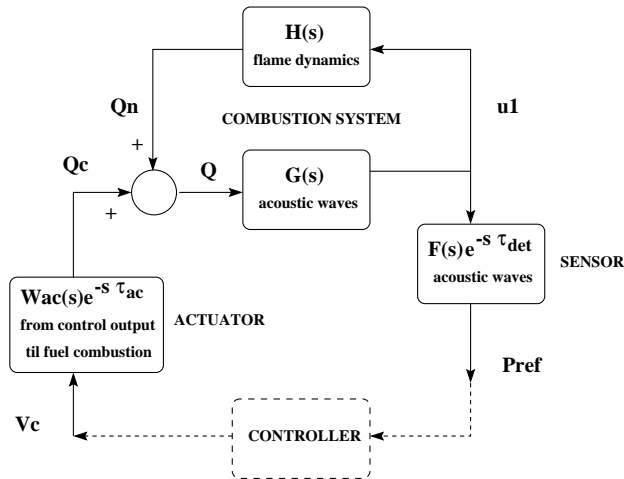


Figure 2: Block-diagram representation of the open-loop process (solid line), which can be controlled by a feedback controller (dashed line).

These three structural properties of the general open-loop process $W_0(s)$ have been exploited to design a Self-Tuning Regulator (STR) which is guaranteed to stabilise the general class of combustion systems shown in figure 1.

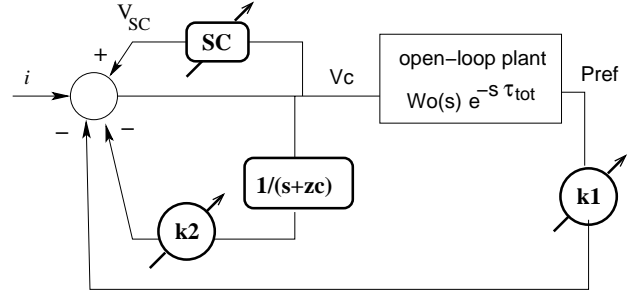


Figure 3: Self-Tuning Regulator (STR) structure. The arrow across a circle indicates an adaptive control parameter.

Results are briefly summarized here. The controller consists of a phase lead compensator associated with a Smith Controller (SC),¹⁸ as indicated in figure 3. The phase lead compensator is essentially a gain (tuned by the parameter k_1) and a phase shift (tuned by k_2). The SC is a series of taps λ_i which compensate for the time delay τ_{tot} . This controller structure of combining a SC with a low-order feedback element is novel, and root locus arguments show that it can stabilise the combustion system. Then an adaptive law for the controller coefficients ($k_1, k_2, \lambda_1, \dots, \lambda_N$) has been derived from a Lyapunov stability analysis. Such a law is guaranteed to lead to stabilizing values for the controller parameters, since it ensures the decay in time of an energy or Lyapunov function, of the adaptive system. The implementation of the STR is straightforward since only one real time pressure measurement P_{ref} is required. The controller algorithm uses P_{ref} , V_c and their past values over a finite time integral, and is summarized below:[†]

When the relative degree n^* of the open-loop process $W_0(s)$ is equal to one, the controller is given by

$$V_c(t) = \mathbf{k}^T(t)\mathbf{d}(t) \quad (5)$$

$$\dot{\mathbf{k}}(t) = -P_{ref}(t)\mathbf{d}(t) \quad (6)$$

[†]the notation $f(s)[.]$ denotes an operator of the variable $s = \frac{d}{dt}$. For instance, $x(t) = \frac{1}{s+1}[y(t)]$ means that $\frac{dx}{dt}(t) + x(t) = y(t)$. Also, throughout the paper vectors are denoted in bold characters.

and when $n^* = 2$, the controller is given by

$$V_c(t) = \mathbf{k}^T(t) \cdot \mathbf{d}(t) + \dot{\mathbf{k}}(t)^T \cdot \mathbf{d}_a(t) \quad (7)$$

$$\dot{\mathbf{k}}(t) = -P_{ref}(t) \mathbf{d}_a(t - \tau_{tot}). \quad (8)$$

where

$$\mathbf{k}(t)^T = [-k_1(t), -k_2(t), \lambda_N(t), \dots, \lambda_1(t)] \quad (9)$$

$$\mathbf{d}(t)^T = [P_{ref}(t), V(t), V_c(t - Ndt), \dots, V_c(t - dt)] \quad (10)$$

$$V(t) = \frac{1}{s + z_c} [V_c(t)] \quad (11)$$

$$\mathbf{d}_a(t) = \frac{1}{s + a} [\mathbf{d}(t)] \quad (12)$$

with $Ndt = \tau_{tot}$, and a and z_c some positive constants.

For the particular case $\tau_{tot} = 0$, the STR reduces to a simple phase lead compensator (the SC transfer function is then unity and the coefficients λ_i equal to zero), that coincides with the adaptive controller developed by Annaswamy et al¹ for a particular combustion system. We showed that such a controller can be used on a much wider class of combustion systems having no time delay ($\tau_{tot} = 0$).

3 STR design with amplitude saturation

The main difficulty due to the imposition of a saturation constraint is the introduction of a nonlinearity which needs to be accommodated in the control design. We show below that closed-loop stability can still be guaranteed even when the amplitude of V_c is saturated (as shown in figure 4), provided that the adaptive law (6) (when $n^* = 2$) or (8) (when $n^* = 2$) is further modified, and that the saturation constraint is not too strong in comparison with the initial level of oscillation.

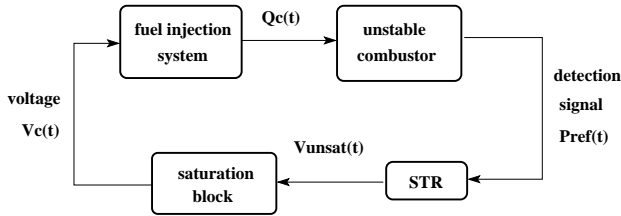


Figure 4: Unstable combustor stabilised by a STR in the presence of amplitude saturation on V_c

Essentially, in the presence of saturation, the unsaturated control signal, denoted V_{unsat} , is still obtained using equation (7) when $n^* = 2$, ie

$$V_{unsat}(t) = \mathbf{k}^T(t) \cdot \mathbf{d}(t) + \dot{\mathbf{k}}(t)^T \cdot \mathbf{d}_a(t) \quad (13)$$

where \mathbf{d} and \mathbf{d}_a are based on the saturated signal V_c , ie $\mathbf{d} = [P_{ref}, V]$ with $V = \frac{1}{s+z_c} [V_c]$, and $\mathbf{d}_a = \frac{1}{s+a} [\mathbf{d}]$. For the case $n^* = 1$, equation (13) reduces to

$$V_{unsat}(t) = \mathbf{k}^T(t) \cdot \mathbf{d}(t) \quad (14)$$

However, the actual signal sent to the fuel injection system is the saturated signal V_c , which is defined as follows:

$$V_c(t) = \begin{cases} V_{unsat}(t) & \text{if } |V_{unsat}(t)| < V_{lim} \\ \text{sign}(V_{unsat}) \cdot V_{lim} & \text{if } |V_{unsat}(t)| \geq V_{lim} \end{cases} \quad (15)$$

where V_{lim} is a constant positive value chosen to limit as required the amount of fuel used for control.

In the following, we show using a Lyapunov stability analysis how the adaptive law (8) for the controller parameters must be modified to guarantee the stability of the closed-loop system in the presence of the amplitude saturation on V_c .

3.1 $\tau_{tot} = 0$

For the particular case of a combustion system without time delay (ie $\tau_{tot} = 0$), a STR design which takes into account an amplitude saturation on V_c has already been proposed.⁵ Results are reproduced briefly here, but details of the mathematical proof can be found in Evesque.⁵ Essentially, the controller structure is kept identical to the non-saturation case (ie two control parameters k_1 and k_2), but the adaptive law (8) is modified as follows:

When $n^* = 2$,

$$\dot{\mathbf{k}}(t) = -P_{new}(t) \mathbf{d}_a(t) \quad (16)$$

with

$$P_{new}(t) = P_{ref}(t) - W_m(s) [\nu_a(t)] \quad (17)$$

$$\nu_a = \frac{1}{s+a} [\nu] \quad (18)$$

$$\nu = V_c - V_{unsat} \quad (19)$$

The corresponding equations for $n^* = 1$ can be derived, which are of the form of (16) and (17), with \mathbf{d}_a and ν_a replaced by \mathbf{d} and ν , respectively. In equation (17), $W_m(s)$ represents the transfer function of

the closed-loop combustion system after being stabilised (ie $W_m(s)$ has no unstable poles, contrary to the open-loop system $W_0(s)$). Physically, the pressure signal in the modified adaptive rule (16) must represent an accurate measure of the instability to be damped. Therefore, P_{new} is used rather than P_{ref} because in P_{new} the component due to the saturation block in the feedback loop has been removed.

However, equations (16)-(17) can be applied only if the transfer function $W_m(s)$ is known. This is not the case in general, but very often the frequencies of the potentially unstable modes are known. Therefore, the adaptive rule for \mathbf{k} can be implemented using an approximate expression of $W_m(s)$, denoted $W_{me}(s)$, where $W_{me}(s)$ is a second-order rational transfer function with stable poles whose real parts are equal to the frequencies of the most unstable modes of the combustion system. In a nutshell, a more general adaptive law to be used in the presence of saturation when $W_m(s)$ is unknown follows:⁵

When $n^* = 1$,

$$\dot{\mathbf{k}}(t) = -P_{new}(t)\mathbf{d}(t) - \sigma_k(\mathbf{k})\mathbf{k}(t) \quad (20)$$

where

$$P_{new}(t) = P_{ref}(t) - W_{me}(s) [\nu(t)] \quad (21)$$

and

$$\sigma_k(\mathbf{k}) = \begin{cases} 0 & \text{if } \|\mathbf{k}\| < k_{lim} \\ \sigma & \text{if } \|\mathbf{k}\| \geq k_{lim} \end{cases} \quad (22)$$

where k_{lim} and σ are some positive constants. σ is called a *leakage coefficient* whose role is to ‘compensate’ for the error caused by using an approximate $W_{me}(s)$ instead of the true $W_m(s)$ in the adaptive law (20). When $n^* = 2$, similar adaptive laws can be derived, with \mathbf{d} and ν replaced by \mathbf{d}_a and ν_a , respectively.

3.2 $\tau_{tot} \neq 0$

This corresponds to most practical combustors. Annaswamy et al² describes a high order controller which stabilizes plants having a time delay τ_{tot} and in the presence of saturation on the control signal V_c . This controller dynamic is of the same order as the plant to be stabilised, ie has n coefficients. However, unstable combustion systems usually have a very high order dynamics (ie n can be very high) therefore

the high order controllers suggested by Annaswamy et al² could be very hard to implement to control unstable combustion systems due to the controller complexity. A further difficulty in implementing the controller of Annaswamy et al² is that the plant (more precisely, $W_m(s)$), needs to be known exactly. Therefore, our aim is to develop a low order controller (ie whose number of coefficients does not depend on n but only on n^* which is small), requiring very little knowledge of the plant, and which can achieve stabilization in the presence of saturation and time delay τ_{tot} . For this purpose, we suggest to use the controller structure which was proposed by Evesque et al⁷ for combustion systems with time delay but in the absence of saturation. This controller is described in figure 3. Then, as for the delay free case (section 3.1), the idea is simply to modify the adaptive law (6) (when $n^* = 1$) for the controller parameters in order to take into account the amplitude saturation on the control signal V_c . We suggest therefore the following modified adaptive rule:

$$\dot{\mathbf{k}}(t) = -P_{new}(t)\mathbf{d}(t - \tau_{tot}) \quad (23)$$

where

$$P_{new}(t) = P_{ref}(t) - W_m(s) [\nu(t - \tau_{tot})]. \quad (24)$$

In appendix A, it is shown that the adaptive law (23) guarantees the boundedness of all signals that start within a certain domain and that P_{new} tends to zero. For ease of exposition, we assume that the relative degree n^* of $W_0(s)$ is one. Similar arguments can be extended to the case when $n^* = 2$.

Again, as for the delay free case, the adaptive law (23) can be applied only if $W_m(s)$ is known. Since this is not the case in general, we propose a more generally applicable adaptive law which requires only an approximate expression $W_{me}(s)$ of $W_m(s)$:

$$\dot{\mathbf{k}}(t) = -P_{new}(t)\mathbf{d}(t - \tau_{tot}) - \sigma_k(\mathbf{k})\mathbf{k}(t) \quad (25)$$

where

$$P_{new}(t) = P_{ref}(t) - W_{me}(s) [\nu(t - \tau_{tot})] \quad (26)$$

and

$$\sigma_k(\mathbf{k}) = \begin{cases} 0 & \text{if } \|\mathbf{k}\| < k_{lim} \\ \sigma & \text{if } \|\mathbf{k}\| \geq k_{lim} \end{cases} \quad (27)$$

In appendix B we demonstrate that the adaptive law (25) is guaranteed to stabilise the combustion system for $\|\mathbf{d}\|$ and $\|\mathbf{k}\|$ initially less than some bounds (see equations (51)). For high levels of V_{lim} , these initial bounds on $\|\mathbf{d}\|$ and $\|\mathbf{k}\|$ are not a serious constraint and are satisfied in a practical combustor when control is switched on while the pressure limit cycle is already established, and with the control parameter \mathbf{k} set to zero initially. However, when the amplitude constraint becomes more severe (ie when V_{lim} is reduced) or when the time delay τ_{tot} becomes too large, the details of the mathematical stability proof indicate that the stability domain is reduced: only small amplitude initial signals can be controlled. These qualitative results make sense: successful control cannot be expected when the maximum amplitude of the control signal is too small in comparison with the amplitude of the oscillations to be damped, which means that in such a case control must be implemented before the oscillations reach a high amplitude level. Further, the original STR design (described in section 2) was already guaranteed to stabilise the general class of combustion systems considered only for a time delay τ_{tot} not too large. But simulation and experiment results^{5,7} have confirmed that the STR successfully eliminate combustion instabilities for time delays up to 3 cycles of oscillation, which is more than enough for most full scale applications.

As detailed in appendix B, the use of an approximated expression $W_{me}(s)$ instead of the true $W_m(s)$ in the adaptive law for \mathbf{k} has for main consequence that the unsteady pressure P_{ref} is guaranteed to converge to a small value and not strictly to zero. As in appendix A, for ease of exposition, the relative degree of $W_0(s)$ is assumed to be one.

4 Application to a premixed ducted flame

4.1 Nonlinear flame model used in the simulation

A model for nonlinear oscillations of a ducted flame developed by Dowling⁴ is used to verify the STR performance. The theory involves extension of the flame model of Fleifil et al⁸ to include a flame holder at the centre of the duct and nonlinear effects. The limit cycle behaviour of the uncontrolled system is illustrated in figure 5.

4.2 Adaptive regulator design

The flame model described in Dowling⁴ satisfies

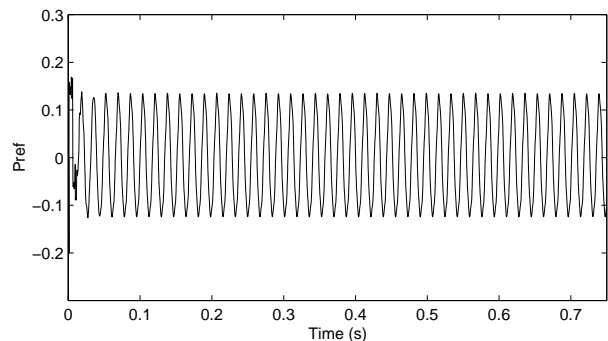


Figure 5: Pressure response of the uncontrolled combustion system at a Mach number $\bar{M}_1 = 0.08$, equivalence ratio $\phi = 0.7$, and mean upstream temperature $\bar{T}_1 = 287.6$ K.

the assumptions (I)-(IV) (see Evesque et al⁷ for details). In particular, the flame transfer function, after linearisation for small perturbations, is given by (see Dowling⁴):

$$H(s) = \frac{Q_n(s)}{u_G(s)} = \frac{2\eta\Delta H e^{-s\tau_2}}{s\tau_f \bar{c}_1^2 (b+a)} \left[a - b e^{-s\tau_f} + \frac{b-a}{s\tau_f} (1 - e^{-s\tau_f}) \right] \quad (28)$$

Therefore, $H \rightarrow 0$ as $s \rightarrow \infty$, and $H(s)$ has no poles: hence assumptions (II) and (III) are satisfied. The actuator chosen in the simulation is a fuel injection system which introduces an additional heat input Q_c , that we model as follows:

$$\frac{Q_c(s)}{V_c(s)} = \frac{e^{-s\tau_{ac}}}{s/\omega_c + 1} \quad (29)$$

where ω_c^{-1} is the time constant of the fuel injector. From equation (29), we deduce that the relative degree of $W_{ac}(s)$, hence of $W_0(s)$, is 1.

A convergence coefficient μ is added in the adaptive law for each controller parameter. The controller parameter vector \mathbf{k} is initialized to zero. Simulations results are given in the next section.

4.3 Simulation results

Control is switched on at $t = 0.15$ s, when the self-excited oscillations are already established. A saturation constraint is imposed on the voltage V_c : $|V_c| \leq 0.10$, which means that the fuel used for control is limited to 10 % of the main fuel.

First, the original STR (equations (5)-(6)) is implemented. The corresponding time evolution of the

unsteady pressure P_{ref} , control input V_c and control parameter k_1 are plotted in dotted line in figure 6. V_c saturates for a while and then decreases once P_{ref} has been sufficiently reduced. Then, under the same operating conditions, the modified STR (equations (14),(15),(19),(25) and (26) with $V_{lim} = 0.10$) is implemented. As indicated by the continuous line signals on figure 6, convergence, hence control of the oscillations, is obtained earlier than with the traditional STR design. This result is not surprising: the modified STR *knows* that the control signal V_c is saturated, and hence can adjust its response accordingly. Therefore, it is clearly demonstrated that a saturation constraint on the control signal V_c needs to be taken into account in the adaptive control design for better control results. The results shown in figure 6 were obtained for a total time delay in the open-loop combustion system $\tau_{tot} = 8$ ms, an equivalence ratio of 0.7 and a mean Mach number of 0.08.

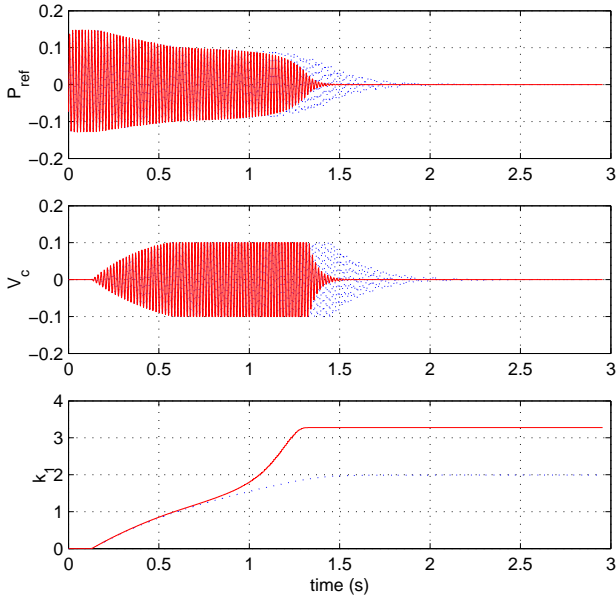


Figure 6: Simulation result based on Dowling’s ducted flame model with $\phi = 0.7$, $\bar{M}_1 = 0.08$, $\tau_{tot} = 8$ ms. Control ON at 0.15 s, $V_{lim} = 0.10$. \dots : original STR (adaptive law (6)), — : modified STR (adaptive law (25)). Benefit of including the amplitude saturation of the control signal V_c into the STR design.

In figure 7, the original and modified STR have been tested for various levels of saturation V_{lim} , under similar operating conditions. It can be seen that, when V_{lim} is large (first case: $V_{lim} = 0.2$), the control signal

V_c does not hit the saturation limit, and therefore the two STR produce identical results. On the contrary, when V_{lim} is very small (third case: $V_{lim} = 0.05$), then the control signal V_c has not a sufficient amplitude to eliminate completely the oscillation P_{ref} , and both STR achieve only a small reduction in the amplitude of P_{ref} . For values of V_{lim} in between these two limiting cases the modified STR always leads to a quicker control of the oscillations than the original STR (second case: $V_{lim} = 0.1$).

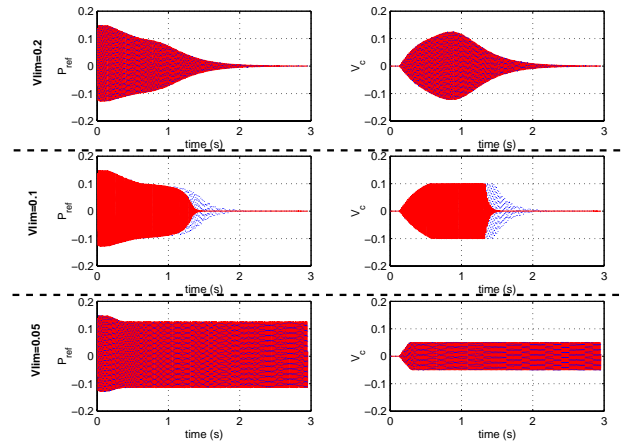


Figure 7: Simulation result based on Dowling’s ducted flame model with $\phi = 0.7$, $\bar{M}_1 = 0.08$, $\tau_{tot} = 8$ ms. Control ON at 0.15 s. Compared performance of the original STR (continuous line) and modified STR (dotted line) for various levels of saturation V_{lim} .

5 Conclusions

An adaptive controller, the STR, had been developed to safely control a general class of combustion systems, using fuel actuation. This attractive adaptive control design has been improved in this paper to ensure the stability of the combustion system and its control loop when the amount of fuel used to damp the instabilities is limited in amplitude. Such an amplitude saturation would be desirable in practice due to stringent emission requirements or the danger of flame extension.

A detailed mathematical analysis of the impact of saturation on the stability domain was provided. It shows that, in the presence of saturation, the modified STR design increases the stability margins of the actuated system more than the original STR. The modified STR remains easy to implement, and simulation results confirm the benefit of using the modified STR

design in the presence of saturation constraints on the control signal V_c . In particular, it is shown clearly that the oscillations are more rapidly damped when the controller *knows* that there is an amplitude saturation.

ACKNOWLEDGEMENTS

The financial support of Trinity College, Cambridge, and the Office of Naval Research, under grant N00014-99-1-0448, is gratefully acknowledged.

APPENDIX A: $W_m(s)$ known

The so-called underlying error model of the adaptive system in the presence of saturation is:

$$P_{ref} = W_m(s)[\tilde{\mathbf{k}}^T(t - \tau_{tot})\mathbf{d}(t - \tau_{tot}) + \nu(t - \tau_{tot})] \quad (30)$$

To make this equation more readable, we introduce a new notation and rewrite equation (30) as follows:

$$P_{ref} = W_m(s)[\tilde{\mathbf{k}}_\tau^T \mathbf{d}_\tau + \nu_\tau] \quad (31)$$

Equations (24) and (31) can be combined to give

$$P_{new} = W_m(s)[\tilde{\mathbf{k}}_\tau^T \mathbf{d}_\tau] \quad (32)$$

We can introduce a matrix A and vectors \mathbf{b} and \mathbf{h} to write a time domain representation of equation (32), symbolically as follows:

$$\dot{\mathbf{e}} = A\mathbf{e} + \mathbf{b}(\tilde{\mathbf{k}}_\tau^T \mathbf{d}_\tau) \quad (33)$$

$$P_{new} = \mathbf{h}^T \mathbf{e} \quad (34)$$

where \mathbf{e} is the state vector. $W_m(s)$ is a stable transfer function and of relative degree one, hence is Strictly Positive Real (SPR).¹⁴ According to lemma 2.1 of Narendra&Annaswamy,¹⁴ given a matrix Q_m symmetric strictly positive, there exists a matrix P_m symmetric strictly positive, such that

$$A^T P_m + P_m^T A = -Q_m, \quad P_m \mathbf{b} = \mathbf{h} \quad (35)$$

Our Lyapunov function candidate is the positive definite function

$$V_l = \mathbf{e}(t)^T P_m \mathbf{e}(t) + \tilde{\mathbf{k}}^T(t) \tilde{\mathbf{k}}(t) + \int_{-\tau_{tot}}^0 \int_{t+\nu}^t \tilde{\mathbf{k}}^T \dot{\tilde{\mathbf{k}}} d\xi d\nu \quad (36)$$

From now on, we can follow the proof for the case without saturation given in given in Evesque et al⁷ to obtain that

$$\dot{V}_l \leq -\mathbf{e}^T (Q_m - 2\tau_{tot} \|\mathbf{d}(t - \tau_{tot})\|^2 \mathbf{h}\mathbf{h}^T) \mathbf{e}. \quad (37)$$

Equation (37) shows that if the norm of $\|\mathbf{d}\|$ is small compared to the minimum eigenvalue of Q_m , then V_l is nonincreasing. More formally, starting from equation (37), it can be shown, using similar arguments to those of Annaswamy et al,² that \mathbf{e} and $\tilde{\mathbf{k}}$ are bounded starting from time t_0 at which control is switched on, and that P_{ref} tends to zero asymptotically if the following conditions are satisfied:⁵

$$\begin{aligned} \tau_{tot} &\leq \min(\tau_1, \tau_2) \\ 2\tau_1 \gamma_d \mathbf{h}\mathbf{h}^T &< Q_m \\ 2\tau_2 \frac{\|C\|^2 V_l(t_0)}{\lambda_{min-P_m}} \mathbf{h}\mathbf{h}^T &< Q_m \\ \sup_{t \in [t_0 - \tau_{tot}, t_0]} \|\mathbf{d}(t)\|^2 &\leq \gamma_d \\ \|\tilde{\mathbf{k}}(t_0)\| &\leq \sqrt{V_l(t_0)} \leq \frac{\lambda_{min-Q_m}}{2\|P_m \mathbf{b}\| \|C\|} \\ \sup_{t \in [t_0 - \tau_{tot}, t_0]} \|\mathbf{X}(t)\| &\leq \frac{V_{lim}}{\|\mathbf{k}^*\| \|C\|} \end{aligned} \quad (38)$$

where γ_d is a positive constant, λ_{min-P_m} and λ_{min-Q_m} are the smallest eigenvalues of the matrices P_m and Q_m respectively, C is the constant matrix such that $\mathbf{d} = C\mathbf{X}$, \mathbf{X} is the state vector corresponding to the time representation of equation (31), and \mathbf{k}^* is a value of the control parameter \mathbf{k} achieving control. Physically, equation (38) means that control can be achieved only if τ_{tot} is not too large and if initially the state \mathbf{X} , hence P_{ref} , is small compared to the maximum control effort allowed V_{lim} . Further, for a given value V_{lim} , starting from a control parameter set to zero ($\mathbf{k}(t_0) = 0$), control will be achieved if the plant is not too 'far' from a stable plant (ie $\|\mathbf{k}^*\|$ not too large). The conditions given in equation (38) are conservative because, as it can be seen in the proof details given by Annaswamy et al,² the worst case, ie V_c hitting repeatedly the saturation bound V_{lim} , has been considered to derive equation (38). But they still show that the stability domain is increased when the saturation constraint is included in the STR design.

APPENDIX B: $W_m(s)$ unknown

We start from equation (31). We denote

$$\Delta W_m(s) = 1 - W_m^{-1}(s)W_{me}(s). \quad (39)$$

Equations (26),(31) and (39) lead to

$$\begin{aligned} P_{new} &= P_{ref} - W_{me}(s)[\nu_\tau] \\ &= W_m(s)[\tilde{\mathbf{k}}_\tau^T \mathbf{d}_\tau + \nu_{b\tau}] \end{aligned} \quad (40)$$

where

$$\nu_{b\tau} = \Delta W_m(s)[\nu_\tau] \quad (41)$$

plays the role of an extra disturbance in the system. A time domain representation of equation (40) follows:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{b}(\tilde{\mathbf{k}}_\tau^T \mathbf{d}_\tau + \nu_{b\tau}) \quad (42)$$

$$P_{new} = \mathbf{h}^T \mathbf{e} \quad (43)$$

where \mathbf{e} is the state vector.

Since $W_m(s)$ is SPR, lemma 2.1 of Narendra&Annaswamy¹⁴ can be applied: given a matrix Q_m symmetric strictly positive, there exists a matrix P_m symmetric strictly positive, such that

$$\mathbf{A}^T P_m + P_m^T \mathbf{A} = -Q_m, \quad P_m \mathbf{b} = \mathbf{h} \quad (44)$$

Our Lyapunov function candidate is the positive definite function

$$\begin{aligned} V_l &= \mathbf{e}(t)^T P_m \mathbf{e}(t) + \tilde{\mathbf{k}}^T(t) \tilde{\mathbf{k}}(t) \\ &+ \int_{-\tau_{tot}}^0 \int_{t+\nu}^t \|P_{new}(\xi) \mathbf{d}_\tau(\xi)\|^2 d\xi d\nu \end{aligned} \quad (45)$$

We rewrite equation (42) as follows:

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{A}\mathbf{e} + \mathbf{b}^T [\tilde{\mathbf{k}}^T \mathbf{d}_\tau] \\ &- \mathbf{b}^T \left[\mathbf{d}_\tau^T \int_{-\tau_{tot}}^0 \dot{\tilde{\mathbf{k}}}(t+\nu) d\nu \right] + \nu_{b\tau} \mathbf{b} \end{aligned} \quad (46)$$

Using equations (45),(46) and (43), we obtain

$$\begin{aligned} \dot{V}_l &= -\mathbf{e}^T Q_m \mathbf{e} + 2P_{new} \nu_{b\tau} - 2\tilde{\mathbf{k}}^T \mathbf{k} \sigma_k(\mathbf{k}) \\ &- 2P_{new} \mathbf{d}_\tau^T \left[\int_{-\tau_{tot}}^0 \dot{\tilde{\mathbf{k}}}(t+\nu) d\nu \right] \end{aligned}$$

$$+ \int_{-\tau_{tot}}^0 (\|P_{new} \mathbf{d}_\tau\|^2 - \|P_{new}(t+\nu) \mathbf{d}_\tau(t+\nu)\|^2) d\nu \quad (47)$$

Denoting

$$\begin{aligned} \mathbf{y} &= P_{ref}(t) \mathbf{d}(t - \tau_{tot}), \\ \mathbf{w} &= P_{ref}(t + \nu) \mathbf{d}(t + \nu - \tau_{tot}), \end{aligned} \quad (48)$$

and using equation (25), equation (47) can be rewritten as

$$\begin{aligned} \dot{V}_l &= -\mathbf{e}^T Q_m \mathbf{e} + 2P_{new} \nu_{b\tau} - 2\tilde{\mathbf{k}}^T \mathbf{k} \sigma_k(\mathbf{k}) \\ &+ 2 \int_{-\tau_{tot}}^0 P_{new} \mathbf{d}_\tau^T \mathbf{k}(t+\nu) \sigma(\mathbf{k}) d\nu \\ &- \int_{-\tau_{tot}}^0 (\|\mathbf{w} - \mathbf{y}\|^2 - 2\mathbf{y}^T \mathbf{y}) d\nu \end{aligned} \quad (49)$$

Noting that $|\nu_{b\tau}| \leq \epsilon(\alpha_a + \alpha_b \|\mathbf{e}\|)$ where ϵ is a small positive gain occurring due to the error in using $W_{me}(s)$ instead of $W_m(s)$ in the adaptive law for \mathbf{k} , and α_a and α_b some positive constants[‡], equation (49) leads to the following inequality:

$$\begin{aligned} |\dot{V}_l| &\leq -\|\mathbf{e}\| [(\lambda_{min-Q_m} + 2\epsilon \|P_m \mathbf{b}\| \alpha_b \\ &+ 2\tau_{tot} \|P_m \mathbf{b}\|^2 \|\mathbf{d}(t - \tau_{tot})\|^2) \|\mathbf{e}\| \\ &- 2\epsilon \|P_m \mathbf{b}\| \alpha_a \\ &- 2\sigma \|P_m \mathbf{b}\| \|\mathbf{d}(t - \tau_{tot})\| \int_{-\tau_{tot}}^0 \|\mathbf{k}(t+\nu)\| d\nu] \\ &- 2\tilde{\mathbf{k}}^T \mathbf{k} \sigma_k(\mathbf{k}) \end{aligned} \quad (50)$$

where λ_{min-Q_m} is the smallest eigenvalue of the matrix Q_m . The inequality (50) is not easy to check at any time t . We show below that it can be replaced by bounds on $\|\mathbf{d}\|$ and on $\|\mathbf{k}\|$ over the interval $[t_0 - \tau_{tot}, t_0)$, where t_0 is the time at which control is switched on.

Suppose that over $[t_0 - \tau_{tot}, t_0)$, \mathbf{d} and \mathbf{k} satisfy

$$\begin{aligned} \sup_{t \in [t_0 - \tau_{tot}, t_0)} \|\mathbf{d}(t)\|^2 &\leq \gamma_d \\ \sup_{t \in [t_0 - \tau_{tot}, t_0)} \|\mathbf{k}(t)\|^2 &\leq \gamma_k \end{aligned} \quad (51)$$

[‡] details on the derivation of the bound on $|\nu_{b\tau}|$ can be found in Evesque⁵

where γ_d and γ_k are some positive real constants. Then, over $[t_0 - \tau_{tot}, t_0)$, the inequality (50) is satisfied if

$$\|\mathbf{e}\| \geq \frac{2\|P_m \mathbf{b}\| (\epsilon \alpha_a + \sigma \gamma_d \tau_{tot} \gamma_k)}{\lambda_{min-Q_m} + 2\epsilon \|P_m \mathbf{b}\| \alpha_b + 2\tau_{tot} \|P_m \mathbf{b}\|^2 \gamma_d^2} \quad (52)$$

or

$$\|\tilde{\mathbf{k}}\| \geq \text{a function of } \epsilon, \tau_{tot}, \gamma_d \text{ and } \gamma_k \quad (53)$$

In other words, over $[t_0 - \tau_{tot}, t_0)$, the inequality (50) is satisfied if $\tau_{tot} \leq \tau_1$ where τ_1 is the largest time delay satisfying (52) over $[t_0 - \tau_{tot}, t_0)$. This means that for any $\tau_{tot} \leq \tau_1$, $\dot{V}_i \leq 0$ over $[t_0 - \tau_{tot}, t_0)$, ie $V_i(t)$ is non increasing for $t \in [t_0, t_0 + \tau_{tot})$, as long as $\tau_{tot} \leq \tau_1$. Therefore, from the definition of V_i in (45), we have

$$\mathbf{e}(t)^T P_m \mathbf{e}(t) \leq V_i(t) \leq V_i(t_0) \quad (54)$$

for all $t \in [t_0, t_0 + \tau_{tot})$, which means that \mathbf{e} is bounded over $[t_0, t_0 + \tau_{tot})$ and the corresponding bound is given by

$$\sup_{t \in [t_0, t_0 + \tau_{tot})} \|\mathbf{e}(t)\|^2 \leq \frac{V_i(t_0)}{\lambda_{min-P_m}} \quad (55)$$

where λ_{min-P_m} is the smallest eigen value of the positive symmetric matrix P_m . A constant matrix C exists such that $\mathbf{d} = C\mathbf{e}$ (this essentially means that \mathbf{d} is a representation of part of the states of the system). Therefore we have

$$\sup_{t \in [t_0, t_0 + \tau_{tot})} \|\mathbf{d}(t)\|^2 \leq \frac{\|C\|^2 V_i(t_0)}{\lambda_{min-P_m}} \quad (56)$$

which does NOT depend on γ_d . Similarly, we have

$$\tilde{\mathbf{k}}(t)^T \tilde{\mathbf{k}}(t) \leq V_i(t) \leq V_i(t_0) \quad (57)$$

for all $t \in [t_0, t_0 + \tau_{tot})$. This implies that

$$\sup_{t \in [t_0, t_0 + \tau_{tot})} \|\tilde{\mathbf{k}}(t)\|^2 \leq V_i(t_0) \quad (58)$$

which does NOT depend on γ_k .

Now, let us consider \dot{V}_i over the second delay interval $[t_0 + \tau_{tot}, t_0 + 2\tau_{tot})$. Then $\dot{V}_i \leq 0$ over this interval if $\tau_{tot} \leq \tau_2$, where τ_2 is the largest time delay which satisfies the inequality

$$\|\mathbf{e}\| \geq \frac{2\|P_m \mathbf{b}\| (\epsilon \alpha_a + \sigma \frac{\|C\|^2 V_i(t_0)}{\lambda_{min-P_m}} \tau_{tot} V_i(t_0))}{\lambda_{min-Q_m} + 2\epsilon \|P_m \mathbf{b}\| \alpha_b + 2\tau_{tot} \|P_m \mathbf{b}\|^2 \left(\frac{\|C\|^2 V_i(t_0)}{\lambda_{min-P_m}} \right)^2} \quad (59)$$

over $[t_0 + \tau_{tot}, t_0 + 2\tau_{tot})$. Then, \dot{V}_i is non-increasing at time $[t_0 + \tau_{tot}, t_0 + 2\tau_{tot})$ if $\tau_{tot} \leq \tau_2$.

By repeating the process, it can be shown that the constructions above also hold on the next delay intervals $[t_0 + k\tau_{tot}, t_0 + (k+1)\tau_{tot})$ for any positive integer $k > 2$. We conclude that for a time delay τ_{tot} smaller than $\tau_3 = \min(\tau_1, \tau_2)$ and for \mathbf{d} and \mathbf{k} satisfying the initial condition (51), equation (52) is satisfied at any time $t > t_0$, ie $\dot{V}_i(t)$ is negative at any time $t > t_0$ outside a compact set in the $(\mathbf{e}, \tilde{\mathbf{k}})$ -space, whose size increases with ϵ , τ_3 , γ_d and γ_k . Then, application of lemma 2.12¹⁴ guarantees that \mathbf{e} , hence P_{new} , tends asymptotically to a value of the order $O(\epsilon + \tau_3 \gamma_d \gamma_k)$ and that $\|\tilde{\mathbf{k}}\|$ is bounded for all time t , as long as the delay τ_{tot} is smaller than τ_3 and for \mathbf{d} and \mathbf{k} satisfying the initial condition (51). Note that these initial conditions are satisfied in a practical self-excited combustion system when control is switched on while the pressure limit cycle is already established, and with an initial control parameter set to zero ($\mathbf{k}(t_0) = 0$).

Finally, similarly to the $W_m(s)$ known case (appendix A), it can be shown that P_{ref} tends asymptotically to a small value of the order $O(\epsilon + \tau_3 \gamma_d \gamma_k)$ if τ_{tot} is not too large, if $\mathbf{d}(t)$, hence $P_{ref}(t)$, for t in $[t_0 - \tau_{tot}, t_0)$, is small compared to the maximum control effort allowed in V_{lim} , and if \mathbf{k} is initially not very far from a stabilizing value \mathbf{k}^* . Essentially, the case $W_m(s)$ unknown differs from the case $W_m(s)$ known by the fact that P_{ref} is not guaranteed to tend to zero exactly but to a small value.

References

- [1] AM Annaswamy, OM El Rifai, M Fleifil, JP Hathout, and AF Ghoniem. A model-based self-tuning controller for thermoacoustic instability. *Combust. Sci. and Tech.*, 135:213–240, 1998.
- [2] AM Annaswamy, S Evesque, S Niculescu, and AP Dowling. Adaptive control of a class of time-delay systems in the presence of saturation. In *Adaptive Control of Nonsmooth Dynamic Systems*, Springer-Verlag, New York, NY, 2001.
- [3] G Billoud, MA Galland, C Huynh Huu, and S Candel. Adaptive control of combustion instabilities. *Combust Sci Tech*, 81:257–283, 1992.
- [4] AP Dowling. A kinematic model of a ducted flame. *J. Fluid Mech.*, 394:51–72, 1999.

- [5] S Evesque. *Adaptive control of combustion oscillations*. PhD thesis, University of Cambridge, Department of Engineering, 2000.
- [6] S. Evesque, Y.C. Chu, A.P. Dowling, and K. Glover. Feedback control of a premixed ducted flame. In *Proceedings of the ISABE XIV International Symposium*, Florence, Italy, September 1999. Paper IS-7187.
- [7] S Evesque, AP Dowling, and AM Annaswamy. Adaptive algorithms for control of combustion. In *Proceedings of the AVT Symposium on Active Control Technology for Enhanced Performance Operation Capabilities*, Braunschweig, Germany, May 2000. Paper 22.
- [8] M Fleifil, AM Annaswamy, Z Ghoniem, and AF Ghoniem. Response of a laminar premixed flame to flow oscillations : a kinematic model and thermoacoustic instabilities results. *Combust Flame*, 106:487–510, 1996.
- [9] C Hantschk, J Hermann, and D Vortmeyer. Active instability control with direct-drive servo valves in liquid-fueled combustion systems. In *Twenty-Sixth Symposium (International) on Combustion*, pages 2835–2841. The Combustion Institute, 1996.
- [10] S Hubbard and AP Dowling. Acoustic instabilities in premix burners. 4th AIAA/CEAS Aeroacoustics Conference, Toulouse, June 1998, 1998.
- [11] A Kemal and CT Bowman. Real time adaptive feedback control of combustion instability. In *Twenty-Sixth Symposium (International) on Combustion*, pages 2803–2809, 1996.
- [12] T Lieuwen and BT Zinn. The role of equivalence ratio oscillations in driving combustion instabilities in low nox gas turbines. In *Proceedings of the 27th Symposium (International) on Combustion*. The Combustion Institute, Pittsburgh, Pennsylvania, 1998.
- [13] AJ Moran, D Steele, and AP Dowling. Active control of combustion and its applications. In *Proceedings of the AVT Symposium on Active Control Technology for Enhanced Performance Operation Capabilities*, Braunschweig, Germany, May 2000.
- [14] K.S. Narendra and A.M. Annaswamy. *Stable Adaptive Systems*. Prentice-Hall International, 1989.
- [15] AA Peracchio and WM Proscia. Nonlinear heat-release/acoustic model for thermoacoustic instability in lean premixed combustors. In *International Gas Turbine & Aeroengine Congress & Exhibition*. American Society of Mechanical Engineers, June 1998. Paper 98-GT-269.
- [16] B Schuermans, W Polifke, and CO Paschereit. Modeling transfer matrices of premixed flames and comparison with experimental results. In *ASME/IGTI Turbo Expo*, Indianapolis, USA, 1999. Paper 99-GT-132.
- [17] JR Seume, N Vortmeyer, W Krause, J Hermann, CC Hantschk, P Zangl, S Gleis, D Vortmeyer, and A Orthmann. Application of active combustion instability control to a heavy duty gas turbine. *Transactions of the ASME, J. Eng for Gas Turbines and Power*, 120:721–726, October 1998.
- [18] OJM Smith. A controller to overcome dead time. *ISA*, 6(2):28–33, 1959.