An Adaptive Smith-Controller for Time-delay Systems with Relative Degree $n^* \leq 2$

Silviu-Iulian Niculescu
HEUDIASYC (UMR CNRS 6599),
Université de Technologie de Compiègne,
Centre de Recherche de Royallieu,
BP. 20529, 60205, Compiègne, France
E-mail: `silviu@hds.utc.fr`

Anuradha M. Annaswamy
Dept. of Mechanical Engineering
Massachussets Institute of Technology
Cambridge, MA 02139-4307, USA
E-mail: `aanna@mit.edu`

Keywords: Adaptive control, stability, Time-delay Systems, strictly positive real, Lyapunov-Krasovskii functionals

Abstract

This paper addresses the control of time-delay systems whose relative degree does not exceed two. An adaptive Smith controller together with an adaptive law similar to the delay-free case is proposed. By using Lyapunov-Krasovskii functionals for an appropriate model transformation of the original system, semiglobal stability of the closed-loop system and asymptotic convergence of the output error is established. Strict positive realness together with the low relative degree of the plant is exploited to establish the stability properties. Robustness properties of the adaptive controller are briefly discussed.

1 Introduction

It is well known that the stabilization of systems involving delays in the states or in the inputs is a difficult problem since the existence of a delay in a system model may induce instability or bad performances for the closed-loop schemes (see, e.g. [9, 12] for further references). A unique approach for controlling systems with known time-delay was originated by Otto Smith in the fifties [17] by compensating for the delayed output using input values stored over a time window of $[t-\tau, t]$ and estimating the plant output using a model of the plant (see also [15] and references therein). In [10], this idea was extended to include unstable plants as well using finite-time integrals of the delayed input values thereby avoiding unstable pole-zero cancellations that may occur in Smith's controller. In [8, 14], adaptive versions of [10] were developed and was shown that the plant can be adaptively controlled in a stable manner in the large. Ortega and Lozano improved the results of [8] in [14] by making use of an augmented error approach and showing that the resulting controller is globally stable for a general linear plant with arbitrary relative degree and minimum phase zeros. In this paper, we consider a smaller class of plants where the relative degree of the plant is less than or equal to two. We show that a fairly simple adaptive controller can be used to stabilize the system. The advantages of the proposed controller over those in [8, 14] are two-fold: The first is that for the cases when $W_m(s)$ is unknown or too difficult to determine, which may occur in cases where the plant is of a very high order, the controller proposed here can be determined unlike the controllers in [8, 14] which require $W_m(s)$. The second is that the controller proposed here is much simpler, which is enabled by making use of the positive realness of the underlying model as well as the knowledge of the time-delay. We also note that the controller proposed here has been directly utilized in both simulation and experimental studies of a practical system and has been shown to be successful in the presence of fairly large delays [4].

The controller structure is similar to that in [14], which is motivated by the Smith controller in [17] and its improvements in [10]. The adaptive law is however along the lines of those used for error model 3 in [11] for delay-free plants. Instead of using the standard quadratic Lyapunov function, an appropriate Lyapunov-Krasovskii functional is added, and is instrumental in deriving the stability properties. With the modified functional, it is shown that for all initial conditions within a bounded domain, and a value of the time-delay that depends on the size of the domain, the closed-loop system will remain bounded. This domain is shown to encompass the entire state-space as $\tau \to 0$. The case when $n^* = 1$ is treated in more detail while the case when $n^* = 2$ is briefly outlined.

The adaptive controller is presented in section 2, with the controller structure in section 2.1, and its adaptive counterpart in section 2.2. In section 3, the main stability result is stated and proved. The robustness properties of the controller are discussed.

2 The Adaptive Smith Controller

The problem is the control of a plant given by the input-output description

$$y(t) \quad = \quad W_p(s)[u(t-\tau)], \qquad W_p(s) \;=\; \frac{k_p Z_p(s)}{R_p(s)} \tag{2.1}$$

where $W_p(s)$ is the transfer function of a finite-dimensional system whose order n is known, relative degree n^* is known and less than or equal to two, zeros are in $\mathbb{C}^-$, and its high frequency gain is known.The time-delay τ is assumed to be known as well. The plant poles and zeros are unknown and it is assumed that all poles have multiplicity one. It is also assumed that the states are not accessible and only input-output measurements are available.

It is required that the plant output follow the output of a reference model with a transfer function

$$y_m(t) \quad = \quad W_m(s)[r(t-\tau)], \qquad W_m(s) \;=\; \frac{k_m}{R_m(s)} \tag{2.2}$$

where R_m is a monic Hurwitz polynomial of degree $n - n^*$, and $k_m > 0$. Our goal is to determine a stable adaptive controller for this class of problems. For ease of exposition, in what follows, we assume that the high frequency gain is known with $k_p = k_m = 1$.

2.1 The Smith Controller

The controller that we propose is an adaptive version of the Smith controller and utilize the finite-time integrals as in [14, 8]. To facilitate the derivation of the adaptive controller, we describe the fixed controller structure in this section.

Since states are not accessible, an standard pole-placement controller is utilized. The presence of the time-delay motivates the use of an additional signal (denoted as u_1 below) which attempts to anticipate the future outputs using a model of the plant. The resulting controller structure can be described as follows [8]:

$$\begin{aligned} u(t) &= \frac{c(s)}{\Lambda(s)}u(t-\tau) + \frac{d(s)}{\Lambda(s)}y(t) + u_1(t) + r(t) \\ u_1(t) &= \frac{n_1(s)}{R_p(s)}u(t) + \frac{n_2(s)}{R_p(s)}u(t-\tau) \end{aligned} \tag{2.3}$$

where $\Lambda(s)$ is a hurwitz polynomial of degree n, c, d, and n_1, are polynomials of degree $n-1$, respectively, which satisfy the relations

$$c(s)R_p(s) + d(s)k_pZ_p(s) = -n_2(s)\Lambda(s) \tag{2.4}$$

$$n_1(s) = R_p(s) - R_m(s)k_pZ_p(s) \tag{2.5}$$

$R_m(s)$ is a monic Hurwitz polynomial of degree n^* and represents the desired closed-loop poles of the plant, and $n_2(s)$ is an $n-1$ degree polynomial. From Bezout identity, it can be easily shown that c, d, and n_1 exist that satisfy (2.4) and (2.5). The controller structure in (2.3) can be shown to result in a closed-loop system with the transfer function $W_m(s)e^{-s\tau}$ where

$$W_m(s) = \frac{1}{R_m(s)}. \tag{2.6}$$

The controller in (2.3) can be implemented as follows:

$$u = \theta_1^{*T}\omega_1 + \theta_2^{*T}\omega_2 + u_1 + r \tag{2.7}$$

$$\dot{\omega}_1 = \Lambda_0\omega_1 + \ell u(t-\tau) \tag{2.8}$$

$$\dot{\omega}_2 = \Lambda_0\omega_2 + \ell y(t) \tag{2.9}$$

$$u_1(t) = \sum_{i=1}^{n}\alpha_i^*\left(\int_{-\tau}^{0} e^{-\lambda_i\sigma}u(t+\sigma)d\sigma\right) \triangleq \int_{-\tau}^{0}\lambda^*(\sigma)u(t+\sigma)d\sigma \tag{2.10}$$

where Λ_0 is an asymptotically stable $n\times n$ matrix,

$$\theta_1^{*T}(sI-\Lambda_0)^{-1}\ell = \frac{c(s)}{\Lambda(s)}, \qquad \theta_2^{*T}(sI-\Lambda_0)^{-1}\ell = \frac{d(s)}{\Lambda(s)}, \qquad R_p(s) = \Pi_{i=1}^{n}(s-r_{p_i}),$$

$$\sum_{i=1}^{n}\frac{\alpha_i^*}{s-r_{p_i}} = \frac{n_1(s)}{R_p(s)}, \qquad \sum_{i=1}^{n}\frac{\alpha_i^* e^{r_{p_i}\tau}}{s-r_{p_i}} = \frac{n_2(s)}{R_p(s)}$$

The realization of $u_1(t)$ as in (2.10) is needed to ensure that no unstable pole-zero cancellations can occur for the case when the plant is open-loop unstable [10, 8]. Such a representation using finite-time integrals requires that the plant poles have multiplicity one.

The discussions in [8] demonstrate that the controller given by (2.7)-(2.3) together with (2.4) and (2.5) stabilizes the plant in (2.1) if all plant parameters including the time-delay are known. It can also be shown that the controller provides stability robustness to uncertainties in the plant parameters including the time-delay [13]. In the next section, we show how an adaptive controller can be developed in the presence of a time-delay and parametric uncertainties.

2.2 The adaptive controller

We now consider the control of the plant in (2.1) when the transfer function $W_p(s)$ has unknown coefficients and the time-delay τ is known. The form of the controller in (2.7)-(2.10) can be directly utilized to develop the adaptive controller, as was done in [8, 14]:

$$\begin{aligned} \dot{\omega}_1 &= \Lambda_0 \omega_1 + \ell u(t-\tau) \\ \dot{\omega}_2 &= \Lambda_0 \omega_2 + \ell y(t) \\ u &= \theta_1^T(t)\omega_1 + \theta_2^T(t)\omega_2 + r(t) + \int_{-\tau}^{0} \lambda(t,\sigma) u(t+\sigma) d\sigma \end{aligned} \tag{2.11}$$

Let

$$\omega = [\omega_1^T,\ \omega_2^T]^T,\ \ \theta = [\theta_1^T,\ \theta_2^T]^T,\ \ \theta^* = [\theta_1^{*T},\ \theta_2^{*T}]]^T,\ \ \widetilde{\theta} = \theta - \theta^*,\ \ \widetilde{\lambda}(t,\sigma) = \lambda(t,\sigma) - \lambda^*(\sigma) \tag{2.12}$$

where θ^* and λ^* are the control parameter values that correspond to the desired closed-loop. That is, with the control input of the form

$$u = r + \theta^{*T}\omega + \int_{-\tau}^{0} \lambda^*(\sigma) u(t+\sigma) d\sigma + \widetilde{\theta}^T \omega + \int_{-\tau}^{0} \widetilde{\lambda}(t,\sigma) u(t+\sigma) d\sigma \tag{2.13}$$

if the parameter errors $\widetilde{\theta} \equiv 0$, and $\widetilde{\lambda}(t,\sigma) \equiv 0$, then the closed-loop transfer function matches that of the model and is given by $W_m(s)e^{-s\tau}$. When the parameter errors are present, it follows that the closed-loop system equations are of the form

$$y = W_m(s)e^{-s\tau}\left(r + \widetilde{\theta}^T\omega + \int_{-\tau}^{0} \widetilde{\lambda}(t,\sigma) u(t+\sigma) d\sigma\right) \tag{2.14}$$

Denoting the output error as $e_1 = y - y_m$, it follows that the error equation is of the form

$$e_1(t) = W_m(s)e^{-s\tau}\left[\widetilde{\theta}^T(t)\omega(t) + \int_{-\tau}^{0} \widetilde{\lambda}(t,\sigma) u(t+\sigma) d\sigma\right] \tag{2.15}$$

The state-space representation of Eq. (2.15) is useful for the stability analysis and is derived below. The finite-dimensional representation of u_1 is given by

$$\begin{aligned} \dot{\omega}_3 &= A_p\omega_3 + b_p u(t) && (2.16)\\ \dot{\omega}_4 &= A_p\omega_4 + b_p u(t-\tau) && (2.17)\\ u_1 &= \theta_3^{*T}\omega_3 + \theta_4^{*T}\omega_4 = \int_{-\tau}^{0}\lambda^*(\sigma)u(t+\sigma)d\sigma && (2.18)\end{aligned}$$

where

$$\theta_3^{*T}(sI-A_p)^{-1}b_p = \frac{n_1(s)}{R_p(s)} \qquad \theta_4^{*T}(sI-A_p)^{-1}b_p = \frac{n_2(s)}{R_p(s)}.$$

We note that a controller defined by Eqs. (2.7)-(2.9) together with Eqs. (2.16)-(2.18) stabilizes the plant in (2.1) for suitable values of θ_i^*, $i=1,\ldots,4$ that are such that Eqs. (2.4) and (2.5) are satisfied. In addition, it follows that at these parameter values, the closed-loop transfer function of the plant together with the controller is given by $W_m(s)e^{-\tau s}$. For such a system, using a plant representation given by

$$\dot{x}_p = A_p x_p + b_p u(t-\tau), \qquad y = c_p^T x_p$$

and the controller in (2.11) which can be represented as

$$u = \overline{\theta}^T(t)\overline{\omega}(t) + r$$

where $\overline{\omega} = [\omega_1^T, \omega_2^T, \omega_3^T, \omega_4^T]^T$, $\overline{\theta} = [\theta_1^T, \theta_2^T, \theta_3^T, \theta_4^T]^T$, the closed-loop system can be derived to be

$$\dot{X} = AX + b\left(\widetilde{\overline{\theta}}^T(t-\tau)\overline{\omega}(t-\tau) + r(t-\tau)\right) \qquad y = c^T X \tag{2.19}$$

where

$$X = [x_p^T, \overline{\omega}^T]^T,\ \overline{\theta}^* = [\theta_1^{*T}, \theta_2^{*T}, \theta_3^{*T}, \theta_4^{*T}]^T, \qquad \widetilde{\overline{\theta}} = \overline{\theta} - \overline{\theta}^*,\ \text{and } c^T(sI-A)^{-1}b = W_m(s). \tag{2.20}$$

Defining X_m as the model state corresponding to X when the parameter error $\widetilde{\theta}$ in (2.19) is zero, and the state error e as $e = X - X_m$, we obtain that the underlying error model is of the form

$$\dot{e} = Ae(t) + b\widetilde{\overline{\theta}}^T(t-\tau)\overline{\omega}(t-\tau), \qquad e_1 = c^T e \tag{2.21}$$

Since Eqs. (2.16)-(2.18) is an alternative representation of u_1 in (2.10), Eq. (2.21) can be equivalently described as

$$\begin{aligned} \dot{e} &= Ae(t) + b\left[\widetilde{\theta}^T(t-\tau)\omega(t-\tau) + \int_{-\tau}^{0}\widetilde{\lambda}(t-\tau,\sigma)u(t-\tau+\sigma)d\sigma\right]\\ e_1 &= c^T e && (2.22)\end{aligned}$$

We note therefore that (2.22) is a state-space representation of the error equation (2.15). It is also easy to see that when $\tau = 0$, Eq. (2.22) collapses to the standard error equations in adaptive control.

An additional property of the above error equation is the relation between ω, $u(t+\sigma)$, $\sigma \in [-\tau, 0]$, and X, and is useful in proving the main result of this paper and is shown below. Eqs. (2.10) and (2.18) show that

$$\theta_3^{*T}\omega_3 + \theta_4^{*T}\omega_4 = \int_{-\tau}^{0} \lambda^*(\sigma)u(t+\sigma)d\sigma \tag{2.23}$$

Since ω_i is a subcomponent of X in (2.20), we have that

$$\|\omega_i(t)\| \leq \|X(t)\| \qquad i = 1, 2, 3, 4. \tag{2.24}$$

Equations (2.23) and (2.24) imply that

$$|u(t+\sigma)| \leq k\|X(t)\| \qquad \forall\, \sigma \in [-\tau, 0] \tag{2.25}$$

where k is a finite positive constant. Eqs. (2.24) and (2.25) are useful in proving the main result.

When the relative degree n^* is equal to unity, it is easy to see that we can find a $W_m(s)$ that is strictly positive real. When $n^* = 2$, an addition of an input u_2 to u as

$$\begin{aligned} u_2(t) &= \dot{\theta}^T(t)\omega'(t) + \int_{-\tau}^{0} \dot{\lambda}(t,\sigma)u'(t+\sigma)d\sigma \\ \dot{\omega}' &= -aI\omega' + \omega, \qquad \dot{u}' = -au' + u, \qquad a > 0 \end{aligned}$$

can be used to derive yet another error equation of the form [11]

$$e_1(t) = W_m(s)(s+a)e^{-s\tau}\left[\widetilde{\theta}^T(t)\omega'(t) + \int_{-\tau}^{0} \lambda(t,\sigma)u'(t+\sigma)d\sigma\right] \tag{2.26}$$

where $a > 0$ is chosen such that $(s+a)W_m(s)$ is strictly positive real. Therefore, it suffices to consider the stability of Eq. (2.22) and show that a stable adaptive law can be derived to adjust $\overline{\theta}$ when $W_m(s)$ is positive real despite the presence of the time-delay; the results can then be extended to the case when $n^* = 2$ by making use of the additional input u_2.

3 Main result

The underlying error equation to be analyzed is given by (2.22) and is of the form

$$\begin{cases} \dot{e}(t) = Ae(t) + b\left[\widetilde{\theta}^T(t-\tau)\omega(t-\tau) + \int_{-\tau}^{0} \widetilde{\lambda}(t-\tau,\sigma)u(t-\tau+\sigma)d\sigma\right] \\ e_1(t) = c^Te(t), \end{cases}$$

under an appropriate initial vector-valued function defined on the interval $[-\tau, 0]$ [7, 9].

Suppose the *adaptive law* is chosen, as in the delay-free case as

$$\begin{aligned}\dot{\tilde{\theta}}(t) &= -e_1(t)\omega(t-\tau)\\ \dot{\tilde{\lambda}}(t,\sigma) &= -e_1(t)u(t-\tau+\sigma), \qquad \sigma\in[-\tau,0].\end{aligned} \tag{3.27}$$

We shall rewrite the error equation as follows:

$$\begin{aligned}\dot{e}(t) &= Ae(t)+b\left[\tilde{\theta}^T(t)\omega(t-\tau)+\int_{-\tau}^{0}\tilde{\lambda}(t,\sigma)u(t-\tau+\sigma)d\sigma\right] \qquad (3.28)\\ &\quad -b\left[\tilde{\theta}^T(t)-\tilde{\theta}^T(t-\tau)\right]\omega(t-\tau)-b\int_{-\tau}^{0}\left[\tilde{\lambda}(t,\sigma)-\tilde{\lambda}(t-\tau,\sigma)\right]u(t-\tau+\sigma)d\sigma.\end{aligned}$$

Using the Leibniz-Newton formula, we have:

$$\begin{aligned}\tilde{\theta}^T(t)-\tilde{\theta}^T(t-\tau) &= \int_{-\tau}^{0}\dot{\tilde{\theta}}^T(t+\nu)d\nu = -\left(\int_{-\tau}^{0}c^Te(t+\nu)\omega(t+\nu-\tau)d\nu\right)^T,\\ \tilde{\lambda}(t,\sigma)-\tilde{\lambda}(t-\tau,\sigma) &= \int_{-\tau}^{0}\dot{\tilde{\lambda}}(t+\nu,\sigma)d\sigma = -\left(\int_{-\tau}^{0}c^Te(t+\nu)u(t-\tau+\nu+\sigma)d\sigma\right).\end{aligned} \tag{3.29}$$

Some simple, but tedious computations lead to the following error equation:

$$\begin{aligned}\dot{e}(t) &= Ae(t)+b\left[\tilde{\theta}^T(t)\omega(t-\tau)+\int_{-\tau}^{0}\tilde{\lambda}(t,\sigma)u(t-\tau+\sigma)d\sigma\right] \qquad (3.30)\\ &\quad +b\omega^T(t-\tau)\int_{-\tau}^{0}c^Te(t+\nu)\omega(t+\nu-\tau)d\nu\\ &\quad +b\int_{-\tau}^{0}\left(\int_{-\tau}^{0}c^Te(t+\nu)\omega(t+\nu+\sigma-\tau)d\nu\right)u(t-\tau+\sigma)d\sigma\\ \dot{\tilde{\theta}}(t) &= -e_1(t)\omega(t-\tau)\\ \dot{\tilde{\lambda}}(t,\sigma) &= -e_1(t)u(t-\tau+\sigma), \quad -\tau\le\sigma\le 0.\end{aligned} \tag{3.31}$$

Remark 1: The *model transformation* technique used above was largely used in the control literature for deriving *delay-dependent* (e.g. including information on the delay size) stability conditions. A brief overview and further comments may be found in [12].

The correspondence between the solutions of the systems (2.22) and (3.31) can be done similarly to Rasvan [16], based on a 'step-by-step' method idea proposed in Halanay [6] for computing the solutions of the corresponding differential equation. Using the same steps as in [12], we may prove the following stability results:

Proposition 1 *Consider $\omega\in\mathcal{L}^\infty$, and $u_t(\cdot)\in\mathcal{L}^\infty([-\tau,0],\mathbf{R}^n)$. Then the stability of the system (3.31) for any delay $\tau\in[0,\tau^*)$ under the inputs ω and u implies the stability of the original system on the same delay interval and under the same inputs.*

It is quite evident that in the delay-free case ($\tau \equiv 0$), if the corresponding system is SPR, we may find a Lyapunov function of the form

$$V_1(e, \widetilde{\theta}, \widetilde{\lambda}) \;\; = \;\; e(t)^T P e(t) + \widetilde{\theta}^T(t)\widetilde{\theta}(t) + \int_{-\tau}^{0} \widetilde{\lambda}^2(t,\sigma)d\sigma \tag{3.32}$$

where $P = P^T > 0$ satisfies the Kalman-Yakubovich-Popov (KYP) lemma

$$\begin{cases} A^T P + PA + Q & = \;\; 0, \\ Pb & = \;\; c. \end{cases} \tag{3.33}$$

for some $Q = Q^T > 0$. Based on it, introduce now the following Lyapunov-Krasovskii functional:

$$V(e, \widetilde{\theta}, \widetilde{\lambda}, \dot{\widetilde{\theta}}, \dot{\widetilde{\lambda}}) \;\; = \;\; V_1(e, \widetilde{\theta}, \widetilde{\lambda}) + V_2(\dot{\widetilde{\theta}}) + V_3(\dot{\widetilde{\lambda}}), \tag{3.34}$$

where the additional terms are given by:

$$\begin{aligned} V_2(\dot{\widetilde{\theta}}) &= \int_{-\tau}^{0}\left(\int_{t+\nu}^{t} \dot{\widetilde{\theta}}(\xi)^T \dot{\widetilde{\theta}}(\xi) d\xi\right) d\theta, \\ V_3(\dot{\widetilde{\lambda}}) &= \int_{-\tau}^{0}\left[\int_{t+\nu}^{t}\left(\int_{-\tau}^{0}\left(\dot{\widetilde{\lambda}}(\xi,\sigma)\right)^2 d\sigma\right) d\xi\right] d\nu. \end{aligned} \tag{3.35}$$

Remark 2: Further remarks on such constructions may be found in [12]. The idea is to use V_2 and V_3 in order to complete the square for a negative-definite derivative.

It is clear that V in (3.34) is positive definite and has an infinitesimal upper bound defined appropriately by the corresponding "sup" norm in the space $\mathbf{R}^n \times \mathcal{L}^2([-\tau, 0], \mathbf{R}^m)$. We shall now compute the derivative of V along the solutions of the model transformation (3.31) and use (3.33). Some simple computations lead to the derivatives:

$$\begin{aligned} \dot{V}_1 &= e(t)^T(A^T P + PA)e(t) + 2e(t)^T Pb\widetilde{\theta}(t)^T \omega(t-\tau) \\ &\quad + 2e(t)^T Pb\omega(t-\tau)^T \int_{-\tau}^{0} e_1(t+\nu)\omega(t+\nu-\tau)d\nu \\ &\quad + 2e(t)^T Pb \int_{-\tau}^{0} \widetilde{\lambda}(t,\sigma)u(t-\tau+\sigma)d\sigma \\ &\quad + 2e(t)^T Pb \int_{-\tau}^{0}\left(\int_{-\tau}^{0} e_1(t+\nu)u(t+\nu-\tau+\sigma)u(t-\tau+\sigma)d\nu\right) d\sigma \\ &\quad - 2e_1(t)\widetilde{\theta}(t)^T \omega(t-\tau) - 2\int_{-\tau}^{0} \widetilde{\lambda}(t,\sigma)e_1(t)u(t-\tau+\sigma)d\sigma \\ \dot{V}_2 &= \tau\|e_1(t)\omega(t-\tau)\|^2 - \int_{-\tau}^{0} \|e_1(t+\nu)\omega(t+\nu-\tau)\|^2 d\nu. \\ \dot{V}_3 &= \int_{-\tau}^{0}\left[\int_{-\tau}^{0}\left(\dot{\widetilde{\lambda}}(t,\sigma)\right)^2 d\sigma - \int_{-\tau}^{0}\left(\dot{\widetilde{\lambda}}(t+\nu,\sigma)\right)^2 d\sigma\right] d\nu \\ &= \tau\int_{-\tau}^{0} \|e_1(t)u(t-\tau+\sigma)\|^2 d\sigma - \int_{-\tau}^{0}\int_{-\tau}^{0} \|e_1(t+\nu)u(t+\nu-\tau+\sigma)\|^2 d\sigma d\nu. \end{aligned}$$

Using the KYP lemma, it follows that:

$$\begin{aligned}\dot{V} &= -e(t)^T Q e(t) + 2e_1(t)\omega(t-\tau)^T \int_{-\tau}^{0} e_1(t+\nu)\omega(t+\nu-\tau)d\nu \\ &+2e_1(t)\int_{-\tau}^{0}\left(\int_{-\tau}^{0} e_1(t+\nu)u(t+\nu-\tau+\sigma)u(t-\tau+\sigma)d\nu\right)d\sigma \\ &+\tau\|e_1(t)\omega(t-\tau)\|^2 - \int_{-\tau}^{0}\|e_1(t+\nu)\omega(t+\nu-\tau)\|^2 d\nu \\ &+\tau\int_{-\tau}^{0}\|e_1(t)u(t-\tau+\sigma)\|^2 d\sigma - \int_{-\tau}^{0}\int_{-\tau}^{0}\|e_1(t+\nu)u(t+\nu-\tau+\sigma)\|^2 d\sigma d\nu. \end{aligned} \tag{3.36}$$

Denoting

$$\bar{a} = e_1(t)\omega(t-\tau), \qquad \bar{b} = e_1(t+\nu)\omega(t+\nu-\tau), \qquad \bar{c} = e_1(t)u(t-\tau+\sigma), \qquad \bar{d} = e_1(t+\nu)u(t+\nu-\tau+\sigma),$$

Eq. (3.36) can be rewritten as

$$\begin{aligned}\dot{V} &= -e^T Q e + \int_{-\tau}^{0}\left[2\bar{a}^T\bar{b} + \bar{a}^T\bar{a} - \bar{b}^T\bar{b}\right]d\nu + \int_{-\tau}^{0}\int_{-\tau}^{0}\left[2\bar{c}^T\bar{d} + \bar{c}^T\bar{c} - \bar{d}^T\bar{d}\right]d\sigma d\nu \\ &\leq -e(t)^T\left[Q - 2\tau\left(\|\omega(t-\tau)\|^2 + \int_{-\tau}^{0}\|u(t-\tau+\sigma)\|^2 d\sigma\right)cc^T\right]e(t). \end{aligned} \tag{3.37}$$

For bounded signals ω and u satisfying at time "t" the matrix inequality:

$$Q - 2\tau\left(\|\omega(t-\tau)\|^2 + \int_{-\tau}^{0}\|u(t-\tau+\sigma)\|^2 d\sigma\right)cc^T \quad > \quad 0. \tag{3.38}$$

Since ω and u are dependent variables, condition (3.38) may not be easy to check. Note however that the bound above on $\dot{V}$ at time "t" is given by some bounds on ω defined at $t-\tau$ and on u defined on the whole interval $[t-2\tau, t-\tau]$. We show below that this condition can be replaced by bounds on states at time t_0 and over the interval $[t_0-2\tau, t_0]$ so that the domain of attraction over which $\dot{V} \leq 0$ can be delineated more precisely.

Suppose the values of ω and u over $[t_0-\tau, t_0)$ and $[t_0-2\tau, t_0)$, respectively, are such that

$$\sup_{\theta\in[t_0-\tau,t_0)} \|\omega(\theta)\|^2 \quad \leq \quad \gamma_1, \tag{3.39}$$

$$\sup_{\theta\in[t_0-2\tau,t_0)} \|u(\theta)\|^2 \quad \leq \quad \gamma_2, \tag{3.40}$$

for some real positive γ_1, γ_2, and a delay value $\bar{\tau}_1$ is such that

$$2\bar{\tau}_1(\gamma_1 + \gamma_2\bar{\tau}_1)cc^T \quad < \quad Q. \tag{3.41}$$

Then using the *step-by-step* type argument for the construction of the solution of the associated FDE with persistent perturbation [6], it follows that combining (3.39)-(3.41) on the interval $[t_0, t_0+\tau)$, the following inequality

$$2\tau\left(\|\omega(\zeta-\tau)\|^2 + \int_{-\tau}^{0}\|u(\zeta-\tau+\sigma)\|^2 d\sigma\right)cc^T \quad < \quad Q,\ \forall\zeta\in[t_0, t_0+\tau),\ \forall\tau\in(0,\bar{\tau}_1) \tag{3.42}$$

is satisfied, and it follows that the Lyapunov-Krasovskii functional V is *non-increasing* on the interval $[t_0, t_0 + \tau)$, if the bound on the delay τ is given by $\bar{\tau}_1$. Thus:

$$\lambda_{min}(P)\|e(\zeta)\|^2 \quad \leq V(e_0, \widetilde{\theta}_0, \widetilde{\lambda}_0, \dot{\widetilde{\theta}}_{t_0}, \dot{\widetilde{\lambda}}_{t_0}), \quad \forall \zeta \in [t_0, t_0 + \tau), \tag{3.43}$$

where $e(t_0) = e_0$, $\widetilde{\theta}(t_0) = \widetilde{\theta}_0$, and $\widetilde{\lambda}_0 = \widetilde{\lambda}(t_0)$. Defining

$$X_0^2 = \frac{V(e_0, \widetilde{\theta}_0, \dot{\widetilde{\theta}}_{t_0})}{\lambda_{min}(P)} + X_{mo}$$

where X_{mo} is a bounded quantity that depends on the model initial conditions, it follows from (2.24) and (2.25) that

$$\|\omega(\zeta)\| \quad \leq \quad X_0 \quad \forall \zeta \in [t_0, t_0 + \tau) \tag{3.44}$$

$$\|u(\zeta + \sigma)\| \quad \leq \quad kX_0 \quad \forall \zeta \in [t_0, t_0 + \tau), \quad \sigma \in [-\tau, 0). \tag{3.45}$$

It should be noted that X_0 does not depend on the values γ_1, γ_2 but only on the system initial conditions. Let $\overline{\tau}_2$ be a positive value such that the following inequality holds:

$$2\overline{\tau}_2(X_0 + \gamma_2\overline{\tau}_2)cc^T \quad < \quad Q, \tag{3.46}$$

using the same argument as in the previous step, it follows that V is a *non-increasing* function on the interval $[t_0, t_0 + 2min\{\overline{\tau}_1(\gamma_1, \gamma_2)), \overline{\tau}_2(\gamma_2)\})$. Suppose

$$\bar{\tau} \quad = \quad min\left\{\overline{\tau}_1(\gamma_1, \gamma_2), \overline{\tau}_2(\gamma_2), \overline{\tau}_3\right\}. \tag{3.47}$$

where $\overline{\tau}_3$ satisfies the inequality

$$2\overline{\tau}_3(1 + k\overline{\tau}_3)X_0 cc^T \quad < \quad Q, \tag{3.48}$$

which, once again, depends on X_0 and not on γ_1 or γ_2. Then, for $\tau \leq \overline{\tau}$, inequality (3.38) continues to remain satisfied over an interval $[t_0 + \tau, t_0]$ and hence $\dot{V}$ is nonpositive over this interval. By applying the arguments repeatedly, and since t_0 is arbitrary, it follows that for all ω and u that satisfy (3.39) and (3.40) and all time-delays that satisfies (3.47), $V(t)$ is non-increasing for all $t \geq t_0$. This in turn implies that all signals are bounded. Using the same arguments as in [11], it can be shown that $\lim\limits_{t \to +\infty} \|e(t)\| = 0$. This leads to our main result outlined below:

Theorem 1 *Consider the closed-loop system defined by the plant in (2.1), the controller in (2.11), and the adaptive law in (3.27), with the model as in (2.2) where $W_m(s)$ is SPR. Then for any* bounded signals ω *and u satisfying the inequalities (3.39) and (3.40), respectively, and for any* delay τ *that satisfies (3.47), the following two properties hold:*

(i) $\lim_{t\to+\infty}\|e(t)\| = 0$,

(ii) all other signals of the closed-loop system are bounded.

Remark 3: Our first remark concerns the bound $\bar{\tau}$ on the allowable time-delay. $\bar{\tau}$ can be interpreted as a *design parameter* in that if one wants to have more degree-of-freedom on the signal ω on $[t_0-\tau, t_0]$ by increasing γ_i ($i = 1,2$), then $\overline{\tau}_1, \overline{\tau}_2$ has to be smaller, i.e. $\bar{\tau}$ smaller, which is in concordance with the continuity principle. We also note that $\overline{\tau}_3$ is completely *independent* of γ_1, γ_2, but depends only on the initial values of the states x and $\widetilde{\theta}$ of the adaptive system. The larger the mismatch between the plant and model parameters, the larger the $\widetilde{\theta}_0$, and smaller the allowable delay.

Remark 4: Theorem 1 shows that the stability semi-global with respect ot τ. For a given atttraction domain, we may compute a *maximal delay value* guaranteeing *stability* of e ($\lim\limits_{t\to+\infty}\|e(t)\| = 0$) and *boundedness* for all other signals. Reciprocally, an *imposed delay value* may generate a *"maximal" attraction domain* guaranteeing the corresponding properties. As is, this proves attractive in a number of applications, where even when the plant is open-loop unstable, inherent nonlinearities present in the plant ensure the boundedness of the output (for example, active combustion control [3]) with the problem being regulation of the output to zero. It should also be noted from (3.38) that when $\tau \to 0$, the stability domain reaches $\mathbf{R}^n$, and hence we recover the stability result derived for systems free of delay.

Remark 5: Even though $\widetilde{\theta}(t-\tau)$ is used in the error model, Theorem 1 demonstrates that there is no harm in adjusting the derivative of $\widetilde{\theta}$ at t in the adaptive law and that stability can still be guaranteed by finding an appropriate Lyapunov function. Note also that Lyapunov function in eq. (3.34) suggests that the evolution of $\widetilde{\theta}$ over the interval $[-\tau, 0]$, $\dot{\widetilde{\theta}}$, should also be used. The triple integral is due to the fact that $e_1(t)$ and $\omega(t)$ evolve independently in $\dot{\widetilde{\theta}}$, and the fact that $u(\sigma)$ is a distributed quantity defined over the whole interval $[t-\tau, t]$. As specified in Remark 2, V_2 and V_3 are used to complete the square by an appropriate "weight" in "t" with respect to all evolutions over some ν-delay interval $[t+\nu, t]$, with $\nu \in [-\tau, 0]$ as a parameter.

Remark 6: The method in [13] is quite similar to the one proposed here. Note however that the controller in [13] makes use of some discretization procedure of u_1 over one delay interval, which simplifies the adaptive scheme to:

$$\begin{cases} \dot{e}(t) = Ae(t) + b\widetilde{\theta}^T(t-\tau)\omega(t-\tau) \\ e_1(t) = c^T e(t) \end{cases},$$

for a suitably redefined $\widetilde{\theta}$ and ω with the adaptive law

$$\dot{\widetilde{\theta}}(t) = -e_1(t)\omega(t-\tau)$$

leading to similar conclusions as in Theorem 1. For the sake of brevity, the corresponding results are not discussed here.

Remark 7: The results are still valid for the more general system:

$$\begin{cases} \dot{e}(t) = Ae(t) + b\left[\widetilde{\theta}^T(t-\tau_1)\omega(t-\tau_2) + \int_{-\tau_3}^{0} \widetilde{\lambda}(t-\tau_4,\sigma)u(t-\tau_5+\sigma)d\sigma\right] \\ e_1(t) = c^T e(t), \end{cases}$$

with the *adaptive law*

$$\begin{aligned} \dot{\widetilde{\theta}}(t) &= -e_1(t)\omega(t-\tau_2) \\ \dot{\widetilde{\lambda}}(t,\sigma) &= -e_1(t)u(t-\tau_5+\sigma), \qquad \sigma \in [-\tau_3, 0]. \end{aligned} \tag{3.49}$$

In fact, the corresponding Lyapunov functional allows a *'mixed' delay-independent/delay-dependent stability* [12] result which is *delay-independent* with respect to τ_2 and τ_5, and *delay-dependent* with respect to τ_1, τ_3 and τ_4. This aspect proves the "decoupling" property between $\widetilde{\theta}$ and ω. Furthermore, it becomes more coherent that the choice of τ_1 in the model is *strongly* connected with the parameter γ_1, γ_2 for characterizing the signals ω and u on $[t_0 - \hat{\tau}, t_0)$, where $\hat{\tau} = max\{\tau_1, \tau_2, \tau_3 + max\{\tau_4, \tau_5\}\}$. This aspect explains better why we mentioned before the use of the delay τ (τ_1 and/or τ_4 here) as a (possible) design parameter.

Remark 8: A more complicated adaptive law including an augmented error was proposed in [8, 14] and requires explicit construction of $W_m(s)$ in the generation of the adaptive law. In several problems such as regulation at a desired operating point, however, the choice of the reference model is often unclear though it can be shown to exist and have a desired behavior. In these cases, making full use of any positive realness that may be present in the underlying system leads to a low-order adaptive controller with few adjustable parameters. Such a control strategy has recently been shown to be very useful in applications related to active combustion control [3].

3.1 Robustness of the Adaptive Controller

As in adaptive control theory for systems in the delay-free case [11], the adaptive controller proposed here can be shown to be robust with respect to disturbances and unmodeled dynamics by introducing modifications to the adaptive law. A brief example of this property is mentioned below where a scalar error equation with an unknown parameter $\lambda^*(\sigma)$, $\sigma \in [0, \tau]$, is discussed for ease of exposition.

If a disturbance $d(t)$ is present in the system, and a σ-modification as in [18] is used, the underlying error equations are typically modified as

$$\begin{cases} \dot{e}_1(t) = -e_1(t) + \int_{-\tau}^{0} \widetilde{\lambda}(t-\tau,\sigma)u(t-\tau+\sigma)d\sigma + d(t) \\ \dot{\widetilde{\lambda}}(t,\sigma) = -e_1(t)u(t-\tau+\sigma) - \sigma_0\left(\widetilde{\lambda}(t,\sigma) - \lambda^*(\sigma)\right), \qquad \sigma_0 > 0 \end{cases} \tag{3.50}$$

For a function $V = V_1(e_1, \widetilde{\lambda}) + V_3(\dot{\widetilde{\lambda}})$, where

$$\begin{aligned} V_1 &= e_1^2(t) + \int_{-\tau}^{0} \widetilde{\lambda}^2(t,\sigma)d\sigma, \\ V_3 &= \int_{-\tau}^{0}\int_{t+\nu}^{t}\left[\int_{-\tau}^{0} \|e_1(\xi)u(\xi-\tau+\sigma)\|^2 d\sigma\right]d\xi d\nu + \frac{\sigma}{\tau}\int_{-\tau}^{0}\int_{t+\nu}^{t}\left[\int_{-\tau}^{0}(\widetilde{\lambda}(\xi,\sigma)-\lambda^*(\sigma))^2 d\sigma\right]d\xi d\nu \end{aligned}$$

after some simplifications, we obtain the time-derivative

$$\begin{aligned}\dot{V} &\leq -2e_1(t)^2\left(1-\tau(1+\tau\sigma_0)\int_{-\tau}^{0}\|u(t-\tau+\sigma)\|^2 d\sigma\right)+2e_1(t)d(t)\\ &\quad +\sigma_0\int_{-\tau}^{0}\left[(\lambda^*(\sigma))^2-\widetilde{\lambda}(t,\sigma)^2\right]d\sigma. \end{aligned} \tag{3.51}$$

Thus, if

$$1-\tau(1+\tau\sigma_0)\int_{-\tau}^{0}\|u(t-\tau+\sigma)\|^2 d\sigma \;>\; 0, \tag{3.52}$$

then $\dot{V} \leq 0$ for all D^c, where D is a compact set in the $(e_1, \widetilde{\lambda}(\cdot,\cdot))$ space. The condition on (3.52) depends only on τ, σ_0 and past values over one delay interval of the input u, and hence the solutions of the adaptive system can be shown to be bounded if $u(\theta)$, $\theta \in [0,\tau)$ satisfies (3.52), using the same arguments as in the disturbance-free case. The experimental investigations in [4] used such modifications in the adaptive law and led to a successful implementation.

4 Summary

In this paper, a simple continuous-time adaptive controller is proposed for time-delay systems whose relative degree does not exceed two. The controller structure is motivated by the Smith Controller and modified so as to accommodate plants that may be unstable in the open-loop. Motivated by strict positive realness of an underlying transfer function, a simple adaptive law as that in the third error model for delay-free plants is used. A novel positive definite function that consists of an appropriate Lyapunov-Krasovskii functional derived for a model transformation of the original system is utilized to derive stability properties of the closed-loop system. The latter is shown to be semi-global in the time-delay τ, and leads to asymptotic convergence of the output error to zero. The controller is also shown to be robust to disturbances.

Acknowledgement

This work was supported by a joint NSF/CNRS project under grant No. INT-9603271. We gratefully acknowledge the support of Dr. Rogelio Lozano, Director, HEUDIASYC, UTC. and for fostering this collaborative research.

References

[1] G.A. Baker Jr. and P. Graves-Morris, *Pade approximants. Second Edition*, McGraw Hill, Inc., New York, NY, 1996.

[2] R. Bakker and A.M. Annaswamy, “Stability and Robustness Properties of a Simple Adaptive Controller”, *IEEE Trans. Automat. Contr.*, **AC-41** (1996) 1352-1358.

[3] S. Evesque, A.P. Dowling, and A.M. Annaswamy, *Self-tuning regulator for combustion instabilities in the presence of time-delay,* NATO RTO/AVT Symposium on Active Control Technology for Maneuvering in Land, Air, and Sea Vehicles, Braunschweig, Germany, 2000.

[4] S. Evesque, *Adaptive Control of Combustion Oscillations*, PhD Thesis, Cambridge University, U.K., November 2000.

[5] K. Gu and S. -I. Niculescu, “Additional dynamics in transformed time-delay systems,” *IEEE Trans. Automat. Contr.* **45** (2000) 572-576.

[6] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, 1966.

[7] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[8] K. Ichikawa, “Adaptive control of delay system,” *Int. J. Contr.*, **41** (1986) 1653-1659.

[9] V. B. Kolmanovskii and A. D. Myshkis, *Applied Theory of functional differential equations*, Kluwer, Dordrecht, 1992.

[10] A.Z. Manitius and A.W. Olbrot, “Finite Spectrum Assignment Problem for Systems with Delays,” *IEEE Transactions on Automatic Control* **AC-24** (1979) 541-553.

[11] K. S. Narendra and A. M. Annaswamy, *Stable adaptive systems*, Englewood Cliffs, NJ: Prentice Hall, 1989.

[12] S. -I. Niculescu, *Delay effects on stability. A robust control approach* (Springer-Verlag: Heidelberg, LNCIS, vol. **269**, May 2001).

[13] S.-I. Niculescu and A.M. Annaswamy, “A Simple Adaptive Controller for Positive-Real Systems with Time-delay,” Proceedings of the *2000 American Control Conference,* Chicago, IL, July 2000.

[14] R. Ortega and R. Lozano, “Globally stable adaptive controller for systems with delay,” *Int. J. Contr.* **47** (1988) 17-23.

[15] Z. J. Palmor, “Time-delay compensation – Smith predictor and its modifications,” in *The Control Handbook* (W. S. Levine, Eds.), CRC Press (1996) 224-237.

[16] Vl. Rasvan, *Absolute stability of automatic control systems with delays* (in Romanian), Eds. Academiei RSR, Bucharest, Romania, 1975 (Russian revised edition by Nauka, Moscow, 1983).

[17] O. J. M. Smith, “A controller to overcome dead time,” *ISA* vol. 6, No. 2, (1959) 28-33.

[18] P.A. Ioannou and P.V. Kokotovic, *Adaptive Systems with Reduced Models*, Springer-Verlag, New York, 1983.