PAPPALARDO SERIES IN MECHANICAL ENGINEERING

An Introduction to Finite Elasticity

Volume III of Lecture Notes on the Mechanics of Elastic Solids

Version 1.1

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http://web.mit.edu/abeyaratne/lecture_notes.html

Published online: March 31, 2022

Electronic Publication

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Abeyaratne, Rohan, An Introduction to Finite Elasticity. Volume III of Lecture Notes on the Mechanics of Elastic Solids. / Rohan Abeyaratne – 1st Edition – Cambridge, MA:

ISBN-13: 978-0-9791865-2-3
ISBN-10: 0-9791865-2-8

QC

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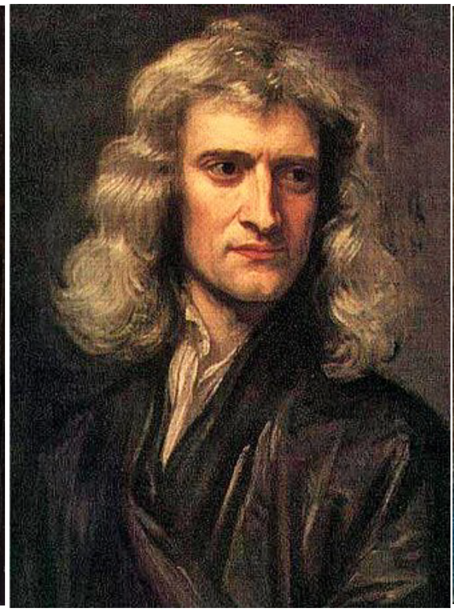
Updated March 18, 2022; January 7, 2024.

To the many students I have had the privilege and joy journeying with during their time at MIT: you have taught me so much.

Galileo Galilei
(1564-1642)



Isaac Newton
(1642-1727)

Leonard Euler
(1707-1783)

Augustin-Louis Cauchy
(1789-1857)

Ronald Rivlin
(1915-2005)

From Wikipedia

PREFACE

The MIT Department of Mechanical Engineering offers a series of graduate-level subjects on the Mechanics of Solids and Structures that in recent years has included:

2.071: Mechanics of Solid Materials,
2.072: Mechanics of Continuous Media,
2.074: Solid Mechanics: Elasticity (formerly 2.083),
2.073: Solid Mechanics: Plasticity and Inelastic Deformation,
2.075: Mechanics of Soft Materials,
2.080: Structural Mechanics,
2.094: Finite Element Analysis of Solids and Fluids,
2.095: Molecular Modeling and Simulation for Mechanics, and
2.099: Computational Mechanics of Materials.

I have taught the second and third of these subjects on several occasions and the current four volumes comprise the notes I developed for them. These are *notes, not textbooks*. The earliest rough drafts were written in 1987 and 1988 and they have been expanded and refined on every subsequent occasion when I taught these classes. They are organized as follows:

Volume I: A Brief Review of Some Mathematical Preliminaries

Volume II: Continuum Mechanics

Volume III: An Introduction to Finite Elasticity

Volume IV: Linear Elasticity (Not as-yet ready.)

This is Volume III.

Until 2018, the subject 2.074 on elasticity treated only the linear(ized) theory. In recent years, several students doing research on "soft materials" and "biomaterials" asked for references to books where they could learn the nonlinear theory on their own and I would direct them to one of the books listed below. In fall 2018 I decided to devote the first part of 2.074 to the nonlinear theory and the rest to the linearized theory. Volume III consists of the notes from the first part (with somewhat more detail than what I actually cover). Due to the limitation of time – less than one semester – the treatment is special in many ways, e.g. inertial effects are not considered. While there is some duplication of material between Volumes II and III, the more narrow focus here should be helpful to the student encountering this material for the first time. An expanded treatment of the underlying theory can be found in the relevant chapters of Volume II.

The content of these notes is entirely classical, in the best sense of the word. While the material covered is not original, some of it is not usually emphasized in textbooks. They include the several boundary-value problems focused on *illustrating nonlinear phenomena* (Chapter 5), strain-energy functions with multiple energy-wells used in the study solid-to-solid phase transitions (Chapter 7), and Cauchy's lattice-based theory of elasticity (Chapter 8).

One of the few positive outcomes of the COVID-19 pandemic was that in the fall of 2020, when 2.074 was taught remotely, I recorded a *few* (amateur) videos on some particular topics. Links to them are provided in the text.

In case you wonder why "an introduction" is about 700 pages long, it is because of the numerous examples and exercises that are included in almost every chapter. They are an essential part of these notes. Many of these problems illustrate general concepts through particular examples. Some provide further details on items touched on in the text. Others generalize previously described special cases. Some concern proofs of results that had simply been quoted before, or they refer to results that will be used in what follows.

The problems are numbered as follows: Problem 2.6 for example can be found at the end of Chapter 2 in the section on Exercises, while Problem 2.6.2 is located within Section 2.6. This distinction between problems identified by two numbers (e.g. 2.6) versus three (e.g. 2.6.2) has been adopted throughout.

My appreciation for mechanics was nucleated by Professors Douglas Amarasekara and Munidasa Ranaweera of the (then) University of Ceylon, and was subsequently shaped and grew substantially under the influence of Professors James K. Knowles and Eli Sternberg of the California Institute of Technology. I have been most fortunate to have had the opportunity to apprentice under these inspiring and distinctive scholars.

I would especially like to acknowledge the innumerable illuminating and stimulating interactions with my mentor, colleague and friend the late Jim Knowles. His influence on me cannot be overstated.

I am also indebted to the many MIT students who have given me enormous fulfillment and joy to be part of their education and for their feedback on these notes.

My understanding of elasticity has benefitted greatly from numerous conversations with many colleagues including Kaushik Bhattacharya, Janet Blume, Eliot Fried, Morton E. Gurtin, Richard D. James, Stelios Kyriakides, David M. Parks, Sensei Phoebus Rosakis,

Stewart Silling and Nicolas Triantafyllidis. My grateful thanks to them all.

I have drawn on a number of sources over the years as I prepared my lectures. I cannot recall every one of them but they certainly include those listed at the end of each chapter. I have found the following articles and books particularly useful:

Volume III: An Introduction to Finite Elasticity

- J. M. Ball, Some recent developments in nonlinear elasticity and its applications to materials science, in *Nonlinear Mathematics and Its Applications*, edited by P.J. Aston, pp. 93–119. Cambridge University Press, 1996.
- P. Chadwick, *Continuum Mechanics: Concise Theory and Problems*, Wiley, 1976. Reprinted by Dover,1999.
- A. Goriely, A. Erlich and C. Goodbrake, C5.1 Solid Mechanics, Online problem sheets, Oxford University. The 2020 version was at https://courses-archive.maths.ox.ac.uk/node/view_material/52105.
- M.E. Gurtin, E. Fried and L. Anand, *The Mechanics and Thermodynamics of Continua*, Cambridge University Press, 2010.
- J. K. Knowles and E. Sternberg, *(Unpublished) Lecture Notes for AM136: Finite Elasticity*, California Institute of Technology, Pasadena, CA 1978.
- R.W. Ogden, *Nonlinear Elastic Deformations*, Ellis Horwood, 1984. Reprinted by Dover, 1997.
- D.J. Steigmann, *Finite Elasticity Theory*, Oxford, 2017.

For a treatment of the rigorous mathematical underpinnings, the student may refer to:

- S. S. Antman, *Nonlinear Problems of Elasticity*, Springer-Verlag, 1995.
- J.E. Marsden and T.J.R. Hughes, *Mathematical Foundations of Elasticity*, Prentice-Hall, 1983. Reprinted by Dover 1994.

The following notation will be used in Volume III, though there will be a few lapses (for reasons of tradition):

- Greek letters will denote scalars;
- lowercase boldface Latin letters will denote vectors; and
- uppercase boldface Latin letters will denote linear transformations (tensors).

Thus, for example, $\alpha, \beta, \gamma ...$ will denote scalars (real numbers); $\mathbf{a}, \mathbf{b}, \mathbf{c}, ...$ will denote vectors; and $\mathbf{A}, \mathbf{B}, \mathbf{C}, ...$ will denote tensors.

One consequence of this notational convention is that I will *not* use the uppercase boldface letter $\mathbf{X}$ to denote the position vector of a particle in the reference configuration (as many authors do). Being a boldface uppercase letter, my convention would dictate that $\mathbf{X}$ represent some tensor. Instead, I use the lowercase boldface letters $\mathbf{x}$ and $\mathbf{y}$ to denote the respective position *vectors* of a particle in the reference and current configurations. I sometimes lightheartedly refer to this as the "Caltech-Minnesota notation".

I have been frequently asked whether I intend to publish these notes in the form of a traditional textbook, and my answer has aways been "no". These notes are being made available primarily for students who like me, when I was studying in Sri Lanka, could not afford the cost of purchasing a textbook. I therefore intend to make these notes available for free online.

Finally, I would like to express my grateful thanks to Jane and Neil Pappalardo for their friendship and support over many years. The writing of Volume III was supported by the MIT-Pappalardo Series in Mechanical Engineering.

List of symbols

Table 1: Kinematics

Quantity	**Symbol**
Position vector of particle in reference configuration	$\mathbf{x}$
Position vector of particle in deformed configuration	$\mathbf{y}$
Deformation field	$\mathbf{y}(\mathbf{x})$
Displacement field	$\mathbf{u}(\mathbf{x})$
Infinitesimal material fiber in reference configuration	$\mathrm{d}\mathbf{x}$
Infinitesimal material fiber in deformed configuration	$\mathrm{d}\mathbf{y}$
Volume of an infinitesimal part in reference configuration	$\mathrm{d}V_x$
Volume of an infinitesimal part in deformed configuration	$\mathrm{d}V_y$
Infinitesimal (vector) area in reference configuration	$\mathrm{d}A_x\,\mathbf{n}_\mathrm{R}$
Infinitesimal (vector) area in deformed configuration	$\mathrm{d}A_y\,\mathbf{n}$
Deformation gradient tensor	$\mathbf{F} = \mathrm{Grad}\,\mathbf{y}$
Jacobian determinant	$\mathrm{J} = \det\mathbf{F}$
Displacement gradient tensor	$\mathbf{H} = \mathrm{Grad}\,\mathbf{u}$
Right (Lagrangian) stretch tensor	$\mathbf{U}$
Left (Eulerian) stretch tensor	$\mathbf{V}$
Rotation tensor	$\mathbf{R}$
Principal stretches	$\lambda_1, \lambda_2, \lambda_3$
Principal directions of Lagrangian stretch	$\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$
Principal directions of Eulerian stretch	$\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3$
Right Cauchy-Green deformation tensor	$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$
Left Cauchy-Green deformation tensor	$\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T$
Principal scalar invariants of $\mathbf{C}$	$\mathrm{I}_1(\mathbf{C}), \mathrm{I}_2(\mathbf{C}), \mathrm{I}_3(\mathbf{C})$
General Lagrangian strain tensor	$\mathbf{E}^{(n)}$
Green Saint-Venant strain tensor	$\mathbf{E}$
Particle velocity	$\mathbf{v}$
Velocity gradient tensor	$\mathbf{L} = \mathrm{grad}\,\mathbf{v}(\mathbf{y}, t)$

Table 2: Mechanics

Quantity	**Symbol**
Cauchy (true) traction vector	$\mathbf{t}(\mathbf{y}, \mathbf{n})$
Normal stress	T_{normal}
Magnitude of resultant shear stress	T_{shear}
Cauchy (true) stress tensor field	$\mathbf{T}(\mathbf{y})$
Principal Cauchy stresses	τ_1, τ_2, τ_3
Principal directions of Cauchy stress	$\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3$
Piola traction vector	$\mathbf{s}(\mathbf{x}, \mathbf{n}_\mathrm{R})$
Piola stress tensor field	$\mathbf{S}(\mathbf{x})$
Stress tensor work conjugate to strain tensor $\mathbf{E}^{(n)}$	$\mathbf{S}^{(n)}$
Biot stress tensor	$\mathbf{S}^{(1)}$
2nd Piola-Kirchhoff stress tensor	$\mathbf{S}^{(2)}$
Body force per unit deformed volume	$\mathbf{b}$
Body force per unit reference volume	$\mathbf{b}_\mathrm{R}$
Mass density in deformed configuration	ρ
Mass density in reference configuration	ρ_R

Table 3: Constitutive Description

Quantity	**Symbol**
Strain energy function	$\widehat{W}(\mathbf{F})$
Strain energy function	$\overline{W}(\mathbf{C})$
Strain energy function	$\widetilde{W}(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}))$
Strain energy function	$W^*(\lambda_1, \lambda_2, \lambda_3)$
Reactive stress due to an internal material constraint	$\mathbf{N}$
Reactive pressure due to incompressibility constraint	q
Fiber directions in anisotropic material	$\mathbf{m}_\mathrm{R}, \mathbf{m}$
Structural tensor for anisotropic material	$\mathbf{M}$
Additional invariants for anisotropic material	$I_4(\mathbf{C}, \mathbf{M}), I_5(\mathbf{C}, \mathbf{M}), \ldots$

Some Useful Formulae

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1. Mathematical background.

$$\text{Orthogonal matrix } [Q]: \qquad Q_{ik}Q_{jk} = Q_{ki}Q_{kj} = \delta_{ij} \tag{1.1}$$

$$\mathbb{T}_{pqr\ldots i\ldots z}\,\delta_{ij} = \mathbb{T}_{pqr\ldots j\ldots z} \tag{1.2}$$

$$\det[A] = e_{ijk}A_{1i}A_{2j}A_{3k} = e_{ijk}A_{i1}A_{j2}A_{k3} = \frac{1}{6}e_{ijk}e_{pqr}A_{ip}A_{jq}A_{kr}. \tag{1.3}$$

$$e_{pqr}\det[A] \ = \ e_{ijk}A_{ip}A_{jq}A_{kr}. \tag{1.4}$$

$$e_{pij}e_{pk\ell} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}. \tag{1.5}$$

$$e_{ijk} = -e_{jik}, \qquad e_{ijk} = -e_{ikj} \tag{1.6}$$

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} \tag{1.7}$$

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}} \tag{1.8}$$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \tag{1.9}$$

$$|\mathbf{u}| = 0 \qquad \Leftrightarrow \qquad \mathbf{u} = \mathbf{o} \tag{1.10}$$

$$\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u} \tag{1.11}$$

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,\sin\theta, \qquad (\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0 \tag{1.12}$$

$$\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}. \tag{1.13}$$

$$\mathbf{e}_i\times\mathbf{e}_j = e_{ijk}\,\mathbf{e}_k. \tag{1.14}$$

$$\mathbf{v} \ = \ v_i\mathbf{e}_i, \qquad v_i = \mathbf{v}\cdot\mathbf{e}_i \tag{1.15}$$

$$\mathbf{u}\cdot\mathbf{v} = u_iv_i. \tag{1.16}$$

$$|\mathbf{u}| = (\mathbf{u}\cdot\mathbf{u})^{1/2} = (u_ku_k)^{1/2}. \tag{1.17}$$

$$(\mathbf{u}\times\mathbf{v})_i = e_{ijk}u_jv_k. \tag{1.18}$$

$$\mathbf{I}\,\mathbf{u} = \mathbf{u}, \qquad \mathbf{0}\mathbf{u} = \mathbf{o} \qquad \text{for all vectors } \mathbf{u} \tag{1.19}$$

$$\mathbf{A}\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{A}^T\mathbf{v} \qquad \text{for all vectors } \mathbf{u},\mathbf{v} \tag{1.20}$$

$$(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T \tag{1.21}$$

$$\text{Symmetric tensor } \mathbf{A}: \qquad \mathbf{A} = \mathbf{A}^T \tag{1.22}$$

$$\text{Skew-symmetric tensor } \mathbf{A}: \qquad \mathbf{A} = -\mathbf{A}^T, \tag{1.23}$$

$$\text{Positive definite tensor } \mathbf{A}: \qquad \mathbf{A}\mathbf{u}\cdot\mathbf{u} > 0 \qquad \text{for all vectors } \mathbf{u} \neq \mathbf{0} \tag{1.24}$$

$$\text{Nonsingular tensor } \mathbf{A}: \qquad \mathbf{A}\mathbf{u} = \mathbf{o} \quad \text{if and only if} \quad \mathbf{u} = \mathbf{o} \tag{1.25}$$

$$\text{Nonsingular tensor } \mathbf{A}: \qquad \mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{1.26}$$

$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{1.27}$$

$$\text{Orthogonal tensor } \mathbf{Q}: \qquad \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{1.28}$$

$$\text{Orthogonal tensor } \mathbf{Q}: \qquad |\mathbf{Q}\mathbf{u}| = |\mathbf{u}| \qquad \text{for all vectors } \mathbf{u} \tag{1.29}$$

$$\mathbf{A} = \mathbf{S} + \mathbf{W}, \qquad \mathbf{S} = \mathbf{S}^T = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T), \quad \mathbf{W} = -\mathbf{W}^T = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) \tag{1.30}$$

$$\text{Nonsingular tensor } \mathbf{F}: \quad \mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \quad \mathbf{R} \text{ orthogonal}, \quad \mathbf{U}, \mathbf{V} \text{ symmetric positive definite} \tag{1.31}$$

$$(\mathbf{a}\otimes\mathbf{b})\mathbf{x} = (\mathbf{x}\cdot\mathbf{b})\mathbf{a} \qquad \text{for all vectors } \mathbf{x} \tag{1.32}$$

$$(\mathbf{a}\otimes\mathbf{b})^T = \mathbf{b}\otimes\mathbf{a}, \qquad (\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\otimes\mathbf{d}) = (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\otimes\mathbf{d}). \tag{1.33}$$

$$\mathbf{A}(\mathbf{a}\otimes\mathbf{b}) = (\mathbf{A}\mathbf{a})\otimes\mathbf{b}, \qquad (\mathbf{a}\otimes\mathbf{b})\mathbf{A} = \mathbf{a}\otimes(\mathbf{A}^T\mathbf{b}). \tag{1.34}$$

$$\mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2 + \mathbf{e}_3\otimes\mathbf{e}_3 = \mathbf{I}. \tag{1.35}$$

$$\mathbf{A}\mathbf{e}_j = A_{ij}\,\mathbf{e}_i, \qquad A_{ij} = (\mathbf{A}\mathbf{e}_j)\cdot\mathbf{e}_i. \tag{1.36}$$

$$\mathbf{A} = A_{ij}\mathbf{e}_i\otimes\mathbf{e}_j. \tag{1.37}$$

$$\operatorname{tr}\mathbf{A} = \operatorname{tr}[A] = A_{ii}, \qquad \det\mathbf{A} = \det[A]. \tag{1.38}$$

$$\det(\mathbf{A}\mathbf{B}) = \det\mathbf{A}\;\det\mathbf{B}. \tag{1.39}$$

$$\det(\alpha\mathbf{A}) = \alpha^3\,\det\,\mathbf{A} \tag{1.40}$$

$$\operatorname{tr}(\mathbf{A}\mathbf{B}) = \operatorname{tr}(\mathbf{B}\mathbf{A}) \tag{1.41}$$

$$(\mathbf{F}\mathbf{a}\times\mathbf{F}\mathbf{b})\cdot\mathbf{F}\mathbf{c} = \det\mathbf{F}\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \tag{1.42}$$

$$\mathbf{F}\mathbf{a}\times\mathbf{F}\mathbf{b} = \det\mathbf{F}\,\mathbf{F}^{-T}(\mathbf{a}\times\mathbf{b}) \tag{1.43}$$

$$\mathbf{A}\cdot\mathbf{B} = \operatorname{tr}(\mathbf{A}\mathbf{B}^T) = A_{ij}B_{ij}. \tag{1.44}$$

$$\mathbf{A}\mathbf{B}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{A}^T\mathbf{C} = \mathbf{A}\cdot\mathbf{C}\mathbf{B}^T, \tag{1.45}$$

$$|\mathbf{A}| = (\mathbf{A}\cdot\mathbf{A})^{1/2} = \left[\operatorname{tr}\,(\mathbf{A}\mathbf{A}^T)\right]^{1/2} = (A_{ij}A_{ij})^{1/2}, \tag{1.46}$$

$$(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j. \tag{1.47}$$

$$(\mathbf{A}\mathbf{e}_i)\otimes\mathbf{e}_i = \mathbf{A}. \tag{1.48}$$

$$\operatorname{tr}(\mathbf{a}\otimes\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} \tag{1.49}$$

$$\det(\mathbf{I} + \mathbf{a}\otimes\mathbf{b}) = 1 + \mathbf{a}\cdot\mathbf{b} \tag{1.50}$$

$$(\mathbf{I}+\mathbf{a}\otimes\mathbf{b})^{-1}=1-\frac{\mathbf{a}\otimes\mathbf{b}}{1+\mathbf{a}\cdot\mathbf{b}} \qquad (\text{provided } \mathbf{a}\cdot\mathbf{b}\neq -1) \tag{1.51}$$

$$v'_i = Q_{ij}v_j, \qquad \{v'\}=[Q]\{v\} \qquad \text{where } Q_{ij}=\mathbf{e}'_i\cdot\mathbf{e}_j \tag{1.52}$$

$$A'_{ij}=Q_{ip}Q_{jq}A_{pq}, \qquad \{A'\}=[Q][A][Q]^T \qquad \text{where } Q_{ij}=\mathbf{e}'_i\cdot\mathbf{e}_j \tag{1.53}$$

$$\det(\mathbf{A}-\mu\mathbf{I}) \;=\; -\mu^3+I_1(\mathbf{A})\mu^2-I_2(\mathbf{A})\mu+I_3(\mathbf{A}) \tag{1.54}$$

where

$$I_1(\mathbf{A})=\mathrm{tr}(\mathbf{A}), \qquad I_2(\mathbf{A})=\frac{1}{2}[(\mathrm{tr}(\mathbf{A}))^2-\mathrm{tr}(\mathbf{A}^2)], \qquad I_3(\mathbf{A})=\det(\mathbf{A}). \tag{1.55}$$

Eigenvalues and eigenvectors

$$\mathbf{A}\mathbf{a}=\alpha\mathbf{a} \tag{1.56}$$

$$\text{Symmetric tensor } \mathbf{S}: \qquad \mathbf{S}=\sigma_1\mathbf{s}_1\otimes\mathbf{s}_1+\sigma_2\mathbf{s}_2\otimes\mathbf{s}_2+\sigma_3\mathbf{s}_3\otimes\mathbf{s}_3 \tag{1.57}$$

$$\frac{dJ}{dt}=J\,\mathbf{F}^{-T}\cdot\frac{d\mathbf{F}}{dt}, \qquad \mathrm{J}(t)=\det\mathbf{F}(t) \tag{1.58}$$

$$\frac{\partial \mathrm{J}}{\partial\mathbf{F}}=\mathrm{J}\mathbf{F}^{-\mathrm{T}}, \qquad \mathrm{J}(\mathbf{F})=\det\mathbf{F} \tag{1.59}$$

$$\frac{d}{dt}(\mathbf{F}^{-1})=-\mathbf{F}^{-1}\,\frac{d\mathbf{F}}{dt}\;\mathbf{F}^{-1}, \qquad \mathbf{F}=\mathbf{F}(t) \tag{1.60}$$

If $W(\mathbf{C})$ is defined for all symmetric tensors $\mathbf{C}$ then

$$\left(\frac{\partial W}{\partial\mathbf{C}}\right)_{ij}=\frac{1}{2}\left(\frac{\partial W}{\partial C_{ij}}+\frac{\partial W}{\partial C_{ji}}\right) \tag{1.61}$$

$$\big(\nabla\phi\big)_i=(\mathrm{grad}\,\phi)_i=\frac{\partial\phi}{\partial x_i} \tag{1.62}$$

$$\phi(\mathbf{x}+\delta\mathbf{x})=\phi(\mathbf{x})+(\nabla\phi)\cdot\delta\mathbf{x}+o(|\delta\mathbf{x}|) \quad \text{as } |\delta\mathbf{x}|\to 0. \tag{1.63}$$

$$\big(\nabla\mathbf{v}\big)_{ij}=(\mathrm{grad}\,\mathbf{v})_{ij}=\frac{\partial v_i}{\partial x_j} \tag{1.64}$$

$$\mathbf{v}(\mathbf{x}+\delta\mathbf{x})=\mathbf{v}(\mathbf{x})+(\nabla\mathbf{v})\delta\mathbf{x}+o(|\delta\mathbf{x}|) \quad \text{as } |\delta\mathbf{x}|\to 0. \tag{1.65}$$

$$\mathrm{div}\,\mathbf{v}=\frac{\partial v_i}{\partial x_i}=\mathrm{tr}\,(\nabla\mathbf{v}) \tag{1.66}$$

$$\big(\mathrm{curl}\;\mathbf{v}\big)_i=e_{ijk}\frac{\partial v_k}{\partial x_j}. \tag{1.67}$$

$$\big(\mathrm{div}\;\mathbf{T}\big)_i=\frac{\partial T_{ij}}{\partial x_j}, \tag{1.68}$$

$$\big(\mathrm{curl}\ \mathbf{T}\big)_{ij} = e_{ipq}\frac{\partial T_{jq}}{\partial x_p}. \tag{1.69}$$

$$\int_{\partial\mathcal{R}} T_{jk\ldots z}\, n_i\ dA = \int_{\mathcal{R}} \frac{\partial}{\partial x_i} T_{jk\ldots z}\ \ dV \tag{1.70}$$

$$\nabla\phi = \frac{\partial\phi}{\partial R}\,\mathbf{e}_R + \frac{1}{R}\,\frac{\partial\phi}{\partial\Theta}\,\mathbf{e}_\Theta + \frac{\partial\phi}{\partial Z}\,\mathbf{e}_Z. \tag{1.71}$$

$$\begin{aligned}
\nabla\mathbf{u} \quad = \quad & \frac{\partial u_R}{\partial R}\,(\mathbf{e}_R\otimes\mathbf{e}_R) + \frac{\partial u_\Theta}{\partial R}\,(\mathbf{e}_\Theta\otimes\mathbf{e}_R) + \frac{\partial u_Z}{\partial R}\,(\mathbf{e}_Z\otimes\mathbf{e}_R) + \\
+ \quad & \frac{1}{R}\left(\frac{\partial u_R}{\partial\Theta} - u_\Theta\right)(\mathbf{e}_R\otimes\mathbf{e}_\Theta) + \frac{1}{R}\left(\frac{\partial u_\Theta}{\partial\Theta} + u_R\right)(\mathbf{e}_\Theta\otimes\mathbf{e}_\Theta) + \frac{1}{R}\frac{\partial u_Z}{\partial\Theta}\,(\mathbf{e}_Z\otimes\mathbf{e}_\Theta) + \\
+ \quad & \frac{\partial u_R}{\partial Z}\,(\mathbf{e}_R\otimes\mathbf{e}_Z) + \frac{\partial u_\Theta}{\partial Z}\,(\mathbf{e}_\Theta\otimes\mathbf{e}_Z) + \frac{\partial u_Z}{\partial Z}\,(\mathbf{e}_Z\otimes\mathbf{e}_Z).
\end{aligned} \tag{1.72}$$

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2. Kinematics

$$\text{Deformation :} \qquad \mathbf{y} = \mathbf{y}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}) \tag{2.1}$$

$$\mathbf{F} = \nabla \mathbf{y} \qquad F_{ij} = \frac{\partial y_i}{\partial x_j} \tag{2.2}$$

$$d\mathbf{y} = \mathbf{F} d\mathbf{x}, \tag{2.3}$$

$$\lambda(\mathbf{m}_{\mathrm{R}}) = |\mathbf{F}\mathbf{m}_{\mathrm{R}}| \tag{2.4}$$

$$dV_y = J \; dV_x, \qquad J = \det \; \mathbf{F} \tag{2.5}$$

$$\mathrm{d}A_y \; \mathbf{n} = \mathrm{d}A_x \; J \, \mathbf{F}^{-T} \; \mathbf{n}_{\mathrm{R}} \; . \tag{2.6}$$

$$\mathbf{F} = \mathbf{R}\,\mathbf{U}, \qquad \mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}, \qquad \mathbf{R} = \mathbf{F}\mathbf{U}^{-1}. \tag{2.7}$$

$$\mathbf{F} = \mathbf{V}\mathbf{R}, \qquad \mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T} \qquad \mathbf{R} = \mathbf{V}^{-1}\mathbf{F}. \tag{2.8}$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 \tag{2.9}$$

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{r}_i \otimes \mathbf{r}_i, \qquad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i; \tag{2.10}$$

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \boldsymbol{\ell}_i \otimes \mathbf{r}_i, \qquad \mathbf{R} = \sum_{i=1}^{3} \boldsymbol{\ell}_i \otimes \mathbf{r}_i. \tag{2.11}$$

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \, (\mathbf{r}_i \otimes \mathbf{r}_i), \qquad \mathbf{B} = \sum_{i=1}^{3} \lambda_i^2 \, (\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i). \tag{2.12}$$

$$I_1(\mathbf{C}) = \operatorname{tr}\mathbf{C}, \qquad I_2(\mathbf{C}) = \frac{1}{2}\Big[\big(\operatorname{tr}\mathbf{C}\big)^2 - \operatorname{tr}\mathbf{C}^2\Big], \qquad I_3(\mathbf{C}) = \det\mathbf{C}, \tag{2.13}$$

$$I_k(\mathbf{C}) = I_k(\mathbf{Q}\mathbf{C}\mathbf{Q}^T) \tag{2.14}$$

$$I_1(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2(\mathbf{C}) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \qquad I_3(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.15}$$

$$\mathbf{E}(\mathbf{U}) = e(\lambda_1)\,\mathbf{r}_1 \otimes \mathbf{r}_1 \; + \; e(\lambda_2)\,\mathbf{r}_2 \otimes \mathbf{r}_2 \; + \; e(\lambda_3)\,\mathbf{r}_3 \otimes \mathbf{r}_3. \tag{2.16}$$

$$\mathbf{E}^{(n)} = \frac{1}{n}\big(\mathbf{U}^n - \mathbf{I}\big) \tag{2.17}$$

$$\text{Biot strain tensor:} \qquad \mathbf{E}^{(1)} = \mathbf{U} - \mathbf{I} \tag{2.18}$$

$$\text{Green-Saint Venant strain tensor:} \qquad \mathbf{E}^{(2)} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) \tag{2.19}$$

$$\mathcal{E}(\mathbf{V}) = e(\lambda_1)\,\boldsymbol{\ell}_1 \otimes \boldsymbol{\ell}_1 \; + \; e(\lambda_2)\,\boldsymbol{\ell}_2 \otimes \boldsymbol{\ell}_2 \; + \; e(\lambda_3)\,\boldsymbol{\ell}_3 \otimes \boldsymbol{\ell}_3. \tag{2.20}$$

$$\text{Piola identity:} \qquad \operatorname{Div}\Big(J\,\mathbf{F}^{-\mathrm{T}}\Big) = \mathbf{o}, \qquad \frac{\partial}{\partial x_j}\big(JF_{ji}^{-1}\big) = 0 \tag{2.21}$$

$$\text{Piola identity:} \qquad \operatorname{div}\Big(J^{-1}\,\mathbf{F}^{\mathrm{T}}\Big) = \mathbf{o}, \qquad \frac{\partial}{\partial y_j}\big(J^{-1}F_{ji}\big) = 0 \tag{2.22}$$

$$
\begin{aligned}
\mathbf{F} \;&= \frac{\partial r}{\partial R}(\mathbf{e}_r \otimes \mathbf{e}_R) + \frac{1}{R}\frac{\partial r}{\partial \Theta}(\mathbf{e}_r \otimes \mathbf{e}_\Theta) + \frac{\partial r}{\partial Z}(\mathbf{e}_r \otimes \mathbf{e}_Z) + \\
&+ r\frac{\partial \theta}{\partial R}(\mathbf{e}_\theta \otimes \mathbf{e}_R) + \frac{r}{R}\frac{\partial \theta}{\partial \Theta}(\mathbf{e}_\theta \otimes \mathbf{e}_\Theta) + r\frac{\partial \theta}{\partial Z}(\mathbf{e}_\theta \otimes \mathbf{e}_Z) + \\
&+ \frac{\partial z}{\partial R}(\mathbf{e}_z \otimes \mathbf{e}_R) \;+ \frac{1}{R}\frac{\partial z}{\partial \Theta}(\mathbf{e}_z \otimes \mathbf{e}_\Theta) \;+ \frac{\partial z}{\partial Z}(\mathbf{e}_z \otimes \mathbf{e}_Z)\,.
\end{aligned}
\tag{2.23}
$$

$$
\begin{aligned}
\mathbf{B} = \quad & B_{rr}\mathbf{e}_r \otimes \mathbf{e}_r \;+\; B_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta \;+\; B_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \;+ \\
& + B_{r\theta}(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) \;+\; B_{rz}(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) \;+ \\
& + B_{\theta z}(\mathbf{e}_z \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_z),
\end{aligned}
\tag{2.24}
$$

where

$$
\left.
\begin{aligned}
B_{rr} \;&= \left(\frac{\partial r}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial r}{\partial \Theta}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2, \\
B_{\theta\theta} \;&= r^2\left[\left(\frac{\partial \theta}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial \theta}{\partial \Theta}\right)^2 + \left(\frac{\partial \theta}{\partial Z}\right)^2\right], \\
B_{zz} \;&= \left(\frac{\partial z}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial z}{\partial \Theta}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2, \\
B_{r\theta} \;&= B_{\theta r} = r\left[\frac{\partial r}{\partial R}\frac{\partial \theta}{\partial R} + \frac{1}{R^2}\frac{\partial r}{\partial \Theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial r}{\partial Z}\frac{\partial \theta}{\partial Z}\right], \\
B_{rz} \;&= B_{zr} = \frac{\partial r}{\partial R}\frac{\partial z}{\partial R} + \frac{1}{R^2}\frac{\partial r}{\partial \Theta}\frac{\partial z}{\partial \Theta} + \frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z}, \\
B_{\theta z} \;&= B_{z\theta} = r\left[\frac{\partial \theta}{\partial R}\frac{\partial z}{\partial R} + \frac{1}{R^2}\frac{\partial z}{\partial \Theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \theta}{\partial Z}\frac{\partial z}{\partial Z}\right].
\end{aligned}
\right\}
\tag{2.25}
$$

$$
\begin{aligned}
\mathbf{F} \;&= \frac{\partial r}{\partial R}(\mathbf{e}_r \otimes \mathbf{e}_R) + \frac{1}{R}\frac{\partial r}{\partial \Theta}(\mathbf{e}_r \otimes \mathbf{e}_\Theta) + \frac{1}{R\sin\Theta}\frac{\partial r}{\partial \Phi}(\mathbf{e}_r \otimes \mathbf{e}_\Phi) + \\
&+ r\frac{\partial \theta}{\partial R}(\mathbf{e}_\theta \otimes \mathbf{e}_R) + \frac{r}{R}\frac{\partial \theta}{\partial \Theta}(\mathbf{e}_\theta \otimes \mathbf{e}_\Theta) + \frac{r}{R\sin\Theta}\frac{\partial \theta}{\partial \Phi}(\mathbf{e}_\theta \otimes \mathbf{e}_\Phi) + \\
&+ r\sin\theta\frac{\partial \varphi}{\partial R}(\mathbf{e}_\varphi \otimes \mathbf{e}_R) \;+ \frac{r\sin\theta}{R}\frac{\partial \varphi}{\partial \Theta}(\mathbf{e}_\varphi \otimes \mathbf{e}_\Theta) \;+ \frac{r\sin\theta}{R\sin\Theta}\frac{\partial \varphi}{\partial \Phi}(\mathbf{e}_\varphi \otimes \mathbf{e}_\Phi)\,.
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
\mathbf{B} = \quad & B_{rr}\mathbf{e}_r \otimes \mathbf{e}_r \;+\; B_{\vartheta\vartheta}\mathbf{e}_\vartheta \otimes \mathbf{e}_\vartheta \;+\; B_{\varphi\varphi}\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi \;+ \\
& + B_{r\vartheta}(\mathbf{e}_r \otimes \mathbf{e}_\vartheta + \mathbf{e}_\vartheta \otimes \mathbf{e}_r) \;+\; B_{r\varphi}(\mathbf{e}_r \otimes \mathbf{e}_\varphi + \mathbf{e}_\varphi \otimes \mathbf{e}_r) \;+ \\
& + B_{\vartheta\varphi}(\mathbf{e}_\varphi \otimes \mathbf{e}_\vartheta + \mathbf{e}_\vartheta \otimes \mathbf{e}_\varphi),
\end{aligned}
\tag{2.27}
$$

where

$$\left.\begin{aligned}
B_{rr} &= \left(\frac{\partial r}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial r}{\partial \Theta}\right)^2 + \frac{1}{R^2 \sin^2\Theta}\left(\frac{\partial r}{\partial \Phi}\right)^2, \\
B_{\vartheta\vartheta} &= r^2\left[\left(\frac{\partial \vartheta}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial \vartheta}{\partial \Theta}\right)^2 + \frac{1}{R^2 \sin^2\Theta}\left(\frac{\partial \vartheta}{\partial \Phi}\right)^2\right], \\
B_{\varphi\varphi} &= r^2\,\sin^2\,\vartheta\left[\left(\frac{\partial \varphi}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial \varphi}{\partial \Theta}\right)^2 + \frac{1}{R^2 \sin^2\Theta}\left(\frac{\partial \varphi}{\partial \Phi}\right)^2\right] \\
B_{r\vartheta} &= B_{\vartheta r} = r\left[\frac{\partial r}{\partial R}\frac{\partial \vartheta}{\partial R} + \frac{1}{R^2}\frac{\partial r}{\partial \Theta}\frac{\partial \vartheta}{\partial \Theta} + \frac{1}{R^2 \sin^2\Theta}\frac{\partial r}{\partial \Phi}\frac{\partial \vartheta}{\partial \Phi}\right], \\
B_{r\varphi} &= B_{\varphi r} = r\,\sin\,\vartheta\left[\frac{\partial r}{\partial R}\frac{\partial \varphi}{\partial R} + \frac{1}{R^2}\frac{\partial r}{\partial \Theta}\frac{\partial \varphi}{\partial \Theta} + \frac{1}{R^2 \sin^2\Theta}\frac{\partial r}{\partial \Phi}\frac{\partial \varphi}{\partial \Phi}\right], \\
B_{\vartheta\varphi} &= B_{\varphi\vartheta} = r^2\,\sin\,\vartheta\left[\frac{\partial \vartheta}{\partial R}\frac{\partial \varphi}{\partial R} + \frac{1}{R^2}\frac{\partial \varphi}{\partial \Theta}\frac{\partial \vartheta}{\partial \Theta} + \frac{1}{R^2 \sin^2\Theta}\frac{\partial \vartheta}{\partial \Phi}\frac{\partial \varphi}{\partial \Phi}\right].
\end{aligned}\right\} \tag{2.28}$$

---

3. Traction, stress, equilibrium.

$$\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n} \tag{3.1}$$

$$T_{ij} = t_i(\mathbf{e}_j) = \mathbf{T}\mathbf{e}_j \cdot \mathbf{e}_i, \tag{3.2}$$

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \qquad \mathbf{T} = \sum_{i=1}^{3} \tau_i\, \mathbf{t}_i \otimes \mathbf{t}_i. \tag{3.3}$$

$$T_{\mathrm{normal}}(\mathbf{n}) = \mathbf{t}(\mathbf{n}) \cdot \mathbf{n} \; = \mathbf{T}\mathbf{n} \cdot \mathbf{n} = T_{ij}n_in_j \; = \tau_1\, n_1^2 + \tau_2\, n_2^2 + \tau_3\, n_3^2 \tag{3.4}$$

$$T_{\mathrm{shear}}(\mathbf{n}) = \sqrt{|\mathbf{t}|^2 - T_{\mathrm{normal}}^2} = \sqrt{\big[\mathbf{t}(\mathbf{n}) \cdot \mathbf{t}(\mathbf{n})\big] - \big[\mathbf{t}(\mathbf{n}) \cdot \mathbf{n}\big]^2}. \tag{3.5}$$

$$T_{\mathrm{shear}}^2 = |\mathbf{t}(\mathbf{n})|^2 - T_{\mathrm{normal}}^2 = \tau_1^2 n_1^2 + \tau_2^2 n_2^2 + \tau_3^2 n_3^2 - (\tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2)^2. \tag{3.6}$$

$$\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{0}, \qquad \big(\operatorname{div}\mathbf{T}\big)_i = \frac{\partial T_{ij}}{\partial y_j} \tag{3.7}$$

$$\frac{\partial T_{ij}}{\partial y_j} + b_i \;=\; 0\;, \tag{3.8}$$

$$\mathbf{T} = \mathbf{T}^T \tag{3.9}$$

$$\text{Pure shear stress:} \quad \mathbf{T} = \tau(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}) \tag{3.10}$$

$$\text{Uniaxial stress:} \quad \mathbf{T} = \sigma\; \mathbf{m} \otimes \mathbf{m} \tag{3.11}$$

$$\mathbf{s}\, dA_x = \mathbf{t}\, dA_y \tag{3.12}$$

$$\mathbf{s} = \mathbf{S}\mathbf{n}_{\mathrm{R}} \tag{3.13}$$

$$\mathbf{S}\mathbf{n}_{\mathrm{R}}\, dA_x = \mathbf{T}\mathbf{n}\, dA_y \tag{3.14}$$

$$\text{Piola stress tensor:} \qquad \mathbf{S} = J\mathbf{T}\mathbf{F}^{-T} \tag{3.15}$$

$$\mathbf{S} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \qquad S_{ij} = s_i(\mathbf{e}_j) \tag{3.16}$$

$$\operatorname{Div}\mathbf{S} + \mathbf{b}_{\mathrm{R}} = \mathbf{0}, \qquad \big(\operatorname{Div}\mathbf{S}\big)_i = \frac{\partial S_{ij}}{\partial x_j} \tag{3.17}$$

$$\frac{\partial S_{ij}}{\partial x_j} + b_i^{\mathrm{R}} \;=\; 0\;, \tag{3.18}$$

$$\mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T \tag{3.19}$$

$$\text{Stress power density} = \mathbf{S} \cdot \dot{\mathbf{F}} = J\; \mathbf{T} \cdot \mathbf{D} \tag{3.20}$$

$$\mathbf{L} = \operatorname{grad}\mathbf{v}, \qquad L_{ij} = \frac{\partial v_i}{\partial y_j}. \tag{3.21}$$

$$\mathbf{D} = \frac{1}{2}\left(\operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^T\right), \qquad D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right) \tag{3.22}$$

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F} \tag{3.23}$$

$$\text{Biot stress tensor:} \qquad \mathbf{S}^{(1)} = \frac{1}{2}(\mathbf{S}\mathbf{R} + \mathbf{R}^T\mathbf{S}) \tag{3.24}$$

$$2^{\text{nd}}\ \text{Piola-Kirchhoff stress tensor:} \qquad \mathbf{S}^{(2)} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} = \mathbf{F}^{-1}\mathbf{S} \tag{3.25}$$

$$
\begin{aligned}
\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r} + b_r &= \rho a_r, \\
\frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{T_{r\theta} + T_{\theta r}}{r} + b_\theta &= \rho a_\theta, \\
\frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r} + b_z &= \rho a_z \ ,
\end{aligned}
\tag{3.26}
$$

$$
\begin{aligned}
\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\phi}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{2T_{rr} - T_{\phi\phi} - T_{\theta\theta} + T_{r\phi}\cot\phi}{r} + b_r &= \rho a_r, \\
\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\theta}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{3T_{r\theta} + 2T_{\theta\phi}\cot\phi}{r} + b_\theta &= \rho a_\theta, \\
\frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{\phi\theta}}{\partial \theta} + \frac{3T_{r\phi} + (T_{\phi\phi} - T_{\theta\theta})\cot\phi}{r} + b_\phi &= \rho a_\phi.
\end{aligned}
\tag{3.27}
$$

4. Constitutive relation.

$$\mathbf{S} \cdot \dot{\mathbf{F}} \;=\; \dot{W} \tag{4.1}$$

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}, \qquad \mathbf{T} = \mathrm{J}^{-1}\mathbf{S}\mathbf{F}^T = \frac{1}{J}\frac{\partial W}{\partial \mathbf{F}}\,\mathbf{F}^T. \tag{4.2}$$

$$\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{QF}) \tag{4.3}$$

$$\widehat{W}(\mathbf{F}) = \overline{W}(\mathbf{C}) \qquad \text{where } \mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{4.4}$$

$$\mathbf{S} = 2\mathbf{F}\,\frac{\partial \overline{W}}{\partial \mathbf{C}}, \qquad \mathbf{T} = \frac{2}{J}\,\mathbf{F}\,\frac{\partial \overline{W}}{\partial \mathbf{C}}\,\mathbf{F}^T. \tag{4.5}$$

$$\mathcal{G} = \{\mathbf{Q} : \;\mathbf{Q}\mathbf{Q}^T = \mathbf{I},\; \det \mathbf{Q} = 1,\; \widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{FQ}) \text{ for all nonsingular } \mathbf{F}\}.$$

$$\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{FQ}) \quad \text{for all } \mathbf{Q} \in \mathcal{G} \text{ and all nonsingular } \mathbf{F}. \tag{4.6}$$

$$\overline{W}(\mathbf{C}) = \overline{W}\big(\mathbf{QCQ}^T\big) \quad \text{for all } \mathbf{Q} \in \mathcal{G} \text{ and all symmetric positive definite } \mathbf{C}. \tag{4.7}$$

$$I_1(\mathbf{C}) = \operatorname{tr}\mathbf{C}, \qquad I_2(\mathbf{C}) = \frac{1}{2}\big[\big(\operatorname{tr}\mathbf{C}\big)^2 - \operatorname{tr}\mathbf{C}^2\big], \qquad I_3(\mathbf{C}) = \det\,\mathbf{C} \tag{4.8}$$

$$W = \widetilde{W}\big(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C})\big) \tag{4.9}$$

$$\left.\begin{aligned} \mathbf{T} \;&=\; 2J\frac{\partial \widetilde{W}}{\partial I_3}\,\mathbf{I} \;+\; \frac{2}{J}\left[\frac{\partial \widetilde{W}}{\partial I_1} + I_1\frac{\partial \widetilde{W}}{\partial I_2}\right]\mathbf{B} \;-\; \frac{2}{J}\frac{\partial \widetilde{W}}{\partial I_2}\mathbf{B}^2, \\ \mathbf{S} \;&=\; 2I_3\frac{\partial \widetilde{W}}{\partial I_3}\,\mathbf{F}^{-T} \;+\; 2\left[\frac{\partial \widetilde{W}}{\partial I_1} + I_1\frac{\partial \widetilde{W}}{\partial I_2}\right]\mathbf{F} \;-\; 2\frac{\partial \widetilde{W}}{\partial I_2}\mathbf{BF}. \end{aligned}\right\} \tag{4.10}$$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \qquad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{4.11}$$

$$W = W^*(\lambda_1, \lambda_2, \lambda_3) \tag{4.12}$$

$$W^*(\lambda_1, \lambda_2, \lambda_3) = W^*(\lambda_2, \lambda_1, \lambda_3) = W^*(\lambda_1, \lambda_3, \lambda_2) = \ldots. \tag{4.13}$$

$$\mathbf{T} = \tau_1\,\boldsymbol{\ell}_1 \otimes \boldsymbol{\ell}_1 + \tau_2\,\boldsymbol{\ell}_2 \otimes \boldsymbol{\ell}_2 + \tau_3\,\boldsymbol{\ell}_3 \otimes \boldsymbol{\ell}_3 \tag{4.14}$$

$$\tau_1 = \frac{\lambda_1}{\lambda_1\lambda_2\lambda_3}\,\frac{\partial W^*}{\partial \lambda_1}, \qquad \tau_2 = \frac{\lambda_2}{\lambda_1\lambda_2\lambda_3}\,\frac{\partial W^*}{\partial \lambda_2}, \qquad \tau_3 = \frac{\lambda_3}{\lambda_1\lambda_2\lambda_3}\,\frac{\partial W^*}{\partial \lambda_3}. \tag{4.15}$$

$$\mathbf{S} = \sigma_1\,\boldsymbol{\ell}_1 \otimes \mathbf{r}_1 + \sigma_2\,\boldsymbol{\ell}_2 \otimes \mathbf{r}_2 + \sigma_3\,\boldsymbol{\ell}_3 \otimes \mathbf{r}_3 \tag{4.16}$$

$$\sigma_1 = \frac{\partial W^*}{\partial \lambda_1}, \qquad \sigma_2 = \frac{\partial W^*}{\partial \lambda_2}, \qquad \sigma_3 = \frac{\partial W^*}{\partial \lambda_3}. \tag{4.17}$$

$$\tau_i = \lambda_i\sigma_i/J \qquad (\text{no sum on } i) \tag{4.18}$$

$$\dot{\phi} = \frac{\partial \phi}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} = 0, \tag{4.19}$$

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - q\,\frac{\partial \phi}{\partial \mathbf{F}}, \qquad \mathbf{T} = \frac{1}{J}\,\frac{\partial W}{\partial \mathbf{F}}\,\mathbf{F}^T - \frac{q}{J}\,\frac{\partial \phi}{\partial \mathbf{F}}\,\mathbf{F}^T. \tag{4.20}$$

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - q\mathbf{F}^{-T}, \qquad \mathbf{T} = \frac{\partial W}{\partial \mathbf{F}}\mathbf{F}^T - q\mathbf{I} \tag{4.21}$$

$$\mathbf{T} = -q\,\mathbf{I} \;+\; 2\left[\frac{\partial \widetilde{W}}{\partial I_1} + I_1\frac{\partial \widetilde{W}}{\partial I_2}\right]\mathbf{B} \;-\; 2\frac{\partial \widetilde{W}}{\partial I_2}\mathbf{B}^2, \tag{4.22}$$

$$\mathbf{S} = -q\,\mathbf{F}^{-T} \;+\; 2\left[\frac{\partial\widetilde{W}}{\partial I_1} + I_1\frac{\partial\widetilde{W}}{\partial I_2}\right]\mathbf{F} \;-\; 2\frac{\partial\widetilde{W}}{\partial I_2}\mathbf{BF}. \tag{4.23}$$

$$W = W^*(\lambda_1, \lambda_2, \lambda_3) \tag{4.24}$$

$$W^*(\lambda_1, \lambda_2, \lambda_3) = W^*(\lambda_2, \lambda_1, \lambda_3) = W^*(\lambda_1, \lambda_3, \lambda_2) = \ldots. \tag{4.25}$$

$$\tau_1 = \lambda_1\,\frac{\partial W^*}{\partial\lambda_1} - q, \qquad \tau_2 = \lambda_2\,\frac{\partial W^*}{\partial\lambda_2} - q, \qquad \tau_3 = \lambda_3\,\frac{\partial W^*}{\partial\lambda_3} - q. \tag{4.26}$$

$$\overline{W}(\mathbf{C},\mathbf{M}) = \widetilde{W}(I_1, I_2, I_3, I_4, I_5), \qquad W_i = \frac{\partial\widetilde{W}}{\partial I_i},\; i = 1,2,3,4,5. \tag{4.27}$$

$$I_4(\mathbf{C},\mathbf{M}) = \mathbf{C}\cdot\mathbf{M}, \qquad I_5(\mathbf{C},\mathbf{M}) = \mathbf{C}^2\cdot\mathbf{M}. \qquad \mathbf{M} = \mathbf{m}_{\mathrm{R}}\otimes\mathbf{m}_{\mathrm{R}} \tag{4.28}$$

$$\begin{aligned}\mathbf{T} &= 2JW_3\,\mathbf{I} \;+\; \frac{2}{J}\left[W_1 + I_1 W_2\right]\mathbf{B} \;-\; \frac{2}{J}W_2\mathbf{B}^2 + \\ &+\frac{2}{J}W_4(\mathbf{Fm}_{\mathrm{R}}\otimes\mathbf{Fm}_{\mathrm{R}}) \;+\; \frac{2}{J}W_5\left[(\mathbf{Fm}_{\mathrm{R}}\otimes\mathbf{BFm}_{\mathrm{R}}) + (\mathbf{BFm}_{\mathrm{R}}\otimes\mathbf{Fm}_{\mathrm{R}})\right],\end{aligned} \tag{4.29}$$

$$\overline{W}(\mathbf{C},\mathbf{M}) = \widetilde{W}(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8), \qquad W_i = \frac{\partial\widetilde{W}}{\partial I_i},\; i = 1,2,\ldots 8. \tag{4.30}$$

$$I_6 = \mathbf{Cm}'_{\mathrm{R}}\cdot\mathbf{m}'_{\mathrm{R}} \qquad I_7 = \mathbf{C}^2\mathbf{m}'_{\mathrm{R}}\cdot\mathbf{m}'_{\mathrm{R}}, \qquad I_8 = \mathbf{Cm}'_{\mathrm{R}}\cdot\mathbf{m}_{\mathrm{R}}. \tag{4.31}$$

$$\begin{aligned}\mathbf{T} &= -q\mathbf{I} + 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2) + \\ &+2W_4\mathbf{Fm}_{\mathrm{R}}\otimes\mathbf{Fm}_{\mathrm{R}} + 2W_6\mathbf{Fm}'_{\mathrm{R}}\otimes\mathbf{Fm}'_{\mathrm{R}} + \\ &+2W_5(\mathbf{Fm}_{\mathrm{R}}\otimes\mathbf{BFm}_{\mathrm{R}} + \mathbf{BFm}_{\mathrm{R}}\otimes\mathbf{Fm}_{\mathrm{R}}) + 2W_7(\mathbf{Fm}'_{\mathrm{R}}\otimes\mathbf{BFm}'_{\mathrm{R}} + \mathbf{BFm}'_{\mathrm{R}}\otimes\mathbf{Fm}'_{\mathrm{R}}) + \\ &+W_8(\mathbf{Fm}_{\mathrm{R}}\otimes\mathbf{Fm}'_{\mathrm{R}} + \mathbf{Fm}'_{\mathrm{R}}\otimes\mathbf{Fm}_{\mathrm{R}})\end{aligned} \tag{4.32}$$

$$\int_{\partial\mathcal{R}_t}\mathbf{t}\cdot\mathbf{v}\,dA_y + \int_{\mathcal{R}_t}\mathbf{b}\cdot\mathbf{v}\,dV_y = \frac{d}{dt}\int_{\mathcal{R}_t}\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\,dV_y + \frac{d}{dt}\int_{\mathcal{R}_{\mathrm{R}}}W\,dV_x \tag{4.33}$$

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10. Potential energy functional.

$$\Phi\{\mathbf{z}\} = \int_{\mathcal{R}_{\mathrm{R}}} W(\nabla\mathbf{z})\,dV_x - \int_{\mathcal{R}_{\mathrm{R}}} \mathbf{b}_R \cdot \mathbf{z}\,dV_x - \int_{\mathcal{S}_2} \widehat{\mathbf{s}} \cdot \mathbf{z}\,dA_x. \tag{10.1}$$

Contents

Chapter 1

BRIEF REVIEW OF MATHEMATICAL PRELIMINARIES

When studying the response of a body subjected to some loading, we will encounter entities such as displacement $\mathbf{u}$ and traction $\mathbf{t}$ that are vectors, and deformation gradient $\mathbf{F}$ and stress $\mathbf{S}$ that are tensors. We will need to carry out various calculations involving them that require us to use *vector and tensor algebra*. We will sometimes work with the components of these vectors and tensors in a basis, and these are represented as column and square *matrices* respectively, e.g. $\{u\}, \{t\}, [F]$ and $[S]$. Calculations involving matrices can often be carried out expeditiously using *indicial notation*. Finally suppose that a typical particle of an undeformed body is located at $\mathbf{x}$, and that the displacement of this particle is $\mathbf{u}$. Since the displacement varies from particle to particle, $\mathbf{u}$ will be a function of $\mathbf{x}$ and so we have the displacement *field* $\mathbf{u}(\mathbf{x})$. Characterizing how the displacement varies with position requires us to calculate the gradient of the displacement with respect to position, $\nabla\mathbf{u}$, and for this we must rely on *the calculus of vector and tensor fields.*

What follows is mostly a list of definitions and properties pertaining to the main mathematical concepts and methods that we will use in these notes. Some proofs are given in the worked examples and exercises. A more detailed treatment of this material can be found in Volume I as well as in the references listed at the end of this chapter. The reader who is familiar with the material in Chapters 1-6 of Volume I can skip this chapter entirely.

The four main topics to be reviewed are (not in this order)

- vector and tensor algebra,
- representation of vectors and tensors in terms of matrices (having chosen a basis),
- the use of indicial notation to simplify calculations, and the
- calculus of vector and tensor fields and of functions of tensors.

Links to some short introductory videos on indicial notation and tensor algebra can be found in Sections 1.2 and 1.3 respectively.

In this chapter we will (*almost*) always use the following convention regarding notation:

		Lowercase Greek letters:
α	.....	scalar
		Lowercase latin letters:
$\{a\}$	.....	3×1 column matrix
$\mathbf{a}$	.....	vector
a_i	.....	i^{th} component of the vector $\mathbf{a}$ in some basis; or
		i^{th} element of the column matrix $\{a\}$
		Uppercase latin letters:
$[A]$	.....	3×3 square matrix
$\mathbf{A}$	.....	second-order tensor (2-tensor) (linear transformation)
A_{ij}	.....	i, j component of the 2-tensor $\mathbf{A}$ in some basis; or
		i, j element of the square matrix $[A]$
		Blackboard bold letters:
$\mathbb{C}$	.....	fourth-order tensor (4-tensor)
$\mathbb{C}_{ijk\ell}$	.....	i, j, k, ℓ component of $\mathbb{C}$ in some basis

While we will closely follow this same notational convention in the *subsequent* chapters as well, there will be a few notable exceptions: for example, we will use the symbol W rather than a lower case Greek letter to denote the (scalar-valued) strain energy function.

1.1 Matrices.

Our discussion here is limited to 3×1 column matrices and 3×3 square matrices.

The element[1] in the ith row and jth column of a 3×3 matrix $[A]$ is denoted by A_{ij} and

[1] We speak of the *elements* of a matrix and the *components* of a vector or tensor.

the element in the ith row of a 3×1 column matrix $\{x\}$ is denoted by x_i:

$$[A] = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \qquad \{x\} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

– The magnitude of a column matrix $\{x\}$ is

$$|\{x\}| := (x_1^2 + x_2^2 + x_3^2)^{1/2}, \tag{1.1}$$

and the magnitude of a square matrix $[A]$ is

$$|[A]| := \left[A_{11}^2 + A_{12}^2 + A_{13}^2 + A_{21}^2 + \ldots + A_{33}^2\right]^{1/2}. \tag{1.2}$$

– The product of a 3×3 matrix $[A]$ with a 3×1 matrix $\{x\}$ is a third 3×1 matrix $\{y\}$ whose element in the ith row is the sum of the pairwise products of the elements in the ith row of $[A]$ and the elements of $\{x\}$:

$$\{y\} = [A]\{x\} \qquad \Rightarrow \qquad y_i = \sum_{k=1}^{3} A_{ik}x_k \qquad \text{for each } i = 1, 2, 3. \tag{1.3}$$

– The product of two matrices $[A]$ and $[B]$ is a third matrix $[C]$ whose element in the ith row and jth column is the sum of the pairwise products of the elements in the ith row of $[A]$ and the jth column of $[B]$:

$$[C] = [A][B] \qquad \Rightarrow \qquad C_{ij} = \sum_{k=1}^{3} A_{ik}B_{kj} \qquad \text{for each } i, j = 1, 2, 3. \tag{1.4}$$

While it may be preferable to say "for each $i = 1, 2, 3$ and each $j = 1, 2, 3$" we write it as above for brevity.

– The product of two matrices is not commutative in general: $[A][B] \neq [B][A]$.

– The *transpose* of a matrix $[A]$ is denoted by $[A]^T$. If the element in the ith row and jth column of $[A]$ is A_{ij}, then the element in the ith row and jth column of $[A]^T$ is A_{ji}.

– The transpose of the product of two matrices has the property

$$([A][B])^T = [B]^T[A]^T. \tag{1.5}$$

– A matrix is *symmetric* if

$$[A] = [A]^T, \qquad\qquad A_{ij} = A_{ji} \quad \text{for each } i, j = 1, 2, 3, \tag{1.6}$$

and *skew- (or anti)-symmetric* if

$$[A] = -[A]^T, \qquad\qquad A_{ij} = -A_{ji} \quad \text{for each } i, j = 1, 2, 3. \tag{1.7}$$

If $[A]$ is skew-symmetric, it follows from (1.7)$_2$ that $A_{11} = A_{22} = A_{33} = 0$ and $A_{12} = -A_{21}, A_{23} = -A_{32}, A_{31} = -A_{13}$. Therefore there are only three independent elements in a skew-symmetric matrix and so there is a one-to-one correspondence between skew-symmetric matrices and column matrices.

– Every matrix can be uniquely decomposed into the sum of a symmetric and skew-symmetric matrix:

$$\begin{aligned}
&[A] = [S] + [W] \qquad \text{where } [S] = \tfrac{1}{2}([A] + [A]^T), \quad [W] = \tfrac{1}{2}([A] - [A]^T);\\
&A_{ij} = S_{ij} + W_{ij} \qquad \text{where } S_{ij} = \tfrac{1}{2}(A_{ij} + A_{ji}), \quad W_{ij} = \tfrac{1}{2}(A_{ij} - A_{ji});
\end{aligned} \tag{1.8}$$

the second row of (1.8) holds for each $i, j = 1, 2, 3$ and so represents 9 scalar equations.

– The trace and determinant are two scalar-valued functions of a matrix that are encountered frequently. They are defined by

$$\text{tr } [A] := A_{11} + A_{22} + A_{33} = \sum_{k=1}^{3} A_{kk}, \tag{1.9}$$

$$\begin{aligned}
\det [A] \;\; := \;\; & A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + \\
& + A_{13}(A_{21}A_{32} - A_{22}A_{31}).
\end{aligned} \tag{1.10}$$

– The determinant of the product of two matrices equals the product of the individual determinants of the two matrices:

$$\det([A][B]) = \det[A] \det[B]. \tag{1.11}$$

The determinant of a matrix is unchanged by transposition:

$$\det\left([A]^T\right) = \det[A]. \tag{1.12}$$

– The *identity matrix* $[I]$ has the property

$$[A][I] = [I][A] = [A], \qquad [I]\{x\} = \{x\}$$

for all square matrices $[A]$ and column matrices $\{x\}$. Also, $\det[I] = 1$ and $\text{tr}[I] = 3$.

– A matrix $[A]$ is *nonsingular (or invertible)* if $\det[A] \neq 0$; *singular* if $\det[A] = 0$.

If $[A]$ is *nonsingular*, then the only column matrix $\{x\}$ for which $[A]\{x\} = \{0\}$ is $\{x\} = \{0\}$.

If $[A]$ is *nonsingular*, it is invertible in the sense that there is a matrix denoted by $[A]^{-1}$ and called the *inverse* of $[A]$ for which

$$[A][A]^{-1} = [I], \qquad [A]^{-1}[A] = [I]. \tag{1.13}$$

The inverse of the product of two nonsingular matrices obeys

$$\big([A][B]\big)^{-1} = [B]^{-1}[A]^{-1}. \tag{1.14}$$

– A matrix $[Q]$ is *orthogonal* if it is nonsingular and

$$[Q]^{-1} = [Q]^T. \tag{1.15}$$

It follows that

$$[Q][Q]^T = [I], \quad [Q]^T[Q] = [I], \tag{1.16}$$

$$\det\,[Q] = \pm 1. \tag{1.17}$$

An orthogonal matrix whose determinant is $+1$ is said to be *proper orthogonal* and represents a rotation. An orthogonal matrix whose determinant is -1 is said to be *improper orthogonal* and represents a reflection.

– If $\{y\}$ is a 3×1 (column) matrix, then $\{y\}^T$ is the associated 1×3 (row) matrix. The element in the ith column of $\{y\}^T$ equals the element in the ith row of $\{y\}$. If $\{x\}$ is a second column matrix, then

$$\{y\}^T\{x\} := y_1x_1 + y_2x_2 + y_3x_3 = \sum_{i=1}^{3} y_ix_i. \tag{1.18}$$

– The column matrices $\{x\}$ and $\{y\}$ are said to be *orthogonal* if

$$\{y\}^T\{x\} = \sum_{i=1}^{3} y_ix_i = 0. \tag{1.19}$$

– A matrix $[A]$ is *positive definite* if

$$\{x\}^T[A]\{x\} \;=\; \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}x_ix_j > 0 \tag{1.20}$$

for all nonzero column matrices $\{x\}$.

– A scalar α and a column matrix $\{a\}\,(\neq \{0\})$ for which

$$[A]\{a\} = \alpha\{a\} \tag{1.21}$$

are said to be an *eigenvalue* and *eigen"vector"* of $[A]$.

– If $[A]$ is symmetric, then it has three real eigenvalues $\alpha_1, \alpha_2, \alpha_3$ and three corresponding eigenvectors $\{a^{(1)}\}, \{a^{(2)}\}, \{a^{(3)}\}$. Without loss of generality the eigenvectors can always be chosen so each has unit magnitude and each is orthogonal to the other two in the sense that

$$\{a^{(i)}\}^T\{a^{(j)}\} = a_1^{(i)}a_1^{(j)} + a_2^{(i)}a_2^{(j)} + a_3^{(i)}a_3^{(j)} = \sum_{k=1}^{3} a_k^{(i)}a_k^{(j)} = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

1.2 Indicial notation.

Three brief videos on indicial notation can be found at
https://www.dropbox.com/sh/bfcvwsnq7k3zefi/AAA-QqgSrOpxxOZZojyFptyYa?dl=0.

Indicial notation is convenient when carrying out calculations involving the elements of matrices. When doing so, it is important that one adhere to certain rules/conventions.

The following terminology will be encountered in our discussion below:

– Free index,

– Dummy (or repeated) index,

– Range convention,

– Summation convention, and

– Substitution rule.

– Consider matrices $[A]$, $\{x\}$ and $\{y\}$ satisfying the matrix equation

$$\{y\} = [A]\{x\} \qquad \Leftrightarrow \qquad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{1.22}$$

On carrying out the matrix multiplication, this is equivalent to the system of 3 scalar equations

$$\left.\begin{aligned} y_1 &= A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = \sum_{k=1}^{3} A_{1k}x_k, \\ y_2 &= A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = \sum_{k=1}^{3} A_{2k}x_k, \\ y_3 &= A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = \sum_{k=1}^{3} A_{3k}x_k. \end{aligned}\right\} \tag{1.23}$$

This system of scalar equations can be written more compactly as

$$y_i = \sum_{k=1}^{3} A_{ik}x_k \quad \text{with } i \text{ taking each value in the range } 1, 2, 3. \tag{1.24}$$

– **Range convention:** We can write (1.24) even more compactly by omitting the phrase "with i taking each value in the range $1, 2, 3$" and simply writing

$$y_i = \sum_{k=1}^{3} A_{ik}x_k \tag{1.25}$$

with the understanding that (1.25) *holds for each value of the index i in the range* $i = 1, 2, 3$. This understanding is referred to as the *range convention*.

Likewise the matrix equation $[C] = [A][B]$ can be written by (1.4) as

$$C_{ij} = \sum_{k=1}^{3} A_{ik}B_{kj}, \tag{1.26}$$

having dropped the phrase "with i and j taking each value in the range $1, 2, 3$".

From here on we shall always use the range convention unless explicitly stated otherwise.

– Observe the distinction between the two types of indices: free indices and repeated (or dummy) indices. The index i in (1.25) is called a *free index* because it is free to take on each value in the range $1, 2, 3$, one at a time. Equation (1.26) involves two free indices i and j, and each, independently, takes each value in the range $1, 2, 3$.

On the other hand the index k (in both equations) is *not* a free index: it is summed over $1, 2, 3$ and is not free to take each value $1, 2, 3$ *one at a time*. Since this index appears twice (in the terms on the right-hand sides), it is called a *repeated index* or (for reasons that will soon become clear) a *dummy index*.

– Now consider the set of equations

$$y_j = \sum_{k=1}^{3} A_{jk} x_k. \tag{1.27}$$

By the range convention, this holds with the free subscript j taking each value in the range $1, 2, 3$. Therefore the set of equations in (1.27) is identical to that in (1.25). This illustrates the fact that *the particular choice of index for the free subscript in an equation is not important provided that the same free subscript appears in every term*[2] *of the equation.*

Likewise the equation

$$C_{pq} = \sum_{k=1}^{3} A_{pk} B_{kq}, \tag{1.28}$$

is equivalent to (1.26) where we have simply used a different pair of free indices p, q.

– If an equation involves n free indices, then it represents 3^n scalar equations. For example (1.25) has 1 free subscript and it represents $3^1 = 3$ scalar equations, while (1.26) has 2 free subscripts and so represents $3^2 = 9$ equations.

– In order to be consistent it is important that *the same free index (or indices) appear once, and only once, in every term of an equation.* For example, the matrix equation $\{y\} = [A]\{x\} + [B]\{x\}$ can be written in scalar form as

$$y_i = \sum_{p=1}^{3} A_{ip} x_p \ + \ \sum_{q=1}^{3} B_{iq} z_q. \tag{1.29}$$

Here we have a free index i on the left-hand side and this same free index i appears in

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[2] It is worth clarifying how the word "term" is used in this section. In an equation such as

$$y_i = \sum_{k=1}^{3} A_{ik} x_k + \sum_{q=1}^{3} a_q x_q z_i,$$

when we say "the first term on the right-hand side" we do not mean A_{ik} but rather $\sum A_{ik} x_k$. Two terms are separated by displayed $+, -$ or $=$ signs. If we wrote this out as

$$y_i = A_{i1} x_1 + A_{i2} x_2 + A_{i3} x_3 + \sum_{q=1}^{3} a_q x_q z_i,$$

the first term on the right-hand side would be $A_{i1} x_1$.

each of the terms on the right-hand side. We will never write an equation such as

$$y_i = \sum_{p=1}^{3} A_{jp} x_p \ + \ \sum_{q=1}^{3} B_{jq} z_q$$

which has the free index i on the left-hand side and the free index j on the right-hand side.

Similarly in (1.26), since the free indices i and j appear on the left-hand side they must also appear on the right-hand side. An equation such as $A_{ij} = B_{pq}$ would violate this consistency requirement.

Observe from (1.29) that the same repeated index does not need to appear in every term.

– **Summation convention:** Next, observe that on the right-hand side of equation (1.25) the subscript k is (a) repeated and (b) there is a sum over it. Likewise in the first term on the right-hand side of equation (1.29) the subscript p is repeated and there is a sum over it, and in the second term the subscript q is repeated and there is a sum over it. In each case there is a summation over the repeated index. An example involving 2 repeated indices is

$$A_{11}x_1^2 + A_{12}x_1x_2 + \ldots + A_{33}x_3^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}x_ix_j.$$

On the right-hand side the subscript i is repeated and there is a summation over it, and likewise the subscript j is repeated and there is a summation over it as well.

In view of this observation we can simplify our writing even further *by agreeing to drop the summation sign and instead imposing the rule that summation is implied over a subscript that appears twice in a term.* With this understanding in force, we would write (1.25), (1.26) and (1.29) as

$$y_i = A_{ik}x_k, \qquad C_{ij} = A_{ik}B_{kj}, \qquad y_i = A_{ip}x_p \ + \ B_{qi}z_q, \tag{1.30}$$

respectively with summation on the subscript k in the first and second, and on p and q in the third being implied.

– Note that we can write

$$\mathrm{tr}[A] = A_{ii}.$$

– Since

$$\sum_{k=1}^{3} A_{ik}x_k = \sum_{j=1}^{3} A_{ij}x_j,$$

it follows that

$$y_i = A_{ij}x_j \tag{1.31}$$

is identical to $(1.30)_1$. Thus we see that the particular choice of index for the repeated subscript is not important: it is a dummy index in this sense.

– In order to avoid ambiguity, no subscript is allowed to appear more than twice in any term in general. Thus we shall not write, for example, $A_{ii}x_i = y_i$ since, if we did, the index i would appear 3 times in the term on the left-hand side. We would not know whether the left-hand side denotes $(A_{11} + A_{22} + A_{33})x_i$ or $A_{11}x_1 + A_{22}x_2 + A_{33}x_3$ or perhaps there is no sum over i at all. On a few occasions, usually involving eigenvalues, we will forced to include a term with the same subscript appearing more than twice. In such cases we will make clear that the summation convention is being suspended and the summation is shown explicitly, e.g. see (1.112) where the subscript i appears 3 times.

– **Summary of Rules**:

1. Lower-case latin subscripts take on values in the range $1, 2, 3$.
2. A given index may appear either once or twice in each term of an equation. If it appears once, it is called a free index and it takes on each value in its range, one at a time. If it appears twice, it is called a dummy index and summation is implied over it.
3. The same index may not appear more than twice in the same term.
4. All terms of an equation must have the same free indices.

– *It is important to emphasize that an equation such as $A_{ij} = B_{ik}C_{kj}$ involves scalar quantities and therefore the order in which the scalar factors appear within a term is not significant.* For example

$$A_{ij} = B_{ik}C_{kj}, \tag{1.32}$$

comprises 3^2 scalar equations. Consider the element A_{11}. On setting $i = 1, j = 1$ in (1.32) we get

$$A_{11} = B_{1k}C_{k1} \qquad \Rightarrow \qquad A_{11} = B_{11}C_{11} + B_{12}C_{21} + B_{13}C_{31}.$$

Clearly we can rearrange terms in the right most expression and write

$$A_{11} = C_{11}B_{11} + C_{21}B_{12} + C_{31}B_{13} \qquad \Rightarrow \qquad A_{11} = C_{k1}B_{1k},$$

and so $A_{11} = B_{1k}C_{k1} = C_{k1}B_{1k}$, and more generally

$$A_{ij} = B_{ik}C_{kj} = C_{kj}B_{ik};$$

i.e. we can move B_{ik} to the back of $B_{ik}C_{kj}$ and write $C_{kj}B_{ik}$.

As a matrix equation $A_{ij} = B_{ik}C_{kj}$ corresponds to $[A] = [B][C]$ and so does $A_{ij} = C_{kj}B_{ik}$. The latter does *not* correspond to $[A] = [C][B]$ (as we shall explain below).

– Frequently, in the course of a calculation, we will have to change our choice of indices. For example suppose $y_i = A_{ij}x_j$ and we want to calculate y_iy_i (which is $y_1^2 + y_2^2 + y_3^2$). We *cannot* write $y_iy_i = (A_{ij}x_j)(A_{ij}x_j) = A_{ij}A_{ij}x_jx_j$ because we then have the index j appearing more than twice in the same term. Instead, we would use the equivalent alternative representations $y_i = A_{ij}x_j$ and $y_i = A_{ik}x_k$ to write $y_iy_i = (A_{ij}x_j)(A_{ik}x_k) = A_{ij}A_{ik}x_jx_k$. Observe that no index appears more than twice in each term.

– The indices in an expression can be changed without altering the meaning of an expression *provided that* (a) the positions of the free and repeated indices does not change and (b) one does not violate the preceding rules. Thus, for example, we can change the free index p on both sides of the equation

$$y_p \;=\; A_{pq}x_q$$

to any other index (except q, why not q?), say k, and equivalently write

$$y_k \;=\; A_{kq}x_q.$$

We can also change the repeated subscript q to some other index (except k), say p, and write

$$y_k \;=\; A_{kp}x_p.$$

In fact, we can even write

$$y_q \;=\; A_{qp}x_q.$$

The four preceding sets of equations are identical.

However, the equations $y_p \;=\; A_{qp}x_q$ and $y_p \;=\; A_{pq}x_q$ are *not* identical even though none of the indicial notation rules are violated. This is because the free and repeated indices do not appear in the same *positions*: the free index p on the right-hand side of $y_p = A_{qp}x_q$ is in the second position in A_{qp}, but it is in the first position in A_{pq} in $y_p = A_{pq}x_q$.

Similarly consider $A_{ij} = B_{ik}C_{kj}$. The fact that the second subscript of B_{ik} is the same as the first subscript of C_{kj} is what tells us that in matrix form this arises from $[A] = [B][C]$; see (1.4). In contrast $A_{ij} = B_{ki}C_{kj}$ does *not* represent $[A] = [B][C]$ because the repeated index is in a different position on the right-hand side. .

– We emphasize that while the factors in a term can be moved around, one cannot arbitrarily move indices. The *location of the indices determines the order in which the associated matrices are multiplied.*

When we write $A_{ij}x_j$ it is the second subscript of A_{ij} that also appears in x_j and this is what tells us that $A_{ij}x_j$ is the ith element of $[A]\{x\}$. For this reason $A_{ji}x_j$ is *not* the ith element of $[A]\{x\}$. But $A_{ji}x_j$ is the ith element of some matrix (since it has one free index i). But of what matrix? In order that this represent the product of a 3×3 matrix with a column matrix, the second subscript of the element associated with the 3×3 matrix must also appear in the element associated with the column matrix. To achieve this we can write $A_{ji}x_j = A^T_{ij}x_j$ which is now in the desired form. Therefore $A_{ji}x_j$ is the ith element of $[A^T]\{x\}$. Here $[A^T]$ is the transpose of $[A]$.

Similarly, observe in the matrix multiplication representation (1.32) that the second subscript of B_{ik} is the same as the first subscript of C_{kj}, whence $B_{ik}C_{kj}$ represents the i, j element of the matrix product $[B][C]$. Suppose we have the expression $B_{ki}C_{kj}$. To put it in the preceding form where adjacent subscripts are repeated, we can write $B_{ki}C_{kj} = B^T_{ik}C_{kj}$. The last subscript of B^T_{ik} is now the same as the first subscript of C_{kj} and so $B_{ki}C_{kj} = B^T_{ik}C_{kj}$ represents the i, j element of $[B]^T[C]$.

Alternatively we can write $B_{ki}C_{kj} = C_{kj}B_{ki} = C^T_{jk}B_{ki} = \big([C]^T[B]\big)_{ji}$. Therefore $B_{ki}C_{kj}$ is the j, i (not i, j) element of $[C]^T[B]$. These alternative representations are of course equivalent since $\big([B]^T[C]\big)^T = [C]^T[B]$ by (1.5).

Kronecker delta: The Kronecker delta, δ_{ij}, is defined as

$$\delta_{ij} := \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad \text{for each } i, j = 1, 2, 3. \tag{1.33}$$

Observe that δ_{ij} is the element in the ith row and jth column of the identity matrix

$$[I] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$\delta_{ii} = 3. \tag{1.34}$$

The Kronecker delta arises, for example, when working with a triplet $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of orthonormal vectors (page 21) and when calculating the derivative $\partial x_i/\partial x_j$ (page 18).

Substitution rule: The following useful property of the Kronecker delta is called the substitution rule. Suppose one wishes to simplify the expression $\delta_{ij}u_j$ for some column matrix $\{u\}$. First note that $\delta_{ij}u_j = \delta_{i1}u_1 + \delta_{i2}u_2 + \delta_{i3}u_3$ where i is a free subscript. Next, consider the choice $i = 1$. Then $\delta_{1j}u_j = \delta_{11}u_1 + \delta_{12}u_2 + \delta_{13}u_3$. Since $\delta_{ij} = 0$ unless $i = j$ and $\delta_{ij} = 1$ if $i = j$ we conclude that the last two terms on the right-hand side of $\delta_{1j}u_j = \delta_{11}u_1 + \cancel{\delta_{12}}u_2 + \cancel{\delta_{13}}u_3$ vanish and so it simplifies to $\delta_{1j}u_j = u_1$.

In a similar manner we see that for any value of the free index i, two terms on the right-hand side of $\delta_{ij}u_j = \delta_{i1}u_1 + \delta_{i2}u_2 + \delta_{i3}u_3$ vanish trivially because the two subscripts of the Kronecker delta in each of those terms will be distinct. The term that remains is the 1$^{\text{st}}$, 2$^{\text{nd}}$ or 3$^{\text{rd}}$ term depending on whether the free index $i = 1, 2$ or 3 respectively. Thus the term that survives on the right-hand side is u_i and so

$$\delta_{ij}u_j = u_i. \tag{1.35}$$

In summary, *(a)* since δ_{ij} is zero unless $j = i$, the expression being simplified has a non-zero value only if $j = i$; and *(b)* when $j = i$, δ_{ij} is unity. Thus we replace the Kronecker delta by unity and simultaneously change the repeated subscript j in the other factor to the non-repeated subscript i of the Kronecker delta. This gives $\delta_{ij}u_j = u_i$.

As a second example suppose that $[A]$ is a square matrix and one wishes to simplify $\delta_{ij}A_{jk}$. Then by the same reasoning[3],

$$\delta_{ij}A_{jk} = \delta_{i1}A_{1k} + \delta_{i2}A_{2k} + \delta_{i3}A_{3k} = A_{ik} \tag{1.36}$$

[3]Observe that these results are immediately apparent by using matrix algebra. In the first example, $\delta_{ij}u_j$ is simply the ith element of the column matrix $[I]\{u\}$. Since $[I]\{u\} = \{u\}$ the result follows at once. Similarly in the second example, $\delta_{ij}A_{jk}$ is simply the i,k-element of the matrix $[I][A]$. Since $[I][A] = [A]$, the result follows.

and so the Kronecker delta has been replaced by unity and the repeated subscript j in A_{jk} has been changed to i.

More generally, if some quantity or expression $\mathbb{T}_{pq\ldots i\ldots z}$ multiplies δ_{ij} with the index i appearing as a subscript in both factors, one simply replaces the Kronecker delta by unity and changes the subscript i in $\mathbb{T}$ to j:

$$\mathbb{T}_{pq\ldots i\ldots z}\,\delta_{ij} = \mathbb{T}_{pq\ldots j\ldots z}. \tag{1.37}$$

Levi-Civita symbol (or alternating symbol or permutation symbol): Consider the following determinant:

$$\det[M] = \det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

$$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

Observe in the final expression that (a) there are no terms in which the subscript $1, 2$ or 3 appears more than once (i.e. the subscripts i, j, k in $a_ib_jc_k$ are distinct); (b) when the subscripts are in cyclic order (i.e. 123, 231, 312) the coefficient is $+1$; and (c) when the subscripts are in anti-cyclic order (i.e. 132, 213, 321) the coefficient is -1. We will encounter this pattern in other settings as well. It is useful therefore to introduce a mathematical entity that captures this. This is the role of the so-called Levi-Civita symbol, also referred to as the alternating or permutation symbol. It is denoted by e_{ijk} and defined as

$$\begin{aligned} e_{ijk} \;&:=\; \begin{cases} 0 & \text{if two or more subscripts i, j, k, are equal,} \\ +1 & \text{if the subscripts i, j, k, are in cyclic order,} \\ -1 & \text{if the subscripts i, j, k, are in anticyclic order,} \end{cases} \\ &=\; \begin{cases} 0 & \text{if two or more subscripts i, j, k, are equal,} \\ +1 & \text{for } (i,j,k) = (1,2,3), (2,3,1), (3,1,2), \\ -1 & \text{for } (i,j,k) = (1,3,2), (2,1,3), (3,2,1). \end{cases} \end{aligned} \tag{1.38}$$

The expression above for the determinant of the matrix $[M]$ can now be written succinctly as

$$\det[M] = e_{ijk}a_ib_jc_k,$$

which can also be written as

$$\det[M] = e_{ijk}M_{1i}M_{2j}M_{3k} = e_{ijk}M_{i1}M_{j2}M_{k3}. \tag{1.39}$$

Other identities involving the determinant include

$$e_{pqr}\det[M] \;=\; e_{ijk}M_{ip}M_{jq}M_{kr}, \tag{1.40}$$

$$\det[M] = \frac{1}{6}e_{ijk}e_{pqr}M_{ip}M_{jq}M_{kr}. \tag{1.41}$$

Another setting in which we shall encounter the Levi-Civita symbol is when working with the vector (cross) product (page 21).

Two useful properties of the Levi-Civita symbol are: (a) the sign of e_{ijk} changes whenever any two adjacent subscripts are switched:

$$e_{ijk} = -e_{jik} = e_{jki} = -e_{kji} = \dots\;, \tag{1.42}$$

(i.e. it is skew-symmetric with respect to every pair of adjacent subscripts) and (b) the Levi-Civita symbol and Kronecker delta are related by

$$e_{pij}e_{pk\ell} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}; \tag{1.43}$$

see Problem 1.3.

1.2.1 Worked examples.

Problem 1.2.1. (Matrices. Indicial notation.) The matrices $[C], [D]$ and $[E]$ are defined in terms of the two matrices $[A]$ and $[B]$ by

$$[C] = [A][B], \qquad [D] = [B][A], \qquad [E] = [A][B]^T. \tag{i}$$

Express the elements of $[C], [D]$ and $[E]$ in terms of the elements of $[A]$ and $[B]$.

Solution: By the rules of matrix multiplication, the element C_{ij} in the i^{th} row and j^{th} column of $[C]$ is obtained by multiplying the elements of the i^{th} row of $[A]$, pairwise, by the respective elements of the j^{th} column of $[B]$ and summing. So, C_{ij} is obtained by multiplying the elements A_{i1}, A_{i2}, A_{i3} by, respectively, B_{1j}, B_{2j}, B_{3j} and summing. Thus

$$C_{ij} = A_{ik}B_{kj}; \tag{ii}$$

note that i and j are both free indices here and so this represents $3^2 = 9$ scalar equations; moreover summation is carried out over the repeated index k. It follows immediately from (ii) that the equation $[D] = [B][A]$ leads to

$$D_{ij} = B_{ik}A_{kj} \qquad \text{or equivalently} \qquad D_{ij} = A_{kj}B_{ik}, \tag{iii}$$

where the second expression was obtained by simply changing the order in which the factors appear in the first expression (since, as noted previously, the order of the factors within a term is insignificant since these are scalar quantities.) In order to calculate E_{ij}, we first use (ii) to directly obtain $E_{ij} = A_{ik}B^T_{kj}$. However, by definition of transposition, the i, j-element of a matrix $[B]^T$ equals the j, i-element of the matrix $[B]$: $B^T_{ij} = B_{ji}$ and so we can write

$$E_{ij} = A_{ik}B_{jk}. \tag{iv}$$

All four expressions here involve the ik, kj or jk elements of $[A]$ and $[B]$. The precise locations of the subscripts vary and the meaning of the terms depend crucially on these locations. It is worth repeating that the location of the repeated subscript k tells us what term multiplies what term.

Problem 1.2.2. (Matrices. Indicial notation.) The matrices $[A]$ and $[B]$ are symmetric and skew-symmetric respectively. Show that

$$A_{ij}B_{ij} = 0. \tag{1.44}$$

Remark: This result will be useful in several later calculations.

Solution: We proceed as follows:

$$A_{ij}B_{ij} \overset{(*)}{=} -A_{ji}B_{ji} \overset{(**)}{=} -A_{ij}B_{ij} \qquad \Rightarrow \qquad 2A_{ij}B_{ij} = 0 \qquad \Rightarrow \qquad A_{ij}B_{ij} = 0 \qquad \square$$

where in step $(*)$ we used the symmetry $A_{ij} = A_{ji}$ and skew-symmetry $B_{ij} = -B_{ji}$, while in step $(**)$ we simply changed the dummy subscripts $i \to j, j \to i$.

Problem 1.2.3. (Matrices. Indicial notation.) Show that

$$e_{ijk}A_{jk} = 0 \tag{1.45}$$

if and only if $[A]$ is a symmetric matrix.

Solution: First suppose that $[A]$ is symmetric. Then, since e_{ijk} is skew-symmetric in j, k while A_{jk} is symmetric in j, k the result follows from (1.44).

Conversely suppose that (1.45) holds. Multiplying it by e_{ipq} gives

$$0 = e_{ipq}e_{ijk}A_{jk} \overset{(1.43)}{=} (\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj})A_{jk} = \delta_{pj}\delta_{qk}A_{jk} - \delta_{pk}\delta_{qj}A_{jk} \overset{(**)}{=} A_{pq} - A_{qp} \quad \Rightarrow A_{pq} = A_{qp} \qquad \square$$

and so $[A]$ is symmetric. In step $(**)$ we used the substitution rule (1.37).

Problem 1.2.4. (Indicial notation.) Simplify the expressions

(a) $A_{ij}\delta_{ij}$,

(b) $A_{ij}\delta_{ip}\delta_{jq}$, and

(c) $\delta_{ij}\delta_{ij}$.

Solution:

(a) In the expression $A_{ij}\delta_{ij}$ we have two repeated subscripts and so we can apply the substitution rule to either. Consider for example the repeated subscript j. Then, according to the substitution rule we replace δ_{ij} by unity and change the repeated subscript j in the other factor (i.e. in A_{ij}) to i. This yields $A_{ij}\delta_{ij} = A_{ii}$.

(b) In simplifying $A_{ij}\delta_{ip}\delta_{jq}$ (for clarity) we proceed in two steps. Consider δ_{ip} and the repeated subscript i. Thus we replace δ_{ip} by unity and change the repeated subscript i in the other factor to p. This yields $A_{ij}\delta_{ip}\delta_{jq} = A_{pj}\delta_{jq}$. We can apply the substitution rule again, this time on the index j, which yields $A_{pj}\delta_{jq} = A_{pq}$. Combing the two steps yields $A_{ij}\delta_{ip}\delta_{jq} = A_{pq}$. We could of course have done this in one step.

(c) The expression $\delta_{ij}\delta_{ij}$ involves two repeated subscripts and so we can apply the substitution rule to either one. If we apply the substitution rule on the repeated subscript i we get $\delta_{ij}\delta_{ij} = \delta_{jj}$. Since $\delta_{jj} = 3$ by (1.34) we can simplify further to get $\delta_{ij}\delta_{ij} = 3$.

Problem 1.2.5. (Indicial notation.) Show that (a) $e_{ijp}e_{ijq} = 2\delta_{pq}$ and (b) $e_{ijk}e_{ijk} = 6$.

Solution

(a) We proceed as follows:

$$e_{ijp}e_{ijq} \overset{(1.43)}{=} \delta_{jj}\delta_{pq} - \delta_{jq}\delta_{pj} \overset{(1.34)}{=} 3\delta_{pq} - \delta_{jq}\delta_{pj} \overset{(1.37)}{=} 3\delta_{pq} - \delta_{pq} = 2\delta_{pq}. \qquad \square \tag{i}$$

(b) Set $p = q = k$ in (i):

$$e_{ijk}e_{ijk} = 2\delta_{kk} \overset{(1.34)}{=} 6. \qquad \square \tag{ii}$$

Problem 1.2.6. (Matrices. Indicial notation.) A matrix $[A]$ has the property that the magnitude of the column matrix $[A]\{x\}$ equals the magnitude of the column matrix $\{x\}$ for all column matrices $\{x\}$. Show that $[A]$ is an orthogonal matrix.

(We can view the matrix $[A]$ as mapping the column matrix $\{x\}$ into the column matrix $[A]\{x\}$. The particular matrix $[A]$ in this problem preserves the magnitude of a column matrix under this mapping.)

Solution: Let $\{y\} = [A]\{x\}$. We are told that

$$\{y\}^T\{y\} = \{x\}^T\{x\} \qquad \Leftrightarrow \qquad y_iy_i = x_ix_i. \tag{i}$$

Since $y_i = A_{ij}x_j$, $(i)_2$ can be written as

$$A_{ij}x_jA_{ik}x_k = x_ix_i \qquad \Leftrightarrow \qquad A_{ij}A_{ik}x_jx_k = x_ix_i. \tag{ii}$$

We are told that $(i)_1$ holds for all $\{x\}$. Therefore $(ii)_2$ holds for all x_p and so we can differentiate $(ii)_2$ with respect to x_p. This yields

$$\frac{\partial}{\partial x_p}\big(A_{ij}A_{ik}x_jx_k\big) = \frac{\partial}{\partial x_p}\big(x_ix_i\big) \qquad \Rightarrow \qquad A_{ij}A_{ik}\frac{\partial x_j}{\partial x_p}x_k + A_{ij}A_{ik}x_j\frac{\partial x_k}{\partial x_p} = \frac{\partial x_i}{\partial x_p}x_i + x_i\frac{\partial x_i}{\partial x_p}. \qquad (iii)$$

Since x_1, x_2, x_3 are independent variables, it follows that $\partial x_1/\partial x_1 = 1, \partial x_1/\partial x_2 = 0$ etc., i.e. $\partial x_p/\partial x_q = 1$ if $p = q$ and $\partial x_p/\partial x_q = 0$ if $p \neq q$. Thus

$$\frac{\partial x_p}{\partial x_q} = \delta_{pq}. \qquad (1.46)$$

Substituting (1.46) into (iii) gives

$$A_{ij}A_{ik}\delta_{jp}x_k + A_{ij}A_{ik}x_j\delta_{kp} = 2\delta_{ip}x_i,$$

which by the substitution rule reduces to

$$A_{ip}A_{ik}x_k + A_{ij}A_{ip}x_j = 2x_p \qquad \Leftrightarrow \qquad A_{ip}A_{ik}x_k + A_{ik}A_{ip}x_k = 2x_p, \qquad (iv)$$

where in getting to the second equation we changed the dummy subscript $j \to k$. Changing the order of the factors in the second term on the left-hand side and simplifying leads to:

$$(iv) \quad \Rightarrow \quad A_{ip}A_{ik}x_k + A_{ip}A_{ik}x_k = 2x_p \quad \Rightarrow \quad 2A_{ip}A_{ik}x_k = 2x_p \quad \Rightarrow \quad A_{ip}A_{ik}x_k = x_p. \qquad (v)$$

Since (v) holds for all x_q we can differentiate it with respect to x_q to get

$$A_{ip}A_{ik}\frac{\partial x_k}{\partial x_q} = \frac{\partial x_p}{\partial x_q} \quad \stackrel{(1.46)}{\Rightarrow} \quad A_{ip}A_{ik}\delta_{kq} = \delta_{pq} \quad \Rightarrow \quad A_{ip}A_{iq} = \delta_{pq}, \qquad (vi)$$

where we again used the substitution rule in getting to the last expression. In matrix form, $(vi)_3$ can be written as

$$[A]^T[A] = [I] \qquad \square$$

which shows that $[A]$ is orthogonal.

1.3 Vector algebra.

– Perhaps the most familiar example of a vector is an "arrow", a geometric entity with both length and direction. Figure 2.1 shows the position *vectors* of a particle in a body, $\mathbf{x}$ in the undeformed configuration and $\mathbf{y}$ in the deformed configuration, the displacement *vector* of this particle being $\mathbf{u}$.

  Other commonly encountered vectors in mechanics include velocity, linear momentum, angular velocity, angular momentum, force, traction and torque.

– A collection of vectors (denoted by V), together with certain operations pertaining to addition, multiplication by a scalar and the null vector, is called a *vector space*.

– The null vector $\mathbf{o}$ has the property that $\mathbf{x} + \mathbf{o} = \mathbf{x}$ for all vectors $\mathbf{x}$ in V.

– A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is said to be **linearly independent** if the only scalars $\alpha_1, \alpha_2, \alpha_3$ for which

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 = \mathbf{o}$$

are $\alpha_1 = \alpha_2 = \alpha_3 = 0$. This implies that no vector in this set can be expressed as a linear combination of the other two.

– If V contains 3 linearly independent vectors but does not contain more than 3 linearly independent vectors, we say the **dimension** of V is 3.

– The **scalar product** (or dot product or inner product) of two vectors $\mathbf{u}$ and $\mathbf{v}$ is a scalar that we denote by

$$\mathbf{u} \cdot \mathbf{v}.$$

The scalar product obeys certain rules analogous to those listed in Problem 1.59.

– A vector space endowed with a scalar product is called a **Euclidean vector space**. Unless stated otherwise, we shall always be concerned with 3-dimensional Euclidean vector spaces which we denote by V (also commonly denoted by $\mathbb{E}^3$).

– The *magnitude* or length of a vector $\mathbf{u}$ is

$$|\mathbf{u}| := \sqrt{\mathbf{u} \cdot \mathbf{u}}, \tag{1.47}$$

and the *distance* between two vectors $\mathbf{u}, \mathbf{v}$ is $|\mathbf{u} - \mathbf{v}|$.

– The only vector with zero length is the null vector:

$$|\mathbf{x}| = 0 \qquad \Leftrightarrow \qquad \mathbf{x} = \mathbf{o}, \tag{1.48}$$

where the double arrow (here and throughout these notes) is short-hand for "if and only if".

– The *angle* θ between two vectors $\mathbf{u} \neq \mathbf{o}$ and $\mathbf{v} \neq \mathbf{o}$ is defined by

$$\cos\ \theta := \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}, \qquad 0 \leq \theta \leq \pi. \tag{1.49}$$

In order for this definition of the angle θ to be meaningful, the right-hand side of (1.49) must lie in the interval $[-1, 1]$. The Cauchy-Schwartz inequality in Problem 1.3.5 shows this to be true.

- If $\mathbf{u} \cdot \mathbf{v} = 0$ then $\theta = \pi/2$ and we say the vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* to each other.

- Two vectors $\mathbf{a} \neq \mathbf{o}$ and $\mathbf{b} \neq \mathbf{o}$ are said to be parallel if there is a scalar α for which $\mathbf{a} = \alpha\mathbf{b}$.

- The **vector product** (or cross product) of two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ is a vector that we denote by
$$\mathbf{u} \times \mathbf{v}.$$
Its magnitude is $|\mathbf{u}|\,|\mathbf{v}|\,\sin\theta$, it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, and its sense is given by the right-hand rule. Thus
$$\mathbf{u} \times \mathbf{v} = \big(|\mathbf{u}|\,|\mathbf{v}|\,\sin\theta\big)\,\mathbf{n} \qquad \text{where} \quad \mathbf{n}\cdot\mathbf{u} = \mathbf{n}\cdot\mathbf{v} = 0, \quad \mathbf{n}\cdot(\mathbf{u}\times\mathbf{v}) > 0. \tag{1.50}$$
The unit vector $\mathbf{n}$ is the direction of $\mathbf{u} \times \mathbf{v}$, and the inequality in (1.50) tells us that the triplet of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{n}\}$ is right-handed. The vector product of two linearly dependent vectors is the null vector.

- The scalar and vector products have the respective properties
$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}, \qquad \mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}. \tag{1.51}$$

- The following useful result is addressed in Problem 1.7: three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent if and only if
$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) \neq 0. \tag{1.52}$$
Geometrically, three vectors are linearly independent if they do not lie in the same plane, i.e. they are non-coplanar.

1.3.1 Components of a vector in a basis.

A brief video on the use of indicial notation in vector algebra can be found at https://www.dropbox.com/sh/4cuw28tfqvl0is9/AAAFWr40a2qNfpneJmJrdZiNa?dl=0.

Basis.

– A set of three linearly independent vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ forms a **basis** for a three-dimensional vector space V. An arbitrary vector $\mathbf{v} \in \mathsf{V}$ can be expressed as a unique linear combination of the three basis vectors.

– The basis is said to be **orthonormal** if each basis vector has unit length and each is perpendicular to the other two:

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1, \qquad \mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0.$$

This can be written succinctly as

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{1.53}$$

where δ_{ij} is the Kronecker delta introduced in (1.33). We shall always restrict attention to orthonormal bases unless explicitly stated otherwise, and so we will drop the adjective "orthonormal" (except when we wish to emphasize it).

– The basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is *right-handed* if

$$\mathbf{e}_1 \times \mathbf{e}_2 = +\mathbf{e}_3 \qquad \Leftrightarrow \qquad (\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = +1.$$

For a right-handed basis it can be readily verified that $\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{o}$, $\mathbf{e}_1 \times \mathbf{e}_2 = +\mathbf{e}_3$, $\mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2$ etc. and so we again encounter the numbers $0, +1$ and -1 (as we did when looking at the determinant of a matrix). It is not surprising therefore that we can express $\mathbf{e}_i \times \mathbf{e}_j$ succinctly in terms of the Levi-Civita symbol e_{ijk} introduced in (1.38). Indeed, for a right-handed basis (Exercise),

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\, \mathbf{e}_k, \qquad\qquad e_{ijk} = (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k. \tag{1.54}$$

Components of a vector.

– Given a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for a three-dimensional vector space V, an arbitrary vector $\mathbf{v} \in \mathsf{V}$ can always be expressed as a unique linear combination of the three basis vectors:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \;\; = \;\; v_i\mathbf{e}_i. \tag{1.55}$$

The scalars v_i are called the **components** of $\mathbf{v}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

– When the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is orthonormal, the components v_1, v_2, v_3 of the vector $\mathbf{v}$ can be calculated from

$$v_i = \mathbf{v} \cdot \mathbf{e}_i. \tag{1.56}$$

This follows by taking the scalar product of (1.55) with $\mathbf{e}_j$. Also, see Figure 1.1.

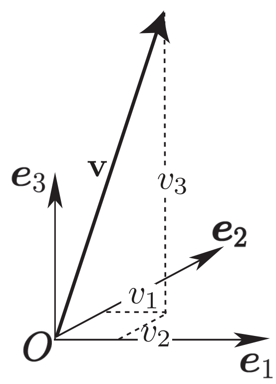

Figure 1.1: Components $\{v_1, v_2, v_3\}$ of a vector $\mathbf{v}$ in an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

– Observe from (1.56) that the components v_i of the vector depend on both the vector $\mathbf{v}$ *and* the choice of basis. If we change the basis the components v_i will change even if we don't change the vector $\mathbf{v}$.

– The components v_1, v_2, v_3 of $\mathbf{v}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ may be assembled into a column matrix

$$\{v\} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{1.57}$$

Thus *a vector, together with a basis, allows the vector to be represented as a column matrix.*

– Once a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is chosen and fixed, there is a unique column matrix $\{v\}$ associated any given vector $\mathbf{v}$ (defined through (1.56), (1.57)); *and conversely,* there is a unique vector $\mathbf{v}$ associated with any given column matrix $\{v\}$ (defined by (1.55), (1.57)) where the components of $\mathbf{v}$ in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are $\{v\}$. *Thus, once the basis is fixed, there is a one-to-one correspondence between column matrices and vectors.*

For example consider the *vector equation*

$$\mathbf{z} = \mathbf{x} + \mathbf{y}.$$

Taking the scalar product of this equation with each basis vector $\mathbf{e}_i$ gives $\mathbf{z} \cdot \mathbf{e}_i = \mathbf{x} \cdot \mathbf{e}_i + \mathbf{y} \cdot \mathbf{e}_i$ so that by (1.56), we obtain the *system of scalar equations*

$$z_i = x_i + y_i \qquad \text{for each } i = 1, 2, 3,$$

where x_i, y_i and z_i are the components of these vectors in the basis at hand. These components can be assembled into column matrices which allows us to express the preceding equation in *matrix form* as

$$\{z\} = \{x\} + \{y\}.$$

Thus, after choosing the basis, we can express the equation $\mathbf{z} = \mathbf{x} + \mathbf{y}$ in three equivalently forms – vector, components and matrix.

– *The fundamental notion of a vector stands on its own without the need to refer to its components in a basis.* For example we can speak of the displacement of a particle or the force acting on a particle without needing to say anything about a basis.

– It follows from (1.55) and (1.53) that the *scalar product* of two vectors $\mathbf{u} = u_i\mathbf{e}_i$ and $\mathbf{v} = v_i\mathbf{e}_i$ can be calculated as $\mathbf{u}\cdot\mathbf{v} = (u_i\mathbf{e}_i)\cdot(v_j\mathbf{e}_j) = u_iv_j(\mathbf{e}_i\cdot\mathbf{e}_j) = u_iv_j\delta_{ij} = u_iv_i$ and therefore expressed as

$$\mathbf{u}\cdot\mathbf{v} = u_iv_i = u_1v_1 + u_2v_2 + u_3v_3. \tag{1.58}$$

The *magnitude* of $\mathbf{u}$ can be written as

$$|\mathbf{u}| = (\mathbf{u}\cdot\mathbf{u})^{1/2} = (u_1^2 + u_2^2 + u_3^2)^{1/2} = (u_ku_k)^{1/2}. \tag{1.59}$$

– Similarly, it follows from (1.55) and (1.54) that the *vector product* of $\mathbf{u} = u_i\mathbf{e}_i$ and $\mathbf{v} = v_i\mathbf{e}_i$ can be expressed as

$$\mathbf{u}\times\mathbf{v} = (u_j\mathbf{e}_j)\times(v_k\mathbf{e}_k) = u_jv_k\,\mathbf{e}_j\times\mathbf{e}_k \overset{(1.54)}{=} u_jv_k\,e_{jki}\mathbf{e}_i = e_{jki}u_jv_k\,\mathbf{e}_i = (e_{ijk}u_jv_k)\;\mathbf{e}_i, \tag{1.60}$$

where e_{ijk} is the Levi-Civita symbol. Equivalently, the ith component of the vector $\mathbf{u}\times\mathbf{v}$ is

$$(\mathbf{u}\times\mathbf{v})_i = e_{ijk}u_jv_k. \tag{1.61}$$

Exercise: Show that $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = e_{ijk}u_iv_jw_k$.

– Finally, we illustrate through an example how one can go back and forth between vectors and their components. For any three vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ we want to show that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{i}$$

In order to prove this vector identity we shall (implicitly) pick and fix a basis for V and express all vectors in terms of their components in that basis. The left-hand side of (i) is a vector. We start from its ith component:

$$\Big[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\Big]_i \overset{(1.61)}{=} e_{ijk}a_j(\mathbf{b}\times\mathbf{c})_k \overset{(1.61)}{=} e_{ijk}a_j(e_{kpq}b_pc_q) =$$
$$\overset{(1.42)}{=} e_{kij}e_{kpq}a_jb_pc_q \overset{(1.43)}{=} (\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp})a_jb_pc_q =$$
$$= \delta_{ip}\delta_{jq}a_jb_pc_q - \delta_{iq}\delta_{jp}a_jb_pc_q \overset{(*)}{=} a_qb_ic_q - a_jb_jc_i =$$
$$\overset{(1.58)}{=} (\mathbf{a}\cdot\mathbf{c})b_i - (\mathbf{a}\cdot\mathbf{b})c_i$$

from which (i) follows. In step $(*)$ we used the substitution rule.

1.3.2 Worked examples.

Problem 1.3.1. (Vector algebra.) Calculate the area of the triangle OAB defined by the linearly independent vectors $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$ shown in Figure 1.2.

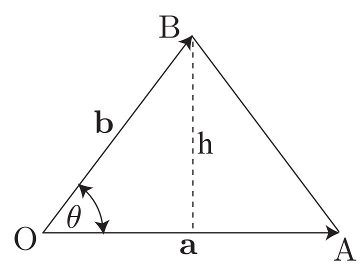

Figure 1.2: Parallelogram OACB.

Solution: The angle $\theta=\angle AOB$ can be calculated from $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$. Suppose $\theta\in(0,\pi/2)$. By geometry, the height h of the triangle is $h=|\mathbf{b}|\sin\theta$. Thus

$$\mathsf{Area\ of\ OAB} = \frac{1}{2}|OA|\,h = \frac{1}{2}|\mathbf{a}|\,h = \frac{1}{2}|\mathbf{a}|\,|\mathbf{b}|\sin\theta \overset{(1.50)}{=} \frac{1}{2}|\mathbf{a}\times\mathbf{b}|.$$

Problem 1.3.2. (Vector algebra.) Show that two vectors $\mathbf{u}$ and $\mathbf{v}$ are equal if and only if

$$\mathbf{u}\cdot\mathbf{x}=\mathbf{v}\cdot\mathbf{x} \qquad \text{for all vectors } \mathbf{x}\in\mathsf{V}. \tag{i}$$

Solution: If $\mathbf{u}=\mathbf{v}$ it is clear that (i) holds. It is the converse that needs to be established. Thus suppose that (i) holds, which we can write as $(\mathbf{u}-\mathbf{v})\cdot\mathbf{x}=0$. Since this holds for all vectors $\mathbf{x}\in\mathsf{V}$ it necessarily holds for the particular choice $\mathbf{x} = \mathbf{u} - \mathbf{v}$. This gives $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0$ whence $|\mathbf{u} - \mathbf{v}| = 0$. However by (1.48), the only vector with zero length is the null vector and so $\mathbf{u} - \mathbf{v} = \mathbf{o}$. $\square$

Problem 1.3.3. (Vector algebra.) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, be three linearly independent (i.e. non-coplanar) vectors with $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} > 0$. Show that the volume V_0 of the tetrahedron formed by them, see Figure 1.11, is

$$V_0 = \frac{1}{6}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

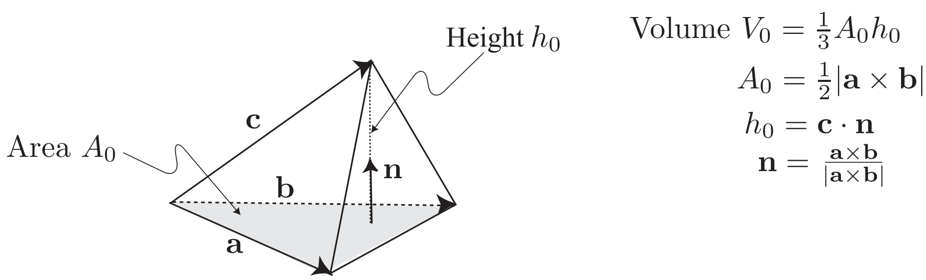

Figure 1.3: Volume of the tetrahedron defined by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Solution: Using the symbols and formulae introduced in Figure 1.3 we have

$$V_0 = \frac{1}{3}A_0 h_0 = \frac{1}{3}\left(\frac{1}{2}|\mathbf{a} \times \mathbf{b}|\right)\frac{\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|} = \frac{1}{6}\,\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

Problem 1.3.4. (Vector algebra) Let $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ be an orthonormal basis and let $\mathbf{b}_1$ and $\mathbf{b}_2$ be two orthogonal vectors that are perpendicular to $\mathbf{a}_2$ and $\mathbf{a}_1$ respectively. Show that either $\mathbf{b}_1$ must be parallel to $\mathbf{a}_1$ or $\mathbf{b}_2$ must be parallel to $\mathbf{a}_2$.

Solution: Expressing the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ in terms of their components in the orthonormal basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ we have

$$\mathbf{b}_1 = \beta_1\mathbf{a}_1 + \cancel{\beta_2}\mathbf{a}_2 + \beta_3\mathbf{a}_3, \qquad \mathbf{b}_2 = \cancel{\gamma_1}\mathbf{a}_1 + \gamma_2\mathbf{a}_2 + \gamma_3\mathbf{a}_3, \tag{i}$$

where we have used $\mathbf{b}_1 \cdot \mathbf{a}_2 = 0$ and $\mathbf{b}_2 \cdot \mathbf{a}_1 = 0$. We are told that $\mathbf{b}_1 \cdot \mathbf{b}_2 = 0$ which, since $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is orthonormal, requires

$$\mathbf{b}_1 \cdot \mathbf{b}_2 = \gamma_3\beta_3 = 0.$$

Therefore either $\gamma_3 = 0$ in which case $(i)_2$ implies that $\mathbf{b}_2$ is parallel to $\mathbf{a}_2$, or $\beta_3 = 0$ in which case $\mathbf{b}_1$ is parallel to $\mathbf{a}_1$ according to $(i)_1$.

Problem 1.3.5. (Vector algebra.) (Cauchy-Schwartz inequality.) Show that

$$|\mathbf{u}|^2\,|\mathbf{v}|^2 \geq (\mathbf{u}\cdot\mathbf{v})^2 \qquad \text{for all } \mathbf{u},\mathbf{v}\in\mathsf{V}. \tag{1.62}$$

Solution: If $\mathbf{u}\cdot\mathbf{v}=0$ the result holds trivially and so consider the case $\mathbf{u}\cdot\mathbf{v}\neq 0$. It then follows that $\mathbf{u}$ and $\mathbf{v}$ are both $\neq \mathbf{o}$ and therefore that $|\mathbf{u}|\neq 0$ and $|\mathbf{v}|\neq 0$. Next, for any scalar ξ we have $|\mathbf{u}-\xi\mathbf{v}|^2\geq 0$ which when expanded out reads

$$|\mathbf{u}|^2 - 2\xi\mathbf{u}\cdot\mathbf{v} + \xi^2|\mathbf{v}|^2 \geq 0.$$

The left-hand side of this expression is a quadratic form in ξ that achieves its smallest value at $\xi = \mathbf{u}\cdot\mathbf{v}/|\mathbf{v}|^2$. Substituting this value of ξ into the preceding inequality and simplifying leads to (1.62).

Exercise: Show that (1.62) holds with equality if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

Problem 1.3.6. (Vectors. Components.) For any four vectors $\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s}$ show, by expressing the vectors in terms of their components in a basis, that

$$(\mathbf{p}\times\mathbf{q})\cdot(\mathbf{r}\times\mathbf{s}) = (\mathbf{p}\cdot\mathbf{r})(\mathbf{q}\cdot\mathbf{s}) - (\mathbf{q}\cdot\mathbf{r})(\mathbf{p}\cdot\mathbf{s}). \tag{i}$$

Hence or otherwise, establish Lagrange's identity

$$|\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2 \qquad \text{for all } \mathbf{a} \text{ and } \mathbf{b}\in\mathsf{V}. \tag{ii}$$

Solution: Both $\mathbf{p}\times\mathbf{q}$ and $\mathbf{r}\times\mathbf{s}$ are vectors and the left-hand side of (i) is the scalar product of these vectors. Thus

$$\begin{aligned}(\mathbf{p}\times\mathbf{q})\cdot(\mathbf{r}\times\mathbf{s}) &\overset{(1.58)}{=} (\mathbf{p}\times\mathbf{q})_i(\mathbf{r}\times\mathbf{s})_i \overset{(1.61)}{=} (e_{ijk}p_jq_k)(e_{imn}r_ms_n) = e_{ijk}e_{imn}p_jq_kr_ms_n = \\ &\overset{(1.43)}{=} (\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})p_jq_kr_ms_n = \delta_{jm}\delta_{kn}p_jq_kr_ms_n - \delta_{jn}\delta_{km}p_jq_kr_ms_n = \\ &\overset{(*)}{=} p_mq_nr_ms_n - p_nq_mr_ms_n = (\mathbf{p}\cdot\mathbf{r})(\mathbf{q}\cdot\mathbf{s}) - (\mathbf{p}\cdot\mathbf{s})(\mathbf{q}\cdot\mathbf{r}), \qquad \square\end{aligned}$$

where in step $(*)$ we used the substitution rule. Pick $\mathbf{p}=\mathbf{r}=\mathbf{a}$ and $\mathbf{q}=\mathbf{s}=\mathbf{b}$ in (i):

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{b}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{b}),$$

which gives the desired result:

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2. \qquad \square$$

1.4 Tensor algebra.

– In this section we consider **linear transformations** from the vector space $\mathsf{V}\to\mathsf{V}$. Such a transformation $\mathbf{A}$ takes each vector $\mathbf{x}\in\mathsf{V}$ and maps it into another vector in

V that we denote by $\mathbf{A}\mathbf{x}$ (where the mapping is subject to certain rules pertaining to addition, multiplication by a scalar and the null linear transformation).

– The most familiar examples of linear transformations are perhaps geometric, say rotation through an angle π about a certain axis which takes each vector into its rotated image, see Figure 1.4.

  Figure 2.6 depicts a deformation that carries an infinitesimal material fiber $d\mathbf{x}$ in an undeformed body into its deformed image $d\mathbf{y} = \mathbf{F}d\mathbf{x}$, $\mathbf{F}$ being the deformation gradient tensor. Other examples from mechanics include the inertia "tensor" $\mathbf{J}$ that takes the angular velocity vector $\boldsymbol{\omega}$ into the angular momentum vector $\mathbf{h} = \mathbf{J}\boldsymbol{\omega}$; and the stress tensor $\mathbf{T}$ that takes a unit normal vector $\mathbf{n}$ into the traction vector $\mathbf{t} = \mathbf{T}\mathbf{n}$.

– Let $\mathbf{F}$ be a function (or transformation) that maps each vector $\mathbf{x} \in \mathsf{V}$ into a second vector $\mathbf{F}(\mathbf{x}) \in \mathsf{V}$:

$$\mathbf{x} \to \mathbf{F}(\mathbf{x}), \qquad \mathbf{x} \in \mathsf{V}, \quad \mathbf{F}(\mathbf{x}) \in \mathsf{V}. \tag{1.63}$$

  It is said to be a *linear* transformation if

$$\mathbf{F}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{F}(\mathbf{x}) + \beta\mathbf{F}(\mathbf{y}) \tag{1.64}$$

  for all scalars α, β and all vectors $\mathbf{x}, \mathbf{y}$ in V. When $\mathbf{F}$ is a linear transformation, we usually omit the parenthesis and write $\mathbf{F}\mathbf{x}$ instead of $\mathbf{F}(\mathbf{x})$. Note that $\mathbf{F}\mathbf{x}$ is a vector, and it is the **image** of $\mathbf{x}$ under the transformation $\mathbf{F}$.

– We shall refer to a linear transformation (from V into V) as a **tensor**.

– Observe that a tensor is defined by the way in which it operates on each vector in V.

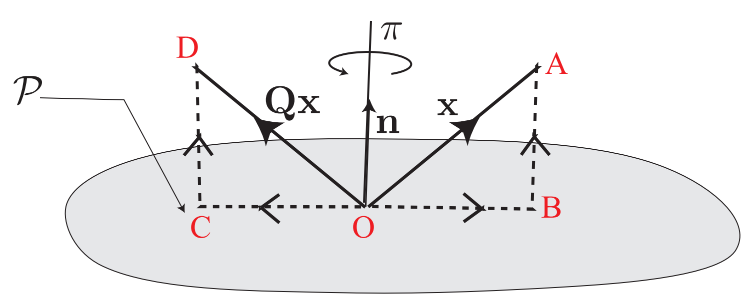

Figure 1.4: The tensor $\mathbf{Q}$ rotates the vector $\mathbf{x} = \overrightarrow{OA}$ through an angle π about the axis $\mathbf{n}$ and takes it to the vector $\mathbf{Qx} = \overrightarrow{OD}$. The plane $\mathcal{P}$ is perpendicular to $\mathbf{n}$

Example: As a geometric example consider the transformation $\mathbf{Q}$ implied by Figure 1.4. It rotates a vector $\mathbf{x}\,(=\overrightarrow{OA})$ through an angle π about an axis $\mathbf{n}$ and carries it into the vector $\mathbf{Qx}\,(=\overrightarrow{OD})$. (Here $\mathbf{x}$ is an arbitrary vector.) The plane $\mathcal{P}$ in the figure is perpendicular to the unit vector $\mathbf{n}$. Observe that $\overrightarrow{OB} = -\overrightarrow{OC}$ and $\overrightarrow{BA} = \overrightarrow{CD}$. Moreover, the magnitude of the vector $\overrightarrow{BA}$ is $\mathbf{x}\cdot\mathbf{n}$ and its direction is $\mathbf{n}$ and therefore $\overrightarrow{BA} = (\mathbf{x}\cdot\mathbf{n})\mathbf{n}$. Thus

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = -\overrightarrow{OB} + \overrightarrow{BA} = -(\overrightarrow{OA} - \overrightarrow{BA}) + \overrightarrow{BA} = -\overrightarrow{OA} + 2\,\overrightarrow{BA},$$

and so we have

$$\mathbf{Qx} = -\mathbf{x} + 2(\mathbf{x}\cdot\mathbf{n})\mathbf{n} \qquad \text{for all } \mathbf{x} \in \mathsf{V}. \tag{1.65}$$

Given any vector $\mathbf{x} \in \mathsf{V}$, the right-hand side of (1.65) tells us how to calculate $\mathbf{Qx}$, i.e. it tell us what $\mathbf{Q}$ does to every vector $\mathbf{x}$, and therefore it defines $\mathbf{Q}$. Observe that $\mathbf{Q}$ is a linear transformation since $\mathbf{Q}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{Qx} + \beta\mathbf{Qy}$.

– The *identity tensor* $\mathbf{I}$ and the *null tensor* $\mathbf{0}$ have the properties

$$\mathbf{Ix} = \mathbf{x}, \qquad \mathbf{0x} = \mathbf{o} \qquad \text{for all } \mathbf{x} \in \mathsf{V}, \tag{1.66}$$

where $\mathbf{o}$ is the null vector.

– Continuing to keep in mind that a tensor is defined by the way it operates on vectors, given two tensors $\mathbf{A}$ and $\mathbf{B}$, their product $\mathbf{AB}$ is defined by

$$(\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{Bx}) \qquad \text{for all } \mathbf{x} \in \mathsf{V};$$

i.e. first the tensor $\mathbf{B}$ operates on the vector $\mathbf{x}$ to produce the vector $\mathbf{Bx}$ and then the tensor $\mathbf{A}$ operates on the vector $\mathbf{Bx}$ to produce the vector $\mathbf{ABx}$. In general, $\mathbf{AB} \neq \mathbf{BA}$.

Tensor product.

– Let $\mathbf{a}$ and $\mathbf{b}$ be two given vectors. Define the associated tensor $\mathbf{T}$ as the transformation that takes an arbitrary vector $\mathbf{x}$ into the vector $\mathbf{Tx}$ defined by

$$\mathbf{Tx} = (\mathbf{x}\cdot\mathbf{b})\mathbf{a} \qquad \text{for all } \mathbf{x} \in \mathsf{V}. \tag{1.67}$$

Observe that corresponding to any vector $\mathbf{x} \in \mathsf{V}$, the right-hand side of (1.67) provides a formula for calculating the vector $\mathbf{Tx}$. Clearly this is a *linear* transformation. This particular tensor $\mathbf{T}$ is called the *tensor product*[4] of the vectors $\mathbf{a}$ and $\mathbf{b}$ and is denoted by

$$\mathbf{T} = \mathbf{a} \otimes \mathbf{b}. \tag{1.68}$$

Thus from (1.67) and (1.68)

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{x} = (\mathbf{x} \cdot \mathbf{b})\mathbf{a} \qquad \text{for all } \mathbf{x} \in \mathsf{V}. \tag{1.69}$$

Note that the vector $(\mathbf{a} \otimes \mathbf{b})\mathbf{x}$ is parallel to the vector $\mathbf{a}$ for all $\mathbf{x}$.

For example observe that we can now write (1.65) as

$$\mathbf{Qx} \;=\; -\mathbf{x} + 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n} \;=\; -\mathbf{x} + 2(\mathbf{n} \otimes \mathbf{n})\mathbf{x} \;=\; \Big[-\mathbf{I} + 2(\mathbf{n} \otimes \mathbf{n}) \Big]\mathbf{x}\,,$$

and so the tensor $\mathbf{Q}$ representing a rotation through an angle π about the axis $\mathbf{n}$ is

$$\mathbf{Q} \;=\; -\mathbf{I} + 2\mathbf{n} \otimes \mathbf{n}. \tag{1.70}$$

If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an arbitrary orthonormal basis for V then (Problem 1.4.1)

$$\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 = \mathbf{I}. \tag{1.71}$$

For all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathsf{V}$, (Problem 1.4.2)

$$(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) \;=\; (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \otimes \mathbf{d}). \tag{1.72}$$

– The set of all tensors (linear transformations from $\mathsf{V} \to \mathsf{V}$) is itself a vector space which we shall denote by Lin ("Lin" standing for linear transformation). It is 9-dimensional. The 9 tensors $\mathbf{e}_i \otimes \mathbf{e}_j, i, j = 1, 2, 3$, are linearly independent and thus form a basis for Lin[5]. Therefore given any tensor $\mathbf{A}$, there is a unique set of nine scalars A_{ij} such that

$$\mathbf{A} = A_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j; \tag{1.73}$$

the A_{ij}'s are the components of $\mathbf{A}$ in this basis. The components A_{ij} of the tensor depend on the basis. If we change the basis, the components A_{ij} will change even if we don't change the tensor $\mathbf{A}$.

Our analysis of tensor algebra in this subsection will *not* depend on the choice of basis and the components of the tensor in that basis. Even so, it will sometimes be useful to refer to tensor components, e.g. in (1.75), though we could have postponed all references to components to Section 1.4.3 where we shall say a lot more about them.

[4] or outer product or dyadic product

[5] After introducing the notion of a scalar product between two tensors in (1.120), we can then speak of the magnitude of a tensor and of two tensors being orthogonal. This will allow us to show that these nine tensors in fact form an *orthonormal basis* for Lin; see Problem 1.4.12.

Problem 1.4.1. (Tensor product.) If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an arbitrary orthonormal basis for V, show that

$$\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 = \mathbf{I}.$$

Solution: For an arbitrary vector $\mathbf{x}$,

$$(\mathbf{e}_i \otimes \mathbf{e}_i)\mathbf{x} \stackrel{(1.69)}{=} (\mathbf{e}_i \cdot \mathbf{x})\mathbf{e}_i \stackrel{(1.56)}{=} x_i \mathbf{e}_i \stackrel{(1.55)}{=} \mathbf{x} \stackrel{(1.66)_1}{=} \mathbf{I}\mathbf{x}$$

Since this holds for all $\mathbf{x} \in \mathsf{V}$ it follows that $\mathbf{e}_i \otimes \mathbf{e}_i = \mathbf{I}$.

Problem 1.4.2. (Tensor product.) For all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathsf{V}$, show that

$$(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) \;=\; (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \otimes \mathbf{d}). \qquad (i)$$

Solution: Before carrying out any calculation it is useful to observe that on the left-hand side we have a tensor (which is the product of the two tensors $\mathbf{a} \otimes \mathbf{b}$ and $\mathbf{c} \otimes \mathbf{d}$). On the right-hand side we also have a tensor, as we must, (and this is the product of the tensor $\mathbf{a} \otimes \mathbf{d}$ by the scalar $\mathbf{b} \cdot \mathbf{c}$). Also, note that if we want to show that two tensors $\mathbf{A}$ and $\mathbf{B}$ are equal, we could show that $\mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{x}$ for all $\mathbf{x} \in \mathsf{V}$. With these in mind, for an arbitrary vector $\mathbf{x}$, we calculate

$$\begin{aligned}
(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d})\mathbf{x} \;&=\; (\mathbf{a} \otimes \mathbf{b})\Big((\mathbf{c} \otimes \mathbf{d})\mathbf{x}\Big) \stackrel{(1.69)}{=} (\mathbf{a} \otimes \mathbf{b})\Big((\mathbf{d} \cdot \mathbf{x})\mathbf{c}\Big) = \\
&\stackrel{(*)}{=} (\mathbf{d} \cdot \mathbf{x})(\mathbf{a} \otimes \mathbf{b})\mathbf{c} \stackrel{(1.69)}{=} (\mathbf{d} \cdot \mathbf{x})(\mathbf{b} \cdot \mathbf{c})\mathbf{a} \stackrel{(**)}{=} (\mathbf{b} \cdot \mathbf{c})(\mathbf{d} \cdot \mathbf{x})\mathbf{a} = \\
&= \stackrel{(1.69)}{=} (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \otimes \mathbf{d})\mathbf{x}
\end{aligned}$$

where in step $(*)$ we used the fact that $\mathbf{d} \cdot \mathbf{x}$ is a scalar and so moved it to the front, and in step $(**)$ we used the fact that $\mathbf{b} \cdot \mathbf{c}$ is a scalar and so moved it to the front. This establishes (i).

The transpose. Symmetric and skew-symmetric tensors.

– Corresponding to any tensor $\mathbf{A}$, there exists a second tensor that we denote by $\mathbf{A}^T$ such that

$$\mathbf{A}\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{A}^T\mathbf{y} \qquad \text{for all vectors } \mathbf{x} \text{ and } \mathbf{y} \in \mathsf{V}; \tag{1.74}$$

$\mathbf{A}^T$ is called the **transpose** of $\mathbf{A}$.

Exercise: Show that

$$\mathbf{A} = A_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j \qquad \Leftrightarrow \qquad \mathbf{A}^T = A_{ji}\, \mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.75}$$

Exercise: Show that

$$(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T \qquad \text{for all tensors } \mathbf{A}, \mathbf{B}. \tag{1.76}$$

For all vectors $\mathbf{a}, \mathbf{b}$ and all tensors $\mathbf{A}$ (Problem 1.4.3)

$$(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}, \tag{1.77}$$

$$\mathbf{A}(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{A}\mathbf{a}) \otimes \mathbf{b}, \qquad (\mathbf{a} \otimes \mathbf{b})\mathbf{A} = \mathbf{a} \otimes (\mathbf{A}^T\mathbf{b}). \tag{1.78}$$

– A tensor $\mathbf{A}$ is **symmetric** if

$$\mathbf{A} = \mathbf{A}^T, \tag{1.79}$$

and **skew-symmetric** (or anti-symmetric) if

$$\mathbf{A} = -\mathbf{A}^T. \tag{1.80}$$

Several tensors that we will encounter including the Cauchy stress tensor $\mathbf{T}$ and the Lagrangian and Eulerian stretch tensors $\mathbf{U}$ and $\mathbf{V}$ will be symmetric.

– Any tensor $\mathbf{A}$ can be uniquely decomposed into the sum of a symmetric tensor $\mathbf{S}$ and a skew-symmetric tensor $\mathbf{W}$:

$$\mathbf{A} = \mathbf{S} + \mathbf{W}, \qquad \mathbf{S} = \mathbf{S}^T, \qquad \mathbf{W} = -\mathbf{W}^T; \tag{1.81}$$

$\mathbf{S}$ is called the *symmetric part* of $\mathbf{A}$ and $\mathbf{W}$ its *skew-symmetric* part. They are given by

$$\mathbf{S} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T), \qquad \mathbf{W} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T). \tag{1.82}$$

– Corresponding to every skew-symmetric tensor $\mathbf{W}$, there is a vector $\boldsymbol{\omega}$ such that

$$\boldsymbol{\omega} \times \mathbf{x} = \mathbf{W}\mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathsf{V}; \tag{1.83}$$

$\boldsymbol{\omega}$ is called the *axial vector* associated with $\mathbf{W}$.

Exercise: If $\mathbf{W}$ is skew-symmetric show that $\mathbf{W}^2$ is symmetric and $\mathbf{W}^3$ skew-symmetric.

Problem 1.4.3. Establish (1.77) and (1.78):

$$(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}, \qquad \mathbf{A}(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{A}\mathbf{a}) \otimes \mathbf{b}, \qquad (\mathbf{a} \otimes \mathbf{b})\mathbf{A} = \mathbf{a} \otimes (\mathbf{A}^T\mathbf{b}). \tag{i}$$

Solution: In order to establish $(i)_1$ we must show, according to (1.74), that $(\mathbf{a} \otimes \mathbf{b})\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot (\mathbf{b} \otimes \mathbf{a})\mathbf{y}$ for all vectors $\mathbf{x}$ and $\mathbf{y}$. This follows from:

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{x} \cdot \mathbf{y} \overset{(1.69)}{=} [(\mathbf{b} \cdot \mathbf{x})\mathbf{a}] \cdot \mathbf{y} = (\mathbf{b} \cdot \mathbf{x})(\mathbf{a} \cdot \mathbf{y}) = (\mathbf{x} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{y}) = \mathbf{x} \cdot [(\mathbf{a} \cdot \mathbf{y})\mathbf{b}] \overset{(1.69)}{=} \mathbf{x} \cdot (\mathbf{b} \otimes \mathbf{a})\mathbf{y}$$

where we have used the fact that $\mathbf{a}\cdot\mathbf{x}$ and $\mathbf{b}\cdot\mathbf{y}$ are scalars.

To show $(i)_2$ we proceed as follows: for any vector $\mathbf{x}$,

$$\mathbf{A}(\mathbf{a}\otimes\mathbf{b})\mathbf{x} \overset{(1.69)}{=} \mathbf{A}[(\mathbf{b}\cdot\mathbf{x})\mathbf{a}] = (\mathbf{b}\cdot\mathbf{x})\mathbf{A}\mathbf{a} \overset{(1.69)}{=} (\mathbf{A}\mathbf{a}\otimes\mathbf{b})\mathbf{x},$$

which establishes $(i)_2$. The result in $(i)_3$ can be obtained similarly.

Nonsingular tensors.

We are frequently interested in linear transformations that are **one-to-one**. For example suppose two particles of a body are located at $\mathbf{x}_1$ and $\mathbf{x}_2$, and that the tensor $\mathbf{F}$ maps ("deforms" the body and takes) them to the locations $\mathbf{F}\mathbf{x}_1$ and $\mathbf{F}\mathbf{x}_2$. We typically want particles not to coalesce: if the particles $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct we want their locations $\mathbf{F}\mathbf{x}_1$ and $\mathbf{F}\mathbf{x}_2$ to be distinct, i.e. we want $\mathbf{x}_1 \neq \mathbf{x}_2$ to imply $\mathbf{F}\mathbf{x}_1 \neq \mathbf{F}\mathbf{x}_2$. In addition we usually want particles not to split: if the locations $\mathbf{F}\mathbf{x}_1$ and $\mathbf{F}\mathbf{x}_2$ are distinct we want them to correspond to distinct particles, i.e. we want $\mathbf{F}\mathbf{x}_1 \neq \mathbf{F}\mathbf{x}_2$ to imply $\mathbf{x}_1 \neq \mathbf{x}_2$. Together, they require $\mathbf{F}\mathbf{x}_1 = \mathbf{F}\mathbf{x}_2 \Leftrightarrow \mathbf{x}_1 = \mathbf{x}_2$, i.e. the linear transformation $\mathbf{F}$ must be one-to-one.

The following statements *are equivalent*: A tensor $\mathbf{F}$ is **nonsingular** (or "one-to-one" or "invertible") if and only if

(a) For all vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathsf{V}$,

$$\mathbf{F}\mathbf{x}_1 = \mathbf{F}\mathbf{x}_2 \qquad \Leftrightarrow \qquad \mathbf{x}_1 = \mathbf{x}_2. \tag{1.84}$$

(b) The only vector $\mathbf{x}$ for which $\mathbf{F}\mathbf{x} = \mathbf{o}$ is the null vector $\mathbf{x} = \mathbf{o}$:

$$\mathbf{F}\mathbf{x} = \mathbf{o} \qquad \Leftrightarrow \qquad \mathbf{x} = \mathbf{o}. \tag{1.85}$$

(c) The only vector $\mathbf{x}$ for which $|\mathbf{F}\mathbf{x}| = 0$ is the null vector $\mathbf{x} = \mathbf{o}$:

$$|\mathbf{F}\mathbf{x}| = 0 \qquad \Leftrightarrow \qquad \mathbf{x} = \mathbf{o}.$$

(d) There exists a unique tensor that we denote by $\mathbf{F}^{-1}$, and call the **inverse** of $\mathbf{F}$, such that

$$\mathbf{F}\mathbf{F}^{-1} = \mathbf{F}^{-1}\mathbf{F} = \mathbf{I}. \tag{1.86}$$

(e) Also (1.94) below.

Exercise: Show that these statements are equivalent; e.g. the example on page 34 shows that (1.94) implies (1.85).

– If $\mathbf{A}$ and $\mathbf{B}$ are both nonsingular, then

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}. \tag{1.87}$$

We shall denote the transpose of the inverse tensor, which equals the inverse of the transpose, by

$$\mathbf{A}^{-T} := (\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}. \tag{1.88}$$

– A tensor $\mathbf{A}$ is **positive definite** if

$$\mathbf{Ax}\cdot\mathbf{x} > 0 \qquad \text{for all vectors } \mathbf{x} \neq \mathbf{o}. \tag{1.89}$$

Exercise: Show that a positive definite tensor is necessarily nonsingular but a nonsingular tensor need not be positive definite.

Exercise: For any nonsingular tensor $\mathbf{F}$ show that the tensor $\mathbf{F}^T\mathbf{F}$ is symmetric and positive definite.

– The **determinant** of a tensor $\mathbf{A}$ is the scalar defined by

$$\det\mathbf{A} := \frac{\mathbf{Ax}\cdot(\mathbf{Ay}\times\mathbf{Az})}{\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})} \qquad \text{for all linearly independent vectors } \mathbf{x},\mathbf{y},\mathbf{z}. \tag{1.90}$$

We know from Problem 1.3.3 that the volume of the tetrahedron defined by three linearly independent vectors $\mathbf{x},\mathbf{y},\mathbf{z}$ is $\frac{1}{6}\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})$ and therefore the volume of its image under the linear transformation $\mathbf{A}$ is $\frac{1}{6}\mathbf{Ax}\cdot(\mathbf{Ay}\times\mathbf{Az})$. The determinant is therefore the ratio between these two volumes. One can show that this *ratio* is independent of the particular choice of $\mathbf{x},\mathbf{y}$ and $\mathbf{z}$ and so depends only on $\mathbf{A}$.

The determinant of the product of two tensors has the property

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det\mathbf{A}\ \det\mathbf{B}, \tag{1.91}$$

and for any scalar α

$$\det(\alpha\mathbf{A}) = \alpha^3\det\mathbf{A}. \tag{1.92}$$

The determinants of $\mathbf{A}$ and $\mathbf{A}^T$ coincide:

$$\det\mathbf{A}^T = \det\mathbf{A}. \tag{1.93}$$

– A tensor $\mathbf{A}$ is nonsingular, as defined previously in (1.85), if and only if

$$\det \mathbf{A} \neq 0. \tag{1.94}$$

Problem 1.4.4. (Nonsingular tensor.) If $\det \mathbf{A} = 0$ show that there is a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{o}$; see (1.94) and (1.85).

Solution: Suppose $\det \mathbf{A} = 0$. Then for any three linearly independent vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, equation (1.90) tells us that $\mathbf{Aa} \cdot (\mathbf{Ab} \times \mathbf{Ac}) = 0$. Therefore by (1.52) the vectors $\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}$ are linearly dependent and so there are scalars α, β, γ such that $\alpha\mathbf{Aa} + \beta\mathbf{Ab} + \gamma\mathbf{Ac} = \mathbf{o}$. Since this can be written as $\mathbf{A}(\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}) = \mathbf{o}$ we conclude that $\mathbf{Ax} = \mathbf{o}$ for $\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$.

Orthogonal tensors.

Orthogonal tensors will play an important role in both the rigid deformation of a body and the mapping between two different bases.

The following statements *are equivalent*: A tensor $\mathbf{Q}$ is **orthogonal** if and only if:

(a) It preserves the length of every vector:

$$|\mathbf{Qx}| = |\mathbf{x}| \qquad \text{for all } \mathbf{x} \in \mathsf{V}. \tag{1.95}$$

(b) It preserves the length of every vector and the angle between every pair of vectors:

$$\mathbf{Qx} \cdot \mathbf{Qy} = \mathbf{x} \cdot \mathbf{y} \qquad \text{for all } \mathbf{x} \text{ and } \mathbf{y} \in \mathsf{V}. \tag{1.96}$$

(c) It is nonsingular and

$$\mathbf{Q}^{-1} = \mathbf{Q}^T, \tag{1.97}$$

from which it follows that

$$\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}. \tag{1.98}$$

Exercise: Show that the preceding statements are equivalent; e.g. Problem 1.4.5 shows that (1.95) implies (1.96); and Problem 1.4.17 shows that (1.95) implies (1.97).

If $\mathbf{Q}$ is orthogonal, then

$$\det \mathbf{Q} = \pm 1. \tag{1.99}$$

(The converse is not true: $\det \mathbf{Q} = \pm 1$ does not imply that $\mathbf{Q}$ is orthogonal.) If $\det \mathbf{Q} = +1$, $\mathbf{Q}$ is said to be *proper orthogonal*. Otherwise it is *improper orthogonal*. If $\mathbf{Q}$ is improper orthogonal, then $-\mathbf{Q}$ is proper orthogonal and vice versa.

Exercise: Show that the tensor

$$-\mathbf{I} + 2\mathbf{n} \otimes \mathbf{n} \tag{1.100}$$

is proper orthogonal. (We saw earlier in (1.70) that it describes a 180^o rotation about the unit vector $\mathbf{n}$). Show that the tensor

$$\mathbf{I} - 2\mathbf{n} \otimes \mathbf{n} \tag{1.101}$$

is improper orthogonal. (It describes a reflection in the plane perpendicular to $\mathbf{n}$; Problem 1.10.)

1.4.1 Worked examples.

Problem 1.4.5. (Orthogonal tensors.) According to (1.95), a tensor $\mathbf{Q}$ is orthogonal if it preserves the length of every vector $\mathbf{x}$:

$$|\mathbf{Q}\mathbf{x}| = |\mathbf{x}| \qquad \text{for all } \mathbf{x} \in \mathsf{V}. \tag{i}$$

Show that such a tensor necessarily preserves the angle between every pair of vectors $\mathbf{x}$ and $\mathbf{y}$ so that

$$\mathbf{Q}\mathbf{x} \cdot \mathbf{Q}\mathbf{y} = \mathbf{x} \cdot \mathbf{y} \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathsf{V}. \tag{ii}$$

This establishes (1.96).

Solution: Since (i) holds for all vectors in V, it necessarily holds for the vectors $\mathbf{y}$ and $\mathbf{x} - \mathbf{y}$, i.e.

$$|\mathbf{Q}\mathbf{y}| = |\mathbf{y}| \qquad \text{and} \qquad |\mathbf{Q}(\mathbf{x} - \mathbf{y})| = |(\mathbf{x} - \mathbf{y})|. \tag{iii}$$

Now evaluate $|\mathbf{Q}(\mathbf{x} - \mathbf{y})|^2$:

$$\begin{aligned} |\mathbf{Q}(\mathbf{x}-\mathbf{y})|^2 &= \mathbf{Q}(\mathbf{x}-\mathbf{y}) \cdot \mathbf{Q}(\mathbf{x}-\mathbf{y}) = (\mathbf{Q}\mathbf{x} - \mathbf{Q}\mathbf{y}) \cdot (\mathbf{Q}\mathbf{x} - \mathbf{Q}\mathbf{y}) = \\ &= \mathbf{Q}\mathbf{x} \cdot \mathbf{Q}\mathbf{x} - \mathbf{Q}\mathbf{y} \cdot \mathbf{Q}\mathbf{x} - \mathbf{Q}\mathbf{x} \cdot \mathbf{Q}\mathbf{y} + \mathbf{Q}\mathbf{y} \cdot \mathbf{Q}\mathbf{y} = \\ &= |\mathbf{Q}\mathbf{x}|^2 - 2\mathbf{Q}\mathbf{x} \cdot \mathbf{Q}\mathbf{y} + |\mathbf{Q}\mathbf{y}|^2 \overset{(i),(iii)_1}{=} |\mathbf{x}|^2 - 2\mathbf{Q}\mathbf{x} \cdot \mathbf{Q}\mathbf{y} + |\mathbf{y}|^2. \end{aligned} \tag{iv}$$

Alternately, by using $(iii)_2$ we can write

$$\begin{aligned} |\mathbf{Q}(\mathbf{x}-\mathbf{y})|^2 &= |\mathbf{x}-\mathbf{y}|^2 = (\mathbf{x}-\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) = \mathbf{x} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} = \\ &= |\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2. \end{aligned} \tag{v}$$

Equating (iv) and (v) yields the desired result.

Problem 1.4.6. (Orthogonal tensors.) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ be two orthonormal bases. (a) Show that the tensor

$$\mathbf{Q} = \mathbf{e}'_1 \otimes \mathbf{e}_1 + \mathbf{e}'_2 \otimes \mathbf{e}_2 + \mathbf{e}'_3 \otimes \mathbf{e}_3 \qquad (i)$$

is orthogonal. In fact, (b) show that $\mathbf{Q}$ maps $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ into $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. (c) If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is right-handed, show that $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is right-handed if and only if $\mathbf{Q}$ is proper orthogonal.

Solution: (a) Since $\mathbf{Q} = \mathbf{e}'_i \otimes \mathbf{e}_i$ it follows from (1.77) that

$$\mathbf{Q}^T = \mathbf{e}_i \otimes \mathbf{e}'_i. \qquad (ii)$$

Thus

$$\mathbf{Q}\mathbf{Q}^T = (\mathbf{e}'_i \otimes \mathbf{e}_i)(\mathbf{e}_j \otimes \mathbf{e}'_j) \overset{(1.72)}{=} (\mathbf{e}_i \cdot \mathbf{e}_j)(\mathbf{e}'_i \otimes \mathbf{e}'_j) \overset{(1.53)}{=} \delta_{ij}(\mathbf{e}'_i \otimes \mathbf{e}'_j) = \mathbf{e}'_i \otimes \mathbf{e}'_i \overset{(1.71)}{=} \mathbf{I}$$

and so

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I}. \qquad (iii)$$

Therefore by (1.98), $\mathbf{Q}$ is orthogonal. Alternatively, for any vector $\mathbf{x}$,

$$\begin{aligned} |\mathbf{Q}\mathbf{x}|^2 \quad &= \quad \mathbf{Q}\mathbf{x} \cdot \mathbf{Q}\mathbf{x} \overset{(i)}{=} \big[(\mathbf{e}'_i \otimes \mathbf{e}_i)\mathbf{x}\big] \cdot \big[(\mathbf{e}'_j \otimes \mathbf{e}_j)\mathbf{x}\big] \overset{(1.69)}{=} \big[(\mathbf{x} \cdot \mathbf{e}_i)\mathbf{e}'_i\big] \cdot \big[(\mathbf{x} \cdot \mathbf{e}_j)\mathbf{e}'_j\big] = \\ &\overset{(1.56)}{=} \quad (x_i \mathbf{e}'_i) \cdot (x_j \mathbf{e}'_j) = x_i x_j (\mathbf{e}'_i \cdot \mathbf{e}'_j) \overset{(1.53)}{=} x_i x_j \delta_{ij} = x_i x_i = \mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2 \end{aligned}$$

and therefore by (1.95), $\mathbf{Q}$ is orthogonal.

(b) Since

$$\mathbf{Q}\mathbf{e}_i = (\mathbf{e}'_j \otimes \mathbf{e}_j)\mathbf{e}_i = (\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{e}'_j = \delta_{ij}\mathbf{e}'_j = \mathbf{e}'_i$$

it follows that $\mathbf{Q}$ maps each $\mathbf{e}_i$ into $\mathbf{e}'_i$.

(c) We are told that $(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 > 0$ and want to determine when $(\mathbf{Q}\mathbf{e}_1 \times \mathbf{Q}\mathbf{e}_2) \cdot \mathbf{Q}\mathbf{e}_3 > 0$. Since

$$(\mathbf{Q}\mathbf{e}_1 \times \mathbf{Q}\mathbf{e}_2) \cdot \mathbf{Q}\mathbf{e}_3 \overset{(1.90)}{=} \det \mathbf{Q}\, (\mathbf{e}_1 \otimes \mathbf{e}_2) \cdot \mathbf{e}_3$$

and $(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 > 0$ it follows that $(\mathbf{Q}\mathbf{e}_1 \times \mathbf{Q}\mathbf{e}_2) \cdot \mathbf{Q}\mathbf{e}_3 > 0$ if and only if $\det \mathbf{Q} > 0$. But from (iii) we know that $\det \mathbf{Q} = \pm 1$ and so the requirement is that $\det \mathbf{Q} = +1$ whence $\mathbf{Q}$ must be proper orthogonal.

Problem 1.4.7. (Orthogonal tensors.) Consider the tensor $\mathbf{R}$ that maps a vector $\mathbf{x}$ into the vector $\mathbf{R}\mathbf{x}$ according to

$$\mathbf{R}\mathbf{x} = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n} \qquad \text{for all } \mathbf{x} \in \mathsf{V}. \qquad (i)$$

Here $\mathbf{n}$ is a given unit vector. Show that

(a) $\mathbf{R}$ is nonsingular,

(b) $\mathbf{R}$ is symmetric,

(c) $\mathbf{R}$ is orthogonal,

(d) $\mathbf{R}$ is in fact improper orthogonal, and

(e) $\mathbf{R}^2 = \mathbf{I}$ and therefore that $\mathbf{R}$ is a square root of the identity tensor $\mathbf{I}$. Note that $\mathbf{R} \neq \mathbf{I}$.

It will be shown in Problem 1.10 that the tensor $\mathbf{R}$ describes reflections in the plane perpendicular to $\mathbf{n}$.

Solution: The solutions below do not use components in a basis. As an exercise, work this problem using components.

Solution 1: By the definition (1.69) of the tensor product we know that $(\mathbf{x} \cdot \mathbf{n})\mathbf{n} = (\mathbf{n} \otimes \mathbf{n})\mathbf{x}$. Therefore we can write (i) equivalently as

$$\mathbf{R}\mathbf{x} = (\mathbf{I} - 2\mathbf{n} \otimes \mathbf{n})\mathbf{x}$$

which holds for all vectors $\mathbf{x}$. Therefore

$$\mathbf{R} = \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}. \tag{1.102}$$

(a) One way to show that $\mathbf{R}$ is nonsingular is to show that $\det \mathbf{R} \neq 0$. From (1.102) and (1.197) (page 94),

$$\det \mathbf{R} = \det(\mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}) = 1 + (-2\mathbf{n}) \cdot \mathbf{n} = -1$$

and so $\det \mathbf{R} \neq 0$ and therefore $\mathbf{R}$ is nonsingular. (Alternatively see part (d) of solution-2 below.)

(b) By using (1.77) and (1.102),

$$\mathbf{R}^T = \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n},$$

and so $\mathbf{R} = \mathbf{R}^T$ whence $\mathbf{R}$ is symmetric.

(c) We proceed as follows

$$\begin{aligned}\mathbf{R}\mathbf{R}^T &\overset{(1.102)}{=} (\mathbf{I} - 2\mathbf{n} \otimes \mathbf{n})(\mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}) = \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n} - 2\mathbf{n} \otimes \mathbf{n} + 4(\mathbf{n} \otimes \mathbf{n})(\mathbf{n} \otimes \mathbf{n}) = \\ &\overset{(1.72)}{=} \mathbf{I} - 4\mathbf{n} \otimes \mathbf{n} + 4(\mathbf{n} \cdot \mathbf{n})(\mathbf{n} \otimes \mathbf{n}) = \mathbf{I} - 4\mathbf{n} \otimes \mathbf{n} + 4(\mathbf{n} \otimes \mathbf{n}) = \mathbf{I}.\end{aligned}$$

Thus $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ and so $\mathbf{R}$ is orthogonal.

(d) Consider a right-handed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_3 = \mathbf{n}$. If $\{\mathbf{R}\mathbf{e}_1, \mathbf{R}\mathbf{e}_2, \mathbf{R}\mathbf{e}_3\}$ is left-handed then $\mathbf{R}$ is improper orthogonal. Since $\mathbf{e}_1 \cdot \mathbf{n} = \mathbf{e}_2 \cdot \mathbf{n} = 0$ and $\mathbf{n} \cdot \mathbf{n} = 1$ it follows that

$$\begin{aligned}\mathbf{R}\mathbf{e}_1 &\overset{(1.102)}{=} (\mathbf{I} - 2\mathbf{n} \otimes \mathbf{n})\mathbf{e}_1 = \mathbf{e}_1 - 2(\mathbf{n} \cdot \mathbf{e}_1)\mathbf{n} = \mathbf{e}_1, \\ \mathbf{R}\mathbf{e}_2 &\overset{(1.102)}{=} (\mathbf{I} - 2\mathbf{n} \otimes \mathbf{n})\mathbf{e}_2 = \mathbf{e}_2 - 2(\mathbf{n} \cdot \mathbf{e}_2)\mathbf{n} = \mathbf{e}_2, \\ \mathbf{R}\mathbf{e}_3 &\overset{(1.102)}{=} \mathbf{R}\mathbf{n} = (\mathbf{I} - 2\mathbf{n} \otimes \mathbf{n})\mathbf{n} = \mathbf{n} - 2(\mathbf{n} \cdot \mathbf{n})\mathbf{n} = -\mathbf{n} = -\mathbf{e}_3.\end{aligned}$$

Therefore the orthogonal tensor $\mathbf{R}$ carries the right-hand triplet of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ into the left-handed triplet of vectors $\{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3\}$ and therefore is improper orthogonal.

(e) Since $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ and $\mathbf{R} = \mathbf{R}^T$ it follows immediately that $\mathbf{R}^2 = \mathbf{I}$. Therefore $\mathbf{R}$ is a square root of $\mathbf{I}$ (but it is not positive definite. Why?)

Solution 2: Here we will not make use of the representation (1.102).

(a) Another way in which to show that $\mathbf{R}$ is nonsingular is to show that the only vector $\mathbf{x}$ for which $\mathbf{Rx} = \mathbf{o}$ is the null vector $\mathbf{x} = \mathbf{o}$. From (i):

$$\mathbf{Rx} \cdot \mathbf{Rx} = \big[\mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n}\big] \cdot \big[\mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n}\big] = \mathbf{x} \cdot \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})^2 - 2(\mathbf{x} \cdot \mathbf{n})^2 + 4(\mathbf{x} \cdot \mathbf{n})^2 = \mathbf{x} \cdot \mathbf{x}$$

Therefore

$$|\mathbf{Rx}| = |\mathbf{x}| \qquad \text{for all vectors } \mathbf{x} \in \mathsf{V}. \tag{ii}$$

Consequently $|\mathbf{Rx}| = 0$ if and only if $|\mathbf{x}| = 0$ which implies that $\mathbf{Rx} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$ (since the only vector whose length vanishes is the null vector). Therefore by (1.85), $\mathbf{R}$ is nonsingular.

(b) According to (1.74) and (1.79) the tensor $\mathbf{R}$ is symmetric if $\mathbf{Rx} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{Ry}$ for all vectors $\mathbf{x}, \mathbf{y}$. From (i),

$$\mathbf{Rx} = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n}, \qquad \mathbf{Ry} = \mathbf{y} - 2(\mathbf{y} \cdot \mathbf{n})\mathbf{n},$$

from which it follows that

$$\mathbf{Rx} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y} - 2(\mathbf{x} \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{y}), \qquad \mathbf{x} \cdot \mathbf{Ry} = \mathbf{x} \cdot \mathbf{y} - 2(\mathbf{y} \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{x}).$$

Thus $\mathbf{Rx} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{Ry}$ for all vectors $\mathbf{x}, \mathbf{y}$ and so $\mathbf{R}$ is symmetric:

$$\mathbf{R} = \mathbf{R}^T. \tag{iii}$$

(c) By (1.95), an orthogonal tensor is one that preserves the length of every vector, and so it follows immediately from (ii) that $\mathbf{R}$ is orthogonal.

(d) To show that $\mathbf{R}$ is improper orthogonal it is sufficient (since we know $\mathbf{R}$ is orthogonal) to show that $\det \mathbf{R} = -1$. Note from (i) that $\mathbf{Rn} = -\mathbf{n}$. Let $\mathbf{a}$ and $\mathbf{b}\,(\neq \mathbf{a})$ be two vectors orthogonal to $\mathbf{n}$. Since $\mathbf{a} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} = 0$ it follows from (i) that $\mathbf{Ra} = \mathbf{a}$ and $\mathbf{Rb} = \mathbf{b}$. Therefore taking $\mathbf{x} = \mathbf{n}, \mathbf{y} = \mathbf{a}, \mathbf{z} = \mathbf{b}$ in the definition (1.90) of the determinant we get

$$\det \mathbf{R} := \frac{\mathbf{Rn} \cdot (\mathbf{Ra} \times \mathbf{Rb})}{\mathbf{n} \cdot (\mathbf{a} \times \mathbf{b})} = \frac{-\mathbf{n} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{n} \cdot (\mathbf{a} \times \mathbf{b})} = -1.$$

Therefore $\mathbf{R}$ is improper orthogonal. (We could have used this to show that $\mathbf{R}$ is nonsingular in part (a).)

(e) Since $\mathbf{RR}^T = \mathbf{I}$ and $\mathbf{R} = \mathbf{R}^T$ it follows immediately that $\mathbf{R}^2 = \mathbf{I}$. Therefore $\mathbf{R}$ is a square root of $\mathbf{I}$.

Eigenvalues and eigenvectors.

– The **trace** of a tensor $\mathbf{A}$ is the scalar defined by

$$\operatorname{tr} \mathbf{A} := \frac{\mathbf{Ax} \cdot (\mathbf{y} \times \mathbf{z}) + \mathbf{x} \cdot (\mathbf{Ay} \times \mathbf{z}) + \mathbf{x} \cdot (\mathbf{y} \times \mathbf{Az})}{\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})} \tag{1.103}$$

which is to hold for all linearly independent vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$. Problem 1.58 asks you to verify that the right-hand side of (1.103) is in fact independent of the choice of the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and therefore depends only on the tensor $\mathbf{A}$, (the dependence being linear).

The trace of the product of two tensors has the property

$$\mathrm{tr}\,(\mathbf{AB}) = \mathrm{tr}\,(\mathbf{BA}). \tag{1.104}$$

Exercise: Show that

$$\mathrm{tr}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}. \tag{1.105}$$

– **Characteristic polynomial:** The characteristic polynomial associated with any tensor $\mathbf{A}$ and all scalars μ is (Problem 1.17)

$$\det(\mathbf{A} - \mu\mathbf{I}) \;=\; -\mu^3 + I_1(\mathbf{A})\mu^2 - I_2(\mathbf{A})\mu + I_3(\mathbf{A}), \tag{1.106}$$

where $I_1(\mathbf{A}), I_2(\mathbf{A})$ and $I_3(\mathbf{A})$ are the scalar-valued functions

$$I_1(\mathbf{A}) \coloneqq \mathrm{tr}\,\mathbf{A}, \qquad I_2(\mathbf{A}) \coloneqq \frac{1}{2}[(\mathrm{tr}\,\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)], \qquad I_3(\mathbf{A}) \coloneqq \det\mathbf{A}. \tag{1.107}$$

These three functions are called the **principal scalar invariants** of $\mathbf{A}$ (for reasons that will be explained in Section 1.5).

According to the *Cayley-Hamilton theorem*

$$-\mathbf{A}^3 + I_1(\mathbf{A})\mathbf{A}^2 - I_2(\mathbf{A})\mathbf{A} + I_3(\mathbf{A})\mathbf{I} = \mathbf{0}. \tag{1.108}$$

– **Eigenvalues and eigenvectors:** A scalar α and vector $\mathbf{a}\,(\neq \mathbf{o})$ are said to be an *eigenvalue* and *eigenvector* of a tensor $\mathbf{A}$ if

$$\mathbf{Aa} = \alpha\mathbf{a}. \tag{1.109}$$

If $\mathbf{a}$ is an eigenvector of $\mathbf{A}$, so is any scalar multiple of $\mathbf{a}$ and so there is no loss of generality in assuming $\mathbf{a}$ to be a unit vector. The eigenvalues α obey the *characteristic equation*

$$\det(\mathbf{A} - \alpha\mathbf{I}) = \; -\alpha^3 + I_1(\mathbf{A})\alpha^2 - I_2(\mathbf{A})\alpha + I_3(\mathbf{A}) = 0 \tag{1.110}$$

which is cubic in α. It has either one or three real roots.

Exercise: Using the Cayley-Hamilton theorem or otherwise, show that

$$\det \mathbf{A} = \frac{1}{6}\Big[[\mathrm{tr}\,\mathbf{A}]^3 - 3(\mathrm{tr}\,\mathbf{A})\,\mathrm{tr}(\mathbf{A}^2) + 2\,\mathrm{tr}(\mathbf{A}^3)\Big]. \tag{1.111}$$

Problem 1.4.8. (Eigenvalues. Eigenvectors.) Calculate the eigenvalues and eigenvectors of the tensor $\mathbf{a}\otimes\mathbf{b}$ where $\mathbf{a}\cdot\mathbf{b} \neq 0$ and $\mathbf{a}, \mathbf{b} \neq \mathbf{o}$.

Solution: Let $\mathbf{x}\,(\neq \mathbf{o})$ be an eigenvector and λ the corresponding eigenvalue. Then by definition

$$(\mathbf{a}\otimes\mathbf{b})\mathbf{x} = \lambda\mathbf{x} \qquad \Rightarrow \qquad (\mathbf{b}\cdot\mathbf{x})\mathbf{a} = \lambda\mathbf{x}. \tag{i}$$

One solution has $\mathbf{x}$ parallel to $\mathbf{a}$ and so we can take $\mathbf{x} = \mathbf{a}/|\mathbf{a}|$. Substituting this back into $(i)_2$ gives the corresponding eigenvalue to be $\lambda = \mathbf{a}\cdot\mathbf{b}$. The only other possibility occurs when $\mathbf{b}\cdot\mathbf{x} = 0$ in which case $\lambda = 0$. Thus the other two eigenvalues are 0 and 0 (i.e. the eigenvalue 0 with multiplicity 2) and any vector perpendicular to $\mathbf{b}$ is a corresponding eigenvector. Alternatively the eigenvalues can be determined from the cubic equation (1.110) using $I_2(\mathbf{a}\otimes\mathbf{b}) = I_3(\mathbf{a}\otimes\mathbf{b}) = 0$ and $I_1(\mathbf{a}\otimes\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}$ as shown in Problem 1.19.

Problem 1.4.9. (Eigenvalues) If $\mathbf{W} \neq \mathbf{0}$ is skew-symmetric show that its characteristic equation is

$$\omega^3 + I_2(\mathbf{W})\omega = 0 \tag{i}$$

and therefore that the only real eigenvalue is 0.

Solution: From the exercise on page 31 we know that $\mathbf{W}^3$ is skew-symmetric. Moreover it can be easily shown that the trace of a skew-symmetric tensor vanishes. Thus $\mathrm{tr}\,\mathbf{W} = \mathrm{tr}\,\mathbf{W}^3 = 0$ and so from (1.111) we conclude that $\det\mathbf{W} = 0$. Therefore by (1.107), the principal scalar invariants of $\mathbf{W}$ are $I_1(\mathbf{W}) = I_3(\mathbf{W}) = 0$ and

$$I_2(\mathbf{W}) = -\frac{1}{2}\mathrm{tr}\,(\mathbf{W}^2) = +\frac{1}{2}\mathbf{W}\cdot\mathbf{W} > 0, \tag{ii}$$

where the strict inequality follows since $\mathbf{W} \neq \mathbf{0}$. Consequently from (1.110) the characteristic equation $\det(\mathbf{W} - \omega\mathbf{I}) = 0$ reduces to (i). Since $I_2(\mathbf{W}) > 0$ by (ii), it follows that the only real eigenvalue is 0, the other two being imaginary.

- **Eigenvalues and eigenvectors of a symmetric tensor:** A *symmetric* linear transformation $\mathbf{S}$ has three *real* eigenvalues $\sigma_1, \sigma_2, \sigma_3$. The corresponding eigenvectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ can be chosen so they are orthonormal. The eigenvectors are referred to as the *principal directions* of $\mathbf{S}$, and the particular basis $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ is called a *principal basis* for $\mathbf{S}$.

– **Spectral representation of a symmetric tensor:** A *symmetric* linear transformation $\mathbf{S}$ can be expressed as

$$\mathbf{S} = \sigma_1\ (\mathbf{s}_1 \otimes \mathbf{s}_1) + \sigma_2\ (\mathbf{s}_2 \otimes \mathbf{s}_2) + \sigma_3\ (\mathbf{s}_3 \otimes \mathbf{s}_3) = \sum_{i=1}^{3} \sigma_i\ (\mathbf{s}_i \otimes \mathbf{s}_i), \tag{1.112}$$

where the orthonormal basis $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ is comprised of eigenvectors. This is called the *spectral representation* of the symmetric tensor $\mathbf{S}$.

Remark: Observe that the subscript i occurs three times in the rightmost expression in (1.112) and so we have suspended the usual summation convention and explicitly displayed the summation on i; see the discussion below (1.31).

For any positive integer n,

$$\mathbf{S}^n = \sum_{i=1}^{3} \sigma_i^n\ (\mathbf{s}_i \otimes \mathbf{s}_i). \tag{1.113}$$

If $\mathbf{S}$ is symmetric and nonsingular, then none of its eigenvalues vanish and

$$\mathbf{S}^{-1} = \sum_{i=1}^{3}\ (1/\sigma_i)\ (\mathbf{s}_i \otimes \mathbf{s}_i). \tag{1.114}$$

If $\mathbf{S}$ is symmetric and positive definite, all three eigenvalues are positive, and there is a unique symmetric positive definite tensor $\mathbf{T}$ such that $\mathbf{T}^2 = \mathbf{S}$. The tensor $\mathbf{T}$ is called the *positive definite square root* of $\mathbf{S}$ and denoted by $\mathbf{T} = \sqrt{\mathbf{S}}$: (Problem 1.25)

$$\sqrt{\mathbf{S}} = \sum_{i=i}^{3} \sqrt{\sigma_i}\ (\mathbf{s}_i \otimes \mathbf{s}_i). \tag{1.115}$$

Exercise: In Problem 1.4.7 we showed that both $\mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}$ and $\mathbf{I}$ are square roots of the identity tensor $\mathbf{I}$. Does this contradict the claim of uniqueness associated with (1.115)?

Exercise: Show that the principal scalar invariants of $\mathbf{S}$ can be written as

$$I_1(\mathbf{S}) = \operatorname{tr}\mathbf{S} = \sigma_1 + \sigma_2 + \sigma_3, \qquad I_2(\mathbf{S}) = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1, \qquad I_3(\mathbf{S}) = \det\,\mathbf{S} = \sigma_1\sigma_2\sigma_3. \tag{1.116}$$

Exercise: If two symmetric tensors $\mathbf{B}$ and $\mathbf{C}$ have the same eigenvalues, we see from (1.116) that they have the same principal scalar invariants. Conversely, if they have the same principal invariants, do they have the same eigenvalues?

Problem 1.4.10. (Spectral representation of a symmetric tensor) A symmetric tensor $\mathbf{A}$ necessarily has three real (not-necessarily distinct) eigenvalues, α_1, α_2 and α_3. They are the roots of the cubic equation $\det(\mathbf{A} - \alpha\mathbf{I}) = 0$. If α obeys $\det(\mathbf{A} - \alpha\mathbf{I}) = 0$ then $\mathbf{A} - \alpha\mathbf{I}$ is singular and therefore by Problem 1.4.4 there is a non-zero vector $\mathbf{a}$ such that $(\mathbf{A} - \alpha\mathbf{I})\mathbf{a} = \mathbf{o}$. Therefore corresponding to an eigenvalue α (defined as a root of the characteristic equation $\det(\mathbf{A} - \alpha\mathbf{I}) = 0$) there exists an eigenvector $\mathbf{a}$ such that $\mathbf{A}\mathbf{a} = \alpha\mathbf{a}$.

(a) If the eigenvalues α_1 and α_2 are distinct, show that the corresponding eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal.

(b) If all three eigenvalues are distinct, $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1$, it follows from part (a) that each eigenvector is orthogonal to the other two and therefore that the eigenvectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ form an orthonormal basis for V. Then from the general representation $\mathbf{A} = A_{ij}\mathbf{a}_i \otimes \mathbf{a}_j$ and $\mathbf{A}\mathbf{a}_i \cdot \mathbf{a}_j = \alpha_i\delta_{ij}$ (no sum on i) one can show that $\mathbf{A} = \alpha_1\mathbf{a}_1 \otimes \mathbf{a}_1 + \alpha_2\mathbf{a}_2 \otimes \mathbf{a}_2 + \alpha_3\mathbf{a}_3 \otimes \mathbf{a}_3$.

(c) Suppose two eigenvalues are coincident and distinct from the third: $\alpha_2 = \alpha_3 \neq \alpha_1$. Show that $\mathbf{A}$ can be expressed as $\mathbf{A} = \alpha_1\mathbf{a}_1 \otimes \mathbf{a}_1 + \alpha_2(\mathbf{I} - \mathbf{a}_1 \otimes \mathbf{a}_1)$ and therefore that every vector perpendicular to $\mathbf{a}_1$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue α_2.

(d) If all three eigenvalues coincide, $\alpha := \alpha_1 = \alpha_2 = \alpha_3$, then $\mathbf{A} = \alpha\mathbf{I}$.

Solution: (a) Take the scalar product of $\mathbf{A}\mathbf{a}_1 = \alpha_1\mathbf{a}_1$ with $\mathbf{a}_2$, and the scalar product of $\mathbf{A}\mathbf{a}_2 = \alpha_2\mathbf{a}_2$ with $\mathbf{a}_1$:

$$\mathbf{A}\mathbf{a}_1 \cdot \mathbf{a}_2 = \alpha_1\mathbf{a}_1 \cdot \mathbf{a}_2, \qquad \mathbf{A}\mathbf{a}_2 \cdot \mathbf{a}_1 = \alpha_2\mathbf{a}_2 \cdot \mathbf{a}_1.$$

Since $\mathbf{A}$ is symmetric we have $\mathbf{A}\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_1 \cdot \mathbf{A}\mathbf{a}_2$ and therefore the preceding equations yield

$$\alpha_1\, \mathbf{a}_1 \cdot \mathbf{a}_2 = \alpha_2\, \mathbf{a}_1 \cdot \mathbf{a}_2 \qquad \Rightarrow \qquad (\alpha_1 - \alpha_2)\, \mathbf{a}_1 \cdot \mathbf{a}_2 = 0 \qquad \Rightarrow \qquad \mathbf{a}_1 \cdot \mathbf{a}_2 = 0, \qquad \square$$

where in the last step we used $\alpha_1 \neq \alpha_2$.

(c) Suppose $\alpha_1 \neq \alpha_2 = \alpha_3$ and

$$\mathbf{A}\mathbf{a}_1 = \alpha_1\mathbf{a}_1, \qquad \mathbf{A}\mathbf{a}_2 = \alpha_2\mathbf{a}_2. \tag{i}$$

Since $\alpha_1 \neq \alpha_2$ it follows from part (a) that $\mathbf{a}_2$ is perpendicular to $\mathbf{a}_1$. Let $\mathbf{a}_3$ be a unit vector perpendicular to both $\mathbf{a}_1$ and $\mathbf{a}_2$. Then $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is an orthonormal basis for V and so we can write

$$\mathbf{A} = A_{ij}\mathbf{a}_i \otimes \mathbf{a}_j. \tag{ii}$$

Operating $\mathbf{A}$ on $\mathbf{a}_1$ yields

$$\mathbf{A}\mathbf{a}_1 \overset{(ii)}{=} (A_{ij}\mathbf{a}_i \otimes \mathbf{a}_j)\mathbf{a}_1 = A_{ij}(\mathbf{a}_j \cdot \mathbf{a}_1)\mathbf{a}_i = A_{ij}\delta_{j1}\mathbf{a}_i = A_{i1}\mathbf{a}_i \overset{(i)_1}{=} \alpha_1\mathbf{a}_1,$$

whence

$$A_{11} = \alpha_1, \qquad A_{21} = A_{31} = 0. \tag{iii}$$

Similarly,

$$\mathbf{A}\mathbf{a}_2 \overset{(ii)}{=} (A_{ij}\mathbf{a}_i \otimes \mathbf{a}_j)\mathbf{a}_2 = A_{ij}(\mathbf{a}_j \cdot \mathbf{a}_2)\mathbf{a}_i = A_{ij}\delta_{j2}\mathbf{a}_i = A_{i2}\mathbf{a}_i \overset{(i)_2}{=} \alpha_2\mathbf{a}_2.$$

and therefore

$$A_{22} = \alpha_2, \qquad A_{12} = A_{32} = 0. \tag{iv}$$

Since $\mathbf{A}$ is symmetric we further have from (iii) and (iv) that

$$A_{13} = A_{23} = 0. \qquad (v)$$

Therefore by $(ii), (iii), (iv)$ and (v) we can express $\mathbf{A}$ as

$$\mathbf{A} = \alpha_1 \mathbf{a}_1 \otimes \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 \otimes \mathbf{a}_2 + A_{33} \mathbf{a}_3 \otimes \mathbf{a}_3. \qquad (vi)$$

Operating $\mathbf{A}$ on $\mathbf{a}_3$ and using $\mathbf{a}_3 \cdot \mathbf{a}_1 = \mathbf{a}_3 \cdot \mathbf{a}_2 = 0$ gives

$$\mathbf{A}\mathbf{a}_3 = A_{33}\mathbf{a}_3.$$

Thus $\mathbf{a}_3$ is an eigenvector of $\mathbf{A}$ and A_{33} is the corresponding eigenvalue. Calculating the trace of (vi) gives $\operatorname{tr}\mathbf{A} = \alpha_1 + \alpha_2 + A_{33}$. However, since the eigenvalues of $\mathbf{A}$ are $\alpha_1, \alpha_2, \alpha_2$, it follows from $(1.116)_1$ that $\operatorname{tr}\mathbf{A} = \alpha_1 + 2\alpha_2$. Thus $A_{33} = \alpha_2$ and so we can write (vi) as

$$\mathbf{A} = \alpha_1 \mathbf{a}_1 \otimes \mathbf{a}_1 + \alpha_2(\mathbf{a}_2 \otimes \mathbf{a}_2 + \mathbf{a}_3 \otimes \mathbf{a}_3).$$

On using the identity $\mathbf{I} = \mathbf{a}_1 \otimes \mathbf{a}_1 + \mathbf{a}_2 \otimes \mathbf{a}_2 + \mathbf{a}_3 \otimes \mathbf{a}_3$ this can be written as

$$\mathbf{A} = \alpha_1 \mathbf{a}_1 \otimes \mathbf{a}_1 + \alpha_2(\mathbf{I} - \mathbf{a}_1 \otimes \mathbf{a}_1). \qquad \square \qquad (vii)$$

Finally, for any vector $\mathbf{x}$ perpendicular to $\mathbf{a}_1$ we have

$$\mathbf{A}\mathbf{x} \stackrel{(vii)}{=} \alpha_1(\mathbf{a}_1 \otimes \mathbf{a}_1)\mathbf{x} + \alpha_2(\mathbf{I} - \mathbf{a}_1 \otimes \mathbf{a}_1)\mathbf{x} = \alpha_1(\mathbf{x} \cdot \mathbf{a}_1)\mathbf{a}_1 + \alpha_2\mathbf{x} - \alpha_2(\mathbf{x} \cdot \mathbf{a}_1)\mathbf{a}_1 = \alpha_2\mathbf{x},$$

where we used $\mathbf{x} \cdot \mathbf{a}_1 = 0$ in the last step. Thus every vector $\mathbf{x}$ perpendicular to $\mathbf{a}_1$ is an eigenvector corresponding to the eigenvalue α_2.

Polar decomposition theorem.

A certain nonsingular tensor $\mathbf{F}$ called the deformation gradient tensor will play a pivotal role in describing the deformation of a body. Part of $\mathbf{F}$ will describe a rotation, the rest a stretch/strain. The polar decomposition theorem tells us how to identify these two parts of $\mathbf{F}$.

- The **polar decomposition theorem** states that, corresponding to any nonsingular tensor $\mathbf{F}$, there exist unique symmetric positive definite tensors $\mathbf{U}$ and $\mathbf{V}$ and a unique orthogonal tensor $\mathbf{R}$ such that[6]

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}. \qquad (1.117)$$

[6]Given an arbitrary (possibly singular) tensor $\mathbf{F} \in \mathsf{Lin}$, there is a (unique) symmetric, positive semi-definite tensor $\mathbf{U}$, and a (not-necessarily unique) orthogonal tensor $\mathbf{R}$, such that $\mathbf{F} = \mathbf{R}\mathbf{U}$; see Halmos [5], Section 83.

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $\mathbf{U}$ and $\mathbf{V}$ coincide. Let the corresponding orthonormal eigenvectors of $\mathbf{U}$ and $\mathbf{V}$ be $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Then

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i, \qquad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i, \qquad \mathbf{R} = \sum_{i=1}^{3} \mathbf{v}_i \otimes \mathbf{u}_i. \tag{1.118}$$

The eigenvectors are related by $\mathbf{v}_i = \mathbf{R}\mathbf{u}_i$. Observe from (1.117) and (1.118), together with (1.72), that $\mathbf{F}$ has the representation

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{u}_i. \tag{1.119}$$

Observe that both bases $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ appear in the right-hand side of (1.119) and so the scalar coefficients of $\mathbf{v}_i \otimes \mathbf{u}_i$ are not the components of $\mathbf{F}$ in either basis (except in the special case where these bases are identical).

Scalar product of two tensors.

As noted previously, the set of all linear transformations on V is a nine-dimensional vector space we denote by Lin. One can define a scalar product on the vector space Lin by

$$\mathbf{A} \cdot \mathbf{B} := \operatorname{tr}(\mathbf{A}\mathbf{B}^T) \qquad \text{for all tensors } \mathbf{A}, \mathbf{B} \in \mathsf{Lin}; \tag{1.120}$$

see Problem 1.59 for a justification of this definition and for various properties of the scalar product.

Notation: Since we use lower case boldface letters to denote vectors and upper case boldface letters to denote tensors, it will usually be clear as to whether a dot between two symbols refers to the scalar product between vectors or between tensors. Some authors denote the scalar product between tensors by a colon as in $\mathbf{A} : \mathbf{B}$.

The *magnitude* (norm) of a tensor $\mathbf{A}$ is denoted by $|\mathbf{A}|$ and defined by

$$|\mathbf{A}| = (\mathbf{A} \cdot \mathbf{A})^{1/2} = \left[\operatorname{tr}(\mathbf{A}\mathbf{A}^T)\right]^{1/2}. \tag{1.121}$$

According to item (d) of Problem 1.59,

$$|\mathbf{A}| = 0 \qquad \text{if and only if} \quad \mathbf{A} = \mathbf{0}; \tag{1.122}$$

see also (1.135). If $\mathbf{A} \cdot \mathbf{B} = 0$ for two non-null tensors $\mathbf{A}$ and $\mathbf{B}$, we say that $\mathbf{A}$ is orthogonal to $\mathbf{B}$.

A useful identity for all tensors $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{Lin}$ is (Problem 1.14)

$$\mathbf{AB} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{A}^T\mathbf{C} = \mathbf{A} \cdot \mathbf{CB}^T. \tag{1.123}$$

Also, for any symmetric tensor $\mathbf{S}$ and skew-symmetric tensor $\mathbf{W}$ (Problem 1.4.16)

$$\mathbf{S} \cdot \mathbf{W} = 0. \tag{1.124}$$

Exercise: For any tensor $\mathbf{A}$ and vectors $\mathbf{x}$ and $\mathbf{y}$ show that

$$\mathbf{A} \cdot (\mathbf{x} \otimes \mathbf{y}) = \mathbf{Ay} \cdot \mathbf{x}, \tag{1.125}$$

where the scalar product on the left is between two tensors while that on the right is between two vectors.

1.4.2 Worked examples.

Problem 1.4.11. (Tensor algebra)

(a) For any skew-symmetric tensor $\mathbf{W}$, show that

$$\mathbf{Wx} \cdot \mathbf{x} = 0 \qquad \text{for all vectors } \mathbf{x} \in \mathsf{V}. \tag{ii}$$

(b) If $\mathbf{S}$ is a symmetric tensor and

$$\mathbf{Sx} \cdot \mathbf{x} = 0 \qquad \text{for all vectors } \mathbf{x} \in \mathsf{V}, \tag{iii}$$

show that $\mathbf{S} = \mathbf{0}$.

Note: In contrast, if $\mathbf{A}$ is an *arbitrary* tensor and $\mathbf{Ax} \cdot \mathbf{x} = 0$ for all vectors $\mathbf{x}$, this does *not* imply that $\mathbf{A} = \mathbf{0}$; only that $\mathbf{A}$ is skew-symmetric.

(c) If the tensor $\mathbf{A}$ obeys

$$\mathbf{Ax} \cdot \mathbf{y} = 0 \qquad \text{for all vectors } \mathbf{x}, \mathbf{y} \in \mathsf{V}, \tag{iv}$$

show that $\mathbf{A} = \mathbf{0}$. Note as an immediate consequence that if

$$\mathbf{Ax} \cdot \mathbf{y} = \mathbf{Bx} \cdot \mathbf{y} \qquad \text{for all vectors } \mathbf{x}, \mathbf{y} \in \mathsf{V}, \tag{v}$$

then $\mathbf{A} = \mathbf{B}$.

(d) If

$$\mathbf{A} \cdot \mathbf{B} = 0 \qquad \text{for all tensors } \mathbf{B} \in \mathsf{Lin} \tag{vi}$$

show that $\mathbf{A} = \mathbf{0}$.

(e) If

$$\mathbf{A} \cdot \mathbf{B} = 0 \qquad \text{for all } \textit{symmetric} \text{ tensors } \mathbf{B} \in \mathsf{Lin}, \tag{vii}$$

show that $\mathbf{A}$ is skew-symmetric. It is important to note that (vii) does *not* imply $\mathbf{A} = \mathbf{0}$.

Solution:

(a) The result follows from the following calculation:

$$\mathbf{Wx}\cdot\mathbf{x} \overset{(1.74)}{=} \mathbf{x}\cdot\mathbf{W}^T\mathbf{x} \overset{(1.80)}{=} -\mathbf{x}\cdot\mathbf{Wx} \overset{(1.51)_1}{=} -\mathbf{Wx}\cdot\mathbf{x}$$

whence (ii) follows.

(b) Since $\mathbf{S}$ is symmetric, it has three real eigenvalues $\sigma_1, \sigma_2, \sigma_3$ and corresponding orthonormal eigenvectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$. Since $\mathbf{Sx}\cdot\mathbf{x} = 0$ for all vectors $\mathbf{x} \in \mathsf{V}$ it must necessarily hold for the choice $\mathbf{x} = \mathbf{s}_1$:

$$\mathbf{Ss}_1\cdot\mathbf{s}_1 = 0 \qquad \Rightarrow \qquad \mathbf{Ss}_1\cdot\mathbf{s}_1 = (\sigma_1\mathbf{s}_1)\cdot\mathbf{s}_1 = \sigma_1(\mathbf{s}_1\cdot\mathbf{s}_1) = \sigma_1 = 0.$$

Thus the eigenvalue $\sigma_1 = 0$. Similarly the other eigenvalues also vanish. This implies that $\mathbf{S} = \mathbf{0}$.

(c) Since (iv) holds for all vectors $\mathbf{y} \in \mathsf{V}$ it necessarily holds for the vector $\mathbf{y} = \mathbf{Ax}$. So we have

$$\mathbf{Ax}\cdot\mathbf{Ax} = 0 \qquad \text{for all vectors } \mathbf{x} \in \mathsf{V}.$$

Thus $|\mathbf{Ax}| = 0$ and since the only vector with zero length is the null vector, $\mathbf{Ax} = \mathbf{o}$. Since this holds for all vectors $\mathbf{x} \in \mathsf{V}$, $\mathbf{A} = \mathbf{0}$ by the definition $(1.66)_2$ of the zero tensor.

(d) Since (vi) holds for all tensors $\mathbf{B} \in \mathsf{Lin}$ it necessarily holds for $\mathbf{B} = \mathbf{A}$ and so $\mathbf{A}\cdot\mathbf{A} = 0$. By (1.122), this implies $\mathbf{A} = \mathbf{0}$.

(e) Using the decomposition (1.81) we can write

$$\mathbf{A} = \mathbf{S} + \mathbf{W} \tag{viii}$$

where

$$\mathbf{S} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T), \qquad \mathbf{W} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T), \tag{ix}$$

and so

$$\mathbf{A}\cdot\mathbf{B} = (\mathbf{S} + \mathbf{W})\cdot\mathbf{B} = \mathbf{S}\cdot\mathbf{B} + \mathbf{W}\cdot\mathbf{B} \overset{(1.124)}{=} \mathbf{S}\cdot\mathbf{B}.$$

Equation (vii) thus tells us that

$$\mathbf{S}\cdot\mathbf{B} = 0 \qquad \text{for all symmetric tensors } \mathbf{B}.$$

Since this is to hold for all symmetric $\mathbf{B}$, and since $\mathbf{S}$ is symmetric, it must necessarily hold for the particular choice $\mathbf{B} = \mathbf{S}$. Therefore

$$\mathbf{S}\cdot\mathbf{S} = 0.$$

By (1.122), this implies $\mathbf{S} = \mathbf{0}$ and so from $(ix)_1$

$$\mathbf{A} = -\mathbf{A}^T$$

which says that $\mathbf{A}$ must be skew-symmetric.

Problem 1.4.12. (Orthonormal basis for Lin.) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis for a Euclidean vector space V and let Lin be the set of all tensors from $\mathsf{V} \to \mathsf{V}$. Show that the nine tensors $\mathbf{e}_i \otimes \mathbf{e}_j, i, j = 1, 2, 3$, are

(a) linearly independent, and

(b) orthonormal.

Therefore if we can show that Lin is 9-dimensional (this is addressed in Problem 1.24), then the nine tensors $\mathbf{e}_i \otimes \mathbf{e}_j, i, j = 1, 2, 3$, form an orthonormal basis for Lin.

Solution:

(a) To show that the tensors $\mathbf{e}_i \otimes \mathbf{e}_j$ are linearly independent we must show that the only scalars α_{ij} for which

$$\alpha_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{0} \qquad (i)$$

are $\alpha_{ij} = 0$. Suppose (i) holds for some scalars α_{ij}. Operating (i) on the vector $\mathbf{e}_k$ gives

$$\alpha_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_k = \mathbf{o} \quad \Rightarrow \quad \alpha_{ij}(\mathbf{e}_j \cdot \mathbf{e}_k)\mathbf{e}_i = \mathbf{o} \quad \Rightarrow \quad \alpha_{ij}\delta_{jk}\mathbf{e}_i = \mathbf{o} \quad \Rightarrow \quad \alpha_{ik}\mathbf{e}_i = \mathbf{o}. \qquad (ii)$$

Taking the scalar product of (ii) with $\mathbf{e}_\ell$ gives $\alpha_{ik}\mathbf{e}_i \cdot \mathbf{e}_\ell = \alpha_{ik}\delta_{i\ell} = \alpha_{\ell k} = 0$. Thus the tensors $\mathbf{e}_i \otimes \mathbf{e}_j$ are linearly independent.

(b) To show that two tensors $\mathbf{A}$ and $\mathbf{B}$ are orthogonal we must show that $\mathbf{A} \cdot \mathbf{B} = 0$ which by (1.120) requires $\mathrm{tr}(\mathbf{A}\mathbf{B}^T) = 0$. Consider the two tensors $\mathbf{e}_i \otimes \mathbf{e}_j$ and $\mathbf{e}_k \otimes \mathbf{e}_\ell$. Their scalar product is

$$\begin{aligned}(\mathbf{e}_i \otimes \mathbf{e}_j) \cdot (\mathbf{e}_k \otimes \mathbf{e}_\ell) \quad &= \mathrm{tr}\Big[(\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{e}_k \otimes \mathbf{e}_\ell)^T\Big] \overset{(1.77)}{=} \mathrm{tr}\Big[(\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{e}_\ell \otimes \mathbf{e}_k)\Big] \overset{(1.72)}{=} \mathrm{tr}\Big[(\mathbf{e}_j \cdot \mathbf{e}_\ell)(\mathbf{e}_i \otimes \mathbf{e}_k)\Big] = \\ &= (\mathbf{e}_j \cdot \mathbf{e}_\ell)\,\mathrm{tr}\Big[(\mathbf{e}_i \otimes \mathbf{e}_k)\Big] \overset{(1.105)}{=} (\mathbf{e}_j \cdot \mathbf{e}_\ell)\,(\mathbf{e}_i \cdot \mathbf{e}_k) = \delta_{ik}\delta_{j\ell}\end{aligned}$$

Thus

$$(\mathbf{e}_i \otimes \mathbf{e}_j) \cdot (\mathbf{e}_k \otimes \mathbf{e}_\ell) = \delta_{ik}\delta_{j\ell}. \qquad (1.126)$$

Therefore if $(\mathbf{e}_i \otimes \mathbf{e}_j) \neq (\mathbf{e}_k \otimes \mathbf{e}_\ell)$, i.e. $i \neq k$ and $j \neq \ell$, we have $(\mathbf{e}_i \otimes \mathbf{e}_j) \cdot (\mathbf{e}_k \otimes \mathbf{e}_\ell) = 0$ and so each of these tensors is orthogonal to the others. On the other hand if $(\mathbf{e}_i \otimes \mathbf{e}_j) = (\mathbf{e}_k \otimes \mathbf{e}_\ell)$, i.e. $i = k$ and $j = \ell$, we have $(\mathbf{e}_i \otimes \mathbf{e}_j) \cdot (\mathbf{e}_k \otimes \mathbf{e}_\ell) = 1$ and so the magnitude of each tensor is unity. Thus the nine tensors $\mathbf{e}_i \otimes \mathbf{e}_j$ are orthonormal.

Problem 1.4.13. (Tensor algebra.) Two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues. This does not imply that $\mathbf{B} = \mathbf{A}$. However it does imply that there is an orthogonal tensor $\mathbf{Q}$ such that $\mathbf{B} = \mathbf{Q}\mathbf{A}\mathbf{Q}^T$. Prove this.

Solution: Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, and let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ be the corresponding eigenvectors. Since $\mathbf{A}$ and $\mathbf{B}$ are symmetric, the eigenvectors can be chosen so each set is orthonormal:

$$\mathbf{a}_i \cdot \mathbf{a}_j = \delta_{ij}, \qquad \mathbf{b}_i \cdot \mathbf{b}_j = \delta_{ij}. \qquad (i)$$

The tensors $\mathbf{A}$ and $\mathbf{B}$ can be represented in spectral form as

$$\mathbf{A} = \sum_{i=1}^{3} \lambda_i \mathbf{a}_i \otimes \mathbf{a}_i, \qquad \mathbf{B} = \sum_{i=1}^{3} \lambda_i \mathbf{b}_i \otimes \mathbf{b}_i. \tag{ii}$$

Whenever an eigenvalue appears in an equation in this problem we will suspend the summation convention (as we have in (ii)) and display the summation explicitly. (We do this because the summation is over an index that appears three times.)

Since the bases $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are orthonormal, it should not be surprising if the tensor that maps one into the other is orthogonal and perhaps this is the tensor $\mathbf{Q}$ we are after. Define the tensor $\mathbf{Q}$ by

$$\mathbf{Q} = \mathbf{b}_k \otimes \mathbf{a}_k, \tag{iii}$$

and observe that

$$\mathbf{Q}\mathbf{a}_i = (\mathbf{b}_k \otimes \mathbf{a}_k)\mathbf{a}_i = (\mathbf{a}_k \cdot \mathbf{a}_i)\mathbf{b}_k = \delta_{ki}\mathbf{b}_k = \mathbf{b}_i. \tag{iv}$$

Therefore $\mathbf{Q}$ maps each $\mathbf{a}_i \to \mathbf{b}_i$ and thus the basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ into the basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. That $\mathbf{Q}$ is orthogonal follows from

$$\begin{aligned} \mathbf{Q}^T\mathbf{Q} \quad &= \quad (\mathbf{b}_i \otimes \mathbf{a}_i)^T (\mathbf{b}_j \otimes \mathbf{a}_j) \overset{(1.77)}{=} (\mathbf{a}_i \otimes \mathbf{b}_i)(\mathbf{b}_j \otimes \mathbf{a}_j) = \\ &\overset{(1.72)}{=} \quad (\mathbf{b}_i \cdot \mathbf{b}_j)(\mathbf{a}_i \otimes \mathbf{a}_j) \overset{(i)}{=} \delta_{ij}(\mathbf{a}_i \otimes \mathbf{a}_j) \overset{(*)}{=} \mathbf{a}_i \otimes \mathbf{a}_i \overset{(1.71)}{=} \mathbf{I}, \end{aligned}$$

where in step $(*)$ we used the substitution rule.

It can now be readily shown that $\mathbf{B} = \mathbf{Q}\mathbf{A}\mathbf{Q}^T$:

$$\mathbf{B} \overset{(ii)}{=} \sum_{i=1}^{3} \lambda_i \mathbf{b}_i \otimes \mathbf{b}_i \overset{(iv)}{=} \sum_{i=1}^{3} \lambda_i \mathbf{Q}\mathbf{a}_i \otimes \mathbf{Q}\mathbf{a}_i \overset{(1.78)}{=} \sum_{i=1}^{3} \lambda_i \mathbf{Q}(\mathbf{a}_i \otimes \mathbf{a}_i)\mathbf{Q}^T = \mathbf{Q}\left(\sum_{i=1}^{3} \lambda_i (\mathbf{a}_i \otimes \mathbf{a}_i)\right)\mathbf{Q}^T \overset{(ii)}{=} \mathbf{Q}\mathbf{A}\mathbf{Q}^T.$$

1.4.3 Components of a tensor in a basis.

A few brief videos on the use of indicial notation in tensor algebra can be found at https://www.dropbox.com/sh/lqj7j139bnwf4k3/AADAVAruud1tPrMA2FU7dN8Fa?dl=0.

– Let Lin be the set of all tensors from the vector space $\mathsf{V} \to \mathsf{V}$. Problem 1.24 shows that the dimension of Lin is 9, and Problem 1.4.12 showed that $\mathbf{e}_i \otimes \mathbf{e}_j, i, j = 1, 2, 3$, are nine orthonormal tensors in Lin (where as usual $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis for V). Therefore these 9 tensors form an orthonormal basis for Lin. Consequently, as noted previously, given any tensor $\mathbf{A}$, there is a unique set of nine scalars A_{ij} such that

$$\mathbf{A} = A_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.127}$$

The A_{ij}'s are the components of $\mathbf{A}$ in this basis.

– Since the basis of nine tensors is orthonormal, one can derive the following formula for the components A_{ij} (show this):

$$A_{ij} = (\mathbf{A}\mathbf{e}_j) \cdot \mathbf{e}_i. \tag{1.128}$$

This says that the ith component of the vector $\mathbf{A}\mathbf{e}_j$ is A_{ij} which can be equivalently stated as

$$\mathbf{A}\mathbf{e}_j = A_{ij}\mathbf{e}_i. \tag{1.129}$$

Remark: We know that a tensor is characterized by the way it transforms every vector in V. Since any vector can always be expressed in terms of the basis vectors, it follows that in order to define a tensor $\mathbf{A}$ it is sufficient to (only) know how $\mathbf{A}$ transforms the basis vectors, i.e. to know $\mathbf{A}\mathbf{e}_j$ for $j = 1, 2, 3$. According to (1.129) (and also (1.128)) the nine scalars A_{ij} do this.

Exercise: If $\mathbf{a}$ and $\mathbf{b}$ have components a_i and b_i, show that the components of the tensor $\mathbf{a} \otimes \mathbf{b}$ are

$$(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j. \tag{1.130}$$

If $\mathbf{A}$ and $\mathbf{x}$ have components A_{ij} and x_i, show that the ith component of the vector $\mathbf{A}\mathbf{x}$ is

$$(\mathbf{A}\mathbf{x})_i = A_{ij} x_j. \tag{1.131}$$

If the tensors $\mathbf{A}$ and $\mathbf{B}$ have components A_{ij} and B_{ij} respectively, show that the i, j component of the tensor $\mathbf{A}\mathbf{B}$ is

$$(\mathbf{A}\mathbf{B})_{ij} = A_{ik} B_{kj}. \tag{1.132}$$

Exercise: If $\mathbf{Q}$ is an orthogonal tensor, show that

$$Q_{ik} Q_{jk} = Q_{ki} Q_{kj} = \delta_{ij}. \tag{1.133}$$

Exercise: If the tensors $\mathbf{A}$ and $\mathbf{B}$ have components A_{ij} and B_{ij} in some basis, show that

$$\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij}, \tag{1.134}$$

$$|\mathbf{A}| = (\mathbf{A} \cdot \mathbf{A})^{1/2} = (A_{ij} A_{ij})^{1/2} = \left(A_{11}^2 + A_{12}^2 + A_{13}^2 + \ldots + A_{33}^3\right)^{1/2}. \tag{1.135}$$

It follows from (1.135) that $|\mathbf{A}| = 0$ if and only if *every* component $A_{ij} = 0$, i.e. if and only if $\mathbf{A} = \mathbf{0}$. Moreover, if $|\mathbf{A}| \to 0$ then each $A_{ij} \to 0$.

– The components of the identity tensor $\mathbf{I}$ in any basis are δ_{ij}. This is reflected in (1.71).

– The components A_{ij} of a tensor $\mathbf{A}$ in a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ can be assembled into a square matrix:

$$[A] = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}. \tag{1.136}$$

The components A_{ij} depend on *both* the tensor $\mathbf{A}$ *and* the choice of basis.

– Once a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is chosen and fixed, there is a unique matrix $[A]$ associated with any given tensor $\mathbf{A}$; *and conversely* there is a unique tensor $\mathbf{A}$ associated with any given square matrix $[A]$ such that the components of $\mathbf{A}$ in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are $[A]$. *Thus, once the basis is fixed, there is a one-to-one correspondence between square matrices and tensors.* Just as for vectors, the fundamental notion of a tensor stands on its own, without the need to refer to its components in a basis. For example, the strain (tensor) at a particle does not depend on a basis.

– If $[F]$ is the matrix of components of $\mathbf{F}$ in some basis, then $[F]^{-1}$ is the matrix of components of $\mathbf{F}^{-1}$ in that basis (with $\mathbf{F}^{-1}$ defined as on page 32).

– If $[A]$ is the matrix of components of $\mathbf{A}$ in some basis, one can show from (1.90), (1.73) and (1.39) that (Problem 1.4.15)

$$\det \mathbf{A} = \det[A]. \tag{1.137}$$

Observe that we used (1.90), not (1.137), as the definition of $\det \mathbf{A}$. This is because the components of a tensor depend on the choice of basis. Therefore $\det[A]$ *may* depend on the choice of basis whereas the definition of $\det \mathbf{A}$ in (1.90) does not. See Problem 1.6.1 for further discussion.

Similarly, one can show that

$$\operatorname{tr} \mathbf{A} = \operatorname{tr}[A] = A_{ii}. \tag{1.138}$$

Again, we used (1.103), not (1.138), to define $\operatorname{tr}\mathbf{A}$ because $\operatorname{tr}[A]$ *may* depend on the choice of basis. See Problem 1.6.1 for further discussion.

– *Various algebraic operations on vectors and tensors correspond exactly to analogous matrix operations on the associated matrices of components* (once a basis has been chosen). As an example suppose that $\mathbf{y} = \mathbf{A}\mathbf{x}$. Then by (1.131),

$$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad \Leftrightarrow \qquad \{y\} = [A]\{x\} \qquad \Leftrightarrow \qquad y_i = A_{ij}x_j, \tag{1.139}$$

where the elements of the column matrices $\{x\}$ and $\{y\}$ are the components of $\mathbf{x}$ and $\mathbf{y}$ respectively. Similarly if $\mathbf{C} = \mathbf{AB}$, it follows from (1.132) that

$$\mathbf{C} = \mathbf{AB} \qquad \Leftrightarrow \qquad [C] = [A][B] \qquad \Leftrightarrow \qquad C_{ij} = A_{ik}B_{kj}. \tag{1.140}$$

– Suppose $[A]$ and $[A']$ are the matrices of components of $\mathbf{A}$ in *any two* orthonormal bases. Then one can show that $\mathrm{tr}[A] = \mathrm{tr}[A']$ and $\det[A] = \det[A']$ (Problem 1.6.1). Consequently, though the respective statements $\det \mathbf{A} = \det[A]$ and $\mathrm{tr}\mathbf{A} = \mathrm{tr}[A]$ in (1.137) and (1.138) involve components in a basis, they are in fact *independent of the choice of basis* and so could have been used to define the trace and determinant.

Exercise: Work Problem 1.4.7 using components.

1.4.4 Worked examples.

Problem 1.4.14. According to Problem 1.11, the tensor $\mathbf{Q}$ describing a rotation through an angle θ about a unit vector $\mathbf{n}$ is defined by

$$\mathbf{Qx} = \cos\theta\,\mathbf{x} + (1-\cos\theta)(\mathbf{n}\cdot\mathbf{x})\mathbf{n} + \sin\theta\,(\mathbf{n}\times\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathsf{V}. \tag{i}$$

Calculate the components of $\mathbf{Q}$ in a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the special case where the axis of rotation $\mathbf{n}$ is $\mathbf{e}_3$.

Solution: We shall use (1.129) to calculate the components Q_{ij}. Taking $\mathbf{x} = \mathbf{e}_j$ and $\mathbf{n} = \mathbf{e}_3$ in (i) yields

$$\mathbf{Qe}_j = \cos\theta\,\mathbf{e}_j + (1-\cos\theta)\delta_{3j}\mathbf{e}_3 + \sin\theta\,e_{3jk}\mathbf{e}_k. \tag{ii}$$

Therefore

$$\left.\begin{aligned}
\mathbf{Qe}_1 &= \cos\theta\,\mathbf{e}_1 + \sin\theta\,e_{31k}\mathbf{e}_k = \cos\theta\,\mathbf{e}_1 + \sin\theta\,e_{312}\mathbf{e}_2 &&= \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2,\\
\mathbf{Qe}_2 &= \cos\theta\,\mathbf{e}_2 + \sin\theta\,e_{32k}\mathbf{e}_k = \cos\theta\,\mathbf{e}_2 + \sin\theta\,e_{321}\mathbf{e}_1 &&= \cos\theta\,\mathbf{e}_2 - \sin\theta\,\mathbf{e}_1,\\
\mathbf{Qe}_3 &= \cos\theta\,\mathbf{e}_3 + (1-\cos\theta)\mathbf{e}_3 + \sin\theta\,e_{33k}\mathbf{e}_k &&= \mathbf{e}_3.
\end{aligned}\right\}$$

Therefore we can read off the components of $\mathbf{Q}$ from this using (1.129) to be

$$[Q] = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

Problem 1.4.15. The determinant of a tensor $\mathbf{A}$ was defined in (1.90). Show that $\det\mathbf{A} = \det[A]$ where $[A]$ is the matrix of components of $\mathbf{A}$ in a basis.

Solution: Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be an arbitrary set of linearly independent vectors, and let a_i, b_i, c_i and A_{ij} be the components of these vectors and $\mathbf{A}$ in a basis. Then (motivated by the numerator of (1.90)) we evaluate the quantity $(\mathbf{Aa} \times \mathbf{Ab}) \cdot \mathbf{Ac}$:

$$
\begin{aligned}
(\mathbf{Aa} \times \mathbf{Ab}) \cdot \mathbf{Ac} \quad &\overset{(1.58)}{=} \quad (\mathbf{Aa} \times \mathbf{Ab})_i \ (\mathbf{Ac})_i \overset{(1.61)}{=} e_{ijk} \ (\mathbf{Aa})_j \ (\mathbf{Ab})_k \ (\mathbf{Ac})_i = \\
&\overset{(1.131)}{=} \quad e_{ijk} \ (A_{jm} \, a_m) \ (A_{kn} \, b_n) \ (A_{is} \, c_s) = e_{ijk} \ A_{is} A_{jm} A_{kn} a_m \, b_n \, c_s.
\end{aligned}
$$

The identity (1.40) for the determinant of a matrix, $e_{smn} \det[A] \ = \ e_{ijk} \, A_{is} A_{jm} A_{kn}$, allows us to write this as

$$
\begin{aligned}
(\mathbf{Aa} \times \mathbf{Ab}) \cdot \mathbf{Ac} \quad &= e_{smn} \det[A] \, a_m \, b_n \, c_s \ = \det[A] \, e_{smn} \, a_m \, b_n \, c_s \\
&\overset{(1.61)}{=} \det[A] \ (\mathbf{a} \times \mathbf{b})_s \, c_s \overset{(1.58)}{=} \det[A] \ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.
\end{aligned}
\qquad (i)
$$

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$, are linearly independent it follows from (1.52) that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \neq 0$. Thus, comparing this with (1.90) shows that $\det \mathbf{A} = \det[A]$.

Aside: Observe from (i) that if $\mathbf{A}$ is nonsingular so that $\det \mathbf{A} \neq 0$, then $(\mathbf{Aa} \times \mathbf{Ab}) \cdot \mathbf{Ac} \neq 0$ and so by (1.52) the three vectors $\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}$ are also linearly independent.

Problem 1.4.16. For any symmetric tensor $\mathbf{S}$ and skew-symmetric tensor $\mathbf{W}$ show that

$$
\mathbf{S} \cdot \mathbf{W} = 0. \qquad (1.141)
$$

Solution: The result follows immediately by writing (1.141) in terms of components and then using the result in Problem 1.2.2.

Problem 1.4.17. An orthogonal tensor was defined in (1.95) as a tensor that preserves length, i.e. $\mathbf{Q}$ is orthogonal if

$$
|\mathbf{Qx}| = |\mathbf{x}| \qquad \text{for all vectors } \mathbf{x} \in \mathsf{V}. \qquad (i)
$$

Show that an orthogonal tensor is nonsingular and that

$$
\mathbf{Q}^T = \mathbf{Q}^{-1}. \qquad (ii)
$$

Solution 1: (Using components in a basis.) It follows from (i) that

$$
\mathbf{Qx} \cdot \mathbf{Qx} = \mathbf{x} \cdot \mathbf{x} \qquad \Rightarrow \qquad Q_{ij} Q_{ik} x_j x_k = x_k x_k.
$$

Since this holds for all x_1, x_2, x_3 we may differentiate it with respect to x_p to get

$$
Q_{ij} Q_{ik} \frac{\partial}{\partial x_p} (x_j x_k) = \frac{\partial}{\partial x_p} (x_k x_k) \qquad \Rightarrow \qquad Q_{ij} Q_{ik} (\delta_{jp} x_k + x_j \delta_{pk}) = 2 \delta_{pk} x_k,
$$

which after using the substitution rule yields $Q_{ip} Q_{ik} x_k + Q_{ij} Q_{ip} x_j = 2 x_p$. Changing the dummy subscript $j \to k$ in the second term now leads to

$$
Q_{ip} Q_{ik} x_k = x_p.
$$

Differentiating this with respect to x_q yields

$$Q_{ip}Q_{ik}\delta_{kq} = \delta_{pq} \qquad \Rightarrow \qquad Q_{ip}Q_{iq} = \delta_{pq} \qquad \Rightarrow \qquad Q^T_{pi}Q_{iq} = \delta_{pq} \qquad \Rightarrow \qquad \mathbf{Q}^T\mathbf{Q} = \mathbf{I}.$$

On taking the determinant of this equation we get $\det \mathbf{Q} = \pm 1$ and so $\det \mathbf{Q} \neq 0$ which implies that $\mathbf{Q}$ is nonsingular. Post-multiplying both sides of the preceding equation by $\mathbf{Q}^{-1}$ now yields (ii).

Solution 2: (Without using components in a basis.) To show that $\mathbf{Q}$ is nonsingular we must show that the only vector $\mathbf{x}$ for which $\mathbf{Qx} = \mathbf{o}$ is $\mathbf{x} = \mathbf{o}$. Suppose $\mathbf{Qx} = \mathbf{o}$. Then $|\mathbf{Qx}| = 0$ and so $|\mathbf{x}| = 0$ by (i). This implies that $\mathbf{x} = \mathbf{o}$ since the only vector with zero length is the null vector; see (1.48). Therefore the only vector $\mathbf{x}$ for which $\mathbf{Qx} = \mathbf{o}$ is the null vector and so by definition (1.85), $\mathbf{Q}$ is nonsingular.

Next we write $|\mathbf{Qx}|^2 = |\mathbf{x}|^2$ as $\mathbf{Qx} \cdot \mathbf{Qx} = \mathbf{x} \cdot \mathbf{x}$ which because of (1.74) implies $\mathbf{Q}^T\mathbf{Qx} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{x}$. Thus

$$(\mathbf{Q}^T\mathbf{Q} - \mathbf{I})\mathbf{x} \cdot \mathbf{x} = 0 \qquad \text{for all vectors } \mathbf{x} \in \mathsf{V}. \tag{iii}$$

Recall from Problem 1.4.11(c) that if $\mathbf{Sx} \cdot \mathbf{x} = 0$ for all vectors $\mathbf{x} \in \mathsf{V}$ and $\mathbf{S}$ is a symmetric tensor then $\mathbf{S} = \mathbf{0}$. Since $\mathbf{Q}^T\mathbf{Q} - \mathbf{I}$ is symmetric, it now follows that

$$\mathbf{Q}^T\mathbf{Q} - \mathbf{I} = \mathbf{0}. \tag{iv}$$

Operating on both sides of (iii) with $\mathbf{Q}^{-1}$ (we know that $\mathbf{Q}$ is nonsingular so $\mathbf{Q}^{-1}$ exists) gives the desired result $\mathbf{Q}^{-1} = \mathbf{Q}^T$.

1.5 Invariance. Isotropic functions.

Consider two orthonormal bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. The orthogonal tensor $\mathbf{Q} = \mathbf{e}'_1 \otimes \mathbf{e}_1 + \mathbf{e}'_2 \otimes \mathbf{e}_2 + \mathbf{e}'_3 \otimes \mathbf{e}_3$ maps the first into the second: $\mathbf{Qe}_k = \mathbf{e}'_k$.

– It can be readily verified that for any vector $\mathbf{v}$, the components of $\mathbf{Qv}$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ equal the components of $\mathbf{v}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. This is not surprising since here we have transformed both the vector $\mathbf{v}$ and the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by the same orthogonal tensor $\mathbf{Q}$ to get $\mathbf{Qv}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$.

– If we think of a basis as an "observer" who sees the vector through its components in that basis, then the observer $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ sees the vector $\mathbf{v}$ exactly as the observer $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ sees the vector $\mathbf{Qv}$.

– Likewise for any tensor $\mathbf{C}$, the components of $\mathbf{QCQ}^T$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ equal the components of $\mathbf{C}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Thus the observer $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ sees the tensor $\mathbf{C}$ exactly as the observer $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ sees the tensor $\mathbf{QCQ}^T$.

Isotropic function of a vector:

– A scalar-valued function $\varphi(\mathbf{v})$ is said to be *isotropic* (or invariant) if

$$\varphi(\mathbf{Q}\mathbf{v}) = \varphi(\mathbf{v}) \qquad \text{for all orthogonal } \mathbf{Q}. \tag{1.142}$$

An example of an isotropic function is

$$\varphi(\mathbf{v}) = |\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}.$$

– Let v_i be the components of $\mathbf{v}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then there is an associated function $\widehat{\varphi}(v_1, v_2, v_3)$ defined on $\mathbb{R}^3$ such that

$$\varphi(\mathbf{v}) = \widehat{\varphi}(v_1, v_2, v_3).$$

For instance in the case of the preceding example $\varphi(\mathbf{v}) = \mathbf{v}\cdot\mathbf{v}$,

$$\widehat{\varphi}(v_1, v_2, v_3) = v_1^2 + v_2^2 + v_3^2.$$

When $\varphi(\mathbf{v})$ is isotropic, the function $\widehat{\varphi}(v_1, v_2, v_3)$ does *not* depend on the basis in the sense that

$$\widehat{\varphi}(v_1', v_2', v_3') = \widehat{\varphi}(v_1, v_2, v_3),$$

where v_1', v_2', v_3' are the components of $\mathbf{v}$ in the basis $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$. Note that it is the same function $\widehat{\varphi}$ that appears on both side of this equation and so every observer $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ sees an isotropic function identically. In the example above,

$$\widehat{\varphi}(v_1', v_2', v_3') = (v_1')^2 + (v_2')^2 + (v_3')^3 = v_1^2 + v_2^2 + v_3^2 = \widehat{\varphi}(v_1, v_2, v_3).$$

– *Representation theorem.* Corresponding to any isotropic scalar-valued function $\varphi(\mathbf{v})$ there exists a function $\overline{\varphi}(\cdot)$ such that

$$\varphi(\mathbf{v}) = \overline{\varphi}(|\mathbf{v}|).$$

Isotropic functions of a tensor:

– A scalar-valued function $\varphi(\mathbf{C})$ defined for all symmetric tensors $\mathbf{C}$ is said to be *isotropic* (or invariant) if

$$\phi(\mathbf{C}) = \phi(\mathbf{Q}\mathbf{C}\mathbf{Q}^T) \qquad \text{for all orthogonal tensors } \mathbf{Q}. \tag{1.143}$$

An example of such a function is (Problem 1.5.2)

$$\varphi(\mathbf{C}) = \operatorname{tr}\mathbf{C}^2 \qquad \text{for all symmetric } \mathbf{C} \in \mathsf{Lin}.$$

– Let C_{ij} be the components of $\mathbf{C}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then there is an associated function $\widehat{\varphi}(C_{11}, C_{12}, \ldots, C_{32}, C_{33})$ defined on $\mathbb{R}^9$ such that

$$\varphi(\mathbf{C}) = \widehat{\varphi}(C_{11}, C_{12}, \ldots, C_{32}, C_{33}).$$

For the example $\varphi(\mathbf{C}) = \operatorname{tr} \mathbf{C}^2$,

$$\widehat{\varphi}(C_{11}, C_{12}, \ldots, C_{32}, C_{33}) = C_{11}^2 + C_{12}^2 + \ldots + C_{32}^2 + C_{33}^2.$$

When $\varphi(\mathbf{C})$ is isotropic, the function $\widehat{\varphi}(C_{11}, C_{12}, \ldots, C_{32}, C_{33})$ does *not* depend on the basis in the sense that

$$\widehat{\varphi}(C_{11}, C_{12}, \ldots, C_{32}, C_{33}) = \widehat{\varphi}(C'_{11}, C'_{12}, \ldots, C'_{32}, C'_{33}),$$

where C'_{ij} are the components of $\mathbf{C}$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. Every observer $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ sees an isotropic function identically.

– The three principal scalar invariants $I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C})$ introduced in (1.107) are isotropic functions. (Problem 1.5.2.) It is because of this invariance that they are called “invariants”, the reason for the adjective “principal” being that they are the particular invariants that appear in the characteristic polynomial (1.106).

– *Representation theorem.* Corresponding to any isotropic scalar-valued function $\varphi(\mathbf{C})$ of a symmetric tensor $\mathbf{C}$ there exists a function $\overline{\varphi}(\cdot, \cdot, \cdot)$ such that

$$\varphi(\mathbf{C}) = \overline{\varphi}(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C})). \tag{1.144}$$

– A scalar-valued function $\varphi(\mathbf{C}, \mathbf{M})$ defined for all symmetric tensors $\mathbf{C}$ and $\mathbf{M}$ is said to be jointly isotropic in its arguments if

$$\varphi(\mathbf{C}, \mathbf{M}) = \varphi(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \mathbf{Q}\mathbf{M}\mathbf{Q}^T) \qquad \text{for all orthogonal } \mathbf{Q}. \tag{1.145}$$

Two examples are given in Problem 1.5.5.

1.5.1 Worked examples.

Problem 1.5.1. Show that the functions

$$(a)\ \phi(\mathbf{C}) = \operatorname{tr} \mathbf{C}^n \text{ where } n \text{ is a positive integer} \qquad \text{and} \qquad (b)\ \phi(\mathbf{C}) = \det \mathbf{C},$$

are isotropic.

Solution: (a) Since

$$\mathbf{C}^n = \underbrace{\mathbf{CC}\ldots\mathbf{CC}}_{n \text{ times}} \tag{i}$$

it follows that

$$(\mathbf{QCQ}^T)^n = \underbrace{\mathbf{QCQ}^T\mathbf{QCQ}^T\ldots\mathbf{QCQ}^T\mathbf{QCQ}^T}_{n \text{ times}} = \mathbf{QC}^n\mathbf{Q}^T, \tag{ii}$$

where we have used $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. Therefore

$$\phi(\mathbf{QCQ}^T) = \operatorname{tr}\left[(\mathbf{QCQ}^T)^n\right] \overset{(ii)}{=} \operatorname{tr}(\mathbf{QC}^n\mathbf{Q}^T) \overset{(1.120)}{=} \mathbf{QC}^n \cdot \mathbf{Q} \overset{(1.123)}{=} \mathbf{C}^n \cdot \mathbf{Q}^T\mathbf{Q} \overset{(1.98)}{=} \mathbf{C}^n \cdot \mathbf{I} = \operatorname{tr}\mathbf{C}^n = \phi(\mathbf{C}).$$

Therefore $\phi(\mathbf{C}) = \operatorname{tr}\mathbf{C}^n$ is isotropic.

(b) In view of (1.91) and (1.97),

$$\phi(\mathbf{QCQ}^T) = \det\,\mathbf{QCQ}^T = \det\mathbf{Q}\,\det\mathbf{C}\,\det\mathbf{Q}^T = \det\,\mathbf{C} = \phi(\mathbf{C}),$$

and so $\phi(\mathbf{C}) = \det\,\mathbf{C}$ is isotropic.

Problem 1.5.2. Show that the principal scalar invariant functions,

$$I_1(\mathbf{C}) = \operatorname{tr}\mathbf{C}, \qquad I_2(\mathbf{C}) = \frac{1}{2}\left[\operatorname{tr}\mathbf{C}^2 - (\operatorname{tr}\mathbf{C})^2\right], \qquad I_3(\mathbf{C}) = \det\,\mathbf{C}, \tag{1.146}$$

are isotropic i.e. show that

$$I_i(\mathbf{C}) = I_i(\mathbf{QCQ}^T),\ \ i = 1, 2, 3, \qquad \text{for all orthogonal } \mathbf{Q}. \tag{1.147}$$

Solution: This follows immediately from the results of Problem 1.5.1.

Problem 1.5.3. Consider the scalar-valued function Φ defined for all symmetric tensors $\mathbf{C}$ by

$$\Phi(\mathbf{C}) = \mathbf{GC}\cdot\mathbf{C} \tag{i}$$

where $\mathbf{G}$ is some constant tensor. If $\Phi(\mathbf{C})$ is isotropic, what does this say about the form of the tensor $\mathbf{G}$?

Solution: Only the symmetric part of $\mathbf{G}$ affects the value of the function Φ and so we might as well assume $\mathbf{G}$ to be symmetric; see Problem (1.15). It follows from (i) that

$$\Phi(\mathbf{QCQ}^T) = \mathbf{GQCQ}^T\cdot\mathbf{QCQ}^T \overset{(1.123)}{=} \mathbf{Q}^T\mathbf{GQCQ}^T\cdot\mathbf{CQ}^T \overset{(1.123)}{=} \mathbf{Q}^T\mathbf{GQCQ}^T\mathbf{Q}\cdot\mathbf{C} \overset{(1.98)}{=} \mathbf{Q}^T\mathbf{GQC}\cdot\mathbf{C}$$

Since Φ is isotropic it follows from (1.145), (i) and this that

$$\mathbf{Q}^T\mathbf{GQC}\cdot\mathbf{C} = \mathbf{GC}\cdot\mathbf{C} \qquad \Rightarrow \qquad (\mathbf{Q}^T\mathbf{GQ} - \mathbf{G})\mathbf{C}\cdot\mathbf{C} = 0.$$

This is to hold for all symmetric tensors $\mathbf{C}$ and so it follows from part (e) of Problem 1.4.11 that $\mathbf{Q}^T\mathbf{GQ} - \mathbf{G}$ must be skew-symmetric:

$$\mathbf{Q}^T\mathbf{GQ} - \mathbf{G} = -\big(\mathbf{Q}^T\mathbf{GQ} - \mathbf{G}\big)^T = -\big(\mathbf{Q}^T\mathbf{GQ}\big)^T - \mathbf{G} \overset{(1.76)}{=} -\mathbf{Q}^T\mathbf{GQ} - \mathbf{G}$$

where we have used the symmetry of $\mathbf{G}$ and $\left(\mathbf{Q}^T\right)^T = \mathbf{Q}$. Therefore

$$\mathbf{Q}^T\mathbf{G}\mathbf{Q} = \mathbf{G}.$$

Since this is to hold for all orthogonal $\mathbf{Q}$ it follows from Problem 1.37 that $\mathbf{G}$ must be a scalar multiple of the identity:

$$\mathbf{G} = \gamma\mathbf{I} \qquad \text{for some scalar constant } \gamma.$$

Remark: Therefore $\Phi(\mathbf{C}) = \mathbf{G}\mathbf{C}\cdot\mathbf{C} = \gamma\mathbf{C}\cdot\mathbf{C} = \gamma\,\mathrm{tr}(\mathbf{C}^2)$ and so this is consistent with the general representation (1.144).

Problem 1.5.4. It was shown in Problem 1.5.1 that $\mathrm{tr}\,\mathbf{C}^n$ is isotropic for all positive integers n. Express the principal scalar invariant $I_3(\mathbf{C})$ in terms of $\mathrm{tr}\,\mathbf{C}$, $\mathrm{tr}\,\mathbf{C}^2$ and $\mathrm{tr}\,\mathbf{C}^3$. Do the same for $\mathrm{tr}\,\mathbf{C}^4$. Hint: Use the Cayley-Hamilton theorem (1.108).

Problem 1.5.5. Show that the functions

$$(a)\ \phi(\mathbf{C},\mathbf{M}) = \mathbf{C}\cdot\mathbf{M}, \qquad \text{and } (b)\ \phi(\mathbf{C},\mathbf{M}) = \mathbf{C}^2\cdot\mathbf{M}.$$

are jointly isotropic in $\mathbf{C}$ and $\mathbf{M}$.

Solution: (a) To show that $\phi(\mathbf{C},\mathbf{M}) = \mathbf{C}\cdot\mathbf{M}$ is isotropic, we proceed as follows:

$$\phi(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \mathbf{Q}\mathbf{M}\mathbf{Q}^T) = \mathbf{Q}\mathbf{C}\mathbf{Q}^T\cdot\mathbf{Q}\mathbf{M}\mathbf{Q}^T \overset{(1.123)}{=} \mathbf{C}\cdot\mathbf{Q}^T\mathbf{Q}\mathbf{M}\mathbf{Q}^T\mathbf{Q} = \mathbf{C}\cdot\mathbf{M} = \phi(\mathbf{C},\mathbf{M}).$$

(b) That $\phi(\mathbf{C},\mathbf{M}) = \mathbf{C}^2\cdot\mathbf{M}$ is isotropic can be seen from

$$\phi(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \mathbf{Q}\mathbf{M}\mathbf{Q}^T) = (\mathbf{Q}\mathbf{C}\mathbf{Q}^T)^2\cdot\mathbf{Q}\mathbf{M}\mathbf{Q}^T = (\mathbf{Q}\mathbf{C}\mathbf{Q}^T\mathbf{Q}\mathbf{C}\mathbf{Q}^T)\cdot\mathbf{Q}\mathbf{M}\mathbf{Q}^T = \mathbf{Q}\mathbf{C}^2\mathbf{Q}^T\cdot\mathbf{Q}\mathbf{M}\mathbf{Q}^T,$$

followed by using (1.123) as in the preceding example.

1.6 Change of basis. Cartesian tensors.

We now look at the components of a vector/tensor in *two* orthonormal bases and examine how these components are related. The vector/tensor stays fixed while the basis changes.

1.6.1 Two orthonormal bases.

We first make some observations on the relation between the two bases. In accordance with the standard way in which this topic is discussed in the literature, let $\mathbf{Q}^T$ be the orthogonal tensor that maps the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ into the orthonormal basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$:

$$\mathbf{e}'_i = \mathbf{Q}^T \mathbf{e}_i, \qquad \mathbf{e}_i = \mathbf{Q}\mathbf{e}'_i. \tag{1.148}$$

One can readily show that the components of $\mathbf{Q}$ in the two bases coincide,

$$\mathbf{Q} = Q_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = Q_{ij}\mathbf{e}'_i \otimes \mathbf{e}'_j,$$

and that

$$Q_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j. \tag{1.149}$$

Given the two bases, equation (1.149) provides a formula for calculating the elements Q_{ij} of the matrix of components $[Q]$. Specifically, Q_{ij} is the cosine of the angle between the basis vectors $\mathbf{e}'_i$ and $\mathbf{e}_j$.

Since $\mathbf{Q}$, and therefore $[Q]$, is orthogonal,

$$[Q][Q]^T = [Q]^T[Q] = [I], \qquad Q_{ik}Q_{jk} = Q_{ki}Q_{kj} = \delta_{ij}. \tag{1.150}$$

If one basis can be rotated into the other as is the case if both bases are right-handed or both are left-handed, then $[Q]$ is proper orthogonal ($\det[Q] = +1$). Otherwise $[Q]$ is improper orthogonal ($\det[Q] = -1$).

Exercise: Show that

$$\mathbf{e}'_i = Q_{ij}\mathbf{e}_j, \qquad \mathbf{e}_i = Q_{ji}\mathbf{e}'_j\ , \tag{1.151}$$

1.6.2 Vectors: 1-tensors.

– Let v_i and v'_i be the components of the same vector $\mathbf{v}$ in the respective bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$:

$$v_i = \mathbf{v} \cdot \mathbf{e}_i, \qquad v'_i = \mathbf{v} \cdot \mathbf{e}'_i.$$

This is illustrated in Figure 1.5.

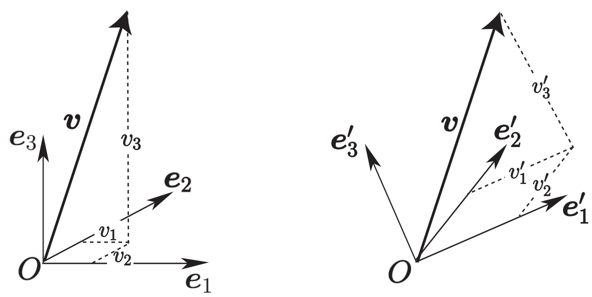

Figure 1.5: Components $\{v_1, v_2, v_3\}$ and $\{v'_1, v'_2, v'_3\}$ of the same vector $\mathbf{v}$ in two different bases.

– These components are related by (Exercise)

$$v'_i = Q_{ij} v_j, \qquad \{v'\} = [Q]\{v\},$$

and equivalently

$$v_i = Q_{ji} v'_j, \qquad \{v\} = [Q]^T \{v'\}.$$

– A quantity whose components v_i and v'_i in (any) two orthonormal bases are related by

$$\left.\begin{array}{ll} v'_i = Q_{ij} v_j, & v_i = Q_{ji} v'_j, \\ \{v'\} = [Q]\{v\}, & \{v\} = [Q]^T \{v'\}, \end{array}\right\} \tag{1.152}$$

is called a 1st-order cartesian tensor or *1-tensor*.

Observe that if we know the components of a 1-tensor in one basis, its components in any other basis can be calculated using (1.152).

1.6.3 Linear transformations: 2-tensors.

– Let A_{ij} and A'_{ij} be the respective components of the same linear transformation $\mathbf{A}$ in the two bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$:

$$A_{ij} = \mathbf{e}_i \cdot (\mathbf{A}\mathbf{e}_j), \qquad A'_{ij} = \mathbf{e}'_i \cdot (\mathbf{A}\mathbf{e}'_j).$$

– These components are related by (Exercise)

$$A'_{ij} \;=\; Q_{ip} Q_{jq} A_{pq}, \qquad [A'] = [Q][A][Q]^T,$$

the inverse relation being

$$A_{ij} = Q_{pi} Q_{qj} A'_{pq}, \qquad [A] = [Q]^T [A'][Q].$$

– An entity whose components A_{ij} and A'_{ij} in every two orthonormal bases are related by

$$\left.\begin{aligned} A'_{ij} = Q_{ip}Q_{jq}A_{pq}, \qquad A_{ij} = Q_{pi}Q_{qj}A'_{pq}, \\ [A'] = [Q][A][Q]^T, \qquad [A] = [Q]^T[A'][Q], \end{aligned}\right\} \tag{1.153}$$

is called a 2^{nd}-order cartesian tensor or a *2-tensor*.

Observe that if we know the components of a 2-tensor in one basis, its components in any other basis can be calculated using (1.153).

Exercise: Show that the components of the identity 2-tensor $\mathbf{I}$ in *every* basis are δ_{ij}.

1.6.4 n-tensors.

The concept of an *n-tensor* can be introduced analogously. Let $\mathbb{T}$ be an entity that is defined in a given basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by a set of 3^n ordered numbers $\mathbb{T}_{i_1 i_2 \ldots. i_n}$. The numbers $\mathbb{T}_{i_1 i_2 \ldots. i_n}$ are called the *components* of $\mathbb{T}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. For example if $\mathbb{T}$ is a scalar, 1-tensor or 2-tensor, it is represented by a set of $3^0, 3^1$ or 3^2 ordered numbers respectively. Let $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ be a second basis related to the first one by an orthogonal matrix [Q] as described by (1.151). Let $\mathbb{T}'_{i_1 i_2 \ldots. i_n}$ be the components of the entity $\mathbb{T}$ in the second basis. If for every choice of bases these two sets of components are related by

$$\mathbb{T}'_{i_1 i_2 \ldots. i_n} \;=\; Q_{i_1 j_1}\, Q_{i_2 j_2}\, \ldots.\, Q_{i_n j_n}\, \mathbb{T}_{j_1 j_2 \ldots. j_n} \tag{1.154}$$

the entity $\mathbb{T}$ is said to be an *n-tensor*. *Thus, the components of a tensor in every basis may be determined if its components in any one basis are known.*

The *quotient rule* states that if $(\mathbb{T}\mathbb{B})_{i_1 i_2 .. i_n}$ is an n-tensor for every m-tensor $\mathbb{B}_{j_1 j_2 \ldots j_m}$, then $\mathbb{T}$ is an ℓ-tensor where

$$(\mathbb{T}\mathbb{B})_{i_1 i_2 .. i_n} = \mathbb{T}_{k_1 k_2 \ldots k_\ell}\, \mathbb{B}_{j_1 j_2 \ldots j_m}. \tag{1.155}$$

Some of the subscripts on the right-hand side of (1.155) maybe repeated. This is called the quotient rule (since it appears to say that the ratio of two tensors is a tensor).

1.6.5 Worked examples.

Problem 1.6.1. Let $[A]$ and $[A']$ be the components of a 2-tensor $\mathbf{A}$ in two orthonormal bases. In general, $[A] \neq [A']$. Show that:

(a) If $[A]$ is symmetric, then so is $[A']$:

$$[A] = [A]^T \qquad \Leftrightarrow \qquad [A'] = [A']^T. \tag{1.156}$$

(b) The trace of the matrices $[A]$ and $[A']$ are equal:

$$\text{tr}\ [A'] = \text{tr}\ [A]. \tag{1.157}$$

(c) The determinants of the matrices $[A]$ and $[A']$ are equal:

$$\det[A'] = \det[A]. \tag{1.158}$$

Remark: Thus, the three characteristics symmetry, trace and determinant are independent of the choice of basis. Therefore there is no ambiguity in defining

(a) a tensor to be symmetric if its matrix of components is symmetric,
(b) the trace of a tensor to be the trace of its matrix of components, and
(c) the determinant of a tensor to be the determinant of its matrix of components.

Solution:

(a) Using the properties $([B][C])^T = [C]^T[B]^T$ and $([B]^T)^T = [B]$ we have

$$[A']^T \stackrel{(1.153)}{=} \big([Q][A][Q]^T\big)^T = [Q][A]^T[Q]^T = [Q][A][Q]^T = [A']. \qquad \square$$

(b)

$$\text{tr}[A'] = A'_{ii} \stackrel{(1.153)}{=} \big([Q][A][Q]^T\big)_{ii} = Q_{ij}A_{jk}Q^T_{ki} = Q_{ij}A_{jk}Q_{ik} = Q_{ij}Q_{ik}A_{jk} \stackrel{(1.133)}{=} \delta_{jk}A_{jk} = A_{jj} = \text{tr}[A]. \quad \square$$

(c) Using the properties $\det([B][C]) = \det[B]\det[C]$ and $\det([B]^T) = \det[B]$ we have

$$\det[A'] \stackrel{(1.153)}{=} \det\big([Q][A][Q]^T\big) = \det[Q]\ \det[A]\ \det[Q]^T = \big(\det[Q]\big)^2\ \det[A] = \det[A]. \qquad \square$$

Problem 1.6.2. The Kronecker delta obeys

$$\delta_{ij} = Q_{ip}Q_{jq}\delta_{pq} \tag{i}$$

for all orthogonal matrices $[Q]$ which follows from the substitution rule and (1.133). Show that the Levi-Civita symbol obeys

$$e_{ijk} = \pm Q_{ip}Q_{jq}Q_{kr}e_{pqr} \tag{ii}$$

for all orthogonal matrices $[Q]$ where the $+$ and $-$ signs hold if $[Q]$ is proper and improper orthogonal respectively.

Remark: If equation (ii) had held with the $\pm$ replaced by $+$ we would have used the term Levi-Civita tensor. Perhaps you have been wondering why we speak of the Levi-Civita *symbol* and not the Levi-Civita tensor!

Solution: Equation (ii) follows immediately upon taking $[A] = [Q]$ in (1.40), multiplying both sides by $Q_{ap}Q_{bq}Q_{cr}$ and using (1.133).

Problem 1.6.3. Consider a scalar-valued function $\phi(\mathbf{A})$. Let $[A]$ be the matrix of components of $\mathbf{A}$ in an arbitrary orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then, there is a function $\widehat{\phi}(\cdot)$ defined on the set of all 3×3 matrices such that $\phi(\mathbf{A}) = \widehat{\phi}([A])$. Note that the function $\widehat{\phi}$ depends on the choice of basis.

Let $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ be a second orthonormal basis and let $[A']$ be the matrix of components of $\mathbf{A}$ in this basis.

If $\phi(\mathbf{A})$ is an *isotropic function*, show that

$$\widehat{\phi}([A]) = \widehat{\phi}([A']),$$

where the function $\widehat{\phi}$ is the *same* on both sides.

Problem 1.6.4. The triplet of orthonormal vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is carried into the set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ by a tensor $\mathbf{R}$:

$$\mathbf{e}'_1 = \mathbf{R}\mathbf{e}_1, \qquad \mathbf{e}'_2 = \mathbf{R}\mathbf{e}_2, \qquad \mathbf{e}'_3 = \mathbf{R}\mathbf{e}_3. \tag{1.159}$$

Show that

(a) the set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is orthonormal if and only if $\mathbf{R}$ is orthogonal.

Assume from hereon that $\mathbf{R}$ is orthogonal.

(b) Show that $\mathbf{R}$ can be expressed as

$$\mathbf{R} = \mathbf{e}'_i \otimes \mathbf{e}_i = \mathbf{e}'_1 \otimes \mathbf{e}_1 + \mathbf{e}'_2 \otimes \mathbf{e}_2 + \mathbf{e}'_3 \otimes \mathbf{e}_3. \tag{1.160}$$

Note that (1.71) is a special case of this.

(c) If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are both right-handed (or both left-handed), show that $\mathbf{R}$ is proper orthogonal (and therefore represents a rotation). If one is right-handed and the other left-handed, show that $\mathbf{R}$ is improper orthogonal.

(d) If $[R]$ and $[R']$ are the matrices of components of the tensor $\mathbf{R}$ in the respective bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, show that $[R] = [R']$; and

Solution:

(a) Suppose that $\mathbf{R}$ is orthogonal. To show that the triplet of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is orthonormal we must show that $\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}$. This follows from

$$\mathbf{e}'_i \cdot \mathbf{e}'_j \overset{(1.159)}{=} \mathbf{R}\mathbf{e}_i \cdot \mathbf{R}\mathbf{e}_j \overset{(1.74)}{=} \mathbf{R}^T\mathbf{R}\mathbf{e}_i \cdot \mathbf{e}_j \overset{(1.98)}{=} \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}. \qquad (i)$$

where in the last step we used the fact that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal set of vectors.

Conversely suppose that $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is orthonormal. Then it is left as an exercise to show that $\mathbf{R}$ is orthogonal.

(b)

$$\mathbf{e}'_i \otimes \mathbf{e}_i \overset{(1.159)}{=} \mathbf{R}\mathbf{e}_i \otimes \mathbf{e}_i = \mathbf{R}\big(\mathbf{e}_i \otimes \mathbf{e}_i\big) \overset{(1.71)}{=} \mathbf{R}\mathbf{I} = \mathbf{R}.$$

(d) By the definition (1.128) of the components of a tensor in a basis,

$$R_{ij} = \mathbf{e}_i \cdot \mathbf{R}\mathbf{e}_j, \qquad R'_{ij} = \mathbf{e}'_i \cdot \mathbf{R}\mathbf{e}'_j. \qquad (ii)$$

Therefore

$$R'_{ij} = \mathbf{e}'_i \cdot \mathbf{R}\mathbf{e}'_j \overset{(1.159)}{=} \mathbf{R}\mathbf{e}_i \cdot \mathbf{R}\mathbf{R}\mathbf{e}_j \overset{(1.74)}{=} \mathbf{e}_i \cdot \mathbf{R}^T\mathbf{R}\mathbf{R}\mathbf{e}_j \overset{(1.98)}{=} \mathbf{e}_i \cdot \mathbf{R}\mathbf{e}_j \overset{(ii)_1}{=} R_{ij}.$$

Problem 1.6.5. (See also Problem 1.32.) When we analyze the bending deformation of a block (see page 167), it will be natural to use rectangular cartesian coordinates in the reference configuration and cylindrical polar coordinates in the deformed configuration. In such settings we will work simultaneously with two bases. That is the motivation for this problem.

Given two orthonormal bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, consider the tensor $\mathbf{F}$ that has the representation

$$\mathbf{F} = \Phi_{ij}\mathbf{e}'_i \otimes \mathbf{e}_j. \qquad (i)$$

Note that *both* bases appear on the right-hand side of (i) whence Φ_{ij} are *not* the components of $\mathbf{F}$ in either basis.

(a) If x_i are the components of a vector $\mathbf{x}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, calculate the components of the vector $\mathbf{y} = \mathbf{F}\mathbf{x}$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$.

(b) Derive a representation analogous to (i) for $\mathbf{F}^T$.

(c) Calculate the components of $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and those of $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$.

(d) Suppose that the two bases are related by $\mathbf{e}'_i = Q_{ij}\mathbf{e}_j$, $\mathbf{e}_i = Q_{ji}\mathbf{e}'_j$ where $[Q]$ is an orthogonal matrix. Calculate the components of $\mathbf{y}$ and $\mathbf{B}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and the components of $\mathbf{C}$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$.

(e) Calculate the components of $\mathbf{F}$ in both bases.

Solution:

(a)

$$\mathbf{y} = \mathbf{F}\mathbf{x} = (\Phi_{ij}\mathbf{e}'_i \otimes \mathbf{e}_j)(x_p\mathbf{e}_p) = \Phi_{ij}x_p(\mathbf{e}'_i \otimes \mathbf{e}_j)\mathbf{e}_p \stackrel{(1.69)}{=} \Phi_{ij}x_p(\mathbf{e}_j \cdot \mathbf{e}_p)\mathbf{e}'_i = \Phi_{ij}x_p\delta_{jp}\mathbf{e}'_i = \Phi_{ip}x_p\ \mathbf{e}'_i, \qquad \square$$

and so we can write

$$\mathbf{y} = y'_i\mathbf{e}'_i \qquad \text{where} \qquad y'_i = \Phi_{ip}x_p, \quad \{y'\} = [\Phi]\{x\};$$

here y'_i are the components of $\mathbf{y}$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. Observe that the vector equation $\mathbf{y} = \mathbf{F}\mathbf{x}$ is equivalent to the matrix equation $\{y'\} = [\Phi]\{x\}$ where $\{y'\}$ are the components of $\mathbf{y}$ in one basis and $\{x\}$ are the components of $\mathbf{x}$ in the other basis.

(b) To determine the transpose we use the definition (1.74), i.e. $\mathbf{F}\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{F}^T\mathbf{y}$. Using (i),

$$\mathbf{F}\mathbf{x} \cdot \mathbf{y} = \Phi_{ij}(\mathbf{e}'_i \otimes \mathbf{e}_j)\mathbf{x} \cdot \mathbf{y} = \Phi_{ij}(\mathbf{e}_j \cdot \mathbf{x})(\mathbf{e}'_i \cdot \mathbf{y}) \stackrel{(*)}{=} \Phi_{ji}(\mathbf{e}_i \cdot \mathbf{x})(\mathbf{e}'_j \cdot \mathbf{y}) = \Phi_{ji}(\mathbf{e}_i \otimes \mathbf{e}'_j)\mathbf{y} \cdot \mathbf{x},$$

where in step $(*)$ we changed the dummy subscripts. Therefore

$$\mathbf{F}^T = \Phi_{ji}(\mathbf{e}_i \otimes \mathbf{e}'_j) \stackrel{(**)}{=} \Phi_{ij}(\mathbf{e}_j \otimes \mathbf{e}'_i), \qquad \square$$

where in step $(**)$ we have changed the dummy subscripts again.

(c) First,

$$\begin{aligned}\mathbf{C} &= \mathbf{F}^T\mathbf{F} = (\Phi_{pq}\mathbf{e}'_p \otimes \mathbf{e}_q)^T(\Phi_{ij}\mathbf{e}'_i \otimes \mathbf{e}_j) = (\Phi_{pq}\mathbf{e}_q \otimes \mathbf{e}'_p)(\Phi_{ij}\mathbf{e}'_i \otimes \mathbf{e}_j) = \Phi_{pq}\Phi_{ij}(\mathbf{e}_q \otimes \mathbf{e}'_p)(\mathbf{e}'_i \otimes \mathbf{e}_j) = \\ &\stackrel{(1.72)}{=} \Phi_{pq}\Phi_{ij}(\mathbf{e}'_p \cdot \mathbf{e}'_i)(\mathbf{e}_q \otimes \mathbf{e}_j) = \Phi_{pq}\Phi_{ij}\delta_{pi}(\mathbf{e}_q \otimes \mathbf{e}_j) = \Phi_{pq}\Phi_{pj}(\mathbf{e}_q \otimes \mathbf{e}_j) = \Phi_{pi}\Phi_{pj}(\mathbf{e}_i \otimes \mathbf{e}_j),\end{aligned}$$

and so we can write

$$\mathbf{C} = C_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j) \qquad \text{where} \qquad C_{ij} = \Phi_{pi}\Phi_{pj}, \quad [C] = [\Phi]^T[\Phi]; \qquad \square$$

here C_{ij} are the components of $\mathbf{C}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Second,

$$\begin{aligned}\mathbf{B} &= \mathbf{F}\mathbf{F}^T = (\Phi_{pq}\mathbf{e}'_p \otimes \mathbf{e}_q)(\Phi_{ij}\mathbf{e}'_i \otimes \mathbf{e}_j)^T = (\Phi_{pq}\mathbf{e}'_p \otimes \mathbf{e}_q)(\Phi_{ij}\mathbf{e}_j \otimes \mathbf{e}'_i). \stackrel{(1.72)}{=} \Phi_{pq}\Phi_{ij}(\mathbf{e}_q \cdot \mathbf{e}_j)(\mathbf{e}'_p \otimes \mathbf{e}'_i) = \\ &= \Phi_{pq}\Phi_{ij}\delta_{qj}(\mathbf{e}'_p \otimes \mathbf{e}'_i) = \Phi_{pq}\Phi_{iq}(\mathbf{e}'_p \otimes \mathbf{e}'_i) = \Phi_{iq}\Phi_{jq}(\mathbf{e}'_i \otimes \mathbf{e}'_j),\end{aligned}$$

and so we can write

$$\mathbf{B} = B'_{ij}(\mathbf{e}'_i \otimes \mathbf{e}'_j) \qquad \text{where} \qquad B'_{ij} = \Phi_{iq}\Phi_{jq}, \quad [B'] = [\Phi][\Phi]^T; \qquad \square$$

here B'_{ij} are the components of $\mathbf{B}$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$.

Observe that the components of $\mathbf{C}$ were naturally expressed in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ while those of $\mathbf{B}$ have been expressed in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. Moreover, even though $[\Phi]$ is *not* the matrix of components of $\mathbf{F}$ in either basis, $[\Phi]^T[\Phi]$ is the matrix of components of $\mathbf{F}^T\mathbf{F}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and $[\Phi][\Phi]^T$ is the matrix of components of $\mathbf{F}\mathbf{F}^T$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$.

(d) Exercise!

(e) In order to determine the components of $\mathbf{F}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we must express $\mathbf{F} = \Phi_{ij}\, \mathbf{e}'_i \otimes \mathbf{e}_j$ in the form $\mathbf{F} = F_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$ by eliminating $\mathbf{e}'_i$ in favor of $\mathbf{e}_i$. Likewise, to determine the components of $\mathbf{F}$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ we must express $\mathbf{F} = \Phi_{ij}\, \mathbf{e}'_i \otimes \mathbf{e}_j$ in the form $\mathbf{F} = F'_{ij}\, \mathbf{e}'_i \otimes \mathbf{e}'_j$ by eliminating $\mathbf{e}_i$ in favor of $\mathbf{e}'_i$. This can be done by using the relations

$$\mathbf{e}'_i = Q_{ij}\mathbf{e}_j, \qquad \mathbf{e}_i = Q_{ji}\mathbf{e}'_j. \tag{ii}$$

We now find the components of $\mathbf{F}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by using $(ii)_1$ to eliminate $\mathbf{e}'_i$:

$$\mathbf{F} = \Phi_{ij}\mathbf{e}'_i \otimes \mathbf{e}_j \stackrel{(vi)_1}{=} \Phi_{ij}(Q_{ip}\mathbf{e}_p) \otimes \mathbf{e}_j \stackrel{(*)}{=} \Phi_{kj}Q_{ki}\mathbf{e}_i \otimes \mathbf{e}_j = Q_{ki}\Phi_{kj}\mathbf{e}_i \otimes \mathbf{e}_j = ([Q]^T[\Phi])_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = F_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

(where in step $(*)$ we changed the dummy subscripts $i \to k$ and $p \to i$) and so

$$[F] = [Q]^T[\Phi]. \qquad \square$$

Likewise we find the components of $\mathbf{F}$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ by using $(ii)_2$ to eliminate $\mathbf{e}_i$:

$$\mathbf{F} = \Phi_{ij}\mathbf{e}'_i \otimes \mathbf{e}_j \stackrel{(vi)_2}{=} \Phi_{ij}\mathbf{e}'_i \otimes (Q_{pj}\mathbf{e}'_p) = \Phi_{ij}Q_{pj}\mathbf{e}'_i \otimes \mathbf{e}'_p \stackrel{(*)}{=} \Phi_{ik}Q_{jk}\mathbf{e}'_i \otimes \mathbf{e}'_j = ([\Phi][Q]^T)_{ij}\mathbf{e}'_i \otimes \mathbf{e}'_j = F'_{ij}\mathbf{e}'_i \otimes \mathbf{e}'_j$$

(where in step $(*)$ we changed the dummy subscripts $j \to k$ and $p \to j$) and therefore

$$[F'] = [\Phi][Q]^T. \qquad \square$$

Observe that $[\Phi] = [Q][F] = [F'][Q]$ i.e. $[F'] = [Q][F][Q]^T$.

1.7 Euclidean point space.

It later chapters, when we study the mechanical response of a body, the body will occupy some region of "physical space". The quantities of interest, for example the stress, will vary from point to point within this region. In this section we briefly touch on the connection of physical space – a Euclidean point space – to a Euclidean vector space.

A *Euclidean point space* $\mathcal{E}$ is a collection of elements that we call points. Corresponding to each ordered pair of points $p, q \in \mathcal{E}$, there is a unique vector $\overrightarrow{pq}$ in an associated Euclidean vector space V with the properties

$$\overrightarrow{pq} = -\overrightarrow{qp} \qquad \text{and} \qquad \overrightarrow{pq} = \overrightarrow{pr} + \overrightarrow{rq},$$

for all points $p, q, r \in \mathcal{E}$. Observe that each point is not a vector but each (ordered) pair of points is associated with a vector. Geometric characteristics in $\mathcal{E}$, such as distance and angle, are derived from the vector space V in a natural way, e.g. the distance between two points $p, q \in \mathcal{E}$ is defined as the magnitude of the vector $\vec{pq}$, etc. There is no notion of the addition of two points.

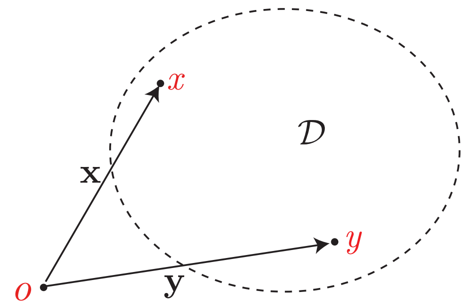

Figure 1.6: Points o, x and y in a Euclidean point space $\mathcal{E}$; ($\mathcal{D}$ is some domain of $\mathcal{E}$). The position vectors of the points x and y relative to the point o – the origin – are the respective vectors $\mathbf{x} = \vec{ox}$ and $\mathbf{y} = \vec{oy}$ of the associated Euclidean vector space.

Pick and fix a point $o \in \mathcal{E}$ as depicted in Figure 1.6. Then, corresponding to each point $x \in \mathcal{E}$ there is a unique vector $\mathbf{x} = \vec{ox} \in \mathsf{V}$. We refer to $\mathbf{x}$ as the **position vector** of point x relative to the *origin* o. Observe that if $\mathbf{x}$ and $\mathbf{y}$ are the position vectors of points x and y relative to an origin o, the vector $\mathbf{y} - \mathbf{x}$ is in fact independent of the choice of origin.

An origin $o \in \mathcal{E}$ together with an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for V is referred to as a **frame** which we denote by $\{o; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The components (x_1, x_2, x_3) of the position vector $\mathbf{x}$ in this basis are called the **coordinates** of the point x in this frame. When the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is fixed, we refer to $\{o; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as a **rectangular cartesian frame**.

Since the quantities of interest will vary from point to point in the region of space occupied by the body, they can be treated as functions of the position vector $\mathbf{x}$ (it being implicit that an origin has been chosen).

Let $\mathcal{D}$ be a domain in physical space (a Euclidean point space) and let $\Phi(x)$ be a scalar-valued function defined at each point $x \in \mathcal{D}$. The function $\Phi(x)$ is referred to as a scalar *field* on $\mathcal{D}$. Vector- and tensor fields are defined analogously.

Once an origin $o \in \mathcal{E}$ has been chosen and fixed, there is a one-to-one correspondence between the points x and the associated position vectors $\mathbf{x} = \vec{ox}$. Let D be the set of all position vectors corresponding to the set of all points in $\mathcal{D}$. Then there is a scalar-valued function $\varphi(\mathbf{x})$ defined at each position vector $\mathbf{x} \in \mathsf{D}$ such that

$$\varphi(\mathbf{x}) = \Phi(x), \qquad \mathbf{x} = \overrightarrow{ox},$$

for all $x \in \mathcal{D}$. Since the effect of a change of origin from say o to o' is to add the vector $\overrightarrow{oo'}$ to all position vectors, the fact that φ depends on the choice of origin is not important. Therefore from hereon we make no distinction between $\varphi, \mathcal{D}$ and Φ, D.

1.8 Calculus.

A brief video on the use of indicial notation in tensor calculus can be found at
https://www.dropbox.com/sh/8qugaq4ru0rk4dz/AACgJHFTe6hnUklwcdmUepg2a?dl=0.

1.8.1 Calculus of scalar, vector and tensor fields.

Throughout this section, $\mathcal{R}$ is a region in three-dimensional space whose boundary is denoted by $\partial\mathcal{R}$. The position vector of a generic point in $\mathcal{R} + \partial\mathcal{R}$ (with respect to some fixed origin) is $\mathbf{x}$. We shall consider a scalar field $\phi(\mathbf{x})$, a vector field $\mathbf{u}(\mathbf{x})$, and a tensor field $\mathbf{T}(\mathbf{x})$, each defined for $\mathbf{x} \in \mathcal{R} + \partial\mathcal{R}$. The region $\mathcal{R} + \partial\mathcal{R}$, and these fields, are assumed to be sufficiently regular so as to permit the calculations carried out below.

In this section we shall work primarily with components in a cartesian (i.e. fixed orthonormal) basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in which $x_i = \mathbf{x} \cdot \mathbf{e}_i$, $u_i = \mathbf{u} \cdot \mathbf{e}_i$ and $T_{ij} = \mathbf{T}\mathbf{e}_j \cdot \mathbf{e}_i$ denote the respective components of $\mathbf{x}, \mathbf{u}$ and $\mathbf{T}$. In Section 1.8.6 we shall derive corresponding expressions in cylindrical polar coordinates.

Gradient of a scalar field. First consider the scalar field $\phi(\mathbf{x})$ and let $\mathbf{g}(\mathbf{x})$ be the vector field defined by

$$\mathbf{g}(\mathbf{x}) \cdot \mathbf{a} := \frac{d}{dt}\phi(\mathbf{x} + t\mathbf{a})\Big|_{t=0} \qquad \text{for all constant vectors } \mathbf{a}, \qquad (i)$$

where t is a scalar parameter, or equivalently by[7]

$$\phi(\mathbf{x}+\mathbf{a}) - \phi(\mathbf{x}) = \mathbf{g}(\mathbf{x})\cdot\mathbf{a} + \mathrm{o}(|\mathbf{a}|) \qquad \text{as } |\mathbf{a}| \to 0. \tag{1.161}$$

(Here the remainder term $\mathrm{o}(|\mathbf{a}|)$ is a term that approaches zero faster than $|\mathbf{a}|$.) One refers to $\mathbf{g}$ as the gradient of ϕ and we introduce the notation $\mathbf{g} = \operatorname{grad}\phi$ or $\nabla\phi$.

Since the partial derivative of ϕ with respect to x_i is the derivative of $\phi(\mathbf{x})$ in the direction $\mathbf{e}_i$ we have

$$\frac{\partial\phi}{\partial x_i} = \frac{d}{dt}\phi(\mathbf{x}+t\mathbf{e}_i)\Big|_{t=0}. \tag{ii}$$

One can show from (i) and (ii) that

$$\mathbf{g} = \frac{\partial\phi}{\partial x_i}\,\mathbf{e}_i. \tag{iii}$$

Equation (iii) is therefore the representation of $\operatorname{grad}\phi$ with respect to the cartesian basis:

$$\operatorname{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x_i}\,\mathbf{e}_i, \qquad\qquad \big(\operatorname{grad}\phi\big)_i = \big(\nabla\phi\big)_i = \frac{\partial\phi}{\partial x_i}. \tag{1.162}$$

Gradient of a vector field. Second, consider the vector field $\mathbf{u}(\mathbf{x})$ and let $\mathbf{G}(\mathbf{x})$ be the tensor field defined by

$$\mathbf{G}(\mathbf{x})\mathbf{a} := \frac{d}{dt}\mathbf{u}(\mathbf{x}+t\mathbf{a})\Big|_{t=0} \qquad \text{for all constant vectors } \mathbf{a}, \tag{iv}$$

where t is a scalar parameter, or equivalently by[8]

$$\mathbf{u}(\mathbf{x}+\mathbf{a}) - \mathbf{u}(\mathbf{x}) = \mathbf{G}(\mathbf{x})\mathbf{a} + \mathrm{o}(|\mathbf{a}|) \qquad \text{as } |\mathbf{a}| \to 0; \tag{1.163}$$

$\mathbf{G}$ is the gradient of $\mathbf{u}$ and we introduce the notation $\mathbf{G} = \operatorname{grad}\mathbf{u}$ or $\nabla\mathbf{u}$.

The partial derivative of $\mathbf{u}$ with respect to x_j is the derivative of $\mathbf{u}(\mathbf{x})$ in the direction $\mathbf{e}_i$ and so

$$\frac{\partial\mathbf{u}}{\partial x_j} = \frac{d}{dt}\mathbf{u}(\mathbf{x}+t\mathbf{e}_j)\Big|_{t=0}. \tag{v}$$

[7] We shall make use of this representation when calculating the gradient of a scalar field in cylindrical polar coordinates in Section 1.8.6 .

[8] We shall make use of this representation when calculating the gradient of a vector field in cylindrical polar coordinates in Section 1.8.6 .

One can show from (iv) and (v) that

$$\mathbf{G} = \frac{\partial \mathbf{u}}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial (u_i \mathbf{e}_i)}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial u_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j. \qquad (vi)$$

Equation (vi) is therefore the representation of grad $\mathbf{u}$ with respect to the cartesian basis:

$$\operatorname{grad} \mathbf{u} = \nabla \mathbf{u} = \frac{\partial u_i}{\partial x_j}\, \mathbf{e}_i \otimes \mathbf{e}_j, \qquad \big(\operatorname{grad} \mathbf{u}\big)_{ij} = \big(\nabla \mathbf{u}\big)_{ij} = \frac{\partial u_i}{\partial x_j}. \qquad (1.164)$$

Observe that the gradient of a n-tensor field is a $n+1$-tensor field, while (as we see next) the divergence of a n-tensor field is a $n-1$-tensor field.

Divergence and curl of a vector field. The divergence of the vector field $\mathbf{u}(\mathbf{x})$ is the scalar field denoted by div $\mathbf{u}$ and defined by

$$\operatorname{div} \mathbf{u} = \operatorname{tr}(\nabla \mathbf{u}). \qquad (1.165)$$

By (1.164) and (1.165), in cartesian components,

$$\operatorname{div} \mathbf{u} = \frac{\partial u_i}{\partial x_i}. \qquad (1.166)$$

The curl of the vector field $\mathbf{u}(\mathbf{x})$ is the vector field denoted by curl $\mathbf{u}$ which in cartesian components is defined by

$$\operatorname{curl} \mathbf{u} = e_{ijk} \frac{\partial u_k}{\partial x_j} \mathbf{e}_i, \qquad \big(\operatorname{curl} \mathbf{u}\big)_i = e_{ijk} \frac{\partial u_k}{\partial x_j}. \qquad (1.167)$$

Divergence and curl of a tensor field. The divergence and curl of the tensor field $\mathbf{T}(\mathbf{x})$ are, respectively, the vector field denoted by div $\mathbf{T}$ and the tensor field denoted by curl $\mathbf{T}$ whose cartesian components are defined by

$$\big(\operatorname{div} \mathbf{T}\big)_i = \frac{\partial T_{ij}}{\partial x_j}, \qquad (1.168)$$

$$\big(\operatorname{curl} \mathbf{T}\big)_{ij} = e_{ipq} \frac{\partial T_{jq}}{\partial x_p}. \qquad (1.169)$$

Exercise: Show that the divergence of a tensor field $\mathbf{T}(\mathbf{x})$ obeys

$$\big(\operatorname{div} \mathbf{T}\big) \cdot \mathbf{v} = \operatorname{div} \big(\mathbf{T}^T \mathbf{v}\big) \qquad \text{for all constant vectors } \mathbf{v},$$

$$\big(\operatorname{div} \mathbf{T}\big) \cdot \mathbf{v} = \operatorname{div}(\mathbf{T}^T \mathbf{v}) - \mathbf{T} \cdot \operatorname{grad} \mathbf{v} \qquad \text{for all vector fields } \mathbf{v}(\mathbf{x}). \qquad (1.170)$$

Note that $\mathbf{T}^T\mathbf{v}$ is a vector and so $\operatorname{div}(\mathbf{T}^T\mathbf{v})$ refers to the divergence of a vector field.

The Laplacian of the scalar field ϕ is denoted by $\nabla^2\phi$ or $\Delta\phi$ and defined as

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_i \partial x_i}. \qquad (1.171)$$

1.8.2 Divergence theorem.

The *divergence theorem* allows one to relate a surface integral on the boundary $\partial\mathcal{R}$ of a closed region $\mathcal{R}$ to a volume integral over $\mathcal{R}$. In particular, for a scalar field $\phi(\mathbf{x})$, a vector field $\mathbf{u}(\mathbf{x})$ and a tensor field $\mathbf{T}(\mathbf{x})$ one has, with $\mathbf{n}(\mathbf{x})$ being the outward pointing unit vector at points $\mathbf{x}$ on the boundary $\partial\mathcal{R}$,

$$\int_{\partial\mathcal{R}} \phi\mathbf{n}\, dA = \int_{\mathcal{R}} \operatorname{grad}\phi\, dV, \tag{1.172}$$

$$\int_{\partial\mathcal{R}} \mathbf{u}\cdot\mathbf{n}\, dA = \int_{\mathcal{R}} \operatorname{div}\mathbf{u}\, dV, \tag{1.173}$$

$$\int_{\partial\mathcal{R}} \mathbf{T}\mathbf{n}\, dA = \int_{\mathcal{R}} \operatorname{div}\mathbf{T}\, dV, \tag{1.174}$$

or, in terms of components,

$$\int_{\partial\mathcal{R}} \phi n_i\, dA = \int_{\mathcal{R}} \frac{\partial\phi}{\partial x_i}\, dV, \tag{1.175}$$

$$\int_{\partial\mathcal{R}} u_i n_i\, dA = \int_{\mathcal{R}} \frac{\partial u_i}{\partial x_i}\, dV, \tag{1.176}$$

$$\int_{\partial\mathcal{R}} T_{ij} n_j\, dA = \int_{\mathcal{R}} \frac{\partial T_{ij}}{\partial x_j}\, dV. \tag{1.177}$$

For a general field $\mathbb{D}_{i_1 i_2 \ldots i_n}(\mathbf{x})$ (which could be the product of various fields) the divergence theorem gives

$$\int_{\partial\mathcal{R}} \mathbb{T}_{i_1 i_2\ldots i_n}\, n_k\, dA = \int_{\mathcal{R}} \frac{\partial}{\partial x_k}(\mathbb{T}_{i_1 i_2\ldots i_n})\, dV. \tag{1.178}$$

1.8.3 Localization.

Let $\mathcal{R}$ be a bounded regular region of three-dimensional space and suppose that the scalar field $\phi(\mathbf{x})$ is defined *and continuous* at all $\mathbf{x}\in\mathcal{R}+\partial\mathcal{R}$. If

$$\int_{\mathcal{D}} \phi(\mathbf{x})\, \mathrm{d}V = 0 \qquad \text{for } \textit{all} \text{ subregions } \mathcal{D}\subset\mathcal{R}, \tag{1.179}$$

then

$$\phi(\mathbf{x}) = 0 \qquad \text{at every point } \mathbf{x}\in\mathcal{R}. \tag{1.180}$$

Since (1.179) holds for all *regions* $\mathcal{D} \subset \mathcal{R}$, it is sometimes said to be a *global* statement, in contrast to (1.180) that holds at each *point* in $\mathcal{R}$ and so is said to be (the associated) *local* statement.

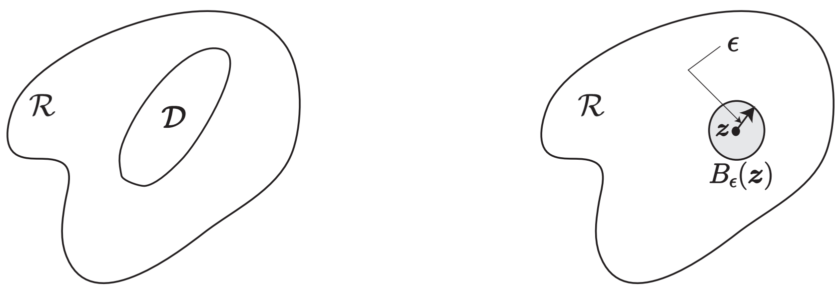

Figure 1.7: The region $\mathcal{R}$, a subregion $\mathcal{D}$, the point $\mathbf{z}$ and a neighborhood $B_\epsilon(\mathbf{z})$ of the point $\mathbf{z}$.

One can prove this by contradiction. Suppose that (1.180) does *not* hold. This implies that there is a point, say $\mathbf{z} \in \mathcal{R}$, at which $\phi(\mathbf{z}) \neq 0$. Suppose that ϕ is positive at this point: $\phi(\mathbf{z}) > 0$. By continuity, ϕ is necessarily (strictly) positive in some neighborhood of $\mathbf{z}$ as well. Let $B_\epsilon(\mathbf{z})$ be a sphere with its center at $\mathbf{z}$ and radius $\epsilon > 0$. We can always choose ϵ sufficiently small (and > 0) so that

$$\phi(\mathbf{x}) > 0 \qquad \text{at all } \mathbf{x} \in B_\epsilon(\mathbf{z});$$

$B_\epsilon(\mathbf{z})$ is a sufficiently small neighborhood of $\mathbf{z}$. Since (1.179) holds for all regions $\mathcal{D}$, we may pick a region $\mathcal{D}$ that is a subset of $B_\epsilon(\mathbf{z})$. Then $\phi(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{D}$. Integrating ϕ over this $\mathcal{D}$ gives

$$\int_{\mathcal{D}} \phi(\mathbf{x})\, \mathrm{d}V > 0$$

thus contradicting (1.179). An entirely analogous calculation can be carried out in the case $\phi(\mathbf{z}) < 0$. Thus the starting assumption must be false and (1.180) must hold.

1.8.4 Function of a tensor.

– Let $\phi(\mathbf{F})$ be a scalar-valued function defined for all tensors $\mathbf{F} \in \mathsf{Lin}$. Then there is a function $\widehat{\phi}$ such that

$$\phi(\mathbf{F}) = \widehat{\phi}(F_{11}, F_{12}, F_{13}, F_{21}, F_{22}, F_{23}, F_{31}, F_{32}, F_{33}), \tag{1.181}$$

where the F_{ij}'s are the components of $\mathbf{F}$ in a fixed orthonormal basis. We let $\partial\phi/\partial\mathbf{F}$ denote the tensor with components

$$\left(\frac{\partial\phi}{\partial\mathbf{F}}\right)_{ij} = \frac{\partial\widehat{\phi}}{\partial F_{ij}}. \tag{1.182}$$

– Let $\phi(\mathbf{C})$ be a scalar-valued function defined for all *symmetric* tensors $\mathbf{C}$. Then $\partial\phi/\partial\mathbf{C}$ must also be symmetric (since ϕ is only defined for symmetric tensors).

Suppose that in terms of components in a fixed basis we have

$$\phi(\mathbf{C}) = \widehat{\phi}(C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23}, C_{31}, C_{32}, C_{33}). \tag{1.183}$$

One must pay some attention when differentiating $\phi(\mathbf{C})$ to ensure that $\partial\phi/\partial\mathbf{C}$ is symmetric. For example consider the function $\widehat{\phi}(C_{11}, C_{12}, \ldots C_{33}) = C_{12}$. If one writes $\partial\widehat{\phi}/\partial C_{12} = 1, \partial\widehat{\phi}/\partial C_{21} = 0$, with all other $\partial\widehat{\phi}/\partial C_{ij} = 0$, the resulting tensor $\partial\phi/\partial\mathbf{C}$ is not symmetric since $\partial\widehat{\phi}/\partial C_{12} \neq \partial\widehat{\phi}/\partial C_{21}$. Instead, one can symmetrize $\widehat{\phi}(C_{11}, C_{12}, \ldots C_{33}) = C_{12}$ by replacing C_{12} by $\frac{1}{2}(C_{12}+C_{21})$ and writing $\widehat{\phi}(C_{11}, C_{12}, \ldots C_{33}) = \frac{1}{2}(C_{12} + C_{21})$. When this is differentiated we get $\partial\widehat{\phi}/\partial C_{12} = 1/2, \partial\widehat{\phi}/\partial C_{21} = 1/2$. Alternatively, we can define $(\partial\phi/\partial\mathbf{C})_{12}$ to be $\frac{1}{2}(\partial\widehat{\phi}/\partial C_{12} + \partial\widehat{\phi}/\partial C_{21})$.

Generalizing this, in the first approach we express the function $\widehat{\phi}$ in "symmetric form" such that

$$\widehat{\phi}(C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23}, C_{31}, C_{32}, C_{33}) = \widehat{\phi}(C_{11}, C_{21}, C_{13}, C_{12}, C_{22}, C_{23}, C_{31}, C_{32}, C_{33}),$$

$$\widehat{\phi}(C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23}, C_{31}, C_{32}, C_{33}) = \widehat{\phi}(C_{11}, C_{12}, C_{31}, C_{21}, C_{22}, C_{23}, C_{13}, C_{32}, C_{33}),$$

$$\widehat{\phi}(C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23}, C_{31}, C_{32}, C_{33}) = \widehat{\phi}(C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{32}, C_{31}, C_{23}, C_{33}).$$

One can achieve such a symmetrization of $\widehat{\phi}$ by replacing C_{ij} by $\frac{1}{2}(C_{ij} + C_{ji})$. Then $\partial\phi/\partial\mathbf{C}$ denotes the *symmetric tensor* with components

$$\left(\frac{\partial\phi}{\partial\mathbf{C}}\right)_{ij} = \frac{\partial\widehat{\phi}}{\partial C_{ij}} = \frac{\partial\widehat{\phi}}{\partial C_{ji}}. \tag{1.184}$$

In the second approach we let $\partial\phi/\partial\mathbf{C}$ be the *symmetric tensor* with components

$$\left(\frac{\partial\phi}{\partial\mathbf{C}}\right)_{ij} = \frac{1}{2}\left(\frac{\partial\phi}{\partial C_{ij}} + \frac{\partial\phi}{\partial C_{ji}}\right). \tag{1.185}$$

In practice it is easier to use (1.185) than to symmetrize the function. See Problem 1.8.6.

1.8.5 Worked examples.

Problem 1.8.1. (Vector and tensor fields.) For any tensor field $\mathbf{A}(\mathbf{x})$ and vector field $\mathbf{u}(\mathbf{x})$ show that

$$\mathbf{u} \cdot \operatorname{div} \mathbf{A} = \operatorname{div}(\mathbf{A}^T \mathbf{u}) - \mathbf{A} \cdot \operatorname{grad} \mathbf{u}. \tag{1.186}$$

Solution:

Since $\mathbf{u}$ and $\operatorname{div} \mathbf{A}$ are vectors, the left-hand side of (1.186) represents their scalar product:

$$\begin{aligned}\mathbf{u} \cdot \operatorname{div} \mathbf{A} = u_i\ (\operatorname{div} \mathbf{A})_i \quad &\overset{(1.168)}{=} u_i \frac{\partial A_{ij}}{\partial x_j} = \\ &= \frac{\partial}{\partial x_j}(A_{ij} u_i) - A_{ij} \frac{\partial u_i}{\partial x_j} \overset{(1.164)}{=} \frac{\partial}{\partial x_j}(\mathbf{A}^T \mathbf{u})_j - A_{ij}\ (\operatorname{grad} \mathbf{u})_{ij} = \\ &\overset{(1.166)}{=} \operatorname{div}(\mathbf{A}^T \mathbf{u}) - \mathbf{A} \cdot \operatorname{grad} \mathbf{u} \qquad \square\end{aligned}$$

Problem 1.8.2. (Vector and tensor fields.) Consider the vector field $\mathbf{u}(\mathbf{x})$:

$$\mathbf{u}(\mathbf{x}) = \beta\ \frac{\mathbf{x}}{r^3}, \qquad r = |\mathbf{x}| \neq 0, \tag{i}$$

where β is a constant. The tensor field $\mathbf{E}(\mathbf{x})$ is related to $\mathbf{u}(\mathbf{x})$ by

$$\mathbf{E} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right), \tag{ii}$$

and the tensor field $\mathbf{S}(\mathbf{x})$ is related to $\mathbf{E}(\mathbf{x})$ by

$$\mathbf{S} = 2\mu \mathbf{E} + \lambda\ \operatorname{tr}(\mathbf{E})\ \mathbf{1}, \tag{iii}$$

where λ and μ are constants. Verify that $\mathbf{S}(\mathbf{x})$ satisfies the differential equation:

$$\operatorname{div} \mathbf{S}(\mathbf{x}) = \mathbf{o}, \qquad |\mathbf{x}| \neq 0. \tag{iv}$$

Solution: We proceed in a straightforward manner by first substituting (i) into (ii) to calculate $\mathbf{E}(\mathbf{x})$; then substituting $\mathbf{E}(\mathbf{x})$ into (iii) to calculate $\mathbf{S}(\mathbf{x})$; and finally checking whether this $\mathbf{S}(\mathbf{x})$ satisfies (iv).

In terms of components, (ii) can be written as

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{v}$$

and so we need to calculate $\partial u_i / \partial x_j$. Since the expression (i) for u_i involves r, it is convenient to start by calculating $\partial r / \partial x_j$. Observe by differentiating $|\mathbf{x}|^2 = r^2 = x_i\, x_i$ that

$$2r \frac{\partial r}{\partial x_j} = 2 \frac{\partial x_i}{\partial x_j}\, x_i = 2\delta_{ij} x_i \ = 2x_j,$$

and therefore

$$\frac{\partial r}{\partial x_j} = \frac{x_j}{r}. \tag{vi}$$

Now differentiating (i), i.e. $u_i = \beta x_i/r^3$, with respect to x_j gives

$$\frac{\partial u_i}{\partial x_j} = \frac{\beta}{r^3}\frac{\partial x_i}{\partial x_j} + \beta x_i \frac{\partial (r^{-3})}{\partial x_j} = \beta\,\frac{\delta_{ij}}{r^3} - 3\beta\frac{x_i}{r^4}\,\frac{\partial r}{\partial x_j} \overset{(vi)}{=} \beta\,\frac{\delta_{ij}}{r^3} - 3\beta\,\frac{x_i}{r^4}\,\frac{x_j}{r} = \beta\,\frac{\delta_{ij}}{r^3} - 3\beta\,\frac{x_i\,x_j}{r^5}.$$

Substituting this into (v) gives us E_{ij}:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \beta\left(\frac{\delta_{ij}}{r^3} - 3\,\frac{x_i\,x_j}{r^5}\right). \qquad (vii)$$

Next, substituting (vii) into (iii) gives S_{ij}:

$$\begin{aligned} S_{ij} &= 2\mu\, E_{ij} + \lambda E_{kk}\delta_{ij} \;=\; 2\mu\beta\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i\,x_j}{r^5}\right) + \lambda\beta\left(\frac{\delta_{kk}}{r^3} - 3\,\frac{x_k\,x_k}{r^5}\right)\delta_{ij} \\ &= 2\mu\beta\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i\,x_j}{r^5}\right) + \lambda\beta\left(\frac{3}{r^3} - 3\,\frac{r^2}{r^5}\right)\delta_{ij} \;=\; 2\mu\beta\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i\,x_j}{r^5}\right). \end{aligned}$$

Finally we use this to calculate $\partial S_{ij}/\partial x_j$

$$\begin{aligned} \frac{1}{2\mu\beta}\,\frac{\partial S_{ij}}{\partial x_j} &= \delta_{ij}\,\frac{\partial}{\partial x_j}\,(r^{-3}) - \frac{3}{r^5}\,\frac{\partial}{\partial x_j}\,(x_i\,x_j) - 3x_i\,x_j\,\frac{\partial}{\partial x_j}(r^{-5}) \\ &= \delta_{ij}\left(-\frac{3}{r^4}\frac{\partial r}{\partial x_j}\right) - \frac{3}{r^5}\,(\delta_{ij}\,x_j + x_i\,\delta_{jj}) - 3x_i\,x_j\left(-\frac{5}{r^6}\,\frac{\partial r}{\partial x_j}\right) \\ &\overset{(vi)}{=} -3\,\frac{\delta_{ij}}{r^4}\,\frac{x_j}{r} - \frac{3}{r^5}\,(x_i + 3x_i) + \frac{15x_i\,x_j}{r^6}\,\frac{x_j}{r} \\ &= 0. \end{aligned}$$

Therefore $\partial S_{ij}/\partial x_j = 0$ which establishes the claim.

Problem 1.8.3. (Divergence theorem.) Show that

$$\int_{\partial\mathcal{R}} \mathbf{x}\otimes\mathbf{n}\,dA = \mathsf{vol}(R)\,\mathbf{I},$$

where $\mathsf{vol}(\mathcal{R})$ is the volume of the region $\mathcal{R}$.

Solution: In terms of components in a fixed basis, we have to show that

$$\int_{\partial\mathcal{R}} x_i n_j\,dA = \mathsf{vol}(\mathcal{R})\,\delta_{ij}.$$

The result follows immediately by using the divergence theorem:

$$\int_{\partial\mathcal{R}} x_i n_j\,dA \overset{(1.176)}{=} \int_{\mathcal{R}} \frac{\partial x_i}{\partial x_j}\,dV = \int_{\mathcal{R}} \delta_{ij}\,dV \quad = \delta_{ij}\int_{\mathcal{R}} dV \quad = \delta_{ij}\mathsf{vol}(\mathcal{R}).$$

Problem 1.8.4. (Functions of a tensor.) The scalar-valued functions

$$I_1(\mathbf{C}) = \operatorname{tr}\mathbf{C}, \qquad I_2(\mathbf{C}) = \frac{1}{2}\big[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}(\mathbf{C}^2)\big], \qquad I_3(\mathbf{C}) = \det\,\mathbf{C},$$

are defined for all nonsingular symmetric tensors $\mathbf{C}$. Show that

$$\frac{\partial I_1}{\partial \mathbf{C}} = \mathbf{I}, \qquad \frac{\partial I_2}{\partial \mathbf{C}} = I_1\,\mathbf{I} - \mathbf{C} \qquad \frac{\partial I_3}{\partial \mathbf{C}} = I_3\mathbf{C}^{-1}. \tag{1.187}$$

Solution: A direct way in which to establish (1.187) is by working with components and we shall take that approach in Problem 1.8.5. Here we use an alternative (often very convenient) approach.

Consider a one-parameter family of symmetric tensors $\mathbf{C}(t)$ depending smoothly on the parameter t and let $\dot{\mathbf{C}} = \frac{d}{dt}\mathbf{C}$. Since $\operatorname{tr}\mathbf{C} = \mathbf{I}\cdot\mathbf{C}$ we can write $I_1(\mathbf{C}) = \mathbf{I}\cdot\mathbf{C}$. Differentiating both sides of this with respect to t gives

$$\frac{\partial I_1}{\partial \mathbf{C}}\cdot\dot{\mathbf{C}} = \mathbf{I}\cdot\dot{\mathbf{C}} \qquad \Rightarrow \qquad \left(\frac{\partial I_1}{\partial \mathbf{C}} - \mathbf{I}\right)\cdot\dot{\mathbf{C}} = 0.$$

Since this must hold for all $\dot{\mathbf{C}}$ and the term inside the parenthesis is symmetric and does not depend on $\dot{\mathbf{C}}$ it follows[9] that the term in the parenthesis must vanish. Thus

$$\frac{\partial I_1}{\partial \mathbf{C}} = \mathbf{I}. \tag{i}$$

Next consider I_2. Since $\operatorname{tr}\mathbf{C}^2 = \mathbf{C}\cdot\mathbf{C}$ we can write I_2 as

$$I_2(\mathbf{C}) = \frac{1}{2}\big(I_1^2(\mathbf{C}) - \mathbf{C}\cdot\mathbf{C}\big).$$

Differentiating both sides with respect to t gives

$$\frac{\partial I_2}{\partial \mathbf{C}}\cdot\dot{\mathbf{C}} = \frac{1}{2}\left(2I_1\frac{\partial I_1}{\partial \mathbf{C}}\cdot\dot{\mathbf{C}} - \dot{\mathbf{C}}\cdot\mathbf{C} - \mathbf{C}\cdot\dot{\mathbf{C}}\right) = \frac{1}{2}\left(2I_1\frac{\partial I_1}{\partial \mathbf{C}}\cdot\dot{\mathbf{C}} - 2\mathbf{C}\cdot\dot{\mathbf{C}}\right) \stackrel{(i)}{=} (I_1\,\mathbf{I} - \mathbf{C})\cdot\dot{\mathbf{C}}$$

which leads to

$$\left(\frac{\partial I_2}{\partial \mathbf{C}} - (I_1\,\mathbf{I} - \mathbf{C})\right)\cdot\dot{\mathbf{C}} = 0 \qquad \Rightarrow \qquad \frac{\partial I_2}{\partial \mathbf{C}} = I_1\,\mathbf{I} - \mathbf{C}.$$

Finally consider I_3. Differentiating both sides of $I_3 = \det\mathbf{C}$ with respect to t gives

$$\frac{\partial I_3}{\partial \mathbf{C}}\cdot\dot{\mathbf{C}} = \frac{d}{dt}\det\mathbf{C} \stackrel{(1.206)}{=} \det\mathbf{C}\;\mathbf{C}^{-T}\cdot\dot{\mathbf{C}} = I_3\;\mathbf{C}^{-1}\cdot\dot{\mathbf{C}} \qquad \Rightarrow \qquad \frac{\partial I_3}{\partial \mathbf{C}} = I_3\;\mathbf{C}^{-1},$$

where we have used the fact that $\mathbf{C}^{-1}$ is a symmetric tensor.

Problem 1.8.5. (Functions of a tensor.) Given a function $\overline{W}(\mathbf{C})$ defined for all symmetric tensors $\mathbf{C}$, define the function $W(\mathbf{F})$ for all tensors $\mathbf{F} \in \mathsf{Lin}$ by

$$W(\mathbf{F}) = \overline{W}(\mathbf{C}), \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{i}$$

Show that

$$\frac{\partial W}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial \overline{W}}{\partial \mathbf{C}}. \tag{ii}$$

[9] This claim needs careful attention, e.g. see Section 3.4 of Gurtin et al. [4]

Solution: In this problem we will work with components. You are encouraged to work this problem using the approach taken in Problem 1.8.4.

Differentiating (i) and using the chain rule,

$$\frac{\partial W}{\partial F_{ij}} = \frac{\partial \overline{W}}{\partial C_{pq}} \frac{\partial C_{pq}}{\partial F_{ij}}. \qquad (iii)$$

However in components, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ reads $C_{pq} = F_{kp}F_{kq}$ and so

$$\frac{\partial C_{pq}}{\partial F_{ij}} = \frac{\partial}{\partial F_{ij}}(F_{kp}F_{kq}) = \frac{\partial F_{kp}}{\partial F_{ij}}F_{kq} + \frac{\partial F_{kq}}{\partial F_{ij}}F_{kp} = \delta_{ki}\delta_{pj}F_{kq} + \delta_{ki}\delta_{qj}F_{kp} = \delta_{pj}F_{iq} + \delta_{qj}F_{ip}. \qquad (iv)$$

Substituting (iv) into (iii) and using the substitution rule gives

$$\frac{\partial W}{\partial F_{ij}} = \frac{\partial \overline{W}}{\partial C_{pq}} \frac{\partial C_{pq}}{\partial F_{ij}} = \frac{\partial \overline{W}}{\partial C_{pq}}\Big[\delta_{pj}F_{iq} + \delta_{qj}F_{ip}\Big] = \frac{\partial \overline{W}}{\partial C_{jq}}F_{iq} + \frac{\partial \overline{W}}{\partial C_{pj}}F_{ip} = 2F_{iq}\frac{\partial \overline{W}}{\partial C_{qj}}.$$

and so

$$\frac{\partial W}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial \overline{W}}{\partial \mathbf{C}}.$$

Problem 1.8.6. (Function of a symmetric tensor.) Let $f(\mathbf{C}) = \mathbf{C}^2\mathbf{m}\cdot\mathbf{m}$ be a scalar-valued function defined for all *symmetric* tensors $\mathbf{C}$, $\mathbf{m}$ being a fixed vector. Calculate $\partial f/\partial \mathbf{C}$.

Solution: We will start by showing why one must proceed with care in order to ensure that $\partial f/\partial \mathbf{C}$ is a symmetric tensor.

On writing $f(\mathbf{C})$ in terms of the components C_{ij} and m_i in some fixed cartesian basis:

$$f(\mathbf{C}) = \mathbf{C}^2\mathbf{m}\cdot\mathbf{m} = \mathbf{Cm}\cdot\mathbf{C}^T\mathbf{m} = \mathbf{Cm}\cdot\mathbf{Cm} = (\mathbf{Cm})_i(\mathbf{Cm})_i = C_{ij}m_jC_{ik}m_k = C_{ij}C_{ik}m_jm_k. \qquad (i)$$

Therefore

$$\begin{aligned}\frac{\partial f}{\partial C_{pq}} &= \frac{\partial}{\partial C_{pq}}(C_{ij}C_{ik})m_jm_k = \delta_{ip}\delta_{jq}C_{ik}m_jm_k + C_{ij}\delta_{ip}\delta_{kq}m_jm_k = \\ &= C_{pk}m_qm_k + C_{pj}m_jm_q = 2C_{pj}m_qm_j = 2(\mathbf{Cm})_pm_q = 2(\mathbf{Cm}\otimes\mathbf{m})_{pq}\end{aligned}$$

and so we might consider writing

$$\frac{\partial f}{\partial \mathbf{C}} = 2\mathbf{Cm}\otimes\mathbf{m}. \qquad (ii)$$

This however would be incorrect since the right-hand side of (ii) is not a symmetric tensor.

To ensure that $\partial f/\partial \mathbf{C}$ is a symmetric tensor we use (1.185) to define $\partial f/\partial \mathbf{C}$ as the symmetric tensor with components

$$\left(\frac{\partial f}{\partial \mathbf{C}}\right)_{pq} = \frac{1}{2}\left(\frac{\partial f}{\partial C_{pq}} + \frac{\partial f}{\partial C_{qp}}\right).$$

Using this on the expression (i) gives

$$\begin{aligned}\frac{\partial f}{\partial C_{pq}} &= \frac{1}{2}\left(\frac{\partial}{\partial C_{pq}}(C_{ij}C_{ik})m_jm_k + \frac{\partial}{\partial C_{qp}}(C_{ij}C_{ik})m_jm_k\right) = \frac{1}{2}(2C_{pj}m_qm_j + 2C_{qj}m_pm_j) = \\ &= C_{pj}m_qm_j + C_{qj}m_pm_j = (\mathbf{Cm})_pm_q + m_p(\mathbf{Cm})_q\end{aligned}$$

which yields

$$\frac{\partial f}{\partial \mathbf{C}} = (\mathbf{Cm}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{Cm})$$

which is symmetric.

1.8.6 Calculus in orthogonal curvilinear coordinates. An example.

In this section we illustrate working in other orthogonal curvilinear coordinate systems through some examples using cylindrical polar coordinates. A general treatment of orthogonal curvilinear coordinates can be found in Chapter 6 of Volume I.

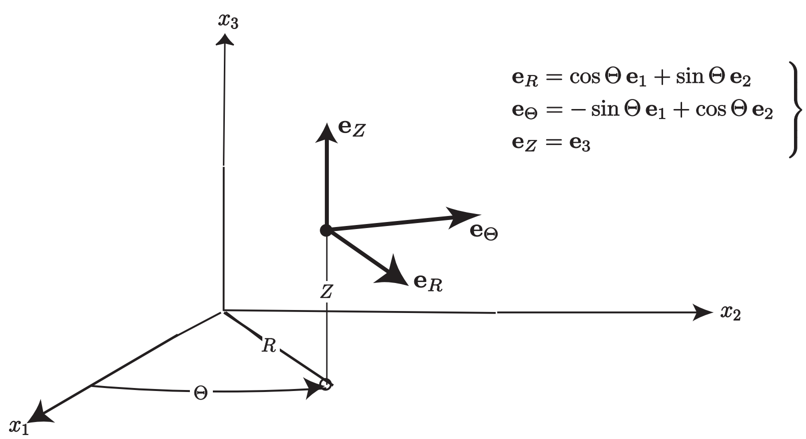

Figure 1.8: Cylindrical polar coordinates (R, Θ, Z) and the associated basis vectors $\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z$.

The rectangular cartesian coordinates (x_1, x_2, x_3) of a point are related to its cylindrical polar coordinates (R, Θ, Z) by

$$x_1 = R\cos\Theta, \quad x_2 = R\sin\Theta, \quad x_3 = Z. \tag{i}$$

As can be seen from Figure 1.8, the basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ associated with the cylindrical polar coordinates is related to the fixed rectangular cartesian basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by

$$\mathbf{e}_R(\Theta) = \cos\Theta\ \mathbf{e}_1 + \sin\Theta\ \mathbf{e}_2, \qquad \mathbf{e}_\Theta(\Theta) = -\sin\Theta\ \mathbf{e}_1 + \cos\Theta\ \mathbf{e}_2, \qquad \mathbf{e}_Z = \mathbf{e}_3. \tag{ii}$$

On differentiating the basis vectors $\mathbf{e}_R, \mathbf{e}_\Theta$ and $\mathbf{e}_Z$ with respect to R, Θ and Z we get

$$\begin{gathered}\frac{\partial \mathbf{e}_R}{\partial \Theta} = \mathbf{e}_\Theta, \qquad \frac{\partial \mathbf{e}_\Theta}{\partial \Theta} = -\mathbf{e}_R, \qquad \frac{\partial \mathbf{e}_Z}{\partial \Theta} = 0, \\ \frac{\partial \mathbf{e}_R}{\partial R} = \frac{\partial \mathbf{e}_\Theta}{\partial R} = \frac{\partial \mathbf{e}_Z}{\partial R} = 0, \qquad \frac{\partial \mathbf{e}_R}{\partial Z} = \frac{\partial \mathbf{e}_\Theta}{\partial Z} = \frac{\partial \mathbf{e}_Z}{\partial Z} = 0.\end{gathered} \tag{iii}$$

From Figure 1.8 we see that the position vector $\mathbf{x}$ of a point can be written as

$$\mathbf{x} = \mathbf{x}(R, \Theta, Z) = R\,\mathbf{e}_R + Z\,\mathbf{e}_Z, \tag{iv}$$

and therefore by the chain rule

$$\begin{aligned} d\mathbf{x} \quad &= \quad \frac{\partial \mathbf{x}}{\partial R} dR + \frac{\partial \mathbf{x}}{\partial \Theta} d\Theta + \frac{\partial \mathbf{x}}{\partial Z} dZ = \\ &\overset{(iv)}{=} \quad \frac{\partial}{\partial R}(R\,\mathbf{e}_R + Z\,\mathbf{e}_Z)dR + \frac{\partial}{\partial \Theta}(R\,\mathbf{e}_R + Z\,\mathbf{e}_Z)d\Theta + \frac{\partial}{\partial Z}(R\,\mathbf{e}_R + Z\,\mathbf{e}_Z)dZ = \\ &\overset{(iii)}{=} \quad \mathbf{e}_R\,dR + R\mathbf{e}_\Theta\,d\Theta + \mathbf{e}_Z\,dZ. \end{aligned}$$

It follows by taking the scalar product of this equation with each unit vector $\mathbf{e}_R$, $\mathbf{e}_\Theta$ and $\mathbf{e}_Z$ that

$$dR = \mathbf{e}_R \cdot d\mathbf{x}, \qquad d\Theta = \frac{1}{R}\mathbf{e}_\Theta \cdot d\mathbf{x}, \qquad dZ = \mathbf{e}_Z \cdot d\mathbf{x}. \tag{v}$$

Example: Gradient of a scalar field. Let $\psi(\mathbf{x}) = \psi(R, \Theta, Z)$ be a scalar-valued field. We wish to calculate its gradient, $\nabla\psi$, which we do by relying on the relation $d\psi = \nabla\psi \cdot d\mathbf{x}$ (see (1.161)). Using the chain-rule on $\psi = \psi(R, \Theta, Z)$ gives

$$\begin{aligned} d\psi \quad &= \frac{\partial \psi}{\partial R} dR + \frac{\partial \psi}{\partial \Theta} d\Theta + \frac{\partial \psi}{\partial Z} dZ = \\ &\overset{(v)}{=} \frac{\partial \psi}{\partial R}(\mathbf{e}_R \cdot d\mathbf{x}) + \frac{1}{R}\frac{\partial \psi}{\partial \Theta}(\mathbf{e}_\Theta \cdot d\mathbf{x}) + \frac{\partial \psi}{\partial Z}(\mathbf{e}_Z \cdot d\mathbf{x}) = \\ &= \left(\frac{\partial \psi}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial \psi}{\partial \Theta}\mathbf{e}_\Theta + \frac{\partial \psi}{\partial Z}\mathbf{e}_Z \right) \cdot d\mathbf{x}, \end{aligned}$$

and therefore, since $d\psi = \nabla\psi \cdot d\mathbf{x}$, we obtain

$$\nabla\psi = \frac{\partial \psi}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial \psi}{\partial \Theta}\mathbf{e}_\Theta + \frac{\partial \psi}{\partial Z}\mathbf{e}_Z. \tag{1.188}$$

Example: Gradient of a vector field. Let $\mathbf{u}(\mathbf{x})$ be a vector field that can be written in component form as

$$\mathbf{u} = u_R(R, \Theta, Z)\mathbf{e}_R + u_\Theta(R, \Theta, Z)\mathbf{e}_\Theta + u_Z(R, \Theta, Z)\mathbf{e}_Z. \tag{vi}$$

We wish to calculate the gradient of $\mathbf{u}(\mathbf{x})$ in cylindrical polar coordinates which we do by making use of the relation $d\mathbf{u} = \nabla\mathbf{u}\,d\mathbf{x}$; see (1.163). First, from (vi) and the chain rule

$$d\mathbf{u} = \frac{\partial \mathbf{u}}{\partial R} dR + \frac{\partial \mathbf{u}}{\partial \Theta} d\Theta + \frac{\partial \mathbf{u}}{\partial Z} dZ. \tag{vii}$$

Next, we calculate each term on the right-hand side of (vii). For example,

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial \Theta}\, d\Theta &\overset{(vi)}{=} \frac{\partial}{\partial \Theta}(u_R \mathbf{e}_R + u_\Theta \mathbf{e}_\Theta + u_Z \mathbf{e}_Z)\, d\Theta = \\
&= \left(\frac{\partial u_R}{\partial \Theta}\,\mathbf{e}_R + u_R\,\frac{\partial \mathbf{e}_R}{\partial \Theta} + \frac{\partial u_\Theta}{\partial \Theta}\,\mathbf{e}_\Theta + u_\Theta\,\frac{\partial \mathbf{e}_\Theta}{\partial \Theta} + \frac{\partial u_Z}{\partial \Theta}\,\mathbf{e}_Z\right) d\Theta = \\
&\overset{(iii)}{=} \left(\frac{\partial u_R}{\partial \Theta}\,\mathbf{e}_R + u_R\,\mathbf{e}_\Theta + \frac{\partial u_\Theta}{\partial \Theta}\,\mathbf{e}_\Theta - u_\Theta\,\mathbf{e}_R + \frac{\partial u_Z}{\partial \Theta}\,\mathbf{e}_Z\right) d\Theta = \\
&= \left(\frac{\partial u_R}{\partial \Theta} - u_\Theta\right) d\Theta\,\mathbf{e}_R + \left(\frac{\partial u_\Theta}{\partial \Theta} + u_R\right) d\Theta\,\mathbf{e}_\Theta + \frac{\partial u_Z}{\partial \Theta}\, d\Theta\,\mathbf{e}_Z = \\
&\overset{(v)}{=} \left(\frac{\partial u_R}{\partial \Theta} - u_\Theta\right)\left(\frac{1}{R}\,\mathbf{e}_\Theta \cdot d\mathbf{x}\right)\mathbf{e}_R + \left(\frac{\partial u_\Theta}{\partial \Theta} + u_R\right)\left(\frac{1}{R}\,\mathbf{e}_\Theta \cdot d\mathbf{x}\right)\mathbf{e}_\Theta + \\
&\qquad + \frac{\partial u_Z}{\partial \Theta}\left(\frac{1}{R}\,\mathbf{e}_\Theta \cdot d\mathbf{x}\right)\mathbf{e}_Z = \\
&= \frac{1}{R}\left(\frac{\partial u_R}{\partial \Theta} - u_\Theta\right)(\mathbf{e}_R \otimes \mathbf{e}_\Theta)d\mathbf{x} + \frac{1}{R}\left(\frac{\partial u_\Theta}{\partial \Theta} + u_R\right)(\mathbf{e}_\Theta \otimes \mathbf{e}_\Theta)d\mathbf{x} + \\
&\qquad + \frac{1}{R}\frac{\partial u_Z}{\partial \Theta}\,(\mathbf{e}_Z \otimes \mathbf{e}_\Theta)d\mathbf{x}.
\end{aligned}
\qquad (viii)
$$

Similarly one finds

$$
\frac{\partial \mathbf{u}}{\partial R}\, dR = \frac{\partial u_R}{\partial R}\,(\mathbf{e}_R \otimes \mathbf{e}_R)d\mathbf{x} + \frac{\partial u_\Theta}{\partial R}\,(\mathbf{e}_\Theta \otimes \mathbf{e}_R)d\mathbf{x} + \frac{\partial u_Z}{\partial R}\,(\mathbf{e}_Z \otimes \mathbf{e}_R)d\mathbf{x}. \qquad (ix)
$$

$$
\frac{\partial \mathbf{u}}{\partial Z}\, dZ = \frac{\partial u_R}{\partial Z}\,(\mathbf{e}_R \otimes \mathbf{e}_Z)d\mathbf{x} + \frac{\partial u_\Theta}{\partial Z}\,(\mathbf{e}_\Theta \otimes \mathbf{e}_Z)d\mathbf{x} + \frac{\partial u_Z}{\partial Z}\,(\mathbf{e}_Z \otimes \mathbf{e}_Z)d\mathbf{x}. \qquad (x)
$$

Therefore combining $(vii), (viii), (ix)$ and (x) yields

$$
\begin{aligned}
d\mathbf{u} &= \frac{\partial u_R}{\partial R}\,(\mathbf{e}_R \otimes \mathbf{e}_R)d\mathbf{x} + \frac{\partial u_\Theta}{\partial R}\,(\mathbf{e}_\Theta \otimes \mathbf{e}_R)d\mathbf{x} + \frac{\partial u_Z}{\partial R}\,(\mathbf{e}_Z \otimes \mathbf{e}_R)d\mathbf{x} + \\
&+ \frac{1}{R}\left(\frac{\partial u_R}{\partial \Theta} - u_\Theta\right)(\mathbf{e}_R \otimes \mathbf{e}_\Theta)d\mathbf{x} + \frac{1}{R}\left(\frac{\partial u_\Theta}{\partial \Theta} + u_R\right)(\mathbf{e}_\Theta \otimes \mathbf{e}_\Theta)d\mathbf{x} + \\
&\qquad + \frac{1}{R}\frac{\partial u_Z}{\partial \Theta}\,(\mathbf{e}_Z \otimes \mathbf{e}_\Theta)d\mathbf{x} + \\
&+ \frac{\partial u_R}{\partial Z}\,(\mathbf{e}_R \otimes \mathbf{e}_Z)d\mathbf{x} + \frac{\partial u_\Theta}{\partial Z}\,(\mathbf{e}_\Theta \otimes \mathbf{e}_Z)d\mathbf{x} + \frac{\partial u_Z}{\partial Z}\,(\mathbf{e}_Z \otimes \mathbf{e}_Z)d\mathbf{x}.
\end{aligned}
\qquad (xi)
$$

Since $d\mathbf{u} = \nabla\mathbf{u}\, d\mathbf{x}$ we can now read off the gradient tensor $\nabla\mathbf{u}$ from (xi) to be

$$\begin{aligned}
\nabla\mathbf{u} \;=\;& \frac{\partial u_R}{\partial R}\,(\mathbf{e}_R\otimes\mathbf{e}_R) + \frac{\partial u_\Theta}{\partial R}\,(\mathbf{e}_\Theta\otimes\mathbf{e}_R) + \frac{\partial u_Z}{\partial R}\,(\mathbf{e}_Z\otimes\mathbf{e}_R) + \\
+\;& \frac{1}{R}\left(\frac{\partial u_R}{\partial \Theta} - u_\Theta\right)(\mathbf{e}_R\otimes\mathbf{e}_\Theta) + \frac{1}{R}\left(\frac{\partial u_\Theta}{\partial \Theta} + u_R\right)(\mathbf{e}_\Theta\otimes\mathbf{e}_\Theta) + \frac{1}{R}\frac{\partial u_Z}{\partial \Theta}\,(\mathbf{e}_Z\otimes\mathbf{e}_\Theta) + \\
+\;& \frac{\partial u_R}{\partial Z}\,(\mathbf{e}_R\otimes\mathbf{e}_Z) + \frac{\partial u_\Theta}{\partial Z}\,(\mathbf{e}_\Theta\otimes\mathbf{e}_Z) + \frac{\partial u_Z}{\partial Z}\,(\mathbf{e}_Z\otimes\mathbf{e}_Z).
\end{aligned} \tag{1.189}$$

Example: Divergence of a vector field. The divergence of the vector field $\mathbf{u}(\mathbf{x})$ can be readily found from (1.189) to be

$$\operatorname{div}\mathbf{u} = \operatorname{tr}(\nabla\mathbf{u}) = \frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\Theta}{\partial \Theta} + \frac{\partial u_Z}{\partial Z}. \tag{1.190}$$

Example: Divergence of a tensor field. The divergence of a tensor field, $\operatorname{div}\mathbf{A}(\mathbf{x})$, is a vector field whose three components can be calculated using the identity (1.170) and taking $\mathbf{v} = \mathbf{e}_R, \mathbf{v} = \mathbf{e}_\Theta$ and $\mathbf{v} = \mathbf{e}_Z$ in turn. This calculation is carried out in Section 3.10.1.

1.9 Exercises

1. Matrices and Indicial Notation

Problem 1.1. (Indicial notation.) Evaluate the following expressions:

$$(a)\ e_{ijk}e_{kji}, \qquad (b)\ \delta_{ij}e_{ijk}, \qquad (c)\delta_{ij}\delta_{ik}\delta_{jk}, \qquad (d)\ e_{ik\ell}e_{jk\ell}.$$

Solution: (a) The calculation proceeds as follows:

$$e_{ijk}e_{kji} \overset{(1.42)}{=} -e_{ijk}e_{kij} \overset{(1.42)}{=} e_{ijk}e_{ikj} \overset{(1.43)}{=} \delta_{jk}\delta_{kj} - \delta_{jj}\delta_{kk} \overset{(1.37)}{=} \delta_{jj} - \delta_{jj}\delta_{kk} \overset{(1.34)}{=} 3 - 9 = -6.$$

In the first two steps we have used the fact that e_{ijk} changes sign when any two adjacent subscripts are switched. Subsequently we have used the substitution rule and $\delta_{kk} = 3$.

(b) By the substitution rule

$$\delta_{ij}e_{ijk} \overset{(1.37)}{=} e_{jjk} \overset{(1.38)}{=} 0$$

and where in the last step we have used the fact that $e_{ijk} = 0$ if two subscripts are equal. Since k is a free index in $\delta_{ij}e_{ijk}$, this result says that $\delta_{ij}e_{ijk} = 0$ for each value of $k = 1, 2, 3$.

(c) By using the substitution rule, first on the repeated index i and then on the repeated index j, we have

$$\delta_{ij}\ \delta_{ik}\ \delta_{jk} \overset{(1.37)}{=} \delta_{jk}\ \delta_{jk} \overset{(1.37)}{=} \delta_{kk} \overset{(1.34)}{=} \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

(d)

$$e_{ik\ell}e_{jk\ell} \overset{(1.42)}{=} e_{\ell ik}e_{\ell jk} \overset{(1.43)}{=} \delta_{ij}\delta_{kk} - \delta_{ik}\delta_{kj} \overset{(1.37),(1.34)}{=} 3\delta_{ij} - \delta_{ij} = 2\delta_{ij}$$

Observe that the final result has two free indices i, j as does the starting expression $e_{ik\ell}e_{jk\ell}$.

Problem 1.2. (Matrices. Indicial notation.) Derive equations (1.40) and (1.41) involving the determinant of a matrix.

Problem 1.3. (Indicial notation.) From the definitions of δ_{ij} and e_{ijk} show that

$$e_{ijk}e_{pqr} = \det\begin{pmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{pmatrix}, \qquad (i)$$

and thus show that

$$e_{ijk}e_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}. \qquad (ii)$$

Solution: First, consider the case where any two of the indices i, j, k or p, q, r are the same. Then, on the right-hand side of (i), either two rows or two columns of the matrix coincide and so the determinant vanishes. On the left-hand side of (i), when two of the indices i, j, k or p, q, r are the same, it also vanishes in view of (1.38).

Second, consider the case $(i, j, k) = (p, q, r) = (1, 2, 3)$. Here the right-hand side of (i) equals unity since the matrix is now the identity matrix. The left-hand side also equals unity by (1.38).

Finally, if any two adjacent indices i, j, k or p, q, r are interchanged, the corresponding Levi-Civita symbol on the left-hand side changes sign by (1.42). On the right-hand side, an interchange of two indices results in an interchange of two rows or two columns in the matrix, thus reversing the sign of the determinant.

This establishes (i).

Now set $r = i$ in (i). Then

$$\begin{aligned}
e_{ijk}e_{pqi} &= \det\begin{pmatrix} \delta_{ip} & \delta_{iq} & \delta_{ii} \\ \delta_{jp} & \delta_{jq} & \delta_{ji} \\ \delta_{kp} & \delta_{kq} & \delta_{ki} \end{pmatrix} \overset{(*)}{=} \det\begin{pmatrix} \delta_{ip} & \delta_{iq} & 3 \\ \delta_{jp} & \delta_{jq} & \delta_{ji} \\ \delta_{kp} & \delta_{kq} & \delta_{ki} \end{pmatrix} = \\
&= \delta_{ip}(\delta_{jq}\delta_{ki} - \delta_{ji}\delta_{kq}) - \delta_{iq}(\delta_{jp}\delta_{ki} - \delta_{ji}\delta_{kp}) + 3(\delta_{jp}\delta_{kq} - \delta_{kp}\delta_{jq}) = \\
&= \delta_{ip}\delta_{jq}\delta_{ki} - \delta_{ip}\delta_{ji}\delta_{kq} - \delta_{iq}\delta_{jp}\delta_{ki} + \delta_{iq}\delta_{ji}\delta_{kp} + 3\delta_{jp}\delta_{kq} - 3\delta_{kp}\delta_{jq} = \\
&\overset{(**)}{=} \delta_{jq}\delta_{kp} - \delta_{jp}\delta_{kq} - \delta_{jp}\delta_{kq} + \delta_{jq}\delta_{kp} + 3\delta_{jp}\delta_{kq} - 3\delta_{kp}\delta_{jq} = \\
&= \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}
\end{aligned}$$

where step $(*)$ we used $\delta_{ii} = 3$ and in step $(**)$ the substitution rule. (By (1.42) we have $e_{pqi} = -e_{piq} = e_{ipq}$ and so $e_{ijk}e_{pqi} = e_{ijk}e_{ipq}$ and so (ii) is established.)

Problem 1.4. (Matrices.) If α_1, α_2 and α_3 are the eigenvalues of a symmetric matrix $[A]$ show that

$$\text{tr}[A] = \alpha_1 + \alpha_2 + \alpha_3, \qquad \det[A] = \alpha_1\alpha_2\alpha_3.$$

Problem 1.5. (Matrices.) Let $[F]$ be a nonsingular matrix, $[R]$ an orthogonal matrix and $[U]$ is symmetric matrix such that $[F] = [R][U]$. Show that

(a) $[U]^2 = [F]^T[F]$, and

(b) if $[F]$ is nonsingular, then $[U]^2$ is positive definite.

Problem 1.6. (Matrices.) Show that

$$\det\Big([Q]^T[A][Q]\Big) = \det[A], \qquad \mathrm{tr}\Big([Q]^T[A][Q]\Big) = \mathrm{tr}[A],$$

for any orthogonal matrix $[Q]$ and arbitrary matrix $[A]$.

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2. Vector and tensor algebra.

Problem 1.7. (Vector algebra.) Show that

(a) $\mathbf{u}\times\mathbf{v} = \mathbf{o}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

(b) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 0$ if and only if $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

Problem 1.8. (Vector algebra.) Show that

$$\mathbf{x}\times(\mathbf{y}\times\mathbf{z}) = (\mathbf{y}\otimes\mathbf{z} - \mathbf{z}\otimes\mathbf{y})\mathbf{x}, \tag{1.191}$$

for all vectors $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$. This says that $\mathbf{y}\times\mathbf{z}$ is the axial vector associated with the skew-symmetric tensor $\mathbf{z}\otimes\mathbf{y} - \mathbf{y}\otimes\mathbf{z}$; see (1.83).

Problem 1.9. (Vector algebra.) Show that

$$(\mathbf{z}\times\mathbf{x})\times(\mathbf{z}\times\mathbf{y}) = (\mathbf{z}\otimes\mathbf{z})\ (\mathbf{x}\times\mathbf{y}), \tag{i}$$

$$\mathbf{n}\cdot\big[(\mathbf{n}\times\mathbf{x})\times(\mathbf{n}\times\mathbf{y})\big] = \mathbf{n}\cdot(\mathbf{x}\times\mathbf{y}), \tag{ii}$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are arbitrary vectors and $\mathbf{n}$ is an arbitrary unit vector.

Solution:

(a) The terms $\mathbf{z}\times\mathbf{x}$ and $\mathbf{z}\times\mathbf{y}$ on the left-hand side of (i) each represents a vector and therefore, so does their cross-product $(\mathbf{z}\times\mathbf{x})\times(\mathbf{z}\times\mathbf{y})$. Its ith component is

$$\begin{aligned}
\Big[(\mathbf{z}\times\mathbf{x})\times(\mathbf{z}\times\mathbf{y})\Big]_i \ &\overset{(1.61)}{=} e_{ijk}(\mathbf{z}\times\mathbf{x})_j(\mathbf{z}\times\mathbf{y})_k = \\
&\overset{(1.61)}{=} e_{ijk}\,\big(e_{jpq}z_px_q\big)\,\big(e_{krs}z_ry_s\big) \overset{(1.42)}{=} e_{kij}e_{krs}\,e_{jpq}z_pz_rx_qy_s = \\
&\overset{(1.43)}{=} (\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})\,e_{jpq}z_pz_rx_qy_s = \\
&\overset{(*)}{=} e_{spq}z_pz_ix_qy_s - e_{rpq}z_pz_rx_qy_i \overset{(**)}{=} e_{spq}z_pz_ix_qy_s = \\
&= z_iz_pe_{spq}x_qy_s \overset{(1.42)}{=} z_iz_pe_{pqs}x_qy_s \overset{(1.61)}{=} z_iz_p(\mathbf{x}\times\mathbf{y})_p = \\
&\overset{(1.130)}{=} (\mathbf{z}\otimes\mathbf{z})_{ip}(\mathbf{x}\times\mathbf{y})_p = \Big[(\mathbf{z}\otimes\mathbf{z})(\mathbf{x}\times\mathbf{y})\Big]_i\ .
\end{aligned}$$

In step $(*)$ we used the substitution rule and in step $(**)$ we used (1.45) keeping in mind that $z_p z_r$ is symmetric in p, r and e_{rpq} is skew-symmetric in p, r.

(b) On taking $\mathbf{z} = \mathbf{n}$ in (i) we get

$$(\mathbf{n} \times \mathbf{x}) \times (\mathbf{n} \times \mathbf{y}) = (\mathbf{n} \otimes \mathbf{n})(\mathbf{x} \times \mathbf{y}). \tag{iii}$$

Each side of this equation is a vector and so we may take its scalar product with $\mathbf{n}$:

$$\mathbf{n} \cdot \Big[(\mathbf{n} \times \mathbf{x}) \times (\mathbf{n} \times \mathbf{y})\Big] = \mathbf{n} \cdot \Big[(\mathbf{n} \otimes \mathbf{n})(\mathbf{x} \times \mathbf{y})\Big]. \tag{iv}$$

Since the left-hand sides of (iv) and (ii) are identical, it remains to show that the right-hand side of (iv) equals the right-hand side of (ii).

Solution 1: using components in a basis. Simplifying the right-hand side of (iv):

$$\mathbf{n} \cdot \Big[(\mathbf{n} \otimes \mathbf{n})(\mathbf{x} \times \mathbf{y})\Big] \stackrel{(1.58)}{=} n_i \Big[(\mathbf{n} \otimes \mathbf{n})(\mathbf{x} \times \mathbf{y})\Big]_i \stackrel{(1.131)}{=} n_i (\mathbf{n} \otimes \mathbf{n})_{ij} (\mathbf{x} \times \mathbf{y})_j \stackrel{(1.130)}{=} n_i n_i n_j (\mathbf{x} \times \mathbf{y})_j =$$
$$= n_j (\mathbf{x} \times \mathbf{y})_j \stackrel{(1.58)}{=} \mathbf{n} \cdot (\mathbf{x} \times \mathbf{y}),$$

which is the right-hand side of (ii). In getting to the second line we used $\mathbf{n} \cdot \mathbf{n} = n_i n_i = 1$.

Solution 2: without using components. Using $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{c} \cdot \mathbf{b})\,\mathbf{c}$ with $\mathbf{a} = \mathbf{b} = \mathbf{n}$ and $\mathbf{c} = \mathbf{x} \times \mathbf{y}$,

$$(\mathbf{n} \otimes \mathbf{n})(\mathbf{x} \times \mathbf{y}) = \big[(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{n}\big]\,\mathbf{n}, \tag{v}$$

and therefore by taking the scalar product of this equation with $\mathbf{n}$,

$$\mathbf{n} \cdot \Big((\mathbf{n} \otimes \mathbf{n})(\mathbf{x} \times \mathbf{y})\Big) = \big[(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{n}\big]\,(\mathbf{n} \cdot \mathbf{n}) = (\mathbf{x} \times \mathbf{y}) \cdot \mathbf{n} \tag{vi}$$

since $\mathbf{n}$ is a unit vector. Thus combining (iv) with (vi) establishes (ii).

Problem 1.10. (Reflection in a plane.) Consider a plane $\mathcal{P}$ and let $\mathbf{n}$ be a unit vector normal to it. The operation of reflection in this plane takes a vector $\mathbf{x}$ into the vector $\mathbf{R}\mathbf{x}$. This is illustrated geometrically in Figure 1.9 where $\mathbf{x} = \overrightarrow{OA}$ and $\mathbf{R}\mathbf{x} = \overrightarrow{OC}$.

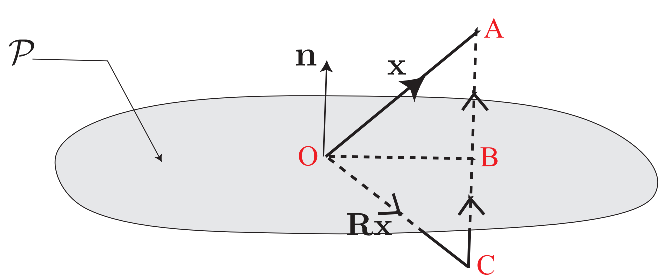

Figure 1.9: The operator $\mathbf{R}$ reflects a vector $\mathbf{x}$ in the plane $\mathcal{P}$.

Determine $\mathbf{R}$ and show that it is precisely the tensor encountered previously in (1.102) of Problem 1.4.7.

Problem 1.11. (Rotation about an axis.) (See also Problem 1.4.14.)

(a) The operation of rotation through an angle θ about a unit vector $\mathbf{n}$ takes a vector $\mathbf{x}$ into the vector $\mathbf{Qx}$ as illustrated geometrically in Figure 1.10. Show that

$$\mathbf{Qx} = \cos\theta\, \mathbf{x} + (1-\cos\theta)(\mathbf{n}\cdot\mathbf{x})\mathbf{n} + \sin\theta\,(\mathbf{n}\times\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathsf{V}. \tag{1.192}$$

Observe that $\mathbf{Q}$ is a *linear* transformation since $\mathbf{Q}(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha\mathbf{Qx} + \beta\mathbf{Qy}$.

(b) Show that $\mathbf{Qn} = +\mathbf{n}$. (What geometric transformation does a tensor $\mathbf{Q}$ with the property $\mathbf{Qn} = -\mathbf{n}$ represent? Hint: Consider Problem 1.10.)

(c) Show from (1.192) that the tensor

$$\mathbf{Q} = -\mathbf{I} + 2\mathbf{n}\otimes\mathbf{n}$$

describes a rotation through an angle π about the axis $\mathbf{n}$.

(d) The components of $\mathbf{Q}$ in a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_3 = \mathbf{n}$ being the axis of rotation were determined in Problem 1.4.14. Hence (or otherwise) show that unity is its only real eigenvalue.

(e) Approximate (1.192) to the case where the angle of rotation is small, $|\theta| \ll 1$.

See also Problem 1.56.

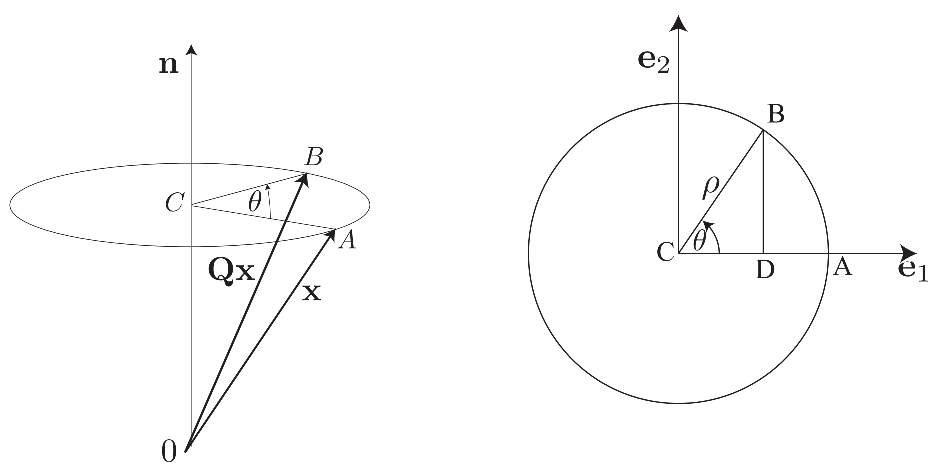

Figure 1.10: Left: The transformation $\mathbf{Q}$ rotates the vector $\mathbf{x} = \overrightarrow{OA}$ through an angle θ about the unit vector $\mathbf{n}$ and takes it to $\mathbf{Qx} = \overrightarrow{OB}$. Right: The plane containing A, B and C looking down the $\mathbf{n}$-axis.(Figure for Problem 1.11.)

Solution:

(a) From the figure we see that

$$\mathbf{x} = \overrightarrow{OA}, \qquad \mathbf{Qx} = \overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CD} + \overrightarrow{DB}, \tag{i}$$

where CD is the projection of CB onto CA as shown in the right-hand figure. First observe that the direction of vector $\overrightarrow{OC}$ is $\mathbf{n}$. Its magnitude $|OC|$ is the projection of $\mathbf{x}$ onto $\mathbf{n}$, i.e. $\mathbf{x}\cdot\mathbf{n}$. Thus

$$\overrightarrow{OC} = (\mathbf{x}\cdot\mathbf{n})\mathbf{n}. \tag{ii}$$

We now have to calculate expressions for $\overrightarrow{CD}$ and $\overrightarrow{DB}$. To this end pick unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ as follows: let $\mathbf{e}_1$ be a unit vector in the direction of $\overrightarrow{CA}$ and let $\mathbf{e}_2$ be a unit vector perpendicular to both $\mathbf{e}_1$ and $\mathbf{n}$ such that $\mathbf{n}\times\mathbf{e}_1 = \mathbf{e}_2$. Let $|CA| = |CB| = \rho$. Then

$$\overrightarrow{CD} = \rho\cos\theta\,\mathbf{e}_1, \qquad \overrightarrow{DB} = \rho\sin\theta\,\mathbf{e}_2 = \rho\sin\theta\,(\mathbf{n}\times\mathbf{e}_1). \tag{iii}$$

We now eliminate ρ by substituting $\overrightarrow{CA} = \rho\mathbf{e}_1$ into (iii) to obtain

$$\overrightarrow{CD} = \cos\theta\ \overrightarrow{CA}, \qquad \overrightarrow{DB} = \sin\theta\,(\mathbf{n}\times \overrightarrow{CA}). \tag{iv}$$

Substituting (ii) and (iv) into $(i)_2$ yields

$$\mathbf{Qx} = (\mathbf{x}\cdot\mathbf{n})\mathbf{n} + \cos\theta\ \overrightarrow{CA} + \sin\theta\,(\mathbf{n}\times \overrightarrow{CA}). \tag{v}$$

Finally we can find $\overrightarrow{CA}$ by geometry:

$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = \mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\mathbf{n}. \tag{vi}$$

Substituting (vi) into (v) yields

$$\mathbf{Qx} = (\mathbf{x}\cdot\mathbf{n})\mathbf{n} + \cos\theta\,\big[\mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\mathbf{n}\big] + \sin\theta\,\mathbf{n}\times\big[\mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\mathbf{n}\big]$$

which after simplification (and using $\mathbf{n}\times\mathbf{n} = \mathbf{o}$): gives

$$\mathbf{Qx} = \cos\theta\,\mathbf{x} + (1-\cos\theta)(\mathbf{x}\cdot\mathbf{n})\mathbf{n} + \sin\theta\,\mathbf{n}\times\mathbf{x}. \qquad \square$$

This defines the tensor $\mathbf{Q}$ since it tells us how it operates on any vector $\mathbf{x}$. This tensor is proper orthogonal, see Problem 1.56.

(b) On setting $\mathbf{x} = \mathbf{n}$ in (1.192) and using $\mathbf{n}\times\mathbf{n} = \mathbf{o}$ and $\mathbf{n}\cdot\mathbf{n} = 1$ we get

$$\mathbf{Qn} = \cos\theta\,\mathbf{n} + (1-\cos\theta)\mathbf{n} = \mathbf{n}.$$

Remark: It follows that $\mathbf{n}$ is an eigenvector of $\mathbf{Q}$ with unity being the corresponding eigenvalue.

(c) Setting $\theta = \pi$ in (1.192) leads to

$$\mathbf{Qx} = -\mathbf{x} + 2(\mathbf{n}\cdot\mathbf{x})\mathbf{n} = \big[-\mathbf{I} + 2\mathbf{n}\otimes\mathbf{n}\big]\mathbf{x}$$

from which the result follows.

(d) The eigenvalues of $\mathbf{Q}$ are given by the roots of the cubic equation $\det(\mathbf{Q} - \lambda\mathbf{I}) = 0$. Using the result from Problem 1.4.14,

$$\det\big([Q] - \lambda[I]\big) = \det\begin{pmatrix} \cos\theta - \lambda & \sin\theta & 0 \\ -\sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (\lambda - 1)(\lambda^2 - 2\lambda\cos\theta + 1) = 0$$

The quadratic factor has a negative discriminant (when $\theta \neq 0, \pi$) and so $\lambda = 1$ is the only real eigenvalue.

(e) For small θ we use $\cos\theta = 1 + \mathrm{O}(\theta^2)$ and $\sin\theta = \theta + \mathrm{O}(\theta^3)$ in (1.192) to get

$$\mathbf{Qx} = \mathbf{x} \ + \theta\, \mathbf{n} \times \mathbf{x},$$

to order θ. We can write this as

$$\mathbf{Qx} = \mathbf{x} - \mathbf{Wx}$$

where the tensor $\mathbf{W}$ is defined by

$$\mathbf{Wx} = -\theta\, \mathbf{n} \times \mathbf{x}.$$

One can show that $\mathbf{W}$ is skew-symmetric. (Exercise.)

Remark: Observe that $\mathbf{Q} = \mathbf{I} - \mathbf{W} + \mathrm{O}(\theta^2)$. Therefore while a finite (i.e. arbitrary, not-necessarily infinitesimal) rotation is represented by a proper orthogonal tensor, an infinitesimal rotation is represented by a skew-symmetric tensor.

Problem 1.12. (Transformation of volume.) The non-singular tensor $\mathbf{F}$ maps the three linearly independent vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ into $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\} = \{\mathbf{Fa},\ \mathbf{Fb},\ \mathbf{Fc}\}$. Show that the volume V_* of the tetrahedron formed by $\mathbf{p}, \mathbf{q}, \mathbf{r}$ is

$$V_* = V_0\ \det\mathbf{F}, \tag{1.193}$$

where V_0 is the volume of the tetrahedron formed by $\mathbf{a}, \mathbf{b}, \mathbf{c}$. (Hint: see Problem 1.3.3 and equation (1.90)).

Solution:

First consider the tetrahedron formed by $\mathbf{a},\ \mathbf{b},\ \mathbf{c}$. Its volume is $V_0 = \frac{1}{3}A_0\, h_0$ where A_0 is the area of its base and h_0 is its height. Taking the triangle defined by the vectors $\mathbf{a}$ and $\mathbf{b}$ to be the base, see Figure 1.11, its area is $A_0 = |\mathbf{a} \times \mathbf{b}|/2$; this follows by geometrically interpreting the cross-product. If $\mathbf{n}$ is a unit vector normal to the base, the height of the tetrahedron is $h_0 = \mathbf{c} \cdot \mathbf{n}$; we may take $\mathbf{n} = (\mathbf{a} \times \mathbf{b})/|\mathbf{a} \times \mathbf{b}|$. Therefore

$$V_0 = \frac{1}{3}A_0 h_0 = \frac{1}{3}\left(\frac{|\mathbf{a} \times \mathbf{b}|}{2}\right)\left(\mathbf{c} \cdot \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}\right) = \frac{1}{6}\ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Similarly, the volume of the tetrahedron formed by $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ is

$$V_* = \frac{1}{6}\ (\mathbf{p} \times \mathbf{q}) \cdot \mathbf{r} = \frac{1}{6}\ (\mathbf{Fa} \times \mathbf{Fb}) \cdot \mathbf{Fc}.$$

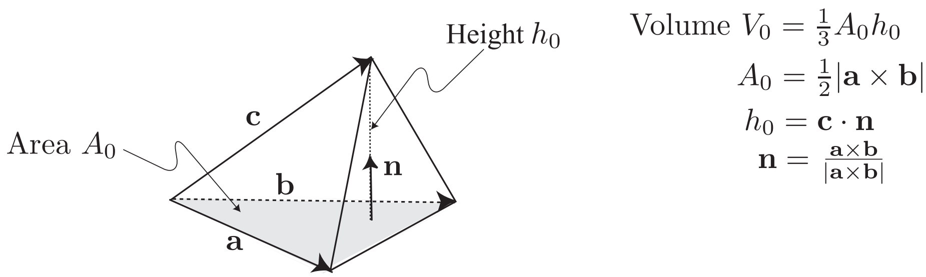

Figure 1.11: Volume of the tetrahedron defined by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

The desired result now follows immediately because of the identity

$$(\mathbf{Fa}\times\mathbf{Fb})\cdot\mathbf{Fc} \ = \ \det\mathbf{F}\ (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

established previously in Problem 1.4.15.

Problem 1.13. (Tensor algebra.) For a nonsingular tensor $\mathbf{F}$ and arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ show that

$$\mathbf{Fa}\times\mathbf{Fb} = \det\mathbf{F}\ \mathbf{F}^{-T}(\mathbf{a}\times\mathbf{b}). \tag{1.194}$$

We know from Problem 1.3.1 that the area of the triangle defined by two linearly independent vectors $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$. The area of its image under the linear transformation $\mathbf{F}$ is therefore $\frac{1}{2}|\mathbf{Fa}\times\mathbf{Fb}|$. Equation (1.194) will be useful when calculating the ratio of these two areas; see Problem 2.47.

Solution:

Without using components in a basis:

$$\det\mathbf{F}\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \overset{(1.90)}{=} (\mathbf{Fa}\times\mathbf{Fb})\cdot\mathbf{Fc} \overset{(1.74)}{=} \mathbf{F}^T(\mathbf{Fa}\times\mathbf{Fb})\cdot\mathbf{c},$$

and therefore

$$\Big(\det\mathbf{F}\,(\mathbf{a}\times\mathbf{b}) - \mathbf{F}^T(\mathbf{Fa}\times\mathbf{Fb})\Big)\cdot\mathbf{c} = 0 \qquad \text{for all vectors } \mathbf{c}.$$

If $\mathbf{x}\cdot\mathbf{c} = 0$ for all vectors $\mathbf{c}$ it follows that $\mathbf{x} = \mathbf{o}$ and thus here,

$$\mathbf{F}^T(\mathbf{Fa}\times\mathbf{Fb}) = \det\mathbf{F}\,(\mathbf{a}\times\mathbf{b}).$$

When $\mathbf{F}$ is nonsingular this implies

$$\mathbf{Fa}\times\mathbf{Fb} = \det\mathbf{F}\ \mathbf{F}^{-T}(\mathbf{a}\times\mathbf{b}). \qquad \square$$

Using components in a basis: To avoid working with the inverse of the tensor $\mathbf{F}^T$, we shall show that $\mathbf{F}^T(\mathbf{Fa}\times\mathbf{Fb}) = \det\mathbf{F}\,(\mathbf{a}\times\mathbf{b})$. The ith component of the left-hand side is:

$$\begin{aligned}\Big[\mathbf{F}^T(\mathbf{Fa}\times\mathbf{Fb})\Big]_i &\overset{(1.139)}{=} F^T_{ij}\big(\mathbf{Fa}\times\mathbf{Fb}\big)_j \overset{(1.61)}{=} F_{ji}\big[e_{jpq}(\mathbf{Fa})_p(\mathbf{Fb})_q\big] \overset{(1.139)}{=} F_{ji}e_{jpq}F_{pm}a_mF_{qn}b_n = \\ &= e_{jpq}F_{ji}F_{pm}F_{qn}a_mb_n \overset{(1.40)}{=} e_{imn}\ \det\mathbf{F}\,a_mb_n = \det\mathbf{F}\ e_{imn}a_mb_n \overset{(1.61)}{=} \det\mathbf{F}\,(\mathbf{a}\times\mathbf{b})_i\end{aligned}$$

Problem 1.14. (Tensor algebra.) Show that

$$\mathbf{AB} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{A}^T\mathbf{C} = \mathbf{A} \cdot \mathbf{CB}^T \tag{1.195}$$

for all tensors $\mathbf{A}, \mathbf{B}$ and $\mathbf{C} \in \mathsf{Lin}$ where the dot between two tensors denotes their scalar product as defined in (1.120).

Solution 1: Without using components:

$$(\mathbf{AB}) \cdot \mathbf{C} \overset{(1.120)}{=} \operatorname{tr} \mathbf{ABC}^T \overset{(1.104)}{=} \operatorname{tr} \mathbf{BC}^T\mathbf{A} \overset{(1.76)}{=} \operatorname{tr} \mathbf{B}(\mathbf{A}^T\mathbf{C})^T \overset{(1.120)}{=} \mathbf{B} \cdot (\mathbf{A}^T\mathbf{C}). \qquad \square$$

Similarly

$$(\mathbf{AB}) \cdot \mathbf{C} \overset{(1.120)}{=} \operatorname{tr} \mathbf{ABC}^T \overset{(1.76)}{=} \operatorname{tr} \mathbf{A}(\mathbf{CB}^T)^T \overset{(1.120)}{=} \mathbf{A} \cdot (\mathbf{CB}^T) \qquad \square$$

Solution 2: Using components. Since $\mathbf{P} \cdot \mathbf{Q} = P_{ij}Q_{ij}$ for any two tensors $\mathbf{P}$ and $\mathbf{Q}$,

$$\mathbf{AB} \cdot \mathbf{C} = (\mathbf{AB})_{ij}C_{ij} = A_{ik}B_{kj}C_{ij} = B_{kj}A_{ik}C_{ij} = B_{kj}A^T_{ki}C_{ij} = B_{kj}(\mathbf{A}^T\mathbf{C})_{kj} = \mathbf{B} \cdot \mathbf{A}^T\mathbf{C}.$$

Likewise

$$\mathbf{AB} \cdot \mathbf{C} = (\mathbf{AB})_{ij}C_{ij} = A_{ik}B_{kj}C_{ij} = A_{ik}C_{ij}B_{kj} = A_{ik}C_{ij}B^T_{jk} = A_{ik}(\mathbf{CB}^T)_{ik} = \mathbf{A} \cdot \mathbf{CB}^T.$$

Problem 1.15. (Tensor algebra.) Consider the scalar-valued function Φ defined for all symmetric tensors $\mathbf{E}$ by

$$\Phi(\mathbf{E}) = \mathbf{CE} \cdot \mathbf{E}; \tag{i}$$

here $\mathbf{C}$ is some constant tensor. Show that "there is no loss of generality in taking $\mathbf{C}$ to be symmetric" (by which we mean that Φ depends only on the symmetric part of $\mathbf{C}$; see (1.82) for what is meant by the symmetric part of a tensor).

Solution: Let $\mathbf{S}$ and $\mathbf{W}$ be the symmetric and skew-symmetric parts of $\mathbf{C}$: $\mathbf{C} = \mathbf{S} + \mathbf{W}$; see (1.81). Then

$$\Phi(\mathbf{E}) = \mathbf{CE} \cdot \mathbf{E} = (\mathbf{S} + \mathbf{W})\mathbf{E} \cdot \mathbf{E} = \mathbf{SE} \cdot \mathbf{E} + \mathbf{WE} \cdot \mathbf{E}. \tag{ii}$$

However

$$\mathbf{WE} \cdot \mathbf{E} \overset{(1.74)}{=} \mathbf{E} \cdot \mathbf{W}^T\mathbf{E} \overset{(1.80)}{=} -\mathbf{E} \cdot \mathbf{WE} = -\mathbf{WE} \cdot \mathbf{E}.$$

It follows that $\mathbf{WE} \cdot \mathbf{E} = 0$ and so (ii) reduces to

$$\Phi(\mathbf{E}) = \mathbf{CE} \cdot \mathbf{E} = \mathbf{SE} \cdot \mathbf{E}.$$

Thus it is only the symmetric part of $\mathbf{C}$ that affects the function Φ and so with no loss of generality we might as well assume $\mathbf{C}$ to be symmetric.

Problem 1.16. (Tensor algebra. Invariants.) Show that

$$\frac{\mathbf{x}\cdot(\mathbf{A}\mathbf{y}\times\mathbf{A}\mathbf{z})+\mathbf{A}\mathbf{x}\cdot(\mathbf{y}\times\mathbf{A}\mathbf{z})+\mathbf{A}\mathbf{x}\cdot(\mathbf{A}\mathbf{y}\times\mathbf{z})}{\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})}=I_2(\mathbf{A}), \tag{1.196}$$

for all linearly independent vectors $\mathbf{x},\mathbf{y},\mathbf{z}$ where $I_2(\mathbf{A})$ is the principal scalar invariant introduced in $(1.107)_2$.

Solution: The fact that the left-hand side of (1.196) is in fact independent of the choice of the linearly independent vectors $\mathbf{x},\mathbf{y},\mathbf{z}$ can be shown using the same procedure as in Problem 1.58. We will not carry out that part of the solution here.

Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be an orthonormal basis. Since the left-hand side of (1.196) is independent of the choice of $\mathbf{x},\mathbf{y},\mathbf{z}$ it necessarily holds for the choice $\mathbf{x}=\mathbf{e}_1,\mathbf{y}=\mathbf{e}_2,\mathbf{z}=\mathbf{e}_3$. Thus

$$\begin{aligned}
&\frac{\mathbf{x}\cdot(\mathbf{A}\mathbf{y}\times\mathbf{A}\mathbf{z})+\mathbf{A}\mathbf{x}\cdot(\mathbf{y}\times\mathbf{A}\mathbf{z})+\mathbf{A}\mathbf{x}\cdot(\mathbf{A}\mathbf{y}\times\mathbf{z})}{\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})}=\\
&=\frac{\mathbf{e}_1\cdot(\mathbf{A}\mathbf{e}_2\times\mathbf{A}\mathbf{e}_3)+\mathbf{A}\mathbf{e}_1\cdot(\mathbf{e}_2\times\mathbf{A}\mathbf{e}_3)+\mathbf{A}\mathbf{e}_1\cdot(\mathbf{A}\mathbf{e}_2\times\mathbf{e}_3)}{\mathbf{e}_1\cdot(\mathbf{e}_2\times\mathbf{e}_3)}=\\
&\overset{(1.129)}{=}\frac{\mathbf{e}_1\cdot(A_{p2}\mathbf{e}_p\times A_{q3}\mathbf{e}_q)+A_{p1}\mathbf{e}_p\cdot(\mathbf{e}_2\times A_{q3}\mathbf{e}_q)+A_{p1}\mathbf{e}_p\cdot(A_{q2}\mathbf{e}_q\times\mathbf{e}_3)}{\mathbf{e}_1\cdot\mathbf{e}_1}=\\
&=A_{p2}A_{q3}\ \mathbf{e}_1\cdot(\mathbf{e}_p\times\mathbf{e}_q)+A_{p1}A_{q3}\ \mathbf{e}_p\cdot(\mathbf{e}_2\times\mathbf{e}_q)+A_{p1}A_{q2}\ \mathbf{e}_p\cdot(\mathbf{e}_q\times\mathbf{e}_3)=\\
&\overset{(1.54)}{=}A_{p2}A_{q3}\ e_{1pq}+A_{p1}A_{q3}\ e_{p2q}+A_{p1}A_{q2}\ e_{pq3}=\\
&\overset{(1.38)}{=}A_{22}A_{33}\ e_{123}+A_{32}A_{23}\ e_{132}+A_{11}A_{33}\ e_{123}+A_{31}A_{13}\ e_{321}+A_{11}A_{12}\ e_{123}+A_{21}A_{12}\ e_{213}=\\
&\overset{(1.38)}{=}A_{22}A_{33}-A_{32}A_{23}+A_{11}A_{33}-A_{31}A_{13}+A_{11}A_{12}-A_{21}A_{12}=\\
&=(A_{11}A_{22}+A_{22}A_{33}+A_{33}A_{11})-(A_{32}A_{23}+A_{31}A_{13}+A_{21}A_{12})=\\
&=\frac{1}{2}(A_{11}+A_{22}+A_{33})^2-\frac{1}{2}(A_{11}^2+A_{22}^2+A_{33}^2+2A_{32}A_{23}+2A_{31}A_{13}+2A_{21}A_{12})=\\
&=\frac{1}{2}(A_{ii})^2-\frac{1}{2}(A_{ij}A_{ji})=\frac{1}{2}\Big[\big(\mathrm{tr}\mathbf{A}\big)^2-\mathrm{tr}\mathbf{A}^2\Big]\overset{(1.107)}{=}I_2(\mathbf{A}) \qquad \square
\end{aligned}$$

Problem 1.17. (Tensor algebra. Invariants.)

(a) Show for any tensor $\mathbf{A}$ and all scalars λ that

$$\det(\mathbf{A}-\lambda\mathbf{I}) \ = \ -\lambda^3+I_1(\mathbf{A})\lambda^2-I_2(\mathbf{A})\lambda+I_3(\mathbf{A}), \tag{i}$$

where

$$I_1(\mathbf{A})=\mathrm{tr}\,\mathbf{A}, \qquad I_2(\mathbf{A})=\frac{1}{2}[(\mathrm{tr}\,\mathbf{A})^2-\mathrm{tr}(\mathbf{A}^2)], \qquad I_3(\mathbf{A})=\det\,\mathbf{A}. \tag{ii}$$

Note that the identity (i) holds for *all* scalars λ not just the eigenvalues. The eigenvalues are the roots of the cubic equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$ – the so-called characteristic equation.

(b) Suppose that $\mathbf{A}$ is symmetric and that $\alpha_1, \alpha_2, \alpha_3$ are its three (necessarily real) eigenvalues. Show that

$$I_1(\mathbf{A}) = \alpha_1 + \alpha_2 + \alpha_3, \qquad I_2(\mathbf{A}) = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1, \qquad I_3(\mathbf{A}) = \alpha_1\alpha_2\alpha_3. \tag{iii}$$

Solution:

(a) According to (1.90), the determinant of a tensor $\mathbf{B}$ is defined by

$$\det \mathbf{B} := \frac{\mathbf{Bx} \cdot (\mathbf{By} \times \mathbf{Bz})}{\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})}, \tag{iv}$$

for all linearly independent vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$. Taking $\mathbf{B} = \mathbf{A} - \lambda\mathbf{I}$,

$$\det(\mathbf{A} - \lambda\mathbf{I}) := \frac{(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} \cdot ((\mathbf{A} - \lambda\mathbf{I})\mathbf{y} \times (\mathbf{A} - \lambda\mathbf{I})\mathbf{z})}{\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})}, \tag{v}$$

on expanding the numerator and using the formulae (1.103) for $I_1(\mathbf{A}) = \operatorname{tr}\mathbf{A}$, (1.196) for $I_2(\mathbf{A})$ and (1.90) for $I_3(\mathbf{A}) = \det\, \mathbf{A}$, we get the desired result (i).

(b) Since $\mathbf{A}$ is symmetric it has three real eigenvalues $\alpha_1, \alpha_2, \alpha_3$ and a corresponding set of orthonormal eigenvectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Consider the orthonormal basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ (a principal basis for $\mathbf{A}$). If A_{ij} are the components of $\mathbf{A}$ in this basis then (with the summation convention suspended)

$$A_{ij} \overset{(1.128)}{=} \mathbf{A}\mathbf{a}_j \cdot \mathbf{a}_i = \lambda_j \mathbf{a}_j \cdot \mathbf{a}_i = \lambda_j \delta_{ij}, \tag{vi}$$

having used $\mathbf{A}\mathbf{a}_j = \lambda_j \mathbf{a}_j$. Therefore the off-diagonal terms of $[A]$ vanish and $A_{ii} = \lambda_i$ whence

$$[A] = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \qquad [A]^2 = \begin{pmatrix} \alpha_1^2 & 0 & 0 \\ 0 & \alpha_2^2 & 0 \\ 0 & 0 & \alpha_3^2 \end{pmatrix}. \tag{vii}$$

Thus

$$I_1(\mathbf{A}) = \operatorname{tr}[A] = \alpha_1 + \alpha_2 + \alpha_3, \qquad \operatorname{tr}[A^2] = \alpha_1^2 + \alpha_2^2 + \alpha_3^2, \qquad I_3(\mathbf{A}) = \det[A] = \alpha_1\alpha_2\alpha_3, \qquad \square$$

and

$$I_2[A] = \frac{1}{2}\left[(\operatorname{tr}\mathbf{A})^2 - \operatorname{tr}(\mathbf{A}^2)\right] = \frac{1}{2}\left[(\alpha_1 + \alpha_2 + \alpha_3)^2 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)\right] = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1. \qquad \square$$

Problem 1.18. (Cayley-Hamilton theorem.) Show that any tensor $\mathbf{A}$ satisfies its own characteristic equation, i.e. show that

$$-\mathbf{A}^3 + I_1(\mathbf{A})\mathbf{A}^2 - I_2(\mathbf{A})\mathbf{A} + I_3(\mathbf{A})\mathbf{I} = \mathbf{0},$$

where the principal scalar invariants $I_i(\mathbf{A}), i = 1, 2, 3$, were defined in (1.107).

Solution: While the Cayley-Hamilton theorem holds for any tensor, the proof below is restricted to symmetric tensors. When $\mathbf{A}$ is symmetric, its eigenvectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ can be selected so that they form an orthonormal basis for V.

Let α and $\mathbf{a}$ be an eigenvalue and corresponding eigenvector of the symmetric tensor $\mathbf{A}$. Then $\mathbf{A}\mathbf{a} = \alpha\mathbf{a}$, and so $\mathbf{A}^2\mathbf{a} = \mathbf{A}(\mathbf{A}\mathbf{a}) = \mathbf{A}(\alpha\mathbf{a}) = \alpha(\mathbf{A}\mathbf{a}) = \alpha(\alpha\mathbf{a}) = \alpha^2\mathbf{a}$, and similarly $\mathbf{A}^3\mathbf{a} = \alpha^3\mathbf{a}$. Therefore

$$\begin{aligned}\big[\mathbf{A}^3 - I_1(\mathbf{A})\mathbf{A}^2 + I_2(\mathbf{A})\mathbf{A} - I_3(\mathbf{A})\mathbf{I}\big]\mathbf{a} &= \mathbf{A}^3\mathbf{a} - I_1(\mathbf{A})\mathbf{A}^2\mathbf{a} + I_2(\mathbf{A})\mathbf{A}\mathbf{a} - I_3(\mathbf{A})\mathbf{a} = \\ &= \big[\alpha^3 - I_1(\mathbf{A})\alpha^2 + I_2(\mathbf{A})\alpha - I_3(\mathbf{A})\big]\mathbf{a}.\end{aligned} \qquad (ii)$$

However, since α is an eigenvalue of $\mathbf{A}$ it obeys the cubic polynomial equation

$$\alpha^3 - I_1(\mathbf{A})\alpha^2 + I_2(\mathbf{A})\alpha - I_3(\mathbf{A}) = 0.$$

Using this in (ii) shows that

$$\big[\mathbf{A}^3 - I_1(\mathbf{A})\mathbf{A}^2 + I_2(\mathbf{A})\mathbf{A} - I_3(\mathbf{A})\mathbf{I}\big]\mathbf{a} = \mathbf{o}. \qquad (iii)$$

Equation (iii) holds for each eigenvector $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Since $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ forms a basis for V this implies that (iii) in fact holds for all vectors $\mathbf{a}$. Thus the term in square brackets in (iii) must be the null tensor which establishes the desired result.

Problem 1.19. (Tensor product.) Let

$$\mathbf{A} = \mathbf{u} \otimes \mathbf{v} \qquad (i)$$

for two vectors $\mathbf{u}$ and $\mathbf{v}$.

(a) Show that for all integers $n \geq 2$,

$$\mathbf{A}^n = (\mathbf{u} \cdot \mathbf{v})^{n-1}\mathbf{A}. \qquad (ii)$$

(b) Calculate the principal scalar invariants of $\mathbf{A}$.

(c) Hence or otherwise show that

$$\det(\mathbf{I} + \mathbf{A}) = 1 + \mathbf{u} \cdot \mathbf{v}. \qquad (iii)$$

(d) Determine the eigenvalues of $\mathbf{A}$.

Solution:

(a) We shall establish this by induction. First we show that (ii) holds for $n = 2$. This follows from

$$\mathbf{A}^2 \overset{(i)}{=} (\mathbf{u} \otimes \mathbf{v})(\mathbf{u} \otimes \mathbf{v}) \overset{(1.72)}{=} (\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \otimes \mathbf{v}) \overset{(i)}{=} (\mathbf{u} \cdot \mathbf{v})\mathbf{A}. \qquad (iv)$$

Next, suppose that (ii) holds for some integer $n = N > 2$:

$$\mathbf{A}^N = (\mathbf{u} \cdot \mathbf{v})^{N-1}\mathbf{A}. \qquad (v)$$

Then we show that (ii) holds for $n = N + 1$. This follows from

$$\mathbf{A}^{N+1} = \mathbf{A}^N\mathbf{A} \overset{(v)}{=} (\mathbf{u}\cdot\mathbf{v})^{N-1}\mathbf{A}\mathbf{A} = (\mathbf{u}\cdot\mathbf{v})^{N-1}\mathbf{A}^2 \overset{(iv)}{=} (\mathbf{u}\cdot\mathbf{v})^{N}\mathbf{A}.$$

Thus if (ii) holds for $n = N$ it necessarily holds for $n = N + 1$. We know it holds for $n = 2$. It thus follows that it holds for all integers $n \geq 2$.

(b) Observe from (i) that

$$\mathrm{tr}\,\mathbf{A} = A_{ii} = u_i v_i = \mathbf{u}\cdot\mathbf{v}, \tag{vi}$$

and from (ii) that

$$\mathrm{tr}\,\mathbf{A}^n = (\mathbf{u}\cdot\mathbf{v})^{n-1}\mathrm{tr}\,\mathbf{A} \overset{(vi)}{=} (\mathbf{u}\cdot\mathbf{v})^{n}. \tag{vii}$$

Therefore in particular,

$$\mathrm{tr}\,\mathbf{A}^2 = (\mathbf{u}\cdot\mathbf{v})^2, \qquad \mathrm{tr}\,\mathbf{A}^3 = (\mathbf{u}\cdot\mathbf{v})^3. \tag{$viii$}$$

Thus the first and second principal scalar invariants of $\mathbf{A}$ are

$$I_1(\mathbf{A}) = \mathrm{tr}\,\mathbf{A} = \mathbf{u}\cdot\mathbf{v}, \qquad \square \tag{ix}$$

$$I_2(\mathbf{A}) = \frac{1}{2}\big[\big(\mathrm{tr}\,\mathbf{A}\big)^2 - \mathrm{tr}\,\mathbf{A}^2\big] = \frac{1}{2}\big[\big(\mathbf{u}\cdot\mathbf{v}\big)^2 - (\mathbf{u}\cdot\mathbf{v})^2\big] = 0. \qquad \square \tag{x}$$

There are several ways in which to calculate the third principal scalar invariant $I_3(\mathbf{A}) = \det\,\mathbf{A}$. *Method 1:* According to the Cayley Hamilton theorem, (1.108),

$$\mathbf{A}^3 - I_1(\mathbf{A})\mathbf{A}^2 + I_2(\mathbf{A})\mathbf{A} - I_3(\mathbf{A})\mathbf{I} = \mathbf{O}. \tag{xi}$$

Taking the trace of this equation gives

$$\mathrm{tr}(\mathbf{A}^3) - I_1(\mathbf{A})\mathrm{tr}(\mathbf{A}^2) + I_2(\mathbf{A})\mathrm{tr}(\mathbf{A}) - 3I_3(\mathbf{A}) = 0.$$

Substituting $(viii), (ix)$ and (x) into this gives

$$(\mathbf{u}\cdot\mathbf{v})^3 - (\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{v})^2 + 0 - 3I_3(\mathbf{A}) = 0 \qquad \Rightarrow \qquad I_3(\mathbf{A}) = 0. \qquad \square \tag{xii}$$

Method 2: Alternatively

$$I_3(\mathbf{A}) \overset{(1.107)_3}{=} \det(\mathbf{A}) \overset{(1.41)}{=} \tfrac{1}{6}e_{ijk}e_{pqr}(\mathbf{A})_{ip}(\mathbf{A})_{jq}(\mathbf{A})_{kr} =$$

$$\overset{(i)}{=} \tfrac{1}{6}e_{ijk}e_{pqr}u_i v_p u_j v_q u_k v_r = \tfrac{1}{6}(e_{ijk}u_i u_j u_k)(e_{pqr}v_p v_q v_r).$$

Next recall from (1.124) that $S_{ij}W_{ij} = 0$ for any symmetric matrix $[S]$ and skew-symmetric matrix $[W]$. Therefore in particular, for any fixed i, since e_{ijk} is skew-symmetric in jk and $u_j u_k$ is symmetric in jk it follows that $e_{ijk}u_j u_k = 0$. Therefore

$$I_3(\mathbf{u}\otimes\mathbf{v}) = 0. \qquad \square.$$

(c) Setting $\mu = 1$ in the identity (1.106), for any tensor $\mathbf{A}$ one has

$$\det(\mathbf{A} + \mathbf{I}) = 1 + I_1(\mathbf{A}) + I_2(\mathbf{A}) + I_3(\mathbf{A}).$$

Thus for $\mathbf{A} = \mathbf{u} \otimes \mathbf{v}$ we get

$$\det(\mathbf{I} + \mathbf{u} \otimes \mathbf{v}) = 1 + I_1(\mathbf{u} \otimes \mathbf{v}) + I_2(\mathbf{u} \otimes \mathbf{v}) + I_3(\mathbf{u} \otimes \mathbf{v}) \overset{(ix)(x)(xii)}{=} 1 + \mathbf{u} \cdot \mathbf{v}.$$

Thus

$$\det(\mathbf{I} + \mathbf{u} \otimes \mathbf{v}) = 1 + \mathbf{u} \cdot \mathbf{v}. \qquad \Box \tag{1.197}$$

(d) By (1.110) and (1.107), the eigenvalues $\alpha_1, \alpha_2, \alpha_3$ of $\mathbf{A}$ are the roots of the cubic equation

$$\det(\mathbf{A} - \alpha\mathbf{I}) = -\alpha^3 + I_1(\mathbf{A})\alpha^2 - I_2(\mathbf{A})\alpha + I_3(\mathbf{A}) = 0,$$

which, because of $(ix), (x)$ and (xii), simplifies to

$$-\alpha^3 + (\mathbf{u} \cdot \mathbf{v})\alpha^2 = 0.$$

Thus the eigenvalues of $\mathbf{A}$ are

$$\alpha_1 = \alpha_2 = 0, \qquad \alpha_3 = \mathbf{u} \cdot \mathbf{v}.$$

Problem 1.20. (Additive decomposition of a tensor.) Show that an arbitrary tensor $\mathbf{T}$ can be *uniquely* decomposed into the sum,

$$\mathbf{T} = \mathbf{A} + \mathbf{B},$$

of a tensor $\mathbf{A}$ with zero trace and a tensor $\mathbf{B}$ that is a scalar multiple of the identity.

Problem 1.21. (Symmetric part of a tensor.) You showed in Problem 1.4.16 that $\mathbf{S} \cdot \mathbf{W} = 0$ for all symmetric tensors $\mathbf{S}$ and skew-symmetric tensors $\mathbf{W}$. Hence or otherwise, show that

$$\mathbf{S} \cdot \mathbf{A} = \mathbf{S} \cdot \mathbf{A}^{\mathrm{sym}} \quad \text{for all tensors } \mathbf{A} \text{ and all symmetric tensors } \mathbf{S}, \tag{1.198}$$

where $\mathbf{A}^{\mathrm{sym}}$ is the symmetric part of $\mathbf{A}$, i.e. $\mathbf{A}^{\mathrm{sym}} := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$.

Problem 1.22. (Coaxial tensors.) Two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$ are said to be coaxial if they have the same principal directions. Show that $\mathbf{A}$ and $\mathbf{B}$ are coaxial if and only if

$$\mathbf{AB} = \mathbf{BA}. \tag{1.199}$$

Problem 1.23. (Tensor algebra.) Let $\mathbf{R}$ and $\mathbf{Q}$ be proper orthogonal tensors, $\mathbf{U}$ a symmetric, positive definite tensor and $\mathbf{a}$ and $\mathbf{n}$ two non-null vectors such that

$$\mathbf{RUQ} = \mathbf{U} + \mathbf{a} \otimes \mathbf{n}. \tag{i}$$

The goal of this problem is to derive three linearly independent scalar equations from (i) that do not involve the rotations $\mathbf{R}$ and $\mathbf{Q}$, and to this end proceed as follows:

(a) By taking the determinant of (i) show that

$$\mathbf{a}\cdot\mathbf{U}^{-1}\mathbf{n} = 0. \qquad \square \tag{ii}$$

(b) Next, show that

$$\mathbf{R}\mathbf{U}^2\mathbf{R}^T = \mathbf{U}^2 + \mathbf{a}\otimes\mathbf{U}\mathbf{n} + \mathbf{U}\mathbf{n}\otimes\mathbf{a} + \mathbf{a}\otimes\mathbf{a}, \tag{iii}$$

and by taking its trace show that

$$2\mathbf{a}\cdot\mathbf{U}\mathbf{n} + |\mathbf{a}|^2 = 0. \qquad \square \tag{iv}$$

Observe from (ii) that

$$(\mathbf{I} + \mathbf{a}\otimes\mathbf{U}^{-1}\mathbf{n})^{-1} = (\mathbf{I} - \mathbf{a}\otimes\mathbf{U}^{-1}\mathbf{n}). \tag{v}$$

(c) Use (i) and (v) to show that

$$\mathbf{R}\mathbf{U}^{-2}\mathbf{R}^T = \mathbf{U}^{-2} - \mathbf{U}^{-1}\mathbf{n}\otimes\mathbf{U}^{-2}\mathbf{a} - \mathbf{U}^{-2}\mathbf{a}\otimes\mathbf{U}_0^{-1}\mathbf{n} + |\mathbf{U}^{-1}\mathbf{a}|^2(\mathbf{U}^{-1}\mathbf{n}\otimes\mathbf{U}^{-1}\mathbf{n}), \tag{vi}$$

and by taking its trace show that

$$2\mathbf{U}^{-2}\mathbf{a}\cdot\mathbf{U}^{-1}\mathbf{n} - |\mathbf{U}^{-1}\mathbf{a}|^2|\mathbf{U}^{-1}\mathbf{n}|^2 = 0. \qquad \square \tag{vii}$$

Problem 1.24. (Nine dimensional vector space Lin.) Show that Lin, the set of all tensors on the vector space V, is nine dimensional.

Solution: Recall from Problem 1.4.12 that $\mathbf{e}_i\otimes\mathbf{e}_j, i,j = 1,2,3$, are nine orthonormal tensors in Lin (where as usual $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is an orthonormal basis for V). Since they are orthonormal, they are necessarily linearly independent. In order to show that Lin is nine dimensional, it is therefore sufficient to demonstrate that any tensor $\mathbf{A}\in\mathsf{Lin}$ can be expressed as a linear combination of the nine tensors $\mathbf{e}_i\otimes\mathbf{e}_j, i,j = 1,2,3$. To show this, define nine scalars A_{ij} by

$$A_{ij} = (\mathbf{A}\mathbf{e}_j)\cdot\mathbf{e}_i. \tag{i}$$

Observe that this says that the ith component of the vector $\mathbf{A}\mathbf{e}_j$ is A_{ij} which can be equivalently stated as

$$\mathbf{A}\mathbf{e}_j = A_{ij}\mathbf{e}_i. \tag{ii}$$

Then for an arbitrary vector $\mathbf{x}\in\mathsf{V}$ we have

$$\mathbf{A}\mathbf{x} = \mathbf{A}(x_j\mathbf{e}_j) = x_j\mathbf{A}\mathbf{e}_j \overset{(ii)}{=} x_jA_{ij}\mathbf{e}_i = A_{ij}x_j\mathbf{e}_i = A_{ij}(\mathbf{x}\cdot\mathbf{e}_j)\mathbf{e}_i = A_{ij}(\mathbf{e}_i\otimes\mathbf{e}_j)\mathbf{x}.$$

Since this holds for all vectors $\mathbf{x}$ it follows that

$$\mathbf{A} = A_{ij}(\mathbf{e}_i\otimes\mathbf{e}_j), \tag{1.200}$$

and so an arbitrary tensor $\mathbf{A}\in\mathsf{Lin}$ can be represented as a linear combination of the nine orthonormal tensors $\mathbf{e}_i\otimes\mathbf{e}_j, i,j = 1,2,3$. Thus Lin is nine dimensional (and the nine aforementioned tensors form an orthonormal basis for it.)

Problem 1.25. (Square root of a symmetric positive definite tensor.) Consider a symmetric positive definite tensor $\mathbf{S}$. Show that it has a *unique* symmetric positive definite square root, i.e. show that there is a unique symmetric positive definite tensor $\mathbf{T}$ for which $\mathbf{T}^2 = \mathbf{S}$.

Solution: *In this solution we shall suspend the summation convention on repeated indices and instead show all summations explicitly. Observe that we will have the same subscript appearing 3 times in some equations below.*

Since $\mathbf{S}$ is symmetric it has three real eigenvalues $\sigma_1, \sigma_2, \sigma_3$ with corresponding eigenvectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ which may be taken to be orthonormal. Furthermore, we know that $\mathbf{S}$ can be represented in its so-called spectral representation

$$\mathbf{S} = \sum_{i=1}^{3} \sigma_i (\mathbf{s}_i \otimes \mathbf{s}_i);$$

see the discussion surrounding (1.112). Since $\mathbf{S}$ is positive definite its eigenvalues are all positive. Hence one can define a linear transformation $\mathbf{T}$ by

$$\mathbf{T} = \sum_{i=1}^{3} \sqrt{\sigma_i} (\mathbf{s}_i \otimes \mathbf{s}_i),$$

and readily verify that $\mathbf{T}$ is symmetric, positive definite and that $\mathbf{T}^2 = \mathbf{S}$. This establishes the existence of a symmetric positive definite square-root of $\mathbf{S}$. What remains is to show uniqueness of this square-root.

Suppose that $\mathbf{S}$ has two symmetric positive definite square roots $\mathbf{T}_1$ and $\mathbf{T}_2$: $\mathbf{S} = \mathbf{T}_1^2 = \mathbf{T}_2^2$. Let $\sigma > 0$ and $\mathbf{s}$ be an eigenvalue and corresponding eigenvector of $\mathbf{S}$. Then $\mathbf{S}\mathbf{s} = \sigma\mathbf{s}$ and so $\mathbf{T}_1^2\mathbf{s} = \sigma\mathbf{s}$. Thus we have

$$\mathbf{T}_1^2\mathbf{s} - \sigma\mathbf{s} = (\mathbf{T}_1^2 - \sigma\mathbf{I})\mathbf{s} = (\mathbf{T}_1 + \sqrt{\sigma}\mathbf{I})(\mathbf{T}_1 - \sqrt{\sigma}\mathbf{I})\mathbf{s} = \mathbf{0}\ .$$

If we set $\mathbf{f} = (\mathbf{T}_1 - \sqrt{\sigma}\mathbf{I})\mathbf{s}$ this can be written as

$$\mathbf{T}_1\mathbf{f} = -\sqrt{\sigma}\mathbf{f}\ .$$

Thus either $\mathbf{f}$ is an eigenvector of $\mathbf{T}_1$ corresponding to the eigenvalue $-\sqrt{\sigma}$ (< 0) or $\mathbf{f} = \mathbf{o}$. Since $\mathbf{T}_1$ is positive definite it cannot have a negative eigenvalue. Thus $\mathbf{f} = \mathbf{o}$ and so

$$\mathbf{T}_1\mathbf{s} = \sqrt{\sigma}\mathbf{s}\ .$$

A similar calculation shows that $\mathbf{T}_2\mathbf{s} = \sqrt{\sigma}\mathbf{s}$. Thus

$$\mathbf{T}_1\mathbf{s} = \mathbf{T}_2\mathbf{s}.$$

This holds for *every* eigenvector $\mathbf{s}$ of $\mathbf{S}$: i.e. $\mathbf{T}_1\mathbf{s}_i = \mathbf{T}_2\mathbf{s}_i, \quad i = 1, 2, 3$. Since the triplet of eigenvectors form a basis for the underlying vector space this implies that $\mathbf{T}_1\mathbf{x} = \mathbf{T}_2\mathbf{x}$ for any vector $\mathbf{x}$. Thus $\mathbf{T}_1 = \mathbf{T}_2$.

Problem 1.26. (Spectral representation.) *The summation convention is suspended in this problem and all summations are shown explicitly.* The polar decomposition theorem (1.117) states that any nonsingular linear transformation $\mathbf{F}$ can be represented uniquely in the forms $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ where $\mathbf{R}$ is orthogonal and $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive definite. Let $\lambda_i, \mathbf{u}_i,\ i = 1, 2, 3$ be the eigenvalues and eigenvectors of $\mathbf{U}$. As stated just below (1.117), the eigenvalues of $\mathbf{V}$ are the same as those of $\mathbf{U}$ and the corresponding eigenvectors $\mathbf{v}_i$ of $\mathbf{V}$ are given by $\mathbf{v}_i = \mathbf{R}\mathbf{u}_i$. [Exercise: show this.] Thus $\mathbf{U}$ and $\mathbf{V}$ have the spectral decompositions

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i\ , \qquad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i\ .$$

Show that

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{u}_i\ , \qquad \mathbf{R} = \sum_{i=1}^{3} \mathbf{v}_i \otimes \mathbf{u}_i\ .$$

Solution: First, by using the property $(1.78)_1$ and $\mathbf{v}_i = \mathbf{R}\mathbf{u}_i$ we have

$$\mathbf{F} = \mathbf{RU} = \mathbf{R}\ \sum_{i=1}^{3}\ \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i = \sum_{i=1}^{3}\ \lambda_i (\mathbf{R}\mathbf{u}_i) \otimes \mathbf{u}_i = \sum_{i=1}^{3}\ \lambda_i \mathbf{v}_i \otimes \mathbf{u}_i. \tag{1.201}$$

Next, since $\mathbf{U}$ is non-singular

$$\mathbf{U}^{-1} = \sum_{i=1}^{3}\ \lambda_i^{-1} \mathbf{u}_i \otimes \mathbf{u}_i.$$

and therefore

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \sum_{i=1}^{3}\ \lambda_i \mathbf{v}_i \otimes \mathbf{u}_i\ \sum_{j=1}^{3}\ \lambda_j^{-1} \mathbf{u}_j \otimes \mathbf{u}_j = \sum_{i=1}^{3}\sum_{j=1}^{3} \lambda_i \lambda_j^{-1} (\mathbf{v}_i \otimes \mathbf{u}_i)(\mathbf{u}_j \otimes \mathbf{u}_j).$$

By using the property $(1.72)_2$ and the fact that $\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$, we have $(\mathbf{v}_i \otimes \mathbf{u}_i)(\mathbf{u}_j \otimes \mathbf{u}_j) = (\mathbf{u}_i \cdot \mathbf{u}_j)(\mathbf{v}_i \otimes \mathbf{u}_j) = \delta_{ij}(\mathbf{v}_i \otimes \mathbf{u}_j)$. Therefore

$$\mathbf{R} = \sum_{i=1}^{3}\sum_{j=1}^{3} \lambda_i \lambda_j^{-1} \delta_{ij} (\mathbf{v}_i \otimes \mathbf{u}_j) = \sum_{i=1}^{3} \lambda_i \lambda_i^{-1} (\mathbf{v}_i \otimes \mathbf{u}_i) = \sum_{i=1}^{3} (\mathbf{v}_i \otimes \mathbf{u}_i). \tag{1.202}$$

Problem 1.27. (Spectral representation.) *The summation convention is suspended in this problem.* Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be two orthonormal sets of vectors. Suppose that a tensor $\mathbf{F}$ has the representation

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{u}_i.$$

If all three λ_i's are non-zero, show that

$$\mathbf{F}^{-1} = \sum_{i=1}^{3} \lambda_i^{-1} \mathbf{u}_i \otimes \mathbf{v}_i.$$

---

3. Change of basis.

Problem 1.28. (See also Problem 1.29.) Consider two bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ and define the scalars Q_{ij} in the usual way by

$$Q_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j. \qquad (i)$$

Show that

$$\mathbf{e}_i = Q_{ji}\mathbf{e}'_j, \qquad \mathbf{e}'_i = Q_{ij}\mathbf{e}_j. \qquad (ii)$$

Let $\mathbf{Q}$ be the tensor whose components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are Q_{ij}. Show that

$$\mathbf{e}'_i = \mathbf{Q}^T\mathbf{e}_i. \qquad (iii)$$

Therefore $\mathbf{Q}^T$ is the orthogonal transformation that carries the first basis into the second (not $\mathbf{Q}$).

Problem 1.29. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ be two orthonormal bases. Define the tensor $\mathbf{R}$ by

$$\mathbf{R} = \mathbf{e}'_i \otimes \mathbf{e}_i = \mathbf{e}'_1 \otimes \mathbf{e}_1 + \mathbf{e}'_2 \otimes \mathbf{e}_2 + \mathbf{e}'_3 \otimes \mathbf{e}_3. \qquad (i)$$

(a) Show that $\mathbf{R}$ is an orthogonal tensor.

(b) Show that $\mathbf{R}$ takes the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ into the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, i.e. show that

$$\mathbf{e}'_i = \mathbf{R}\mathbf{e}_i. \qquad (ii)$$

(c) For any tensor $\mathbf{A}$, let $[A]$ and $[A']$ be its respective matrices of components in the bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. Show that

$$[A'] = [R]^T[A][R], \qquad (iii)$$

where $[R]$ is the matrix of components of $\mathbf{R}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Remark: Note that $\mathbf{R}$ is the transpose of the tensor $\mathbf{Q}$ in Problem 1.28.

Solution:

(a) We use (i) to calculate $\mathbf{R}\mathbf{R}^T$:

$$\mathbf{R}\mathbf{R}^T = (\mathbf{e}'_i \otimes \mathbf{e}_i)(\mathbf{e}_j \otimes \mathbf{e}'_j) = (\mathbf{e}_i \cdot \mathbf{e}_j)(\mathbf{e}'_i \otimes \mathbf{e}'_j) = \delta_{ij}(\mathbf{e}'_i \otimes \mathbf{e}'_j) = \mathbf{e}'_i \otimes \mathbf{e}'_i = \mathbf{I}$$

where in the first equality we used $(\mathbf{a}\otimes\mathbf{b})^T = \mathbf{b}\otimes\mathbf{a}$, in the next step we used $(\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\otimes\mathbf{d}) = (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\otimes\mathbf{d})$, and in the next we used $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Therefore $\mathbf{R}$ is orthogonal.

(b) From (i) we have

$$\mathbf{R}\mathbf{e}_j = (\mathbf{e}'_i \otimes \mathbf{e}_i)\mathbf{e}_j = (\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{e}'_i = \delta_{ij}\mathbf{e}'_i = \mathbf{e}'_j. \qquad (iv)$$

where in the second step we used $(\mathbf{a}\otimes\mathbf{b})\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$.

(c) From the definition of the components of a tensor:

$$\mathbf{R}\mathbf{e}_j = R_{ij}\mathbf{e}_i, \qquad \mathbf{A}\mathbf{e}_j = A_{ij}\mathbf{e}_i, \qquad A'_{ij} = \mathbf{A}\mathbf{e}'_j \cdot \mathbf{e}'_i. \qquad (v)$$

Thus

$$\begin{aligned} A'_{ij} &\overset{(v)_3}{=} \mathbf{A}\mathbf{e}'_j \cdot \mathbf{e}'_i \overset{(iv)}{=} \mathbf{A}\mathbf{R}\mathbf{e}_j \cdot \mathbf{R}\mathbf{e}_i = \mathbf{R}^T\mathbf{A}\mathbf{R}\mathbf{e}_j \cdot \mathbf{e}_i \overset{(v)_1}{=} \mathbf{R}^T\mathbf{A}(R_{pj}\mathbf{e}_p) \cdot \mathbf{e}_i = \\ &= R_{pj}\mathbf{R}^T\mathbf{A}\mathbf{e}_p \cdot \mathbf{e}_i \overset{(v)_2}{=} R_{pj}\mathbf{R}^T(A_{qp}\mathbf{e}_q) \cdot \mathbf{e}_i = R_{pj}A_{qp}\mathbf{R}^T\mathbf{e}_q \cdot \mathbf{e}_i = \\ &= R_{pj}A_{qp}\mathbf{e}_q \cdot \mathbf{R}\mathbf{e}_i \overset{(v)_1}{=} R_{pj}A_{qp}\mathbf{e}_q \cdot (R_{ki}\mathbf{e}_k) = R_{pj}A_{qp}R_{ki}\ \mathbf{e}_q \cdot \mathbf{e}_k = \\ &= R_{pj}A_{qp}R_{ki}\delta_{qk} = R_{pj}A_{qp}R_{qi} = R^T_{iq}A_{qp}R_{pj} = ([R]^T[A][R])_{ij} \end{aligned}$$

which leads to (iii).

Problem 1.30. Suppose that the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is obtained by rotating the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ through an angle θ about the unit vector $\mathbf{e}_3$; see Figure 1.12. Write out the transformation rule (1.153) explicitly for a 2-tensor $\mathbf{A}$ whose matrix of components in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is

$$[A] = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

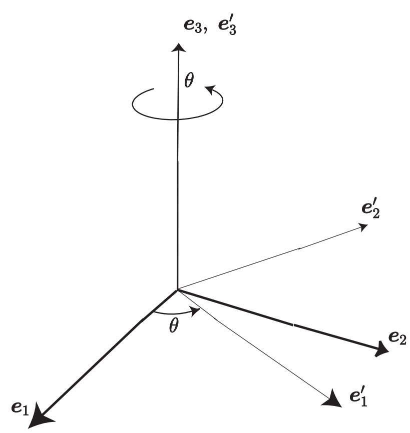

Figure 1.12: A basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ obtained by rotating the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ through an angle θ about the unit vector $\mathbf{e}_3$.

Problem 1.31. Let $[A]$ and $[A']$ be the components of a 2-tensor $\mathbf{A}$ in two bases. Show that the two matrices $[A]$ and $[A']$ have the same eigenvalues.

Hint: The eigenvalues of a matrix are the roots of the characteristic equation. For matrices $[A]$ and $[A']$ the respective characteristic equations are

$$\det([A] - \lambda[I]) = 0 \qquad \text{and} \qquad \det([A'] - \lambda[I]) = 0.$$

To show that these two matrices have the same eigenvalues it is sufficient to show that the two characteristic equations are identical. When the characteristic equations are written out each has the form given in (1.110). Therefore we should aim to show that

$$I_1([A]) = I_1([A']), \qquad I_2([A]) = I_2([A']), \qquad I_3([A]) = I_3([A']).$$

For this we must show that $\mathsf{trace}[A] = \mathsf{trace}[A']$, $\mathsf{trace}([A]^2) = \mathsf{trace}([A']^2)$ and $\mathsf{det}[A] = \mathsf{det}[A']$.

Solution: We know that

$$[A'] = [Q][A][Q]^T, \tag{i}$$

or in components

$$A'_{ij} = Q_{ik}Q_{j\ell}A_{k\ell}. \tag{ii}$$

Therefore

$$\mathsf{trace}[A'] = A'_{jj} \overset{(ii)}{=} Q_{jk}Q_{j\ell}A_{k\ell} \overset{(1.98)}{=} \delta_{k\ell}A_{k\ell} \overset{(1.37)}{=} A_{kk} = \mathsf{trace}[A]. \tag{iii}$$

Similarly consider the trace of $[A']^2$. If we set $[B] = [A']^2$ then $B_{ij} = A'_{ik}A'_{kj}$ and so $B_{jj} = A'_{jk}A'_{kj}$. Thus

$$\begin{aligned} \mathsf{trace}[A']^2 &= A'_{jk}A'_{kj} \overset{(ii)}{=} (Q_{jp}Q_{kq}A_{pq})(Q_{km}Q_{jn}A_{mn}) = \\ &= (Q_{jp}Q_{jn})(Q_{kq}Q_{km})A_{mn} \overset{(1.98)}{=} \delta_{pn}\delta_{qm}A_{pq}A_{mn} = \overset{(1.37)}{=} A_{nm}A_{mn} = \mathsf{trace}[A]^2 \end{aligned} \tag{iv}$$

From $\det[A'] = \det[Q]\det[A]\det[Q^T] = (\det[Q])^2\det[A]$ and $\det[Q] = \pm 1$ we have

$$\det[A'] = \det[A]. \tag{v}$$

It follows from $(iii), (iv), (v)$ that

$$I_1([A]) = I_1([A']), \qquad I_2([A]) = I_2([A']), \qquad I_3([A]) = I_3([A']),$$

and therefore that the eigenvalues of $[A]$ and $[A']$ coincide.

Problem 1.32. (See also Problem 1.6.5.) A tensor $\mathbf{F}$ has the representation

$$\mathbf{F} = \Phi_{ij}\mathbf{e}'_i \otimes \mathbf{e}_j, \tag{i}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are two right-handed orthonormal bases.

(a) Show that $\det \mathbf{F} = \det[\Phi]$ where $[\Phi]$ is the matrix whose i,j-element is Φ_{ij}.

(b) Suppose $\mathbf{F}$ is nonsingular. Since $(\mathbf{e}_i \otimes \mathbf{e}'_j)(\mathbf{e}'_i \otimes \mathbf{e}_j) = \mathbf{e}_i \otimes \mathbf{e}_i = \mathbf{I}$, the tensor $\mathbf{F}^{-1}$ has the representation

$$\mathbf{F}^{-1} = \Psi_{ij}\mathbf{e}_i \otimes \mathbf{e}'_j. \tag{ii}$$

Show that Ψ_{ij} is the i,j-element of the matrix $[\Psi] := [\Phi]^{-1}$.

Solution:

(a) From (i) we obtain $\mathbf{F}\mathbf{e}_k = \Phi_{ij}(\mathbf{e}'_i \otimes \mathbf{e}_j)\mathbf{e}_k = \Phi_{ij}(\mathbf{e}_j \cdot \mathbf{e}_k)\mathbf{e}'_i = \Phi_{ij}\delta_{jk}\mathbf{e}'_i$ and so

$$\mathbf{F}\mathbf{e}_k = \Phi_{ik}\mathbf{e}'_i. \qquad (iii)$$

By taking $\mathbf{x} = \mathbf{e}_1, \mathbf{y} = \mathbf{e}_2, \mathbf{z} = \mathbf{e}_3$ in (1.90) we get

$$\begin{aligned}
\det \mathbf{F} \quad &= \mathbf{F}\mathbf{e}_1 \cdot (\mathbf{F}\mathbf{e}_2 \times \mathbf{F}\mathbf{e}_3) = \\
&\stackrel{(iii)}{=} (\Phi_{i1}\mathbf{e}'_i) \cdot \Big[(\Phi_{j2}\mathbf{e}'_j) \times (\Phi_{k3}\mathbf{e}'_k)\Big] = \Phi_{i1}\Phi_{j2}\Phi_{k3}\, \mathbf{e}'_i \cdot (\mathbf{e}'_j \times \mathbf{e}'_k) = \\
&\stackrel{(1.54)}{=} \Phi_{i1}\Phi_{j2}\Phi_{k3}e_{jki} = e_{ijk}\Phi_{i1}\Phi_{j2}\Phi_{k3} \stackrel{(1.39)}{=} \det[\Phi] \qquad \square
\end{aligned}$$

Alternatively, from Problem 1.6.5, we have $[F] = [Q]^T[\Phi]$ whence $\det \mathbf{F} = \det[F] = \det[\Phi]$ where we have used the fact that $\det[Q] = +1$ which is a consequence of $[Q]$ being proper orthogonal since the two bases are both right-handed and orthonormal.

(b) We can assume $\mathbf{F}^{-1}$ to have the form (ii) and need to show that $[\Psi] = [\Phi]^{-1}$. From (i) and (ii)

$$\mathbf{F}\mathbf{F}^{-1} = (\Phi_{ij}\mathbf{e}'_i \otimes \mathbf{e}_j)(\Psi_{k\ell}\mathbf{e}_k \otimes \mathbf{e}'_\ell) = \Phi_{ij}\Psi_{k\ell}(\mathbf{e}_j \cdot \mathbf{e}_k)(\mathbf{e}'_i \otimes \mathbf{e}'_\ell) = \Phi_{ij}\Psi_{k\ell}\delta_{jk}(\mathbf{e}'_i \otimes \mathbf{e}'_\ell) = \Phi_{ik}\Psi_{k\ell}(\mathbf{e}'_i \otimes \mathbf{e}'_\ell)$$

Since this equals the identity, which we can write as $\mathbf{I} = \mathbf{e}'_i \otimes \mathbf{e}'_i = \delta_{i\ell}\, \mathbf{e}'_i \otimes \mathbf{e}'_\ell$, it follows that

$$\Phi_{ik}\Psi_{k\ell} = \delta_{i\ell}, \qquad [\Phi][\Psi] = [I]. \qquad \square$$

---

4. Invariant (isotropic) functions.

Problem 1.33.

(a) A scalar-valued function $\phi(\mathbf{x}) : \mathsf{V} \to \mathbb{R}$ is said to be isotropic if $\phi(\mathbf{x}) = \phi(\mathbf{Q}\mathbf{x})$ for all orthogonal $\mathbf{Q}$. Show that $\phi(\mathbf{x})$ is isotropic if and only if there is a function $\widehat{\phi}$ such that

$$\phi(\mathbf{x}) = \widehat{\phi}(|\mathbf{x}|) \qquad \text{for all } \mathbf{x} \in \mathsf{V}.$$

(Here $\mathbb{R}$ is the set of all real numbers.)

(b) A vector-valued function $\mathbf{u}(\mathbf{x}) : \mathsf{V} \to \mathsf{V}$ is said to be isotropic if $\mathbf{Q}\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{Q}\mathbf{x})$ for all orthogonal $\mathbf{Q}$. Show that $\mathbf{u}(\mathbf{x})$ is isotropic if and only if there is a function $\widehat{\phi}$ such that

$$\mathbf{u}(\mathbf{x}) = \widehat{\phi}(|\mathbf{x}|)\, \mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathsf{V}.$$

Problem 1.34. Suppose that the principal scalar invariants of two symmetric tensors $\mathbf{B}$ and $\mathbf{C}$ have the same values:

$$I_1(\mathbf{B}) = I_1(\mathbf{C}), \qquad I_2(\mathbf{B}) = I_2(\mathbf{C}), \qquad I_3(\mathbf{B}) = I_3(\mathbf{C}). \tag{i}$$

Show that there necessarily exists an orthogonal tensor $\mathbf{Q}$ such that

$$\mathbf{B} = \mathbf{Q}\mathbf{C}\mathbf{Q}^T. \tag{ii}$$

Solution: Recall from (1.110) and (1.106) that the eigenvalues of a symmetric tensor $\mathbf{A}$ are given by the real roots α of the cubic equation

$$\det(\mathbf{A} - \alpha\mathbf{I}) = -\alpha^3 + I_1(\mathbf{A})\alpha^2 - I_2(\mathbf{A})\alpha + I_3(\mathbf{A}) = 0. \tag{iii}$$

It follows from (i) and (iii) that $\mathbf{B}$ and $\mathbf{C}$ have the same eigenvalues, say β_1, β_2 and β_3, (but possibly different eigenvectors). Let the corresponding orthonormal eigenvectors be $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$. Therefore by (1.112), $\mathbf{B}$ and $\mathbf{C}$ can be expressed as

$$\mathbf{B} = \sum_{i=1}^{3} \beta_i \mathbf{b}_i \otimes \mathbf{b}_i, \qquad \mathbf{C} = \sum_{i=1}^{3} \beta_i \mathbf{c}_i \otimes \mathbf{c}_i. \tag{1.203}$$

Since each of the sets of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is orthonormal, there is an orthogonal tensor $\mathbf{Q}$ that takes $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ into $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$:

$$\mathbf{Q}\mathbf{c}_i = \mathbf{b}_i, \qquad i = 1, 2, 3; \tag{iv}$$

see discussion preceding (1.160). In view of this and the tensor identity $\mathbf{F}(\mathbf{x} \otimes \mathbf{y})\mathbf{G} = (\mathbf{F}\mathbf{x}) \otimes (\mathbf{G}^T\mathbf{y})$

$$\mathbf{Q}(\mathbf{c}_i \otimes \mathbf{c}_i)\mathbf{Q}^T \;=\; (\mathbf{Q}\mathbf{c}_i) \otimes (\mathbf{Q}\mathbf{c}_i) \;\overset{(iv)}{=}\; \mathbf{b}_i \otimes \mathbf{b}_i \qquad \text{(no sum)}. \tag{v}$$

Therefore

$$\mathbf{Q}\mathbf{C}\mathbf{Q}^T \;\overset{(1.203)_2}{=}\; \sum_{i=1}^{3} \beta_i \mathbf{Q}(\mathbf{c}_i \otimes \mathbf{c}_i)\mathbf{Q}^T \;\overset{(v)}{=}\; \sum_{i=1}^{3} \beta_i \mathbf{b}_i \otimes \mathbf{b}_i \;\overset{(1.203)_1}{=}\; \mathbf{B},$$

which establishes the result (ii).

Problem 1.35. Let $\phi(\mathbf{A})$ be an isotropic scalar-valued function defined for all symmetric tensors $\mathbf{A}$. Show that there exists a function $\widehat{\phi}$ such that

$$\phi(\mathbf{A}) = \widehat{\phi}(I_1(\mathbf{A}), I_2(\mathbf{A}), I_3(\mathbf{A}))$$

where the I_i's are the principal scalar invariant functions defined in (1.107).

Solution: Let $\phi(\mathbf{A})$ be an isotropic scalar-valued function:

$$\phi(\mathbf{A}) = \phi(\mathbf{Q}\mathbf{A}\mathbf{Q}^T) \tag{i}$$

for all symmetric tensors $\mathbf{A}$ and orthogonal tensors $\mathbf{Q}$. It is sufficient for us to show that

$$\phi(\mathbf{B}) = \phi(\mathbf{C}) \tag{ii}$$

whenever

$$I_1(\mathbf{B}) = I_1(\mathbf{C}), \qquad I_2(\mathbf{B}) = I_2(\mathbf{C}), \qquad I_3(\mathbf{B}) = I_3(\mathbf{C}). \tag{iii}$$

From the result in Problem 1.34, whenever (iii) holds there is an orthogonal tensor $\mathbf{Q}$ such that

$$\mathbf{B} = \mathbf{QCQ}^T. \tag{iv}$$

Therefore $\phi(\mathbf{B}) \overset{(iv)}{=} \phi(\mathbf{QCQ}^T) \overset{(i)}{=} \phi(\mathbf{C})$ and thus (ii) holds.

Problem 1.36. Let $\phi(\mathbf{A})$ be the function defined for all symmetric tensors $\mathbf{A}$ by

$$\phi(\mathbf{A}) = \frac{1}{2}\mathbb{C}_{ijk\ell}A_{ij}A_{k\ell}. \tag{i}$$

The components here have been taken with respect to a fixed basis and $\mathbb{C}$ is a constant 4-tensor. If ϕ is isotropic, find the most general form of ϕ and also of $\mathbb{C}$.

Solution: We know from the general representation (1.144) of an isotropic function that $\phi(\mathbf{A})$ can be written as a function of the three principal invariants: $\phi(\mathbf{A}) = \widehat{\phi}(I_1(\mathbf{A}), I_2(\mathbf{A}), I_3(\mathbf{A}))$. Observe that (i) is the most general *quadratic* polynomial function of $\mathbf{A}$, and note from (1.107) that $I_1(\mathbf{A})$ is a linear function of $\mathbf{A}$, $I_2(\mathbf{A})$ is a quadratic function of $\mathbf{A}$, and $I_3(\mathbf{A})$ is a cubic function of $\mathbf{A}$. It therefore follows that the most general isotropic quadratic polynomial function can be written as

$$\phi(\mathbf{A}) = c_1\big(I_1(\mathbf{A})\big)^2 + c_2 I_2(\mathbf{A}) \tag{ii}$$

for two constants c_1 and c_2. Since

$$I_1(\mathbf{A}) = \mathrm{tr}(\mathbf{A}) = A_{ii}, \qquad I_2(\mathbf{A}) = \frac{1}{2}[(\mathrm{tr}(\mathbf{A}))^2 - \mathrm{tr}(\mathbf{A}^2)] = \frac{1}{2}\Big(A_{ii}A_{jj} - A_{ik}A_{ki}\Big), \tag{iii}$$

(ii) can be written as

$$\phi(\mathbf{A}) = c_1 A_{ii}A_{jj} + \frac{1}{2}c_2\Big(A_{ii}A_{jj} - A_{ik}A_{ki}\Big),$$

which leads to

$$\phi(\mathbf{A}) = c_3 A_{ii}A_{jj} + c_4 A_{ik}A_{ki} = c_3\big(\mathrm{tr}\mathbf{A}\big)^2 + c_4\mathrm{tr}\big(\mathbf{A}^2\big) \qquad \square \tag{iv}$$

for two other constants c_3 and c_4. This is the most general isotropic function of the form (i).

Observe that

$$A_{ii}A_{jj} = \delta_{pq}\delta_{rs}A_{pq}A_{rs}, \qquad A_{ij}A_{ij} = \frac{1}{2}(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr})A_{pq}A_{rs}$$

and so we can write (iv) as

$$\phi(\mathbf{A}) = c_3\delta_{pq}\delta_{rs}A_{pq}A_{rs} + \frac{1}{2}c_4(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr})A_{pq}A_{rs} = \frac{1}{2}\Big[2c_3\delta_{pq}\delta_{rs} + c_4(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr})\Big]A_{pq}A_{rs}$$

which is of the form (i) with

$$\mathbb{C}_{pqrs} = \frac{1}{2}\Big[2c_3\delta_{pq}\delta_{rs} + c_4(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr})\Big]. \qquad \square$$

We will encounter this 4-tensor when studying isotropic linear elastic materials.

Problem 1.37. A symmetric tensor $\mathbf{B}$ that has the property

$$\mathbf{QBQ}^T = \mathbf{B} \qquad \text{for all orthogonal tensors } \mathbf{Q} \tag{1.204}$$

is said to be *isotropic*. Show that $\mathbf{B}$ is isotropic if and only if it has the representation $\mathbf{B} = \beta\mathbf{I}$ for some scalar β.

Problem 1.38. Let $\mathbf{F}(\mathbf{B})$ be a symmetric tensor-valued function that is defined for all symmetric tensors $\mathbf{B}$. Such a function is said to be *isotropic* if $\mathbf{F}(\mathbf{QBQ}^T) = \mathbf{QF}(\mathbf{B})\mathbf{Q}^T$ for all orthogonal tensors $\mathbf{Q}$. Show that $\mathbf{F}(\mathbf{B})$ is isotropic if and only if it has the representation

$$\widehat{\mathbf{F}}(\mathbf{B}) = \beta_2\mathbf{B}^2 + \beta_1\mathbf{B} + \beta_0\mathbf{I}, \tag{1.205}$$

where the β_j's are functions of the principal scalar invariants of $\mathbf{B}$.

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5. Calculus.

Problem 1.39. Let $\mathbf{F}(t)$ be a one-parameter family of nonsingular tensors that depend smoothly on the parameter t. Show that

(a)

$$\frac{d}{dt}(\det\mathbf{F}) = \det\mathbf{F}\,\mathrm{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1}) = \mathrm{J}\,\mathbf{F}^{-T}\cdot\dot{\mathbf{F}}, \tag{1.206}$$

where $J = \det\mathbf{F}$ and $\dot{\mathbf{F}} = d\mathbf{F}/dt$.

(b) Show also that

$$\frac{d}{dt}\left(\mathbf{F}^{-1}\right) = -\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{1.207}$$

See also Problem 1.47.

Solution:

(a) We write (1.90) as

$$\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})\ \det\mathbf{F} = \mathbf{Fx}\cdot(\mathbf{Fy}\times\mathbf{Fz}) \tag{i}$$

Differentiating (i) with respect to t and letting $\mathbf{A} = \dot{\mathbf{F}}\mathbf{F}^{-1}$

$$\begin{aligned}
\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})\ \tfrac{d}{dt}(\det\mathbf{F}) &= \dot{\mathbf{F}}\mathbf{x}\cdot(\mathbf{Fy}\times\mathbf{Fz}) + \mathbf{Fx}\cdot(\dot{\mathbf{F}}\mathbf{y}\times\mathbf{Fz}) + \mathbf{Fx}\cdot(\mathbf{Fy}\times\dot{\mathbf{F}}\mathbf{z}) = \\
&= \mathbf{AFx}\cdot(\mathbf{Fy}\times\mathbf{Fz}) + \mathbf{Fx}\cdot(\mathbf{AFy}\times\mathbf{Fz}) + \mathbf{Fx}\cdot(\mathbf{Fy}\times\mathbf{AFz}) = \\
&\overset{(1.103)}{=} (\mathrm{tr}\,\mathbf{A})\ \mathbf{Fx}\cdot(\mathbf{Fy}\times\mathbf{Fz}) \overset{(i)}{=} (\mathrm{tr}\,\mathbf{A})\ (\det\mathbf{F})\ \mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})
\end{aligned}$$

Therefore
$$\frac{d}{dt}(\det \mathbf{F}) = (\det \mathbf{F})\ (\mathrm{tr}\, \mathbf{A}) = (\det \mathbf{F})\ (\mathrm{tr}\, \dot{\mathbf{F}}\mathbf{F}^{-1}) = J\ (\mathrm{tr}\, \dot{\mathbf{F}}\mathbf{F}^{-1}) \overset{(1.120)}{=} J\mathbf{F}^{-T} \cdot \dot{\mathbf{F}}. \qquad \square$$

(b) Differentiating
$$\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$$
with respect to t gives
$$\dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}\frac{d}{dt}(\mathbf{F}^{-1}) = \mathbf{0}$$
which yields the desired result (1.207). $\square$

Problem 1.40. Let ϕ, $\mathbf{v}$, and $\mathbf{w}$, be a scalar field and two vector fields, respectively. Show that:

(a) $\mathsf{div}(\phi\mathbf{v}) = \phi\,\mathsf{div}\mathbf{v} + \mathbf{v}\cdot\mathsf{grad}\phi$,

(b) $\mathsf{grad}(\phi\mathbf{v}) = \phi\,\mathsf{grad}\mathbf{v} + \mathbf{v}\otimes\mathsf{grad}\phi$,

(c) $\mathsf{grad}(\mathbf{v}\cdot\mathbf{w}) = (\boldsymbol{\nabla}\mathbf{w})^T\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T\mathbf{w}$,

(d) $\mathsf{div}(\mathbf{v}\otimes\mathbf{w}) = (\mathsf{div}\,\mathbf{w})\mathbf{v} + (\mathsf{div}\,\mathbf{v})\mathbf{w}$.

Problem 1.41. (Gradient in spherical polar coordinates.) Calculate the gradient of a scalar field and the gradient and divergence of a vector field in spherical polar coordinates (R, Θ, Φ) with associated basis vectors $\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi$ as defined by equations (2.80) and (2.81) on page 168.

Problem 1.42. (Localization) "Localization" refers to deriving local equations (i.e. equations that hold at each point in a region $\mathcal{R}$) from a global statement in integral form. In Section 1.8.3 we encountered one circumstance in which localization is possible. Here we look at a second.

Let $\phi(\mathbf{x})$ be a continuous scalar-valued function defined for all $\mathbf{x} \in \mathcal{R}$. Suppose that
$$\int_{\mathcal{R}} \phi(\mathbf{x})\psi(\mathbf{x})\ dV = 0$$
for *all* continuous functions $\psi(\mathbf{x})$ defined on $\mathcal{R}$. Show that this implies $\phi(\mathbf{x}) = 0$ at all points in $\mathcal{R}$.

Remark: In Section 1.8.3 the global statement held for *all* subregions of $\mathcal{R}$; in contrast here, we have a single integral statement that hold on $\mathcal{R}$. On the other hand here, the integrand involves an arbitrary function ψ while such a function was absent in Section 1.8.3.

Problem 1.43. (Weak and strong forms.) Let $\mathbf{A}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ be smooth tensor and vector fields respectively defined on some regular region $\mathcal{R}$. Suppose that

$$\int_{\partial\mathcal{R}} \mathbf{A}\mathbf{n}\cdot\mathbf{w}\,dA + \int_{\mathcal{R}} \mathbf{b}\cdot\mathbf{w}\,dV = \int_{\mathcal{R}} \mathbf{A}\cdot\nabla\mathbf{w}\,dV, \tag{i}$$

for *all* smooth vector fields $\mathbf{w}(\mathbf{x})$. Here $\partial\mathcal{R}$ is the boundary of $\mathcal{R}$. Show by localization that

$$\operatorname{div}\mathbf{A} + \mathbf{b} = \mathbf{o} \qquad \text{at each } \mathbf{x}\in\mathcal{R}. \tag{ii}$$

One often speaks of (i) as being the *weak form* of the differential statement (ii), and (ii) as being the *strong form* of the integral statement (i). Note that (i) does not require $\mathbf{A}(\mathbf{x})$ to be differentiable while (ii) does.

Solution: By the definition $\mathbf{A}\cdot\mathbf{B} = A_{ij}B_{ij}$ of the scalar product of two tensors,

$$\mathbf{A}\cdot\nabla\mathbf{w} = A_{ij}(\nabla\mathbf{w})_{ij} \overset{(1.164)}{=} A_{ij}\,\frac{\partial w_i}{\partial x_j}. \tag{iii}$$

We can now write (i) in terms of components as

$$\int_{\partial\mathcal{R}} A_{ij}n_jw_i\,dA + \int_{\mathcal{R}} b_iw_i\,dV = \int_{\mathcal{R}} A_{ij}\,\frac{\partial w_i}{\partial x_j}\,dV. \tag{iv}$$

By using the divergence theorem (1.178) we can write the first term in (iv) as

$$\int_{\partial\mathcal{R}} A_{ij}w_in_j\,dA = \int_{\mathcal{R}} \frac{\partial}{\partial x_j}(A_{ij}w_i)\,dV = \int_{\mathcal{R}} \left[\frac{\partial A_{ij}}{\partial x_j}\,w_i\; + \frac{\partial w_i}{\partial x_j}\,A_{ij}\right] dV. \tag{v}$$

Substituting (v) into (iv) gives

$$\int_{\mathcal{R}} \left[\frac{\partial A_{ij}}{\partial x_j} + b_i\right] w_i\; dV = 0.$$

We are told that this holds for all smooth vector fields $\mathbf{w}(\mathbf{x})$ and so the (natural generalization of the) localization theorem in Problem 1.42 allows us to conclude that

$$\frac{\partial A_{ij}}{\partial x_j} + b_i = 0 \qquad \text{at each } \mathbf{x}\in\mathcal{R}.$$

When written in basis-free notation (and keeping (1.168) in mind) this yields (ii).

In Chapter 3 we will find that (ii) is the equilibrium equation for a certain stress tensor $\mathbf{A} = \mathbf{S}$ with (i) being a statement of the *principle of virtual work*.

Problem 1.44. (Divergence theorem.) Let $\mathbf{u}(\mathbf{x})$ be a smooth vector field defined on some region $\mathcal{R}$. Suppose that

$$\int_{\partial\mathcal{D}} \mathbf{u}(\mathbf{x})\otimes\mathbf{n}(\mathbf{x})\;\mathrm{d}A \;=\; \mathbf{0} \qquad \text{for all subregions } \mathcal{D}\subset\mathcal{R}, \tag{i}$$

where $\mathbf{n}(\mathbf{x})$ is the unit outward normal vector at a point $\mathbf{x}$ on the boundary $\partial\mathcal{D}$. Show by using the divergence theorem and localization that (i) holds if and only if

$$\boldsymbol{\nabla}\mathbf{u} = \mathbf{0} \qquad \text{at each } \mathbf{x}\in\mathcal{R}. \tag{ii}$$

Problem 1.45. (Divergence theorem. Localization.)

(a) Let $\mathbf{S}(\mathbf{x})$ be a continuously differentiable tensor field on $\mathcal{R}$ with the property

$$\int_{\partial\mathcal{D}} \mathbf{S}(\mathbf{x})\mathbf{n}(\mathbf{x})\ \mathrm{d}A \ = \ \mathbf{o} \qquad \text{for all subregions } \mathcal{D}\subset\mathcal{R}, \tag{i}$$

where $\mathbf{n}(\mathbf{x})$ is the unit outward normal vector at a point $\mathbf{x}$ on the boundary $\partial\mathcal{D}$. Show by using the divergence theorem and localization that (i) implies

$$\mathsf{div}\ \mathbf{S} = \mathbf{o} \qquad \text{at each } \mathbf{x}\in\mathcal{R}. \tag{ii}$$

(b) Conversely, if $\mathbf{S}(\mathbf{x})$ is a tensor field such that (ii) holds, show that then (i) holds.

(c) Suppose that $\mathbf{S}(\mathbf{x})$ is a smooth tensor field that obeys (ii). Show that

$$\int_{\mathcal{D}} \mathbf{Sn}\cdot\mathbf{w}\, dA = \int_{\mathcal{D}} \mathbf{S}\cdot\nabla\mathbf{w}\, dV$$

for any smooth vector field $\mathbf{w}$.

Solution:

(a) In terms of components in a fixed basis, we are told that

$$\int_{\partial\mathcal{D}} S_{ij}n_j\ \mathrm{d}A \ = \ 0 \qquad \text{for all subregions } \mathcal{D}\subset\mathcal{R}. \tag{iv}$$

By using the divergence theorem (1.178), this implies that

$$\int_{\mathcal{D}} S_{ij,j}\ \mathrm{d}V \ = \ 0 \qquad \text{for all subregions } \mathcal{D}\subset\mathcal{R}.$$

We are told that $S_{ij,j}$ is continuous on $\mathcal{R}$, and so the localization result established in Problem 1.8.3 allows us to conclude that

$$S_{ij,j} \ = \ 0 \qquad \text{at all } \mathbf{x}\in\mathcal{R}. \tag{v}$$

(b) Now we start with $S_{ij,j}=0$. Integrating this over D and using the divergence theorem gives (iv).

Problem 1.46. (Divergence theorem. Localization.)

Reconsider the tensor field $\mathbf{S}(\mathbf{x})$ introduced in part (a) of Problem 1.45. Suppose that in addition to equation (i) there, $\mathbf{S}(\mathbf{x})$ also has the property that

$$\int_{\partial\mathcal{D}} \mathbf{x}\times\mathbf{S}(\mathbf{x})\mathbf{n}(\mathbf{x})\ \mathrm{d}A \ = \ \mathbf{o} \qquad \text{for all subregions } \mathcal{D}\subset\mathcal{R}. \tag{iii}$$

Show that if (i) and (iii) hold, then

$$\mathbf{S}=\mathbf{S}^T \qquad \text{at each } \mathbf{x}\in\mathcal{R}. \tag{iv}$$

Hint: Use the divergence theorem, localization and equation (ii) of Problem 1.45.

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6. Functions of a tensor.

Problem 1.47. The function $\mathrm{J}(\mathbf{F}) = \det \mathbf{F}$ is defined for all nonsingular tensors $\mathbf{F}$. Show that

$$\frac{\partial \mathrm{J}}{\partial \mathbf{F}} = \mathrm{J}\mathbf{F}^{-\mathrm{T}}. \tag{1.208}$$

Problem 1.48. Consider the function $W(\mathbf{F})$ defined for all nonsingular tensors $\mathbf{F}$ by $W(\mathbf{F}) = \widehat{W}(I_1(\mathbf{C}))$ where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $I_1(\mathbf{C}) = \operatorname{tr}\mathbf{C}$. Calculate the components of the 4-tensor

$$\frac{\partial^2 W(\mathbf{F})}{\partial F_{ij}\partial F_{k\ell}}.$$

Solution: First note that

$$\frac{\partial F_{ij}}{\partial F_{k\ell}} = \delta_{ik}\delta_{j\ell}, \tag{i}$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \qquad \Rightarrow \qquad C_{ij} = F_{ki}F_{kj}, \tag{ii}$$

$$I_1 = \operatorname{tr}\mathbf{C} = C_{ii} = F_{ki}F_{ki}. \tag{iii}$$

Therefore

$$\frac{\partial I_1}{\partial F_{pq}} \overset{(iii)}{=} \frac{\partial}{\partial F_{pq}}(F_{ki}F_{ki}) \overset{(i)}{=} \delta_{kp}\delta_{iq}F_{ki} + F_{ki}\delta_{kp}\delta_{iq} = F_{pq} + F_{pq} = 2F_{pq}, \tag{iv}$$

and so

$$\frac{\partial W(I_1)}{\partial F_{k\ell}} = W'(I_1)\frac{\partial I_1}{\partial F_{k\ell}} \overset{(iv)}{=} 2W'(I_1)F_{k\ell}. \tag{v}$$

Thus

$$\begin{aligned}\frac{\partial^2 W(I_1)}{\partial F_{ij}\partial F_{k\ell}} = \frac{\partial}{\partial F_{ij}}\left(\frac{\partial W(I_1)}{\partial F_{k\ell}}\right) \quad &\overset{(v)}{=} \frac{\partial}{\partial F_{ij}}(2W'(I_1)F_{k\ell}) \overset{(i)}{=} 2W''(I_1)\frac{\partial I_1}{\partial F_{ij}}F_{k\ell} + 2W'(I_1)\delta_{ki}\delta_{\ell j} = \\ &\overset{(iv)}{=} 4W''(I_1)F_{ij}F_{k\ell} + 2W'(I_1)\delta_{ki}\delta_{\ell j}, \qquad \square\end{aligned}$$

where a prime denotes differentiation with respect to the argument.

Problem 1.49. The scalar valued function $\widehat{W}(\mathbf{F})$ is defined for all nonsingular tensors $\mathbf{F}$, and the scalar valued function $\overline{W}(\mathbf{C})$ is defined for all symmetric positive definite tensors $\mathbf{C}$. Suppose that these two functions are related by

$$\widehat{W}(\mathbf{F}) = \overline{W}(\mathbf{C}) \qquad \text{where} \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{i}$$

Furthermore, suppose the tensors $\mathbf{S}$, $\mathbf{T}$ and $\mathbf{F}$ are related by

$$\mathbf{S} = \frac{\partial \widehat{W}}{\partial \mathbf{F}}(\mathbf{F}), \qquad \mathbf{T} = \frac{1}{J}\mathbf{S}\mathbf{F}^T \quad \text{where} \quad J = \det \mathbf{F}. \tag{ii}$$

Derive an expression for $\mathbf{T}$ in terms of $\partial\overline{W}(\mathbf{C})/\partial\mathbf{C}$ and $\mathbf{F}$. Specialize it to the case where

$$\overline{W}(\mathbf{C}) = \operatorname{tr}\mathbf{C}. \tag{iii}$$

Solution: From the chain rule and $(i)_1$,

$$\frac{\partial \widehat{W}}{\partial F_{ij}} = \frac{\partial \overline{W}}{\partial C_{pq}} \frac{\partial C_{pq}}{\partial F_{ij}}. \tag{iv}$$

We now calculate the last term in (iv) by differentiating $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ with respect to $\mathbf{F}$:

$$\frac{\partial C_{pq}}{\partial F_{ij}} = \frac{\partial}{\partial F_{ij}}(F^T_{pk}F_{kq}) = \frac{\partial}{\partial F_{ij}}(F_{kp}F_{kq}) = \delta_{ki}\delta_{pj}F_{kq} + F_{kp}\delta_{ki}\delta_{qj} = \delta_{pj}F_{iq} + F_{ip}\delta_{qj}.$$

Substituting this back into (iv) gives

$$\frac{\partial \widehat{W}}{\partial F_{ij}} = \delta_{pj}F_{iq}\frac{\partial \overline{W}}{\partial C_{pq}} + F_{ip}\delta_{qj}\frac{\partial \overline{W}}{\partial C_{pq}} = F_{iq}\frac{\partial \overline{W}}{\partial C_{jq}} + F_{ip}\frac{\partial \overline{W}}{\partial C_{pj}} \stackrel{(*)}{=} F_{iq}\frac{\partial \overline{W}}{\partial C_{qj}} + F_{iq}\frac{\partial \overline{W}}{\partial C_{qj}} = 2F_{iq}\frac{\partial \overline{W}}{\partial C_{qj}}$$

where in step $(*)$, we used the symmetry of $\mathbf{C}$ to write $C_{qj} = C_{jq}$ in the first term, and changed the dummy subscript p to q in the second term. Thus we have

$$\frac{\partial \widehat{W}}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial \overline{W}}{\partial \mathbf{C}}. \tag{v}$$

Substituting this into $(ii)_1$ and the result into $(ii)_2$ gives

$$\mathbf{T} = \frac{2}{J}\mathbf{F}\frac{\partial \overline{W}}{\partial \mathbf{C}}\mathbf{F}^T. \qquad \square \tag{vi}$$

Now suppose that $\overline{W}(\mathbf{C}) = \operatorname{tr}\mathbf{C}$. Then

$$\frac{\partial}{\partial C_{ij}}(\operatorname{tr}\mathbf{C}) = \frac{\partial}{\partial C_{ij}}(C_{kk}) = \delta_{ki}\delta_{kj} = \delta_{ij} \qquad \Rightarrow \qquad \frac{\partial}{\partial \mathbf{C}}(\operatorname{tr}\mathbf{C}) = \mathbf{I}.$$

Substituting this into (vi) gives

$$\mathbf{T} = \frac{2}{J}\mathbf{F}\mathbf{I}\mathbf{F}^T = \frac{2}{J}\mathbf{F}\mathbf{F}^T. \tag{vii}$$

Problem 1.50. Let $\mathbb{C}$ be a constant 4-tensor and suppose that the function $\widehat{W}(\mathbf{E})$ is defined for all symmetric 2-tensors $\mathbf{E}$ by

$$\widehat{W}(\mathbf{E}) = W(E_{11}, E_{12}, ...E_{33}) = \frac{1}{2}\,\mathbb{C}_{ijk\ell}E_{ij}E_{k\ell},$$

where $\mathbb{C}_{ijkl}$ and E_{ij} are the components of $\mathbb{C}$ and $\mathbf{E}$ in some fixed basis. Calculate

$$\frac{\partial W}{\partial E_{ij}} \qquad \text{and} \qquad \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}}.$$

Following the discussion in Section 1.8.4, ensure that these derivative have the proper symmetries.

Remark: We will encounter this function W later in the linear theory of elasticity where it will correspond to the strain energy density at a point in the body with $\mathbf{E}$ being the infinitesimal strain tensor and $\mathbb{C}$ a tensor of elastic moduli.

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7. Additional problems.

Problem 1.51. (Ball and James, 1987, DOI:10.1007/BF00281246.) Show that necessary and sufficient for a symmetric tensor $\mathbf{C}$ with eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3$ to be expressible in the form

$$\mathbf{C} = (\mathbf{I} + \mathbf{m} \otimes \mathbf{b})(\mathbf{I} + \mathbf{b} \otimes \mathbf{m}) \tag{i}$$

for linearly independent vectors $\mathbf{b}$ and $\mathbf{m}$ are that

$$\lambda_1 \le 1, \qquad \lambda_2 = 1, \qquad \lambda_3 \ge 1. \tag{ii}$$

Problem 1.52. Consider an N-dimensional Euclidean vector space. In this problem we consider an N-dimensional vector space (rather than a 3-dimensional one) since we will use this result later for the 9-dimensional vector space Lin of all linear transformations. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be two non-null vectors. If $\mathbf{a}_2$ is perpendicular to all vectors perpendicular to $\mathbf{a}_1$, show that $\mathbf{a}_2$ is parallel to $\mathbf{a}_1$.

Solution: Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ with $\mathbf{a}_1$ parallel to, say, the basis vector $\mathbf{e}_N$. Then $\mathbf{a}_1 = \alpha_1 \mathbf{e}_N$ for some nonzero scalar α_1. The set of all vectors perpendicular to $\mathbf{a}_1$ is now the set spanned by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{N-1}\}$. Since the vector $\mathbf{a}_2$ is perpendicular to all vectors perpendicular to $\mathbf{a}_1$, it must be perpendicular to each of these $N-1$ vectors and so it must be of the form $\mathbf{a}_2 = \alpha_2 \mathbf{e}_N$. This shows that $\mathbf{a}_2$ is parallel to $\mathbf{a}_1$.

An alternative proof that doesn't rely on a basis is the following where we proceed in three steps:

(a) First we show that corresponding to *any* two non-null vectors $\mathbf{a}_1, \mathbf{a}_2$ there is a scalar α and a vector $\mathbf{n}$ perpendicular to $\mathbf{a}_1$ such that

$$\mathbf{a}_2 = \alpha \mathbf{a}_1 + \mathbf{n}, \qquad \mathbf{n} \cdot \mathbf{a}_1 = 0. \tag{i}$$

(b) Then we show that the representation (i) is unique.

(c) Finally we use (a) and (b) to establish the desired result.

(a) Given any $\mathbf{a}_1 \ne \mathbf{o}$ and $\mathbf{a}_2$, define α and $\mathbf{n}$ by

$$\alpha = \frac{\mathbf{a}_2 \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1}, \qquad \mathbf{n} = \mathbf{a}_2 - \alpha \mathbf{a}_1. \tag{ii}$$

Then $\mathbf{n} \cdot \mathbf{a}_1 = \mathbf{a}_2 \cdot \mathbf{a}_1 - \alpha \mathbf{a}_1 \cdot \mathbf{a}_1 = \mathbf{o}$ which establishes (i).

(b) Suppose this representation is not unique. Then there exist a scalar β and vector $\mathbf{m}$ such that

$$\mathbf{a}_2 = \beta \mathbf{a}_1 + \mathbf{m}, \qquad \mathbf{m} \cdot \mathbf{a}_1 = 0. \tag{iii}$$

Subtracting (iii) from (i) gives

$$(\alpha - \beta)\mathbf{a}_1 + (\mathbf{n} - \mathbf{m}) = \mathbf{o}. \tag{iv}$$

Taking the scalar product of this with $\mathbf{a}_1$, and then using $(i)_2$ and $(iii)_2$ shows that $\beta = \alpha$. Then (iv) reduces to $\mathbf{m} = \mathbf{n}$. Thus if both (i) and (iii) hold then necessarily $\beta = \alpha$ and $\mathbf{m} = \mathbf{n}$. Thus the representation (i) is unique.

(c) By the result above

$$\mathbf{a}_2 = \alpha\mathbf{a}_1 + \mathbf{n}, \qquad \mathbf{a}_1 \cdot \mathbf{n} = 0. \tag{v}$$

Now suppose that $\mathbf{a}_2 \cdot \mathbf{x} = 0$ for all $\mathbf{x}$ for which $\mathbf{a}_1 \cdot \mathbf{x} = 0$. By $(v)_2$, one such $\mathbf{x}$ is $\mathbf{x} = \mathbf{n}$. Therefore it follows that $\mathbf{a}_2 \cdot \mathbf{n} = 0$. Thus taking the scalar product of $(v)_1$ with $\mathbf{n}$ yields $\mathbf{n} \cdot \mathbf{n} = 0$. Consequently $\mathbf{n} = \mathbf{o}$ and hence $\mathbf{a}_2 = \alpha\mathbf{a}_1$.

Problem 1.53. (Projection tensor.) The "**projection tensor**" $\mathbf{P}$ projects vectors onto a given plane $\mathcal{P}$. It takes any vector $\mathbf{v} \in \mathsf{V}$ into the vector $\mathbf{P}\mathbf{v} \in \mathcal{P}$ as illustrated geometrically in Figure 1.13. Determine $\mathbf{P}$. Show that $\mathbf{P}$ is singular.

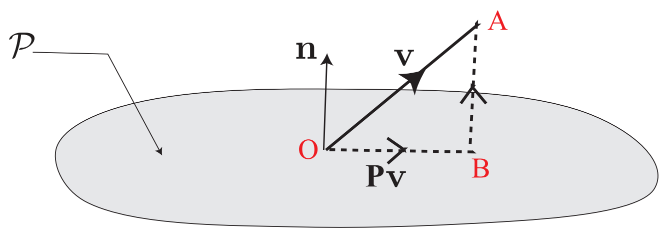

Figure 1.13: The projection $\mathbf{P}\mathbf{v}$ of a vector $\mathbf{v}$ onto the plane $\mathcal{P}$.

Solution: Let $\mathbf{v}$ be an arbitrary vector: $\overrightarrow{OA} = \mathbf{v}$. Its image after projections is $\overrightarrow{OB} = \mathbf{P}\mathbf{v}$. Let $\mathbf{n}$ be a unit vector normal to the plane $\mathcal{P}$. Observe from Figure 1.13 that the vector $\overrightarrow{BA}$ has magnitude $\mathbf{v} \cdot \mathbf{n}$ and is in the direction $\mathbf{n}$. Thus

$$\overrightarrow{BA} = (\mathbf{v} \cdot \mathbf{n})\mathbf{n}.$$

Consequently

$$\mathbf{P}\mathbf{v} \ = \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} - \overrightarrow{BA} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}.$$

Thus

$$\mathbf{P}\mathbf{v} \ = \ \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n} \qquad \text{for all vectors} \quad \mathbf{v} \in \mathsf{V}. \tag{1.209}$$

This completely defines $\mathbf{P}$ since it tells us how it operates on an arbitrary vector $\mathbf{v}$.

Remark: Note that $\mathbf{P}$ is a *linear* operator since $\mathbf{P}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{P}\mathbf{x} + \beta\mathbf{P}\mathbf{y}$. Thus it is a linear transformation (tensor).

Remark: Recall that the tensor product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the tensor, denoted by $\mathbf{a} \otimes \mathbf{b}$, that has the property $(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}$ for all vectors $\mathbf{v} \in \mathsf{V}$. Therefore we can express the projection tensor defined by (1.209) equivalently as

$$\mathbf{P} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}. \tag{1.210}$$

Remark: Observe from (1.209) or (1.210) or geometrically from Figure 1.13 that $\mathbf{P}\mathbf{n} = \mathbf{o}$. Since $\mathbf{n} \neq \mathbf{o}$ it follows from (1.85) that $\mathbf{P}$ is singular.

Remark: The components P_{ij} of $\mathbf{P}$ in a basis will be calculated in Problem 1.54 .

Problem 1.54. (Projection tensor.) Determine the components in an *arbitrary* basis of the projection tensor $\mathbf{P}$ introduced in Problem 1.53.

Solution: If P_{ij}, x_i, n_i are the components of $\mathbf{P}, \mathbf{x}$ and $\mathbf{n}$ in an arbitrary basis then (1.209) $\Leftrightarrow$

$$P_{ij}x_j \;=\; x_i - x_j n_j n_i = (\delta_{ij} - n_i n_j)x_j.$$

Since this must hold for all vectors $\mathbf{x} \in \mathsf{V}$ it follows that

$$P_{ij} \;=\; \delta_{ij} - n_i n_j.$$

Alternatively: Since $\mathbf{Px} = \mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}$ for all vectors $\mathbf{x} \in \mathsf{V}$, it holds for the particular choice $\mathbf{x} = \mathbf{e}_j$:

$$\mathbf{Pe}_j = \mathbf{e}_j - (\mathbf{e}_j \cdot \mathbf{n})\mathbf{n} = \mathbf{e}_j - n_j\,\mathbf{n}.$$

Take the dot product of this equation with $\mathbf{e}_i$:

$$\mathbf{Pe}_j \cdot \mathbf{e}_i = \mathbf{e}_j \cdot \mathbf{e}_i - n_j\,\mathbf{n} \cdot \mathbf{e}_i = \delta_{ij} - n_j n_i.$$

By the definition (1.128) of the components of a tensor, $P_{ij} = \mathbf{Pe}_j \cdot \mathbf{e}_i$.

Problem 1.55. (Skew-symmetric tensor. Axial vector.) Let $\mathbf{W}$ be a skew-symmetric tensor, i.e.

$$\mathbf{W} = -\mathbf{W}^T. \tag{i}$$

Show that there is a vector $\mathbf{w}$ such that

$$\mathbf{Wx} = \mathbf{w} \times \mathbf{x} \qquad \text{for all vectors } \mathbf{x} \in \mathsf{V}. \tag{ii}$$

In terms of components in a basis, show that

$$w_i = -\frac{1}{2} e_{ijk} W_{jk}. \tag{iii}$$

Remark: Recall from the comment below (1.7) that any skew-symmetric matrix $[W]$ has *only three independent elements*, e.g. W_{12}, W_{23}, W_{31}. This is because the elements on the diagonal of a skew-symmetric matrix $[W]$ are zero and $W_{12} = -W_{21}, W_{23} = -W_{32}$ and $W_{31} = -W_{13}$. It is not surprising therefore that one can associate a vector $\mathbf{w}$ (which has three independent components) with each skew-symmetric tensor $\mathbf{W}$.

Problem 1.56. (Rotation tensor.) Consider the rotation tensor $\mathbf{Q}$ introduced in (1.192) (page 85).

(a) Determine the components of $\mathbf{Q}$ in an *arbitrary* basis.

(b) Verify that $\mathbf{Q}$ is proper orthogonal.

(c) Show that

$$\mathsf{trace}\,\mathbf{Q} = 1 + 2\cos\theta \tag{i}$$

and

$$\sin\theta\, n_i = -\frac{1}{2} e_{ipq} Q_{pq}. \tag{ii}$$

Therefore, given a proper orthogonal $\mathbf{Q}$, one can find the rotation angle θ from (i) and thereafter the rotation axis $\mathbf{n}$ from (ii).

(d) Suppose that the axis of rotation coincides with one of the basis vectors, say $\mathbf{n} = \mathbf{e}_3$. Specialize your answer to part (a) and display the matrix $[Q]$.

Solution:

(a) If Q_{ij}, x_i, n_i are the components of $\mathbf{Q}, \mathbf{x}$ and $\mathbf{n}$ in an arbitrary basis then in view of (1.58) and (1.61) we can write (1.192) as

$$\begin{aligned} Q_{ij}x_j &= \cos\theta\, x_i + (1-\cos\theta)(n_j x_j) n_i + \sin\theta\, e_{ijk} n_j x_k = \\ &= \big(\cos\theta\,\delta_{ij} + (1-\cos\theta) n_i n_j + \sin\theta\, e_{ikj} n_k\big) x_j = \\ &= \big(\cos\theta\,\delta_{ij} + (1-\cos\theta) n_i n_j - \sin\theta\, e_{ijk} n_k\big) x_j, \end{aligned}$$

where in going to the second line we used the substitution rule; and in going to the last line we used the fact the alternator changes sign if any two adjacent subscripts are switched. If we take all of the terms to one side this can be written in the form $A_{ij}x_j = 0$, which because it hold for all x_i, implies $A_{ij} = 0$. This leads to

$$Q_{ij} = \cos\theta\,\delta_{ij} + (1-\cos\theta) n_i n_j - \sin\theta\, e_{ijk} n_k, \tag{iii}$$

which are the components of $\mathbf{Q}$ in a basis.

(b) Here we must show that Q_{ij} given in (iii) obeys the orthogonality condition $Q_{ki}Q_{kj} = \delta_{ij}$ (i.e. $[Q]^T[Q] = [I]$). Exercise.

(c) To calculate the trace of $\mathbf{Q}$ we set $i = j$ in (iii):

$$Q_{ii} = \cos\theta\,\delta_{ii} + (1-\cos\theta) n_i n_i - \sin\theta\, e_{iik} n_k = 3\cos\theta\; + (1-\cos\theta) = 1 + 2\cos\theta.$$

where in the second step we have used (a) $\delta_{ii} = 3$, (b) $\mathbf{n}$ is a unit vector whence $n_i n_i = 1$ and (c) e_{ijk} vanishes if two subscripts are equal.

To get (ii) we multiply Q_{pq} in (iii) by e_{ipq}:

$$e_{ipq}Q_{pq} = \cos\theta\,\delta_{pq} e_{ipq} + (1-\cos\theta) n_p n_q e_{ipq} - \sin\theta\, e_{pqk} e_{ipq} n_k.$$

The first two groups of terms on the right-hand side vanish because of the result of Problem 1.2.3. We simplify the last term using $e_{ijk} = -e_{jik}$, the identity (1.43) and the substitution rule:

$$e_{ipq}Q_{pq} = -\sin\theta e_{pqk} e_{ipq} n_k = \sin\theta e_{pqk} e_{piq} n_k = (\delta_{qi}\delta_{kq} - \delta_{qq}\delta_{ki}) \sin\theta n_k = (n_i - 3n_i)\sin\theta = -2\sin\theta,$$

and so (ii) follows.

(d) Suppose that $\mathbf{n} = \mathbf{e}_3$. Then $n_k = \delta_{k3}$ and so (iii) yields

$$Q_{ij} = \cos\theta\ \delta_{ij} + (1-\cos\theta)\delta_{i3}\delta_{j3} - \sin\theta\ e_{ij3}.$$

If we evaluate each term Q_{ij} we find that

$$[Q] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 1.57. (Rotation tensor.) The tensor $\mathbf{Q}$ is proper orthogonal.

(a) Show that $\det(\mathbf{Q} - \mathbf{I}) = 0$ and hence deduce that unity is an eigenvalue of $\mathbf{Q}$.

(b) Let $\mathbf{a}$ be the eigenvector of $\mathbf{Q}$ corresponding to the eigenvalue $\lambda = 1$. Consider an orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. Show that $\mathbf{Qb}$ and $\mathbf{Qc}$ are perpendicular to $\mathbf{a}$ and therefore lie in the plane spanned by $\mathbf{b}$ and $\mathbf{c}$. Hence deduce that for some angle θ,

$$\mathbf{Qb} = \cos\theta\mathbf{b} + \sin\theta\mathbf{c}, \qquad \mathbf{Qc} = -\sin\theta\mathbf{b} + \cos\theta\mathbf{c}.$$

(c) Show that $\mathbf{Q}$ has the representation

$$\mathbf{Q} = \mathbf{a}\otimes\mathbf{a} + \cos\theta(\mathbf{b}\otimes\mathbf{b} + \mathbf{c}\otimes\mathbf{c}) + \sin\theta(\mathbf{c}\otimes\mathbf{b} - \mathbf{b}\otimes\mathbf{c}). \qquad (i)$$

Hence show that (same as (1.192))

$$\mathbf{Qx} = \cos\theta\ \mathbf{x} + (1-\cos\theta)(\mathbf{a}\cdot\mathbf{x})\mathbf{a} + \sin\theta\ (\mathbf{a}\times\mathbf{x}),$$

for any vector $\mathbf{x}$.

(d) Calculate the principal scalar invariants of $\mathbf{Q}$. Hint: use (1.90), (1.103) and (1.196).

(e) Show that $\mathbf{Q}$ has no real eigenvalues other than unity. Hint: use the result from (d).

Solution:

(a) Since $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ it follows that $\mathbf{Q}^T(\mathbf{Q}-\mathbf{I}) = -(\mathbf{Q}^T - \mathbf{I})$ and taking the determinant of both sides and using $\det\mathbf{Q} = 1$ and $\det\mathbf{A}^T = \det\mathbf{A}$, one finds

$$\det(\mathbf{Q}-\mathbf{I}) = -\det(\mathbf{Q}^T - \mathbf{I}) = -\det(\mathbf{Q}-\mathbf{I}) \qquad \Rightarrow \qquad \det(\mathbf{Q}-\mathbf{I}) = 0.$$

It follows that $\lambda = 1$ is an eigenvalue of $\mathbf{Q}$ (since the eigenvalues of $\mathbf{Q}$ are given by $\det(\mathbf{Q} - \lambda\mathbf{I}) = 0$).

(b) Let $\mathbf{a}$ be the eigenvector corresponding to the eigenvalue unity. Then

$$\mathbf{Qa} = \mathbf{a}. \qquad (ii)$$

Note from (iii) that $\mathbf{Q}^T\mathbf{a} = \mathbf{a}$. Thus

$$0 = \mathbf{b}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{Q}^T\mathbf{a} = \mathbf{Q}\mathbf{b}\cdot\mathbf{a},$$

and so $\mathbf{Qb}$ is perpendicular to $\mathbf{a}$. Similarly $\mathbf{Qc}$ is also perpendicular to $\mathbf{a}$. Therefore $\mathbf{Qb}$ and $\mathbf{Qc}$ lie in the plane spanned by $\mathbf{b}$ and $\mathbf{c}$ and therefore they can be expressed as linear combinations of the vectors $\mathbf{b}$ and $\mathbf{c}$. Since $\mathbf{Q}$ is a rotation this representation is, for some angle θ,

$$\mathbf{Qb} = \cos\theta\mathbf{b} + \sin\theta\mathbf{c}, \qquad \mathbf{Qc} = -\sin\theta\mathbf{b} + \cos\theta\mathbf{c}. \tag{iii}$$

(c) Since $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ forms an orthonormal basis, the three equations in (ii) and (iii) completely defines $\mathbf{Q}$. In particular, the coefficients on the right-hand sides of those equations are the components of $\mathbf{Q}$ in the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and so (i) follows.

(d) The principal scalar invariants of $\mathbf{Q}$ can be determined from (i) and (1.90), (1.103), (1.196) to be

$$I_1(\mathbf{Q}) = \mathrm{tr}\mathbf{Q} = 1 + 2\cos\theta, \quad I_2(\mathbf{Q}) = \frac{1}{2}\left[(\mathrm{tr}\mathbf{Q})^2 - \mathrm{tr}\mathbf{Q}^2\right] = 1 + 2\cos\theta, \quad I_3(\mathbf{Q}) = \det\mathbf{Q} = 1. \tag{iv}$$

e) The eigenvalues of $\mathbf{Q}$ are given by the roots of the characteristic equation

$$\lambda^3 - I_1(\mathbf{Q})\lambda^2 + I_2(\mathbf{Q})\lambda - I_3(\mathbf{Q}) = 0 \tag{v}$$

Substituting (iv) into (v) leads to

$$(\lambda - 1)(\lambda^2 - 2\cos\theta\lambda + 1) = 0,$$

the only real root of which is $\lambda = 1$.

Problem 1.58. (Trace of a tensor.) The trace of a tensor was defined in (1.103). Show that the right-hand side of that formula, i.e.

$$\frac{\mathbf{Ax}\cdot(\mathbf{y}\times\mathbf{z}) + \mathbf{x}\cdot(\mathbf{Ay}\times\mathbf{z}) + \mathbf{x}\cdot(\mathbf{y}\times\mathbf{Az})}{\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})}, \tag{i}$$

is independent of the choice of the linearly independent vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

Solution: Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an arbitrary orthonormal basis and let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be arbitrary linearly independent vectors with $\mathbf{x} = x_i\mathbf{e}_i, \mathbf{y} = y_i\mathbf{e}_i$ and $\mathbf{z} = z_i\mathbf{e}_i$. Then

$$\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z}) = x_i\mathbf{e}_i\cdot(y_j\mathbf{e}_j\times z_k\mathbf{e}_k) = x_iy_jz_k\ \mathbf{e}_i\cdot(\mathbf{e}_j\times\mathbf{e}_k) \overset{(1.54)}{=} x_iy_jz_k\ e_{ijk} \tag{ii}$$

and

$$\begin{aligned}
&\mathbf{Ax}\cdot(\mathbf{y}\times\mathbf{z}) + \mathbf{x}\cdot(\mathbf{Ay}\times\mathbf{z}) + \mathbf{x}\cdot(\mathbf{y}\times\mathbf{Az}) = \\
&= x_iy_jz_k\mathbf{Ae}_i\cdot(\mathbf{e}_j\times\mathbf{e}_k) + x_iy_jz_k\mathbf{e}_i\cdot(\mathbf{Ae}_j\times\mathbf{e}_k) + x_iy_jz_k\mathbf{e}_i\cdot(\mathbf{e}_j\times\mathbf{Ae}_k) = \\
&= x_iy_jz_k\Big[\mathbf{Ae}_i\cdot(\mathbf{e}_j\times\mathbf{e}_k) + \mathbf{e}_i\cdot(\mathbf{Ae}_j\times\mathbf{e}_k) + \mathbf{e}_i\cdot(\mathbf{e}_j\times\mathbf{Ae}_k)\Big] = \\
&\overset{(1.38),(1.54)}{=} x_iy_jz_k\ e_{ijk}\Big[\mathbf{Ae}_1\cdot(\mathbf{e}_2\times\mathbf{e}_3) + \mathbf{e}_1\cdot(\mathbf{Ae}_2\times\mathbf{e}_3) + \mathbf{e}_1\cdot(\mathbf{e}_2\times\mathbf{Ae}_3)\Big]
\end{aligned} \tag{iii}$$

It follows from (ii) and (iii) that

$$\frac{\mathbf{Ax}\cdot(\mathbf{y}\times\mathbf{z})+\mathbf{x}\cdot(\mathbf{Ay}\times\mathbf{z})+\mathbf{x}\cdot(\mathbf{y}\times\mathbf{Az})}{\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})}=\mathbf{Ae}_1\cdot(\mathbf{e}_2\times\mathbf{e}_3)+\mathbf{e}_1\cdot(\mathbf{Ae}_2\times\mathbf{e}_3)+\mathbf{e}_1\cdot(\mathbf{e}_2\times\mathbf{Ae}_3)$$

which is independent of the choice of the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

Problem 1.59. (Scalar (dot) product of two tensors.) Consider the scalar-valued function

$$f(\mathbf{A},\mathbf{B}) \ = \ \mathsf{trace}(\mathbf{AB}^T)$$

defined for all tensors $\mathbf{A}$ and $\mathbf{B} \in \mathsf{Lin}$. Show that this function f has the following properties for all tensors $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{Lin}$ and all scalars α:

(a) $f(\mathbf{A},\mathbf{B}) = f(\mathbf{B},\mathbf{A})$,

(b) $f(\alpha\mathbf{A},\mathbf{B}) = \alpha f(\mathbf{A},\mathbf{B})$,

(c) $f(\mathbf{A}+\mathbf{C},\mathbf{B}) = f(\mathbf{A},\mathbf{B}) + f(\mathbf{C},\mathbf{B})$ and

(d) $f(\mathbf{A},\mathbf{A}) > 0$ provided $\mathbf{A} \neq \mathbf{0}$ and $f(\mathbf{0},\mathbf{0}) = 0$.

Solution: Let A_{ij} and B_{ij} be the components of $\mathbf{A}$ and $\mathbf{B}$ in an arbitrary basis. In terms of these components,

$$f(\mathbf{A},\mathbf{B}) \ = \ \mathsf{trace}(\mathbf{AB}^T) = (\mathbf{AB}^T)_{ii} = A_{ij}B^T_{ji} = A_{ij}B_{ij}.$$

It is now trivial to verify that all of the above hold. In confirming (d) we use the fact that $f(\mathbf{A},\mathbf{A}) = A_{ij}A_{ij}$ is the sum of the squares of all the components A_{ij}.

The statements $(a) - (d)$ describe the requirements of a proper definition of a scalar product. See, e.g. Knowles. Therefore for two tensors $\mathbf{A}$ and $\mathbf{B}$, we may define their *scalar-product*, denoted by $\mathbf{A}\cdot\mathbf{B}$, to be

$$\mathbf{A}\cdot\mathbf{B} = \ \mathsf{trace}(\mathbf{AB}^T) = \ A_{ij}B_{ij}.$$

Problem 1.60. Show that

$$(\mathbf{I}+\mathbf{a}\otimes\mathbf{b})^{-1} = \mathbf{I} - \frac{\mathbf{a}\otimes\mathbf{b}}{1+\mathbf{a}\cdot\mathbf{b}} \qquad (\text{provided } \mathbf{a}\cdot\mathbf{b} \neq -1)$$

Remark: It is shown in Problem 1.19 that

$$\det(\mathbf{I}+\mathbf{a}\otimes\mathbf{b}) = 1 + \mathbf{a}\cdot\mathbf{b}.$$

Problem 1.61. (Cofactor.) Given any tensor $\mathbf{A}$, there is a tensor called its cofactor and denoted by $\mathbf{A}^*$ with the property

$$\mathbf{A}^*(\mathbf{a}\times\mathbf{b}) = \mathbf{A}\mathbf{a}\times\mathbf{A}\mathbf{b}$$

for all vectors $\mathbf{a}$ and $\mathbf{b}$; see section 2 of Chadwick [2]. If $\mathbf{A}$ is nonsingular show that

$$\mathbf{A}^* = (\det\mathbf{A})\,\mathbf{A}^{-T}.$$

Solution: We can obtain the desired result if we can show that $\det\mathbf{A}\,(\mathbf{a}\times\mathbf{b}) = \mathbf{A}^T(\mathbf{A}\mathbf{a}\times\mathbf{A}\mathbf{b})$. This is precisely the result established in Problem 1.13.

Problem 1.62. (Convex set of tensors.) Suppose S is a subset of the set of all tensors Lin. The set S is said to be *convex* if all tensors on the line joining every pair of tensors in S lies in S, i.e. if every tensor $\mathbf{A}(\xi) = \xi\mathbf{A}_1 + (1-\xi)\mathbf{A}_2$, $0\le\xi\le 1$, belongs to S whenever $\mathbf{A}_1\in\mathsf{S}$ and $\mathbf{A}_2\in\mathsf{S}$.

(a) Show that the set Symm of all symmetric tensors is a convex set.

(b) Show that the set Lin^+ of all tensors with positive determinant is *not* a convex set.

Solution

(a) Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two arbitrary symmetric tensors. Clearly, $\mathbf{A}(\xi) = \xi\mathbf{A}_1 + (1-\xi)\mathbf{A}_2$ is necessarily symmetric for all $0\le\xi\le 1$. Thus if $\mathbf{A}_1$ and $\mathbf{A}_2$ are in Symm, so is every tensor on the line joining them. This proves that Symm is convex.

(b) (Steigmann) Take $\mathbf{A}_1 = -3\mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2 - \mathbf{e}_3\otimes\mathbf{e}_3$ and $\mathbf{A}_2 = \mathbf{e}_1\otimes\mathbf{e}_1 - 3\mathbf{e}_2\otimes\mathbf{e}_2 - \mathbf{e}_3\otimes\mathbf{e}_3$. Note that $\det\mathbf{A}_1 = \det\mathbf{A}_2 = 3$ and so $\mathbf{A}_1$ and $\mathbf{A}_2$ are both in Lin^+. Consider the tensors on the line joining them:

$$\mathbf{A}(\xi) = \xi\mathbf{A}_1 + (1-\xi)\mathbf{A}_2 = (1-4u)\mathbf{e}_1\otimes\mathbf{e}_1 + (4u-3)\mathbf{e}_2\otimes\mathbf{e}_2 - \mathbf{e}_3\otimes\mathbf{e}_3, \qquad 0\le\xi\le 1.$$

Then

$$\det\mathbf{A}(\xi) = -16(\xi-1/4)(3/4-\xi),$$

which is <0 for $1/4<\xi<3/4$ and so all $\mathbf{A}(\xi)$ on this line are *not* in Lin^+.

Problem 1.63. Let $\mathbf{S}$ be a tensor and $\mathbf{u}$ a vector. Show that $\mathbf{u}$ is an eigenvector of $\mathbf{S}$ if and only if $\mathbf{S}\mathbf{u}\otimes\mathbf{u} = \mathbf{u}\otimes\mathbf{S}\mathbf{u}$.

Solution: Suppose that $\mathbf{u}$ is an eigenvector of $\mathbf{S}$. Then

$$\mathbf{S}\mathbf{u} = \lambda\mathbf{u}, \tag{i}$$

for some scalar (the eigenvalue) λ. Then

$$\mathbf{S}\mathbf{u}\otimes\mathbf{u} \overset{(i)}{=} \lambda\mathbf{u}\otimes\mathbf{u} = \mathbf{u}\otimes(\lambda\mathbf{u}) \overset{(i)}{=} \mathbf{u}\otimes\mathbf{S}\mathbf{u}.$$

Conversely, suppose that

$$\mathbf{Su} \otimes \mathbf{u} = \mathbf{u} \otimes \mathbf{Su}, \qquad (ii)$$

for some non-zero vector $\mathbf{u}$. Operating each side of (ii) on the vector $\mathbf{u}$

$$(\mathbf{Su} \otimes \mathbf{u})\mathbf{u} = (\mathbf{u} \otimes \mathbf{Su})\mathbf{u} \qquad \Rightarrow \qquad (\mathbf{u} \cdot \mathbf{u})\mathbf{Su} = (\mathbf{Su} \cdot \mathbf{u})\mathbf{u} \qquad \Rightarrow \qquad \mathbf{Su} = \underbrace{\frac{(\mathbf{Su} \cdot \mathbf{u})}{\mathbf{u} \cdot \mathbf{u}}}_{\lambda} \mathbf{u}$$

and so $\mathbf{u}$ is an eigenvector of $\mathbf{S}$ (with eigenvalue $\lambda = (\mathbf{Su} \cdot \mathbf{u})/(\mathbf{u} \cdot \mathbf{u})$).

Problem 1.64. (4-tensors.) As before: vectors in the 1-dimensional Euclidean vector space V are denoted by lowercase boldface latin letters; 2-tensors are denoted by uppercase boldface latin letters and a 2-tensor $\mathbf{A}$ is a linear transformation that takes a vector $\mathbf{x} \in \mathsf{V}$ into another vector in V that we denote by $\mathbf{Ax}$ (subject to certain rules); the collection of all 2-tensors is itself a vector space that we denote by Lin; 4-tensors are denoted with uppercase blackboard letters and a 4-tensor $\mathbb{L}$ is a linear transformation that takes a 2-tensor $\mathbf{A} \in \mathsf{Lin}$ into another 2-tensor in Lin that we denote by $\mathbb{L}\mathbf{A}$ (subject to certain rules).

We shall denote the set of all 4-tensors by LinLin.

Reference: G. Del Piero [3].

- The identity 4-tensor $\mathbb{I}$ and null 4-tensor $\mathbb{O}$ obey

$$\mathbb{I}\mathbf{A} = \mathbf{A}, \qquad \mathbb{O}\mathbf{A} = \mathbf{0} \qquad \text{for all 2-tensors } \mathbf{A} \in \mathsf{Lin}.$$

- The product of two 4-tensors $\mathbb{C}$ and $\mathbb{D}$ is defined by

$$(\mathbb{C}\mathbb{D})\mathbf{A} = \mathbb{C}(\mathbb{D}\mathbf{A}) \qquad \text{for all 2-tensors } \mathbf{A} \in \mathsf{Lin}$$

(a) Exercise: Let $\mathbb{T}$ be the particular 4-tensor that takes any 2-tensor $\mathbf{A} \in \mathsf{Lin}$ into the 2-tensor $\mathbf{A}^T \in \mathsf{Lin}$:

$$\mathbb{T}\mathbf{A} = \mathbf{A}^T \qquad \text{for all 2-tensors } \mathbf{A} \in \mathsf{Lin}.$$

It is called the **transposition 4-tensor**.

(a1) Show that $\mathbb{T}$ is invertible and its inverse is $\mathbb{T}$:

$$\mathbb{T}\mathbb{T} = \mathbb{I};$$

(a2) Define the 4-tensor $\mathbb{S}$ by

$$\mathbb{S} = \frac{1}{2}(\mathbb{I} + \mathbb{T}).$$

Show that $\mathbb{S}\mathbb{T} = \mathbb{T}\mathbb{S} = \mathbb{S}$, $\mathbb{S}\mathbb{S} = \mathbb{S}$ and

$$\mathbb{S}\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) \qquad \text{for all 2-tensors } \mathbf{A} \in \mathsf{Lin}.$$

– Define the **transpose** $\mathbb{L}^T$ of the tensor $\mathbb{L}$ by

$$\mathbb{L}^T\mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbb{L}\mathbf{B} \qquad \text{for all 2-tensors } \mathbf{A},\mathbf{B}\in\mathsf{Lin}.$$

(b) Exercise: Show that $(\mathbb{C}\mathbb{D})^T = \mathbb{D}^T\mathbb{C}^T$.

– Define the **tensor product of two 2-tensors A** and **B** to be the 4-tensor, denoted by $\mathbf{A}\boxtimes\mathbf{B}$, for which

$$(\mathbf{A}\boxtimes\mathbf{B})\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{B}^T \qquad \text{for all 2-tensors } \mathbf{X}\in\mathsf{Lin}.$$

(c) Exercise: Show that

(c1) $(\mathbf{A}\boxtimes\mathbf{I})\mathbf{X} = \mathbf{A}\mathbf{X}$

(c2) $(\mathbf{I}\boxtimes\mathbf{A}^T)\mathbf{X} = \mathbf{X}\mathbf{A}$

(c3) $(\mathbf{A}\boxtimes\mathbf{B})^T = \mathbf{A}^T\boxtimes\mathbf{B}^T$.

(c4) $\mathbf{A}\mathbb{L}(\mathbf{X}\mathbf{B}) = (\mathbf{A}\boxtimes\mathbf{I})\mathbb{L}(\mathbf{I}\boxtimes\mathbf{B}^T)\mathbf{X}$

(c5) $(\mathbf{A}\boxtimes\mathbf{B})(\mathbf{C}\boxtimes\mathbf{D}) = \mathbf{A}\mathbf{C}\boxtimes\mathbf{B}\mathbf{D}$

(c6) $\mathbb{T}(\mathbf{A}\boxtimes\mathbf{B}) = (\mathbf{B}\boxtimes\mathbf{A})\mathbb{T}$

(c7) $(\mathbf{A}\boxtimes\mathbf{B})^{-1} = \mathbf{A}^{-1}\boxtimes\mathbf{B}^{-1}$.

– We say that $\mathbb{L}$ has the first minor symmetry if

$$\mathbb{L} = \mathbb{T}\mathbb{L},$$

where $\mathbb{T}$ is the transposition tensor introduced earlier. We say that $\mathbb{L}$ has the second minor symmetry if

$$\mathbb{L} = \mathbb{L}\mathbb{T}$$

Verify that $\mathbb{L}$ has both minor symmetries if

$$\mathbb{L} = \mathbb{S}\mathbb{L}\mathbb{S} \qquad \text{where} \quad \mathbb{S} = \frac{1}{2}(\mathbb{I}+\mathbb{T})$$

We say that $\mathbb{L}$ has the major symmetry (or simply we say that $\mathbb{L}$ is symmetric) if

$$\mathbb{L} = \mathbb{L}^T.$$

(d) Exercise: Show that if $\mathbb{L}$ has the major symmetry and one of the minor symmetries, it necessarily has the other minor symmetry.

– Components: Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be an orthonormal basis for the Euclidean vector space V. Define the 3^4 numbers $\mathbb{L}_{ijk\ell}$ by

$$\mathbb{L}_{ijk\ell} = \big(\mathbb{L}(\mathbf{e}_k\otimes\mathbf{e}_\ell)\big)\cdot(\mathbf{e}_i\otimes\mathbf{e}_j)$$

Show that

$$\mathbb{L} = \mathbb{L}_{ijk\ell}(\mathbf{e}_i\otimes\mathbf{e}_k)\boxtimes(\mathbf{e}_j\otimes\mathbf{e}_\ell)$$

$$\big(\mathbb{L}\mathbf{A}\big)_{ij} = \mathbb{L}_{ijk\ell}A_{k\ell}, \qquad (\mathbf{A}\boxtimes\mathbf{B})_{ijk\ell} = A_{ik}B_{j\ell}.$$

If the components of $\mathbb{L}$ are $\mathbb{L}_{ijk\ell}$ what are the components of $\mathbb{L}^T$? If a tensor $\mathbb{L}$ has in turn (a) the first minor symmetry, (b) the second minor symmetry, and (c) the major symmetry, what does this imply about the components $\mathbb{L}_{ijk\ell}$?

– Let $\mathbf{F}(\mathbf{X})$ be a 2-tensor valued function of all 2-tensors $\mathbf{X}$. Assuming $\mathbf{F}(\mathbf{X})$ is differentiable at $\mathbf{X}$, its gradient is the 4-tensor denoted by $\nabla\mathbf{F}$ for which

$$\mathbf{F}(\mathbf{X}+\mathbf{H}) = \mathbf{F}(\mathbf{X}) + (\nabla\mathbf{F}(\mathbf{X}))\mathbf{H} + o(|\mathbf{H}|) \qquad \text{with} \quad \lim_{|\mathbf{H}|\to 0} \frac{o(|\mathbf{H}|}{|\mathbf{H}|} \to 0.$$

(e) Exercise: Show that the components of this 4-tensor are

$$(\nabla F)_{ijk\ell} = \frac{\partial F_{ij}}{\partial X_{k\ell}}.$$

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Chapter 2

Kinematics: Finite Deformation

Several short videos on the material in Sections 2.3 - 2.6 can be found here.

In this chapter we shall consider *purely geometric issues* ("kinematics") associated with the deformation of a body. At this stage we will not address the *causes* of the deformation, such as what the applied loading is, nor will we discuss the characteristics of the material of which the body is composed, assuming only that it can be described as a continuum. Our focus will be entirely on kinematic considerations[1].

Problem 2.2 shows that the familiar notion of strain as defined in linear theories of solid mechanics is deficient when considering finite (i.e. large) deformations. This is why it is necessary that we devote some time to a careful analysis of the kinematics of large deformations. An even more detailed discussion, especially of time dependent entities such as strain-rate, can be found in the references [1, 3, 5, 6] listed at the end of this chapter.

A roadmap of this chapter is as follows: in Section 2.1 we introduce the notion of a deformation: $\mathbf{y} = \mathbf{y}(\mathbf{x})$. Some homogeneous deformations $\mathbf{y}(\mathbf{x}) = \mathbf{F}\mathbf{x} + \mathbf{b}$ such as a pure stretch, simple shear and a rigid deformation are discussed in Section 2.2. In Section 2.3 we introduce the deformation gradient tensor $\mathbf{F}(\mathbf{x})$, the central ingredient needed to describe the deformation in the neighborhood of a particle $\mathbf{x}$. We then consider in Section 2.4 an

[1]It is worth mentioning that in developing a continuum theory for a material, the appropriate kinematic description of the body is not entirely independent of the nature of the forces. For example, in describing the interaction between particles in a dielectric material subjected to an electric field, one might allow for internal forces *and internal couples* between every pair of points in the body. This in turn requires that the kinematics allow for *independent* displacement *and rotation* fields in the body. In general, the kinematics and the forces must be *conjugate* to each other in order to construct a self-consistent theory.

infinitesimal material curve, material surface and material region in the reference configuration and examine the geometric characteristics of their images in the deformed configuration. The decomposition of a general deformation gradient tensor $\mathbf{F}$ into the product of a rigid rotation $\mathbf{R}$ and pure stretches $\mathbf{U}, \mathbf{V}$ is described in Section 2.5. Section 2.6 introduces the notion of strain. In Section 2.7.1 we calculate the deformation gradient tensor $\mathbf{F}$ and the left Cauchy-Green deformation tensor $\mathbf{B}$ in cylindrical and spherical polar coordinates. We discuss material and spatial descriptions of a field in Section 2.8. Finally we linearize the preceding results in Section 2.9. In the appendix, Section 2.11, we touch on the material time derivative and a transport formula.

2.1 Deformation

In this chapter we examine how the geometric characteristics of one configuration of the body (the "deformed" configuration) *are related* to those of a second configuration (the "undeformed" or "reference" configuration). Thus we are necessarily concerned with two configurations of the body[2].

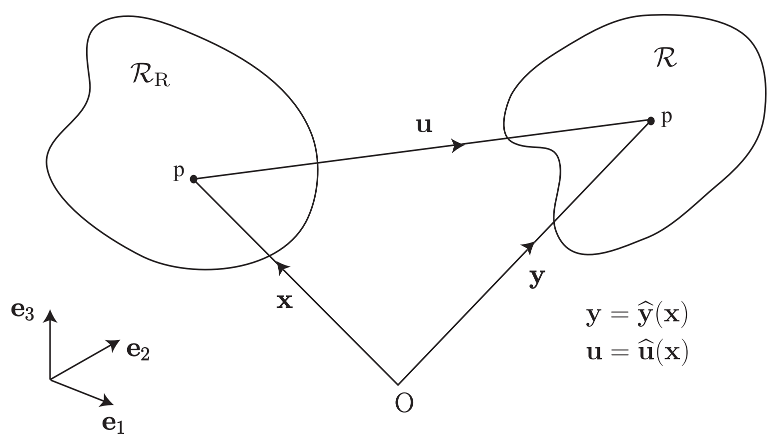


Figure 2.1: The respective regions $\mathcal{R}_R$ and $\mathcal{R}$ are occupied by a body in the reference and deformed configurations; the position vectors of a generic particle p in these two configurations are denoted by $\mathbf{x}$ and $\mathbf{y}$; the displacement of this particle is $\mathbf{u}$.

In the deformed configuration the body occupies a region $\mathcal{R}$ of physical space while the corresponding region in a reference configuration is $\mathcal{R}_R$. The position vector of a generic

[2]This is in contrast to the study of many fluids where only the current configuration needs to be considered.

particle p in the reference configuration is denoted by $\mathbf{x}$ and the deformation takes this particle to the position $\widehat{\mathbf{y}}(\mathbf{x})$ in the deformed configuration. This is illustrated in Figure 2.1. We write

$$\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{R}_{\mathrm{R}}, \quad \mathbf{y} \in \mathcal{R}. \tag{2.1}$$

We refer to $\widehat{\mathbf{y}}(\mathbf{x})$ as the **deformation**. We use the "hat" over $\mathbf{y}$ in order to distinguish the *function* $\widehat{\mathbf{y}}(\cdot)$ from its value $\mathbf{y}$. As we progress through these notes, we will usually omit the "hat" unless the context does not make clear whether we are referring to $\widehat{\mathbf{y}}$ or $\mathbf{y}$, or when we wish to emphasize the distinction.

The reference configuration serves two main purposes. One, geometric *changes*, e.g. the change in length of a fiber, are measured with respect to this configuration. Two, it provides a convenient way in which to "label" particles of the body: since there is a one-to-one correspondence between a particle p and its position $\mathbf{x}$ in the reference configuration[3], we can uniquely identify a particle by $\mathbf{x}$. Whenever there is no confusion in doing so, *we shall speak of "the particle* $\mathbf{x}$*" rather than "the particle located at* $\mathbf{x}$ *in the reference configuration".*

The reference configuration is an *arbitrary* conveniently chosen configuration, the only requirement being that it be a "possible" configuration that the body *can* occupy. It need not, for example, be the initial configuration of a body undergoing a motion. Unless explicitly stated otherwise, we shall always consider one fixed reference configuration.

The **displacement** $\widehat{\mathbf{u}}(\mathbf{x})$ of the particle $\mathbf{x}$ is

$$\widehat{\mathbf{u}}(\mathbf{x}) = \widehat{\mathbf{y}}(\mathbf{x}) - \mathbf{x}, \tag{2.2}$$

as shown in Figure 2.1. The functions $\widehat{\mathbf{y}}$ (and $\widehat{\mathbf{u}}$) are defined on $\mathcal{R}_{\mathrm{R}}$, i.e. at every $\mathbf{x} \in \mathcal{R}_{\mathrm{R}}$.

For physical reasons we require that (a) a single particle $\mathbf{x}$ not split into two particles and occupy two locations $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ in the deformed configuration, and (b) that two particles $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ both not occupy the same location $\mathbf{y}$ in the deformed configuration. Therefore we take the deformation $\mathbf{x} \mapsto \widehat{\mathbf{y}}(\mathbf{x})$ to be one-to-one.

Unless explicitly stated otherwise, we will assume $\widehat{\mathbf{y}}(\mathbf{x})$ to be "smooth", i.e. that it may be differentiated as many times as needed, and that these derivatives are continuous on $\mathcal{R}_{\mathrm{R}}$. There are situations in which this must be relaxed: for example, if we consider a "dislocation" it will be necessary to allow the displacement field to be discontinuous across a surface in the body; or if we consider the deformation of a "two-phase composite material"

[3]See Chapter 1 of Volume II for a more careful discussion of what we mean by "a particle" and "a configuration" in the continuum theory.

we must allow the gradient of the displacement field to be discontinuous across the interface between the two phases.

Finally, we pick and fix (an arbitrary) right-handed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. When referring to components of vector and tensor quantities, it will always be with respect to this basis (unless explicitly stated otherwise). In particular, denoting the components of $\mathbf{x}$ and $\mathbf{y}$ in this basis by $x_i = \mathbf{x} \cdot \mathbf{e}_i$ and $y_i = \mathbf{y} \cdot \mathbf{e}_i$, we write the deformation (2.1) in component form as

$$y_i = \widehat{y}_i(x_1, x_2, x_3) = x_i + \widehat{u}_i(x_1, x_2, x_3). \tag{2.3}$$

The rectangular cartesian coordinates of a particle in the reference and deformed configurations are (x_1, x_2, x_3) and (y_1, y_2, y_3) respectively.

2.2 Some homogeneous deformations.

In a **homogeneous deformation**, the position $\mathbf{y}(\mathbf{x})$ of a particle in the deformed configuration depends linearly on its position $\mathbf{x}$ in the reference configuration and so the deformation has the form

$$\mathbf{y} = \mathbf{y}(\mathbf{x}) = \mathbf{F}\mathbf{x} + \mathbf{b}, \tag{2.4}$$

where $\mathbf{F}$ is a *constant* tensor with positive determinant and $\mathbf{b}$ is a *constant* vector representing a rigid translation. In component form,

$$y_i = F_{ik}x_k + b_i.$$

Note that

$$\frac{\partial y_i}{\partial x_j} = F_{ik}\frac{\partial x_k}{\partial x_j} = F_{ik}\delta_{kj} = F_{ij}.$$

Since $\mathbf{y}(\mathbf{x})$ is the deformation, we refer to $\partial y_i/\partial x_j$ as the components of the deformation gradient tensor. In a homogeneous deformation, the deformation gradient tensor does not depend on $\mathbf{x}$.

Consider two identical cubic subregions of $\mathcal{R}_{\mathrm{R}}$ located at different places in the undeformed body. When the body undergoes a homogeneous deformation, their images in the deformed configuration will also be identical (other than for their locations). This is not true in general in an arbitrary deformation.

Exercise: Show that the set of points lying on a straight line/plane/ellipsoid in the reference configuration are mapped by a homogeneous deformation into a straight line/plane/ellipsoid in the deformed configuration.

2.2.1 Pure stretch.

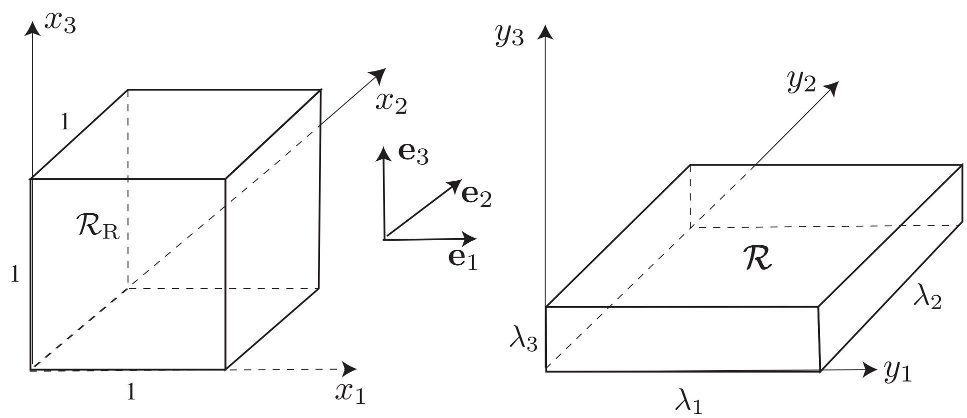


Figure 2.2: Pure homogeneous stretch of a cube. A unit cube in the reference configuration is carried into an orthorhombic region of dimensions $\lambda_1 \times \lambda_2 \times \lambda_3$.

The constant tensor $\mathbf{F}$ in a pure stretch $\mathbf{y}(\mathbf{x}) = \mathbf{F}\mathbf{x}$ is symmetric and positive definite. We will see in Section 2.5 that this implies that such a deformation does not involve a rigid rotational part.

Consider a body that occupies a unit cube in a reference configuration with its edges aligned with the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as shown in Figure 2.2. The body is subjected to the deformation

$$y_1 = \lambda_1 x_1, \qquad y_2 = \lambda_2 x_2, \qquad y_3 = \lambda_3 x_3, \qquad \lambda_i > 0, \tag{2.5}$$

where the three $\lambda_i' s$ are positive constants. This deformation maps the $1 \times 1 \times 1$ undeformed cube $\mathcal{R}_R$ into a $\lambda_1 \times \lambda_2 \times \lambda_3$ orthorhombic region $\mathcal{R}$ as shown in Figure 2.2. The positive constants λ_1, λ_2 and λ_3 represent the ratios by which the three edges of the cube *stretch* in the respective directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. This deformation is called a **pure stretch**.

Observe that a material fiber parallel to an edge of the cube in the reference configuration simply undergoes a stretch and no rotation under this deformation. However, this is not true of all material fibers, e.g. a fiber oriented along a diagonal of a face of the cube will undergo both a length change and a rotation.

The deformation (2.5) can be written in matrix form as

$$\{y\} = [F]\{x\} \qquad \text{where} \qquad [F] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{2.6}$$

and in tensor form as

$$\mathbf{y} = \mathbf{F}\mathbf{x} \qquad \text{where} \qquad \mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 \; + \; \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 \; + \; \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{2.7}$$

The 3×3 matrix $[F]$ in $(2.6)_2$ is the matrix of components of the tensor $\mathbf{F}$ in (2.7). In the special case where the deformed and reference configurations coincide, i.e. the body is undeformed, then $\mathbf{y}(\mathbf{x}) = \mathbf{x}$ and so $\lambda_1 = \lambda_2 = \lambda_3 = 1$ whence $[F] = [I]$, $F_{ij} = \delta_{ij}$ and $\mathbf{F} = \mathbf{I}$.

The deformation gradient tensor (2.7) *in this example* is symmetric and positive definite, with principal directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and corresponding principal stretches $\lambda_1, \lambda_2, \lambda_3$.

We now consider some particular pure stretches:

– **Pure dilatation:** The special case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ of (2.5) describes a pure dilatation of the body. In this case

$$\mathbf{F} = \lambda(\mathbf{e}_1 \otimes \mathbf{e}_1 \; + \; \mathbf{e}_2 \otimes \mathbf{e}_2 \; + \; \mathbf{e}_3 \otimes \mathbf{e}_3) = \lambda \mathbf{I},$$

and so the deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ specializes to $\mathbf{y} = \lambda\, \mathbf{x}$. This shows that *all* dimensions of the body are uniformly scaled by the factor λ.

– **Isochoric pure stretch:** A deformation is said to be isochoric if it is volume preserving at each point of the body. In the case of the pure stretch (2.7) this requires

$$\lambda_1 \lambda_2 \lambda_3 = 1. \tag{2.8}$$

Note that this is a constraint on the three stretches in that their values cannot be prescribed independently.

– **Uniaxial stretch:** The deformation

$$y_1 = \lambda x_1, \qquad y_2 = x_2, \qquad y_3 = x_3, \tag{2.9}$$

is illustrated in Figure 2.3. It describes a uniaxial stretch in the $\mathbf{e}_1$-direction. If $\lambda > 1$ the stretch is an elongation, if $\lambda < 1$ a contraction. (The terms "tensile" and

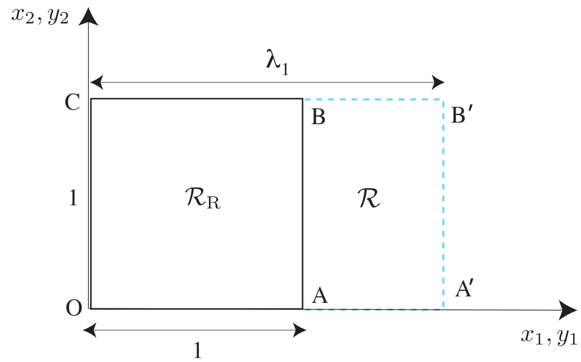


Figure 2.3: Uniaxial stretch in the $\mathbf{e}_1$-direction. A unit cube $\mathcal{R}_R$ in the reference configuration is carried into a $\lambda_1 \times 1 \times 1$ tetragonal region $\mathcal{R}$ in the deformed configuration.

"compressive" refer to stress not deformation.) This deformation can be written in tensor form as $\mathbf{y} = \mathbf{F}\mathbf{x}$ by taking

$$\mathbf{F} = \lambda \mathbf{e}_1 \otimes \mathbf{e}_1 \; + \; \mathbf{e}_2 \otimes \mathbf{e}_2 \; + \; \mathbf{e}_3 \otimes \mathbf{e}_3 = \mathbf{I} + (\lambda - 1)\mathbf{e}_1 \otimes \mathbf{e}_1.$$

More generally, the deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ with

$$\mathbf{F} = \mathbf{I} + (\lambda - 1)\mathbf{m}_R \otimes \mathbf{m}_R \tag{2.10}$$

represents a uniaxial stretch in the direction of the unit vector $\mathbf{m}_R$. If the body is composed of an incompressible material, its volume cannot change and so it cannot undergo a uniaxial stretch of the form (2.9) (except for the trivial one where $\lambda = 1$).

– **Isochoric uniaxial stretch with equal lateral stretch:** If the body undergoes a stretch λ in the x_1-direction and equal lateral stretches λ_2 in the x_2- and x_3-directions, then $y_1 = \lambda x_1, y_2 = \lambda_2 x_2, y_3 = \lambda_2 x_3$. If the deformation is isochoric, then $\lambda_1\lambda_2\lambda_3 = \lambda\lambda_2^2 = 1$ and so $\lambda_2 = \lambda^{-1/2}$ Therefore such a deformation is described by

$$y_1 = \lambda x_1, \qquad y_2 = \lambda^{-1/2} x_2, \qquad y_3 = \lambda^{-1/2} x_3. \tag{2.11}$$

2.2.2 Simple shear.

Consider the homogeneous deformation that carries the cube $\mathcal{R}_R$ into the sheared region $\mathcal{R}$ as shown in Figure 2.4. The displacement field associated with such a **simple shearing deformation** has components

$$u_1 = k x_2, \qquad u_2 = 0, \qquad u_3 = 0,$$

where k is a constant. Observe that the displacement (vector) of every particle has only an $\mathbf{e}_1$-component and the magnitude of this displacement increases linearly with x_2. One refers to a plane x_2 = constant as a *shearing (or glide) plane*, the x_1-direction as the *shearing direction* and the scalar k as the *amount of shear*.

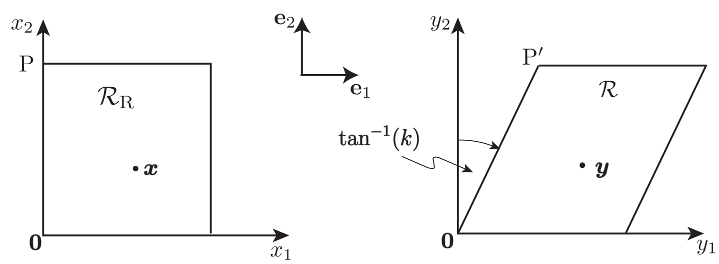


Figure 2.4: Simple shear of a cube. Each plane x_2 = constant undergoes a displacement in the x_1-direction by the amount kx_2.

The deformation $\mathbf{y}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$ associated with a simple shear has components

$$y_1 = x_1 + kx_2, \qquad y_2 = x_2, \qquad y_3 = x_3. \tag{2.12}$$

The deformation (2.12) can be written in matrix form as

$$\{y\} = [F]\{x\} \qquad \text{where} \qquad [F] = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.13}$$

and tensor form as

$$\mathbf{y} = \mathbf{F}\mathbf{x} \qquad \text{where} \qquad \mathbf{F} = \mathbf{I} + k\, \mathbf{e}_1 \otimes \mathbf{e}_2. \tag{2.14}$$

Note that $\det \mathbf{F} = 1$ and therefore a simple shear preserves volume. (That $\det \mathbf{F}$ is a measure of volume change is discussed in Section 2.4.3.)

More generally, if $\mathbf{n}_\mathrm{R}$ and $\mathbf{m}_\mathrm{R}$ are arbitrary unit vectors that are orthogonal, $|\mathbf{m}_\mathrm{R}| = |\mathbf{n}_\mathrm{R}| = 1, \mathbf{m}_\mathrm{R} \cdot \mathbf{n}_\mathrm{R} = 0$, a simple shear whose glide plane normal is $\mathbf{n}_\mathrm{R}$ and shear direction is $\mathbf{m}_\mathrm{R}$ is described by the deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ where

$$\mathbf{F} = \mathbf{I} + k\mathbf{m}_\mathrm{R} \otimes \mathbf{n}_\mathrm{R}. \tag{2.15}$$

One can of course consider combinations of the preceding homogeneous deformations. For example consider

$$\mathbf{y} = \mathbf{F}_1\mathbf{F}_2\mathbf{x} \qquad \text{where} \quad \mathbf{F}_1 = \mathbf{I} + \alpha\mathbf{a} \otimes \mathbf{a}, \quad \mathbf{F}_2 = \mathbf{I} + k\mathbf{m} \otimes \mathbf{n},$$

where the vectors $\mathbf{a}, \mathbf{m}, \mathbf{n}$ have unit length and $\mathbf{m} \cdot \mathbf{n} = 0$. We can examine this deformation in two steps as $\mathbf{y} = \mathbf{F}_1(\mathbf{F}_2\mathbf{x})$: in the first step a particle goes from $\mathbf{x} \mapsto \mathbf{F}_2\mathbf{x}$ corresponding to a simple shearing of the body. In the second step it goes from $\mathbf{F}_2\mathbf{x} \mapsto \mathbf{F}_1(\mathbf{F}_2\mathbf{x})$, and the body undergoes a uniaxial stretching. Figure 2.5 illustrates such a deformation in the particular case $\mathbf{a} = \mathbf{n} = \mathbf{e}_2, \mathbf{m} = \mathbf{e}_1$. *It is worth pointing out that the individual tensors* $\mathbf{F}_1$, $\mathbf{F}_2$ *enter the tensor* $\mathbf{F}$ *multiplicatively (not additively)*, i.e. as $\mathbf{F} = \mathbf{F}_1\mathbf{F}_2$ not $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$.

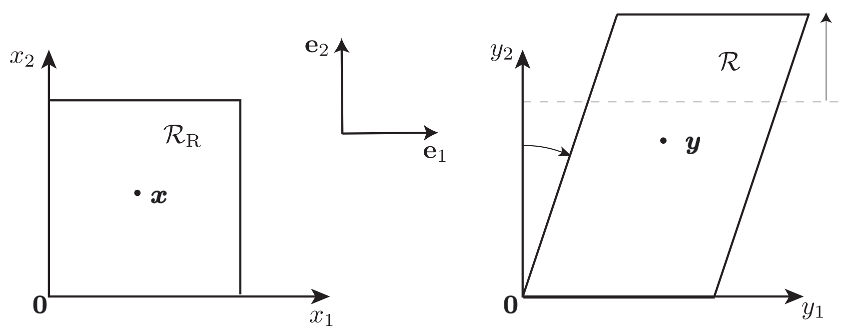


Figure 2.5: A unit cube subjected to a simple shear (with glide plane normal $\mathbf{e}_2$) and a uniaxial stretch in the direction $\mathbf{e}_2$.

2.2.3 Rigid deformation.

A deformation is said to be rigid if the distance between all pairs of particles remains unchanged, i.e. if the distance $|\mathbf{x}_1 - \mathbf{x}_2|$ between any two particles $\mathbf{x}_1$ and $\mathbf{x}_2$ in the reference configuration equals the distance $|\mathbf{y}(\mathbf{x}_2) - \mathbf{y}(\mathbf{x}_1)|$ between them in the deformed configuration:

$$|\mathbf{y}(\mathbf{x}_2) - \mathbf{y}(\mathbf{x}_1)|^2 = |\mathbf{x}_2 - \mathbf{x}_1|^2 \qquad \text{for all } \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{R}_{\mathrm{R}}. \tag{2.16}$$

It can be shown (Problem 2.45) that a deformation is rigid if and only if it has the form

$$\mathbf{y}(\mathbf{x}) = \mathbf{Q}\mathbf{x} + \mathbf{b}, \tag{2.17}$$

where $\mathbf{Q}$ is a constant *orthogonal* tensor and $\mathbf{b}$ is a constant vector[4]. When the deformation preserves orientation, it follows because of (2.25) below that $\det \mathbf{Q} = +1$ and therefore $\mathbf{Q}$ is proper orthogonal and represents a rigid rotation. The vector $\mathbf{b}$ represents a rigid translation.

[4]Recall from Problem 1.4.17 that an orthogonal tensor $\mathbf{Q}$ preserves length, i.e. $|\mathbf{Q}\mathbf{x}| = |\mathbf{x}|$ for all vectors $\mathbf{x}$. Therefore it is immediately clear that the deformation (2.17) obeys (2.16). What requires proof is the converse, that (2.16) implies (2.17).

A rigid *material* (or rigid body) is a material that can *only* undergo rigid deformations.

All of the deformations considered in this section were homogeneous in the sense that they were of the form (2.4) with $\mathbf{F}$ being a *constant* tensor. Most deformations are *not* of this form, a simple example being

$$\left.\begin{aligned} y_1 &= x_1 \cos \alpha x_3 - x_2 \sin \alpha x_3, \\ y_2 &= x_1 \sin \alpha x_3 + x_2 \cos \alpha x_3, \\ y_3 &= x_3, \end{aligned}\right\}$$

where α is a constant. It is readily seen that the deformation gradient tensor components $\partial y_i / \partial x_j$ depend on $\mathbf{x}$. This deformation represents a torsional deformation about the x_3-axis in which each plane $x_3 =$ constant rotates by an angle αx_3. [Exercise: show this.]

2.3 Deformation in the neighborhood of a particle. Deformation gradient tensor.

We now return to a general deformation $\mathbf{y} = \mathbf{y}(\mathbf{x})$. Questions such as "what is the state of stress at a particle $\mathbf{x}$?" depend not only on the deformation at $\mathbf{x}$ but also on the deformation of all particles in a neighborhood of $\mathbf{x}$. Thus we now turn to characterizing the deformation in the *neighborhood* of a generic particle. Intuitively, we expect the deformation of a small ball of material centered at $\mathbf{x}$ to consist of a combination of a rigid translation, a rigid rotation and a "straining", notions that we shall make precise in what follows.

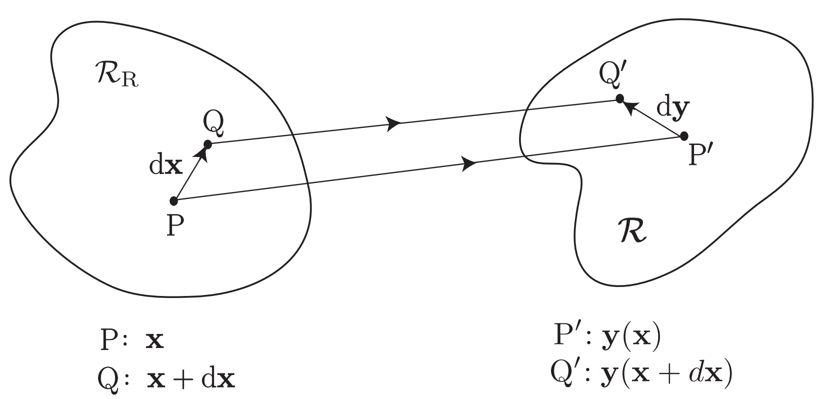


Figure 2.6: An infinitesimal material fiber in the reference and deformed configurations.

Consider two particles $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$ located in the reference configuration at P and Q as depicted in Figure 2.6. The material fiber joining them is $\overrightarrow{PQ} = d\mathbf{x}$. In the deformed configuration these particles are located at P' and Q' with respective position vectors $\mathbf{y}(\mathbf{x})$ and $\mathbf{y}(\mathbf{x}+d\mathbf{x})$. The deformed image of this *material* fiber[5] is

$$\overrightarrow{P'Q'} = d\mathbf{y} = \mathbf{y}(\mathbf{x}+d\mathbf{x}) - \mathbf{y}(\mathbf{x}). \tag{2.18}$$

From (1.163), we have $\mathbf{y}(\mathbf{x}+d\mathbf{x}) = \mathbf{y}(\mathbf{x}) + \nabla\mathbf{y}(\mathbf{x})\,d\mathbf{x} + \mathrm{o}(|d\mathbf{x}|)$ where the tensor $\nabla\mathbf{y}$ is the gradient of the deformation $\mathbf{y}(\mathbf{x})$. Thus when the two particles are *close to each other* we can write

$$d\mathbf{y} = \nabla\mathbf{y}\ d\mathbf{x} + \mathrm{o}(|\mathrm{d}\mathbf{x}|). \tag{2.19}$$

We denote the **deformation gradient tensor** $\nabla\mathbf{y}$ at particle $\mathbf{x}$ by

$$\boxed{\mathbf{F}(\mathbf{x}) \coloneqq \nabla\mathbf{y}(\mathbf{x})}, \tag{2.20}$$

and write (2.19) formally as

$$\boxed{d\mathbf{y} = \mathbf{F}\,d\mathbf{x}.} \tag{2.21}$$

Note that (2.21) *does not assume the deformation or deformation gradient to be "small"; only the two particles to be close to each other.*

The deformation gradient tensor $\mathbf{F}$ carries an infinitesimal material fiber $d\mathbf{x}$ in the undeformed configuration into $d\mathbf{y} = \mathbf{F}\,d\mathbf{x}$ in the deformed configuration. It describes the deformation of *every* infinitesimal material fiber through $\mathbf{x}$, and therefore it describes the deformation of the entire neighborhood of $\mathbf{x}$. Thus one can calculate all local changes in geometry at $\mathbf{x}$ in terms of $\mathbf{F}(\mathbf{x})$, e.g. the change in length of a material fiber, the change in angle between two material fibers etc. We will carry out these calculations in Section 2.4.

The deformation gradient tensor is the principal entity used to study the deformation in the neighborhood of a particle. It characterizes *both* the rigid rotation and the "strain" at $\mathbf{x}$.

The equation $d\mathbf{y} = \mathbf{F}\,d\mathbf{x}$ is the local version in the vicinity of the particle $\mathbf{x}$ of the equation $\mathbf{y} = \mathbf{F}\mathbf{x}$ that we had previously when examining homogeneous deformations in Section 2.2.

Given $\mathbf{F}(\mathbf{x})$, we can calculate the deformation of every material fiber (through $\mathbf{x}$). Conversely, given the deformation of any *three* linearly independent material fibers, one can

[5]The fibers PQ and $P'Q'$ being *material* fibers refers to the fact that they are comprised of the same set of particles.

calculate $\mathbf{F}$, and therefore determine the deformation of all other material fibers (Problem 2.5).

The deformation gradient tensor $\mathbf{F}(\mathbf{x})$ is a 2-tensor field whose cartesian components[6],

$$F_{ij}(\mathbf{x}) = \frac{\partial y_i}{\partial x_j}(\mathbf{x}), \tag{2.22}$$

correspond to the elements of a 3×3 matrix field $[F(\mathbf{x})]$. In terms of components, (2.21) reads

$$dy_i = F_{ij}\, dx_j. \tag{2.23}$$

Problem 2.3.1. Consider two orthonormal bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, with the former used in characterizing the reference configuration, and the latter, the deformed configuration. Thus in particular,

$$\mathbf{x} = x_i\, \mathbf{e}_i, \qquad \mathbf{y} = y_i\, \mathbf{e}'_i.$$

The deformation is described by $y_i = \widehat{y}_i(x_1, x_2, x_3)$. Show that the deformation gradient tensor has the representation

$$\mathbf{F} = F_{ij}\, \mathbf{e}'_i \otimes \mathbf{e}_j \qquad \text{where} \quad F_{ij} = \frac{\partial \widehat{y}_i}{\partial x_j}.$$

In physically realizable deformations we expect (a) a single fiber $d\mathbf{x}$ to not split into two fibers $d\mathbf{y}^{(1)}$ and $d\mathbf{y}^{(2)}$, and (b) two fibers $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ not to coalesce into a single fiber $d\mathbf{y}$. This requires $d\mathbf{y} = \mathbf{F}d\mathbf{x}$ to be a one-to-one relation between $d\mathbf{x}$ and $d\mathbf{y}$ whence $\mathbf{F}$ must be *nonsingular*. The **Jacobian** determinant, J, therefore cannot vanish:

$$J \ := \ \det \mathbf{F} \neq 0. \tag{2.24}$$

Without any further restrictions, a deformation might map a right-handed triplet of vectors into a left-handed triplet of vectors (which would imply that the body has been turned "inside out" like a sock). We say that the deformation *preserves "orientation"* if every right-handed (linearly independent) triplet of material fibers $\{d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}, d\mathbf{x}^{(3)}\}$ is carried into a right-handed triplet of vectors $\{d\mathbf{y}^{(1)}, d\mathbf{y}^{(2)}, d\mathbf{y}^{(3)}\}$. According to Problem 2.46, orientation is preserved if and only if

$$J \ = \ \det \mathbf{F} > 0. \tag{2.25}$$

[6] Given a function $\phi(\mathbf{x})$, the partial derivative $\partial\phi/\partial x_k$ is often denoted by $\phi_{,k}$. If we were to adopt this notation we would write $F_{ij} = y_{i,j}$. However, since we have both referential coordinates x_1, x_2, x_3 and spatial coordinates y_1, y_2, y_3, we will encounter both $\partial/\partial x_i$ and $\partial/\partial y_i$. In order not to confuse one partial derivative with the other, we shall write-out the partial derivatives explicitly and *not* adopt the subscript comma notation.

In these notes we will mostly be concerned with orientation-preserving deformations[7] and therefore, unless explicitly stated otherwise, assume (2.25) to hold.

Caution: In these notes we use the term "orientation-preserving" in two different ways. In the sense of the preceding paragraph, it refers to the preservation of the right-handedness (or left-handedness) of a triplet of vectors. When concerned with a particular material fiber, if its direction (orientation) in the reference and deformed configurations is the same, we shall say its orientation is preserved. The context should make clear the sense in which the term is being used.

In the special case where the deformed and reference configurations coincide $\mathbf{y}(\mathbf{x}) = \mathbf{x}$ and so

$$\mathbf{F}(\mathbf{x}) = \mathbf{I} \qquad \text{at all } \mathbf{x} \in \mathcal{R}_{\mathrm{R}}. \tag{2.26}$$

2.4 Change of length, orientation, angle, volume and area.

The deformation gradient tensor $\mathbf{F}(\mathbf{x})$ characterizes the deformation of *all* (infinitesimal) material fibers $d\mathbf{x}$ at the particle $\mathbf{x}$. We can therefore calculate various geometric quantities of interest (near $\mathbf{x}$) in terms of $\mathbf{F}$. In particular we now calculate the *local*[8] change in length of a fiber, change in angle between two fibers, change in volume of an infinitesimal material region and the change in area of an infinitesimal material surface, all in terms of $\mathbf{F}$.

The change in length is related to the notion of fiber stretch (or normal strain), the change in angle to the notion of shear strain and the change in volume to the notion of volumetric (or dilatational) strain. The change in area is indispensable when calculating the traction (force per unit area) on a surface.

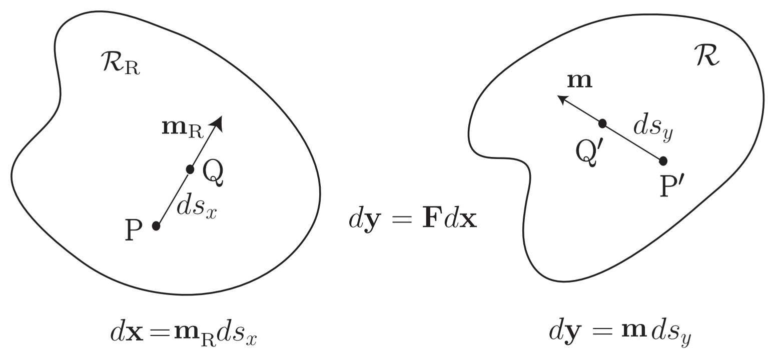


Figure 2.7: The infinitesimal material fiber $\overrightarrow{PQ}$ has length ds_x and direction $\mathbf{m}_\mathrm{R}$ in the reference configuration and length ds_y and direction $\mathbf{m}$ in the deformed configuration.

2.4.1 Change of length and direction.

Consider a material fiber that has length ds_x and direction $\mathbf{m}_\mathrm{R}$ in the reference configuration. Then $d\mathbf{x} = (ds_x)\mathbf{m}_\mathrm{R}$. If ds_y and $\mathbf{m}$ denote its length and direction in the deformed configuration, then $d\mathbf{y} = (ds_y)\mathbf{m}$. Given ds_x and $\mathbf{m}_\mathrm{R}$ we want to calculate ds_y and $\mathbf{m}$.

Since $d\mathbf{y}$ and $d\mathbf{x}$ are related by $d\mathbf{y} = \mathbf{F}d\mathbf{x}$, it follows that

$$(ds_y)\mathbf{m} = (ds_x)\mathbf{F}\mathbf{m}_\mathrm{R}. \tag{2.27}$$

On taking the magnitude of both sides of this vector equation we get $ds_y|\mathbf{m}| = ds_x|\mathbf{F}\mathbf{m}_\mathrm{R}|$ and so the deformed length of the fiber is

$$ds_y = ds_x|\mathbf{F}\mathbf{m}_\mathrm{R}|. \tag{2.28}$$

The **stretch** λ at the particle $\mathbf{x}$ in the direction $\mathbf{m}_\mathrm{R}$ is defined as the ratio

$$\lambda := \lim_{ds_x \to 0} \frac{ds_y}{ds_x}, \tag{2.29}$$

and so

$$\boxed{\lambda = \lambda(\mathbf{m}_\mathrm{R}) = |\mathbf{F}\mathbf{m}_\mathrm{R}|.} \tag{2.30}$$

This is the stretch of the fiber with referential direction $\mathbf{m}_\mathrm{R}$.

[7]Problem 2.19 concerns the eversion of a hollow cylinder where the body is turned "inside out". Such deformations do not preserve orientation. See also Problem 5.14.

[8]These are *local* changes in the sense that they refer to changes of *infinitesimally small* line, area and volume elements at $\mathbf{x}$.

Exercise: among all fibers of all orientations at $\mathbf{x}$, which has the maximum stretch? (Problem 2.22)

The stretch λ is related to the relative change in length by

$$\frac{ds_y - ds_x}{ds_x} = \lambda - 1.$$

We will return to this later in (2.68).

The direction $\mathbf{m}$ of this fiber in the deformed configuration is found from (2.27) and (2.28) to be

$$\mathbf{m} = \frac{\mathbf{F}\mathbf{m}_{\mathrm{R}}}{|\mathbf{F}\mathbf{m}_{\mathrm{R}}|}. \tag{2.31}$$

It is worth noting from (2.30) and (2.31) that

$$\lambda\,\mathbf{m} = \mathbf{F}\mathbf{m}_{\mathrm{R}}. \tag{2.32}$$

Exercise: Determine all directions that remain unstretched in the simple shear deformation (2.14) with shearing direction $\mathbf{e}_1$ and glide plane normal $\mathbf{e}_2$. *Remark*: Clearly, material fibers in the directions $\mathbf{e}_1$ and $\mathbf{e}_3$ remain unstretched. In fact, since each glide plane simply translates rigidly, all material fibers in a glide plane (i.e. fibers normal to $\mathbf{e}_2$) remain unstretched. There are additional directions that remain unstretched – determine them.

2.4.2 Change of angle.

In order to calculate the change in angle between two fibers one can make use of the fact that the angle between two vectors appears in the expression for the scalar product.

Consider two fibers $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ in the reference configuration (Figure 2.8) oriented in the respective directions $\mathbf{m}_{\mathrm{R}}^{(1)}$ and $\mathbf{m}_{\mathrm{R}}^{(2)}$. Let θ_x denote the angle between them. Then, by the definition of the scalar product,

$$\cos\theta_x = \frac{d\mathbf{x}^{(1)}}{|d\mathbf{x}^{(1)}|}\cdot\frac{d\mathbf{x}^{(2)}}{|d\mathbf{x}^{(2)}|} = \mathbf{m}_{\mathrm{R}}^{(1)}\cdot\mathbf{m}_{\mathrm{R}}^{(2)}. \tag{2.33}$$

In the deformed configuration these fibers are characterized by

$$d\mathbf{y}^{(1)} = \mathbf{F}d\mathbf{x}^{(1)}, \qquad d\mathbf{y}^{(2)} = \mathbf{F}d\mathbf{x}^{(2)}. \tag{2.34}$$

Letting θ_y denote the angle between them, again by the definition of the scalar product, we have

$$\cos\theta_y = \frac{d\mathbf{y}^{(1)}}{|d\mathbf{y}^{(1)}|}\ \cdot\ \frac{d\mathbf{y}^{(2)}}{|d\mathbf{y}^{(2)}|}\ =\ \frac{\mathbf{F}\mathbf{m}_{\mathrm{R}}^{(1)}\cdot\mathbf{F}\mathbf{m}_{\mathrm{R}}^{(2)}}{|\mathbf{F}\mathbf{m}_{\mathrm{R}}^{(1)}||\mathbf{F}\mathbf{m}_{\mathrm{R}}^{(2)}|}\ . \tag{2.35}$$

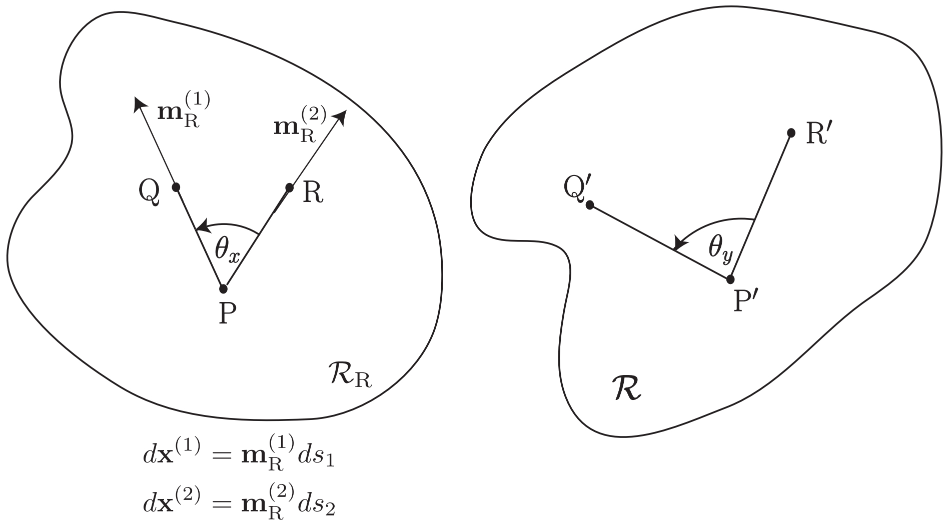


Figure 2.8: In the reference configuration two infinitesimal material fibers are oriented in the directions $\mathbf{m}_{\mathrm{R}}^{(1)}$ and $\mathbf{m}_{\mathrm{R}}^{(2)}$. The angle between them in the reference and deformed configurations are θ_x and θ_y respectively.

Thus, given the deformation gradient tensor $\mathbf{F}$ and the directions $\mathbf{m}_{\mathrm{R}}^{(1)}$ and $\mathbf{m}_{\mathrm{R}}^{(2)}$ of two fibers in the reference configuration, (2.35) gives the angle between them in the deformed configuration.

The decrease in angle, $\gamma := \theta_x - \theta_y$, is the *shear* associated with the directions $\mathbf{m}_{\mathrm{R}}^{(1)}, \mathbf{m}_{\mathrm{R}}^{(2)}$: $\gamma = \gamma(\mathbf{m}_{\mathrm{R}}^{(1)}, \mathbf{m}_{\mathrm{R}}^{(2)})$.

Exercise: among all pairs of fibers at $\mathbf{x}$, which pair undergoes the maximum change in angle, i.e. maximum shear? (Problem 2.10)

2.4.3 Change of volume.

Consider three linearly independent material fibers $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}, d\mathbf{x}^{(3)}$ in the reference configuration that form a tetrahedron of volume dV_x as shown in Figure 2.9. The deformation carries them into $d\mathbf{y}^{(1)} = \mathbf{F}\mathbf{x}^{(1)}, d\mathbf{y}^{(2)} = \mathbf{F}d\mathbf{x}^{(2)}, d\mathbf{y}^{(3)} = \mathbf{F}d\mathbf{x}^{(3)}$. If dV_y denotes the volume of the tetrahedron formed by the deformed fibers, according to Problem 1.12,

$$\boxed{dV_y = J\ dV_x} \qquad \text{where} \quad \mathrm{J} = \det \mathbf{F}, \tag{2.36}$$

having used $\det \mathbf{F} > 0$. This relates the volumes of an infinitesimal part of the body in the reference and deformed configurations.

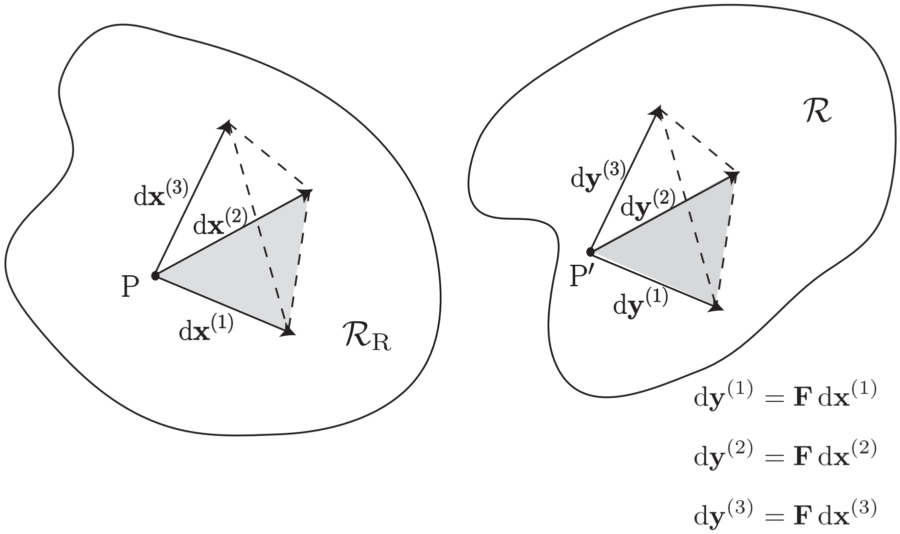


Figure 2.9: Three infinitesimal material fibers defining a tetrahedral region. The volumes of the tetrahedrons in the reference and deformed configurations are dV_x and dV_y respectively.

Observe from (2.36) that a deformation preserves the volume of *every* infinitesimal part of the body if and only if

$$J(\mathbf{x}) = 1 \qquad \text{for all} \quad \mathbf{x} \in \mathcal{R}_{\mathrm{R}}. \tag{2.37}$$

Such a deformation is said to be **isochoric** or locally volume preserving.

An incompressible *material* is a material that can *only* undergo isochoric deformations. Keep in mind that a material that is not incompressible can undergo an isochoric deformation, e.g. a simple shear.

2.4.4 Change of area.

Finally we turn to the relationship between two area elements in the reference and deformed configurations. Consider two linearly independent material fibers $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ in the reference configuration that form a parallelogram as shown in Figure 2.10. Let $\mathrm{d}A_x$ denote its area and let $\mathbf{n}_{\mathrm{R}}$ be a unit vector normal to the plane of the parallelogram and oriented such that $(d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}) \cdot \mathbf{n}_{\mathrm{R}} > 0$. The deformation carries these fibers into $d\mathbf{y}^{(1)} = \mathbf{F}d\mathbf{x}^{(1)}$ and $d\mathbf{y}^{(2)} = \mathbf{F}d\mathbf{x}^{(2)}$. Let $\mathrm{d}A_y$ and $\mathbf{n}$ be the area and unit normal vector respectively of the parallelogram defined by $d\mathbf{y}^{(1)}$ and $d\mathbf{y}^{(2)}$ with $(d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)}) \cdot \mathbf{n} > 0$. According to Problem

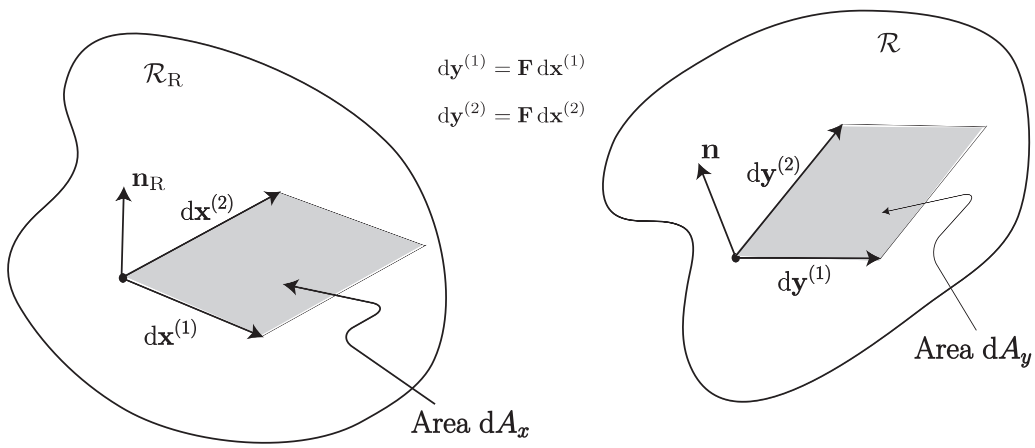


Figure 2.10: The two infinitesimal material fibers $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ define a parallelogram of area dA_x and unit normal vector $\mathbf{n}_R$. The corresponding quantities in the deformed configuration are $d\mathbf{y}^{(1)}$, $d\mathbf{y}^{(2)}$, dA_y and $\mathbf{n}$.

2.47 these two vector areas are related by

$$\boxed{\mathrm{d}A_y\,\mathbf{n} = \mathrm{d}A_x\, J\,\mathbf{F}^{-T}\,\mathbf{n}_R\ .} \tag{2.38}$$

Equation (2.38) is known as *Nanson's formula*. The relation between the scalar areas dA_y and dA_x is found by taking the magnitude of both sides of this vector equation which leads to

$$\mathrm{d}A_y = \mathrm{d}A_x\, J\,|\mathbf{F}^{-T}\,\mathbf{n}_R|. \tag{2.39}$$

The relation between the unit normal vectors $\mathbf{n}_R$ and $\mathbf{n}$ is obtained by using (2.39) in (2.38):

$$\mathbf{n} = \frac{\mathbf{F}^{-T}\mathbf{n}_R}{|\mathbf{F}^{-T}\mathbf{n}_R|}. \tag{2.40}$$

It is worth noting that a material fiber in the direction $\mathbf{n}_R$ in the reference configuration maps into $\mathbf{F}\mathbf{n}_R$, which in general is *not* the direction $\mathbf{n}$ given by (2.40). The vectors $\mathbf{n}_R$ and $\mathbf{n}$ are defined by the fact that they are normal to the particular material surface elements being considered. These vectors are not attached to a material fiber. To see this clearly, consider a simple shear as illustrated in Figure 2.11. Here $P'Q'$ is the image of PQ and the unit vectors $\mathbf{n}_R$ and $\mathbf{n}$ are defined as being normal to PQ and $P'Q'$ respectively. A material fiber in the direction $\mathbf{n}_R$ in the reference configuration (i.e. on the dashed line) remains on the dashed line. Its direction in the deformed configuration is $\mathbf{e}_1$, not $\mathbf{n}$.

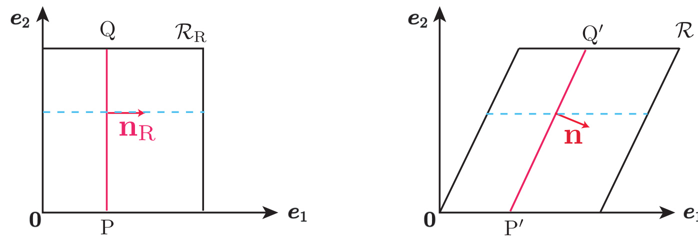


Figure 2.11: $P'Q'$ is the image of PQ. The unit vectors $\mathbf{n}_R$ and $\mathbf{n}$ are normal to PQ and $P'Q'$ respectively. A material fiber that is in the direction $\mathbf{n}_R$ in the reference configuration (i.e. on the dashed line) remains on the dashed line in the deformed configuration – its direction in the deformed configuration is $\mathbf{e}_1$, not $\mathbf{n}$.

2.4.5 Worked examples.

Problem 2.4.1. The region $\mathcal{R}_R$ occupied by a body in a reference configuration is a unit cube.

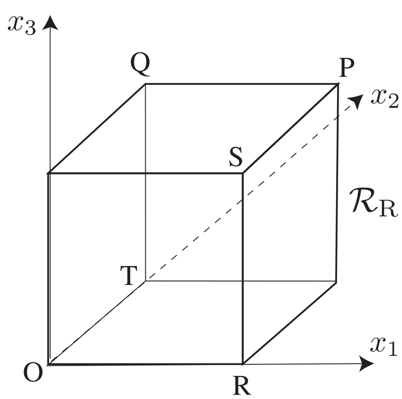


Figure 2.12: Unit cube $\mathcal{R}_R$ occupied by a body in its reference configuration. (Problem 2.4.1)

The body undergoes the pure stretch $\mathbf{y} = \mathbf{F}\mathbf{x}$ described by

$$y_1 = \lambda_1 x_1, \qquad y_2 = \lambda_2 x_2, \qquad y_3 = \lambda_3 x_3, \tag{i}$$

where the components have been taken with respect to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ aligned with the edges of the cube, see Figure 2.12. Derive relationships between the λ's in each of the following cases:

(a) The body is composed of an incompressible material.

(b) The length of the fiber $\overrightarrow{OP}$ remains unchanged by the deformation.

(c) The angle between the fibers $\overrightarrow{OP}$ and $\overrightarrow{QR}$ remains unchanged by the deformation.

(d) The area of the plane $RSQT$ remains unchanged by the deformation.

(e) The orientation of the plane $RSQT$ remains unchanged by the deformation.

Solution: The deformation gradient tensor and its inverse are

$$\mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3, \qquad \mathbf{F}^{-1} = \lambda_1^{-1} \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2^{-1} \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda 1_3 \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{ii}$$

(a) In general, the volumes of an infinitesimal part of the body in the reference and deformed configurations are related by $dV_y = JdV_x$, $J = \det \mathbf{F}$. If the material is incompressible then $dV_y = dV_x$ and so $J = 1$:

$$J = \det \mathbf{F} = \lambda_1 \lambda_2 \lambda_3 = 1. \qquad \square$$

(b) In general, the length ds_y of a deformed material fiber is given by (2.28) where ds_x and $\mathbf{m}_\mathrm{R}$ are the length and direction of the fiber in the reference configuration. Thus if the fiber does not change length, then $ds_x = ds_y$ and so the stretch $\lambda = 1$:

$$|\mathbf{F}\mathbf{m}_\mathrm{R}| = 1. \tag{iii}$$

The fiber of interest $\overrightarrow{\mathrm{OP}}$ can be expressed as $\overrightarrow{\mathrm{OP}} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ and therefore the unit vector $\mathbf{m}_\mathrm{R}$ in the direction of $\overrightarrow{\mathrm{OP}}$ is

$$\mathbf{m}_\mathrm{R} = \frac{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3}{\sqrt{3}}. \tag{iv}$$

Substituting $(ii)_1$ and (iv) into (iii) and simplifying leads to

$$\frac{\lambda_1^2}{3} + \frac{\lambda_2^2}{3} + \frac{\lambda_3^2}{3} = 1. \qquad \square$$

(c) In general, the angle θ_x between two material fibers that are in the directions of the unit vectors $\mathbf{m}_\mathrm{R}^{(1)}$ and $\mathbf{m}_\mathrm{R}^{(2)}$ in the reference configuration is given by (2.33) and the corresponding angle θ_y in the deformed configuration between these same two fibers is given by (2.35). Thus if the angle remains unchanged by the deformation we must have

$$\frac{\mathbf{F}\mathbf{m}_\mathrm{R}^{(1)}}{|\mathbf{F}\mathbf{m}^{(1)}|} \cdot \frac{\mathbf{F}\mathbf{m}^{(2)}}{|\mathbf{F}\mathbf{m}_\mathrm{R}^{(2)}|} = \mathbf{m}_\mathrm{R}^{(1)} \cdot \mathbf{m}_\mathrm{R}^{(2)} . \tag{v}$$

The unit vectors $\mathbf{m}_\mathrm{R}^{(1)}$ and $\mathbf{m}_\mathrm{R}^{(2)}$ in the directions of the material fibers $\overrightarrow{\mathrm{OP}}$ and $\overrightarrow{\mathrm{QR}}$ are

$$\mathbf{m}_\mathrm{R}^{(1)} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}, \qquad \mathbf{m}_\mathrm{R}^{(2)} = (\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)/\sqrt{3}. \tag{vi}$$

Thus substituting $(ii)_1$ and (vi) into (v) and simplifying leads to

$$2\lambda_1^2 = \lambda_2^2 + \lambda_3^2. \qquad \square$$

(d) In general, infinitesimal elements of area in the reference and deformed configurations are related by $dA_y = J|\mathbf{F}^{-T}\mathbf{n}_\mathrm{R}|\, dA_x$. If a particular area element remains unchanged, $dA_y = dA_x$, the deformation must be much that

$$J|\mathbf{F}^{-T}\mathbf{n}_\mathrm{R}| = 1, \tag{vii}$$

where $J = \det \mathbf{F}$ and $\mathbf{n}_\mathrm{R}$ is a unit vector normal to the surface of interest in the reference configuration. The unit vector normal to the plane RSQT is

$$\mathbf{n}_\mathrm{R} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2). \qquad (viii)$$

Substituting $(ii)_2$ and $(viii)$ into (vii) and simplifying leads to

$$\lambda_1\lambda_2\lambda_3\left(\frac{1}{2\lambda_1^2} + \frac{1}{2\lambda_2^2}\right)^{1/2} = 1. \qquad \square$$

(e) In general, the unit vectors $\mathbf{n}_\mathrm{R}$ and $\mathbf{n}$ normal to a surface in the reference and deformed configurations are related by (2.40). If the orientation of this surface does not change, then $\mathbf{n}_\mathrm{R} = \mathbf{n}$ in which case (2.40) yields

$$\mathbf{n}_\mathrm{R} = \mathbf{F}^{-T}\mathbf{n}_\mathrm{R}/|\mathbf{F}^{-T}\mathbf{n}_\mathrm{R}|. \qquad (ix)$$

Substituting $(ii)_2$ and $(viii)$ into (ix) and simplifying yields

$$\lambda_1 = \lambda_2, \qquad \square$$

(which is precisely what one would expect intuitively).

Problem 2.4.2. (Spencer) A body undergoes an arbitrary homogeneous deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$. Consider the set of particles that lie on a sphere of radius b in the deformed configuration. Show that in the undeformed configuration these particles lie on the surface of an ellipsoid and determine the lengths of the three major axes of the ellipsoid. Under what condition on $\mathbf{F}$ is this ellipsoid a sphere of radius a?

Solution: Let $\mathcal{S}$ denote the spherical surface of interest in the deformed configuration, and let $\mathcal{S}_\mathrm{R}$ be its image in the undeformed configuration. We pick the origin to be at the center of $\mathcal{S}$. Then the position vector $\mathbf{y}$ of a point on $\mathcal{S}$ and the radius b of $\mathcal{S}$ are related by

$$\mathcal{S}: \qquad \mathbf{y}\cdot\mathbf{y} = b^2. \qquad (i)$$

Let $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ be a particle on $\mathcal{S}_\mathrm{R}$ and consider the deformation gradient tensor[9]

$$\mathbf{F} = \lambda_1\mathbf{e}_1\otimes\mathbf{e}_1 + \lambda_2\mathbf{e}_2\otimes\mathbf{e}_2 + \lambda_3\mathbf{e}_3\otimes\mathbf{e}_3.$$

The deformation takes $\mathbf{x} \mapsto \mathbf{y}$ according to

$$\mathbf{y} = \mathbf{F}\mathbf{x} = (\lambda_1\mathbf{e}_1\otimes\mathbf{e}_1 + \lambda_2\mathbf{e}_2\otimes\mathbf{e}_2 + \lambda_3\mathbf{e}_3\otimes\mathbf{e}_3)(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = x_1\lambda_1\mathbf{e}_1 + x_2\lambda_2\mathbf{e}_2 + x_3\lambda_3\mathbf{e}_3. \qquad (ii)$$

Substituting (ii) into (i) gives

$$(x_1\lambda_1\mathbf{e}_1 + x_2\lambda_2\mathbf{e}_2 + x_3\lambda_3\mathbf{e}_3)\cdot(x_1\lambda_1\mathbf{e}_1 + x_2\lambda_2\mathbf{e}_2 + x_3\lambda_3\mathbf{e}_3) = b^2$$

[9] Why did we take $\mathbf{F}$ to be a pure stretch?

which when simplified leads to

$$\mathcal{S}_{\rm R}: \qquad \frac{x_1^2}{b^2/\lambda_1^2}+\frac{x_2^2}{b^2/\lambda_2^2}+\frac{x_3^2}{b^2/\lambda_3^2}=1.$$

Therefore the surface $\mathcal{S}_{\rm R}$ is an ellipsoid and the lengths of its (semi)-major axes are $b/\lambda_1, b/\lambda_2$ and b/λ_3.

The surface $\mathcal{S}_{\rm R}$ is a sphere of radius a if $\lambda_1=\lambda_2=\lambda_3=b/a$.

Problem 2.4.3. The region $\mathcal{R}_R$ occupied by a body in a reference configuration is a hollow sphere. In spherical polar coordinates, a general deformation takes the particle located at (R,Θ,Φ) in the reference configuration into the location (r,θ,φ) in the deformed configuration. In a spherically *symmetric* deformation one has

$$r=r(R), \qquad \theta=\Theta, \qquad \varphi=\Phi. \tag{o}$$

(a) In terms of $r(R)$, calculate the stretch of a material fiber in the radial direction? What is the stretch of a material fiber perpendicular to the radial direction?

(b) Determine the function $r(R)$ (to the extent possible) in each of the following cases:

- (b1) the material is incompressible,
- (b2) the material is inextensible in the radial direction (perhaps there are very stiff fibers in the radial direction),
- (b3) the material is inextensible in circumferential directions (perhaps there are very stiff fibers in the circumferential directions).

(c) Suppose that the inner and outer radii of the body are A and B in the reference configuration, and the inner radius is a in the deformed configuration. Calculate the outer radius of the body in each of the preceding cases?

Solution:

(a) In order to calculate the stretch of a radial fiber, consider two particles located in the reference configuration at $A:(R,\Theta,\Phi)$ and $B:(R+dR,\Theta,\Phi)$ as shown in Figure 2.13. The spherically symmetric deformation (o) maps them to $A':(r(R),\Theta,\Phi)$ and $B':(r(R+dR),\Theta,\Phi)$ in the deformed configuration. Thus the deformation takes the infinitesimal radial material fiber

$$\overrightarrow{AB}=d\mathbf{x}=dR\,\mathbf{e}_R \quad\mapsto\quad \overrightarrow{A'B'}=d\mathbf{y}=(r(R+dR)-r(R))\,\mathbf{e}_r,$$

and therefore the stretch of this fiber is

$$\lambda_R=\lim_{|AB|\to 0}\frac{|A'B'|}{|AB|}=\lim_{|d\mathbf{x}|\to 0}\frac{|d\mathbf{y}|}{|d\mathbf{x}|}=\lim_{dR\to 0}\frac{r(R+dR)-r(R)}{dR}=r\,'(R). \qquad \square \tag{i}$$

Next, in order to calculate the stretch of a fiber perpendicular to the radial direction consider two particles located in the reference configuration at $A:(R,\Theta,\Phi)$ and $C:(R,\Theta+d\Theta,\Phi)$ as shown in Figure

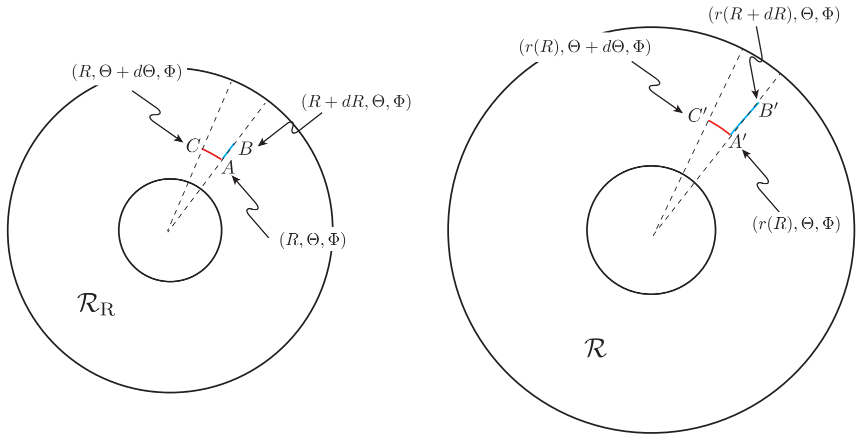


Figure 2.13: The radial and circumferential material fibers $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are mapped by the spherically symmetric deformation into $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ respectively. The spherical polar coordinates of the points A, B, C, A', B' and C' are shown in the figure.

2.13. The spherically symmetric deformation (o) maps them to $A' : (r(R), \Theta, \Phi)$ and $C' : (r(R), \Theta + d\Theta, \Phi)$ in the deformed configuration. The deformation takes the infinitesimal material fiber in the circumferential Θ direction

$$\overrightarrow{AC} = d\mathbf{x} = Rd\Theta\,\mathbf{e}_\Theta \quad \mapsto \quad \overrightarrow{A'C'} = d\mathbf{y} = (r(R)d\Theta)\,\mathbf{e}_\theta,$$

and so the stretch of this fiber is

$$\lambda_\Theta = \lim_{|AC|\to 0} \frac{|A'C'|}{|AC|} = \lim_{|d\mathbf{x}|\to 0} \frac{|d\mathbf{y}|}{|d\mathbf{x}|} = \lim_{d\Theta\to 0} \frac{r(R)d\Theta}{Rd\Theta} = \frac{r(R)}{R}. \qquad \Box \qquad (ii)$$

By symmetry, the stretch of a fiber in the circumferential Φ direction is

$$\lambda_\Phi = \lambda_\Theta = \frac{r(R)}{R}.$$

$(b1)$ Consider a spherical annulus of inner and outer radii R and $R + dR$ respectively in the reference configuration. Its volume is

$$\frac{4}{3}\pi\Big[(R + dR)^3 - R^3\Big] = \frac{4}{3}\pi\Big[R^3 + 3R^2dR - R^3\Big] + \mathrm{O}(|dR|^2) = 4\pi R^2 dR + \mathrm{O}(|dR|^2).$$

The deformation maps this region into the spherical annulus between the radii $r(R)$ and $r(R + dR)$ in the deformed configuration whose volume is

$$\begin{aligned}
\frac{4}{3}\pi\Big[\big(r(R + dR)\big)^3 - \big(r(R)\big)^3\Big] &= \frac{4}{3}\pi\Big[\big(r(R) + r'(R)dR\big)^3 - r^3(R)\Big] + \mathrm{O}(|dR|^2) = \\
&= \frac{4}{3}\pi\Big[3r^2(R)r'(R)dR\Big] + \mathrm{O}(|dR|^2) = \\
&= 4\pi r^2(R)r'(R)dR + \mathrm{O}(|dR|^2).
\end{aligned}$$

When the material is incompressible, these volumes must be equal, and so equating the preceding expressions and taking the limit $dR \to 0$ yields

$$r^2(R)r'(R) = R^2. \qquad (iii)$$

Alternatively (iii) can be obtained directly from $\det \mathbf{F} = \lambda_R\lambda_\Theta\lambda_\Phi = 1$ which follows from (2.46) below. Solving the differential equation (iii) for $r(R)$ gives

$$r(R) = \Big[R^3 + c_1\Big]^{1/3}, \qquad (iv)$$

where c_1 is an arbitrary constant.

Remark: Note that the angle between any pair of the three material fibers in the directions $\mathbf{e}_R, \mathbf{e}_\Theta$ and $\mathbf{e}_\Phi$ is $\pi/2$ in both the reference and deformed configurations. Therefore if we jump ahead to the next section, and in particular Problem 2.5.1, we conclude that the three stretches $\lambda_R, \lambda_\Theta$ and λ_Φ are in fact the principal stretches and the radial and circumferential directions are the principal directions (for both the Lagrangian and Eulerian stretch tensors $\mathbf{U}$ and $\mathbf{V}$ to be introduced in the next section). Moreover, incompressibility requires $\det \mathbf{F} = \lambda_R\lambda_\Theta\lambda_\Phi = 1$ which leads immediately to (iii).

$(b2)$ Inextensibility in the radial direction requires $\lambda_R = 1$:

$$\lambda_R = r\,'(R) = 1 \qquad \Rightarrow \qquad r(R) = R + c_2, \qquad \square \qquad (v)$$

where c_2 is an arbitrary constant.

$(b3)$ Inextensibility in the circumferential direction requires $\lambda_\Theta = 1$:

$$\lambda_\Theta = \frac{r(R)}{R} = 1 \qquad \Rightarrow \qquad r(R) = R. \qquad \square \qquad (vi)$$

(c) Let b denote the (unknown) outer radius of the body in the deformed configuration. Then

$$r(A) = a, \qquad r(B) = b. \qquad (vii)$$

In the incompressible case (iv) and (vii) give

$$b = \big[B^3 - A^3 + a^3\big]^{1/3}. \qquad \square$$

In the radially inextensible case (v) and (vii) give

$$b = B - A + a. \qquad \square$$

In the circumferentially inextensible case (vi) and (vii) give

$$b = B \qquad \text{(and in fact } a = A). \qquad \square$$

Remark: You could have deduced these values of b directly by physical considerations.

2.5 Stretch and rotation.

As mentioned previously, the deformation gradient tensor $\mathbf{F}(\mathbf{x})$ completely characterizes the deformation in the vicinity of the particle $\mathbf{x}$. Part of this deformation is a rigid rotation, the rest a "distortion", i.e. a "stretch/strain". We now explore this decomposition and examine various features of the rotation and stretch.

2.5.1 Right (or Lagrangian) Stretch Tensor U.

– According to the polar decomposition theorem stated in Section 1.4, every nonsingular tensor $\mathbf{F}$ can be written uniquely as the product of an orthogonal tensor $\mathbf{R}$ and a symmetric positive definite tensor $\mathbf{U}$ as

$$\boxed{\mathbf{F} = \mathbf{R}\,\mathbf{U}.} \tag{2.41}$$

When $\det \mathbf{F} > 0$, $\mathbf{R}$ is proper orthogonal and therefore represents a *rotation*. Since the tensor $\mathbf{U}$ is symmetric and positive definite, it describes a pure *stretch*; see the remarks below (2.7). Since these tensors depend on the particle $\mathbf{x}$ in general, $\mathbf{R}(\mathbf{x})$ and $\mathbf{U}(\mathbf{x})$ represent the rotation and stretch *locally* at the particle $\mathbf{x}$.

– The stretch tensor $\mathbf{U}$ can be expressed it in terms of its three real positive eigenvalues λ_1, λ_2 and λ_3 and corresponding orthonormal eigenvectors $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{r}_3$ as

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{r}_i \otimes \mathbf{r}_i, \qquad \lambda_i > 0, i = 1, 2, 3; \tag{2.42}$$

see (1.112). Equivalently, the matrix of components of $\mathbf{U}$ in its principal basis $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is

$$[U] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{2.43}$$

which may be compared with the matrix[10] $[F]$ in (2.6) representing a pure stretch.

The tensor $\mathbf{U}$ is called the **right stretch tensor**[11] and the λ_i's are the **principal stretches**, the $\mathbf{r}_i$'s the corresponding **principal directions**. As we shall see shortly, $\mathbf{U}$ can be viewed as a **Lagrangian stretch tensor**.

– Let λ and $\mathbf{r}$ be one of the eigenvalues and corresponding eigenvectors of $\mathbf{U}$ and consider a referential material fiber that is in the direction $\mathbf{r}$: $d\mathbf{x} = dx\,\mathbf{r}$. When $\mathbf{U}$ operates on this fiber it is carried into $\mathbf{U}d\mathbf{x} = dx\,\mathbf{U}\mathbf{r} = \lambda dx\,\mathbf{r} = \lambda\,d\mathbf{x}$ and so the stretch tensor $\mathbf{U}$ simply stretches this fiber by λ without rotation.

Now consider an infinitesimal rectangular parallelepiped of dimensions $dx_1 \times dx_2 \times dx_3$ with its edges aligned with the principal directions $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$. The tensor $\mathbf{U}$ carries it

[10] In equation (2.6), the deformation was homogeneous and so the matrix of components of $\mathbf{F}$ had this form at every point in the body. Here, (2.43) holds locally, at the point $\mathbf{x}$ under consideration.

[11] "Right" because $\mathbf{U}$ appears on the right-hand side of the expression $\mathbf{R}\mathbf{U}$.

into a rectangular parallelepiped of dimensions $\lambda_1 dx_1 \times \lambda_2 dx_2 \times \lambda_3 dx_3$ (without rotating it). When $\mathbf{R}$ acts on it, this infinitesimal part of the body will rotate rigidly, and so in particular, the angle between the edges remains $\pi/2$.

– The deformation of a generic material fiber $d\mathbf{x}$ can also be viewed in two-steps:

$$d\mathbf{x} \quad \xrightarrow{\text{stretch}} \quad \mathbf{U}d\mathbf{x} \quad \xrightarrow{\text{rigid rotation}} \quad \mathbf{R}(\mathbf{U}d\mathbf{x}) = d\mathbf{y} \tag{2.44}$$

In the first part of the deformation where $\mathrm{d}\mathbf{x} \mapsto \mathbf{U}d\mathbf{x}$, the fiber $d\mathbf{x}$ is subjected to a pure stretch in the directions of the eigenvectors of $\mathbf{U}$, the amounts of stretch being equal to the eigenvalues of $\mathbf{U}$. Since $d\mathbf{x}$ will not be parallel to $\mathbf{U}d\mathbf{x}$ in general, the fiber will also rotate when it undergoes the stretching deformation $d\mathbf{x} \mapsto \mathbf{U}\mathrm{d}\mathbf{x}$. However, this is not a rigid rotation since the length of the fiber changes.

In the second step $\mathbf{U}d\mathbf{x} \mapsto \mathbf{R}(\mathbf{U}d\mathbf{x})$, the stretched fiber is rigidly rotated by $\mathbf{R}$.

The stretch $\lambda(\mathbf{m}_\mathrm{R})$ of a fiber in an arbitrary direction $\mathbf{m}_\mathrm{R}$ can be written in terms of the principal stretches using (2.30) as

$$\lambda^2(\mathbf{m}_\mathrm{R}) = \lambda_1^2 m_1^2 + \lambda_2^2 m_2^2 + \lambda_3^2 m_3^2, \tag{2.45}$$

where the m_i's are the components of $\mathbf{m}_\mathrm{R}$ in a principal basis for $\mathbf{U}$, (Problem 2.22).

– Since the determinant of a proper orthogonal tensor is 1, it follows from $\det \mathbf{F} = \det(\mathbf{R}\mathbf{U}) = \det \mathbf{R} \det \mathbf{U} = \det \mathbf{U}$ that

$$J = \det \mathbf{F} = \lambda_1 \lambda_2 \lambda_3. \tag{2.46}$$

– The right stretch tensor $\mathbf{U}$ is often referred to as the **Lagrangian stretch tensor**. This is because expressions for the length of a fiber, the angle between two fibers, area of a surface element etc. in the deformed configuration can be calculated in terms of just $\mathbf{U}$ and the referential geometry. For example, given a referential material fiber $d\mathbf{x}$, its length in the deformed configuration is $|d\mathbf{y}| = |\mathbf{F}\, d\mathbf{x}| = |\mathbf{R}\mathbf{U}\, d\mathbf{x}| = |\mathbf{U}d\mathbf{x}|$ which shows that this length depends only on $\mathbf{U}$ and $d\mathbf{x}$ and not $\mathbf{R}$.

Exercise: By using $\mathbf{F} = \mathbf{R}\mathbf{U}$ in (2.35), (2.36) and (2.39), show that changes in angle, volume and area can be expressed in terms of $\mathbf{U}$ and the referential geometry without involving $\mathbf{R}$.

On the other hand if we are given the fiber $d\mathbf{y}$ in the deformed configuration, we cannot calculate its undeformed length without knowing both $\mathbf{U}$ and $\mathbf{R}$. This follows by taking the magnitude of the vector equation $d\mathbf{x} = \mathbf{U}^{-1}\mathbf{R}^T d\mathbf{y}$. In this sense, $\mathbf{U}$ is not an Eulerian stretch tensor.

The principal directions $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ of $\mathbf{U}$ are known as the **Lagrangian principal directions** of stretch.

– *Principal scalar invariants.* Looking ahead, in Chapter 4.3 we will encounter (scalar-valued) functions[12] of the Lagrangian stretch tensor, $\varphi = \varphi(\mathbf{U})$, that have the property $\varphi(\mathbf{U}) = \varphi(\mathbf{Q}\mathbf{U}\mathbf{Q}^T)$ for all orthogonal tensors $\mathbf{Q}$. Such functions are called scalar-valued invariants (or isotropic functions). In Section 1.5 we saw that the **principal scalar invariants** of $\mathbf{U}$,

$$I_1(\mathbf{U}) = \operatorname{tr}\mathbf{U}, \qquad I_2(\mathbf{U}) = \frac{1}{2}\Big[\big(\operatorname{tr}\mathbf{U}\big)^2 - \operatorname{tr}\mathbf{U}^2\Big], \qquad I_3(\mathbf{U}) = \det\mathbf{U}, \tag{2.47}$$

have this invariance $I_i(\mathbf{U}) = I_i(\mathbf{Q}\mathbf{U}\mathbf{Q}^T)$.

One can show using (2.42) and (2.47) that the principal scalar invariants of $\mathbf{U}$ can be written in terms of the principal stretches as (Problem 1.17)

$$I_1(\mathbf{U}) = \lambda_1 + \lambda_2 + \lambda_3, \qquad I_2(\mathbf{U}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \qquad I_3(\mathbf{U}) = \lambda_1\lambda_2\lambda_3. \tag{2.48}$$

– Finally we turn to the question of how, given $\mathbf{F}$, one might go about calculating $\mathbf{U}$ and $\mathbf{R}$. From (2.41) we have $\mathbf{F}^T\mathbf{F} = (\mathbf{R}\mathbf{U})^T\mathbf{R}\mathbf{U} = \mathbf{U}\mathbf{R}^T\mathbf{R}\mathbf{U} = \mathbf{U}^2$. The tensor $\mathbf{F}^T\mathbf{F}$ is symmetric and positive definite and therefore $\mathbf{U}$ is its unique, symmetric, positive definite square root (Problem 1.25):

$$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}. \tag{2.49}$$

After determining the stretch $\mathbf{U}$ from (2.49) the rotation $\mathbf{R}$ can be found from

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}. \tag{2.50}$$

In Problem 2.5.2 we shall work out the details of this calculation for a simple shear deformation.

2.5.2 Left (or Eulerian) Stretch Tensor V.

– The alternative version of the polar decomposition theorem (Section 1.4) provides a second representation for $\mathbf{F}$. According to this part of the theorem, every tensor $\mathbf{F}$

[12]representing the energy stored per unit volume in an isotropic elastic body.

with $\det \mathbf{F} > 0$ can be written uniquely as the product of a symmetric positive definite tensor $\mathbf{V}$ and a proper orthogonal tensor $\mathbf{R}$ as

$$\boxed{\mathbf{F} = \mathbf{V}\mathbf{R}.} \tag{2.51}$$

The rotation tensor $\mathbf{R}$ here is identical to that in the preceding representation and $\mathbf{V}$ is the unique, symmetric, positive definite square root

$$\mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T}. \tag{2.52}$$

The tensor $\mathbf{V}$ is called the **left stretch tensor**[13] and as we shall see shortly, can be thought of as an **Eulerian stretch tensor**. The principal values of $\mathbf{V}$ are the same as those of $\mathbf{U}$. The stretch tensor $\mathbf{V}$ can be expressed as

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i, \tag{2.53}$$

where $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ are the principal directions of $\mathbf{V}$. The principal directions of $\mathbf{U}$ and $\mathbf{V}$ are related by

$$\boldsymbol{\ell}_i = \mathbf{R}\mathbf{r}_i. \tag{2.54}$$

The deformation of a generic fiber can now be written as

$$d\mathbf{y} = \mathbf{V}\,(\mathbf{R}\,d\mathbf{x}), \tag{2.55}$$

and so interpreted as a rigid rotation, $d\mathbf{x} \mapsto \mathbf{R}\,d\mathbf{x}$, followed by stretching in the directions $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ by $\lambda_1, \lambda_2, \lambda_3$.

– The tensors $\mathbf{F}$ and $\mathbf{R}$ have the following representations (Problem 1.26):

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \boldsymbol{\ell}_i \otimes \mathbf{r}_i, \qquad \mathbf{R} = \sum_{i=1}^{3} \boldsymbol{\ell}_i \otimes \mathbf{r}_i. \tag{2.56}$$

Exercise: Show that

$$\mathbf{F}^{-1} = \sum_{i=1}^{3} \lambda_i^{-1} \mathbf{r}_i \otimes \boldsymbol{\ell}_i. \tag{2.57}$$

The left stretch tensor $\mathbf{V}$ is also called the **Eulerian stretch tensor**. Given a material fiber $d\mathbf{y}$ in the deformed configuration, its length in the reference configuration can be

[13] "Left" because $\mathbf{V}$ appears on the left-hand side of the expression $\mathbf{V}\mathbf{R}$.

expressed as $|\mathbf{F}^{-1}\,d\mathbf{y}| = |(\mathbf{V}\mathbf{R})^{-1}\,d\mathbf{y}| = |\mathbf{R}^T\mathbf{V}^{-1}\,d\mathbf{y}| = |\mathbf{V}^{-1}d\mathbf{y}|$ showing that it can be calculated in terms of just $\mathbf{V}$ and $d\mathbf{y}$ without involving $\mathbf{R}$.

The principal directions $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ of $\mathbf{V}$ are known as the **Eulerian principal directions** of stretch. Since

$$\mathbf{F}\mathbf{r}_i = \mathbf{R}\mathbf{U}\mathbf{r}_i = \lambda_i\mathbf{R}\mathbf{r}_i = \lambda_i\boldsymbol{\ell}_i \qquad (\text{no sum on } i)$$

it follows that a referential material fiber in a Lagrangian principal direction is mapped by the deformation into a fiber in an Eulerian principal direction.

– The principal scalar invariants of $\mathbf{U}$ and $\mathbf{V}$ coincide: $I_i(\mathbf{U}) = I_i(\mathbf{V})$.

2.5.3 Cauchy–Green deformation tensors.

– In Problem 2.5.2 where we calculate the Lagrangian stretch tensor $\mathbf{U}$ associated with a simple shear deformation, we will see that this calculation is quite tedious, mainly because of having to find the square root of $\mathbf{F}^T\mathbf{F}$. However, since there is a one-to-one relation between $\mathbf{U}$ and $\mathbf{U}^2$, and similarly between $\mathbf{V}$ and $\mathbf{V}^2$, we can just as well use $\mathbf{U}^2$ and $\mathbf{V}^2$ as our measures of stretch and these are much easier to calculate. These two tensors, usually denoted by $\mathbf{C}$ and $\mathbf{B}$,

$$\boxed{\mathbf{C} := \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \qquad \mathbf{B} := \mathbf{F}\mathbf{F}^T = \mathbf{V}^2,} \tag{2.58}$$

are referred to as the **right** and **left Cauchy-Green deformation tensors** respectively[14]. They represent Lagrangian and Eulerian measures of the deformation. Observe from (2.30) that

$$\lambda^2 = \mathbf{F}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{F}\mathbf{m}_{\mathrm{R}} \overset{(1.74)}{=} \mathbf{F}^T\mathbf{F}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}} = \mathbf{C}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}}, \tag{2.59}$$

where $\mathbf{m}_{\mathrm{R}}$ is the direction of a material fiber in the reference configuration and λ is its stretch. Likewise, from $\lambda\mathbf{F}^{-1}\mathbf{m} = \mathbf{m}_{\mathrm{R}}$ it follows that

$$\frac{1}{\lambda^2} = |\mathbf{F}^{-1}\mathbf{m}|^2 = (\mathbf{F}^{-1}\mathbf{m})\cdot(\mathbf{F}^{-1}\mathbf{m}) \overset{(1.74)}{=} (\mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{m})\cdot\mathbf{m} = \mathbf{B}^{-1}\mathbf{m} \cdot \mathbf{m}, \tag{2.60}$$

where $\mathbf{m}$ is the direction of a material fiber in the deformed configuration and λ is its stretch. Exercise: Equivalently, show that

$$|d\mathbf{y}|^2 - |d\mathbf{x}|^2 = (\mathbf{C}-\mathbf{I})d\mathbf{x}\cdot d\mathbf{x}, \qquad |d\mathbf{y}|^2 - |d\mathbf{x}|^2 = (\mathbf{I}-\mathbf{B}^{-1})d\mathbf{y}\cdot d\mathbf{y}. \tag{2.61}$$

[14] Truesdell and Toupin [9] attribute the tensor $\mathbf{B}^{-1}$ to Cauchy (1827) and the tensor $\mathbf{C}$ to Green (1841) (and also Piola (1836)).

Note that the eigenvalues of $\mathbf{C}$ and $\mathbf{B}$ are λ_1^2, λ_2^2 and λ_3^2, where the λ_i's are the principal stretches, and the eigenvectors of $\mathbf{C}$ and $\mathbf{B}$ are the same as those of $\mathbf{U}$ and $\mathbf{V}$ respectively. Thus the two Cauchy-Green tensors admit the spectral representations

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \, (\mathbf{r}_i \otimes \mathbf{r}_i), \qquad \mathbf{B} = \sum_{i=1}^{3} \lambda_i^2 \, (\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i). \tag{2.62}$$

– The principal scalar invariants of $\mathbf{C} = \mathbf{U}^2$ can be written in terms of the principal stretches as

$$I_1(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2(\mathbf{C}) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \qquad I_3(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.63}$$

The principal scalar invariants of $\mathbf{B}$ and $\mathbf{C}$ coincide: $I_i(\mathbf{C}) = I_i(\mathbf{B})$.

2.5.4 Worked examples.

Problem 2.5.1. Consider three referential material fibers oriented in mutually orthogonal directions $\mathbf{m}_R^{(1)}, \mathbf{m}_R^{(2)}, \mathbf{m}_R^{(3)}$. If the angle between each pair of these fibers remains $\pi/2$ in the deformed configuration, show that these directions are principal directions of the Lagrangian stretch tensor $\mathbf{U}$.

Solution: Since a fiber in the $\mathbf{m}_R^{(1)}$ direction remains orthogonal to fibers in the $\mathbf{m}_R^{(2)}$ and $\mathbf{m}_R^{(3)}$ directions it follows from (2.35) that

$$\mathbf{F}\mathbf{m}_R^{(1)} \cdot \mathbf{F}\mathbf{m}_R^{(2)} = 0, \qquad \mathbf{F}\mathbf{m}_R^{(1)} \cdot \mathbf{F}\mathbf{m}_R^{(3)} = 0.$$

On using (1.74) and (2.58) this tells us that

$$\mathbf{C}\mathbf{m}_R^{(1)} \cdot \mathbf{m}_R^{(2)} = 0, \qquad \mathbf{C}\mathbf{m}_R^{(1)} \cdot \mathbf{m}_R^{(3)} = 0,$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Therefore the vector $\mathbf{C}\mathbf{m}_R^{(1)}$ is perpendicular to both $\mathbf{m}_R^{(2)}$ and $\mathbf{m}_R^{(3)}$ and so must be parallel to $\mathbf{m}_R^{(1)}$. Thus $\mathbf{C}\mathbf{m}_R^{(1)} = \gamma \mathbf{m}_R^{(1)}$ for some scalar γ from which it follows that $\mathbf{m}_R^{(1)}$ is an eigenvector of the right Cauchy-Green tensor $\mathbf{C} = \mathbf{U}^2$ and therefore of $\mathbf{U}$.

That $\mathbf{m}_R^{(2)}$ and $\mathbf{m}_R^{(3)}$ are principal directions of $\mathbf{U}$ follows similarly.

Problem 2.5.2. (See also Problems 2.39 and 2.40.) Calculate the principal stretches $\lambda_1, \lambda_2, \lambda_3$ and principal Lagrangian stretch directions $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ associated with the simple shear deformation

$$y_1 = x_1 + kx_2, \quad y_2 = x_2, \quad y_3 = x_3; \qquad k > 0. \tag{i}$$

Here the components have been taken with respect to a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Graphically illustrate the simple shear deformation in the form $\mathbf{y} = \mathbf{F}\mathbf{x} = \mathbf{R}(\mathbf{U}\mathbf{x})$.

Solution: Since[15]

$$\mathbf{U} = \lambda_1 \mathbf{r}_1 \otimes \mathbf{r}_2 + \lambda_2 \mathbf{r}_2 \otimes \mathbf{r}_2 + \lambda_3 \mathbf{r}_3 \otimes \mathbf{r}_3, \qquad \mathbf{C} = \lambda_1^2 \mathbf{r}_1 \otimes \mathbf{r}_2 + \lambda_2^2 \mathbf{r}_2 \otimes \mathbf{r}_2 + \lambda_3^2 \mathbf{r}_3 \otimes \mathbf{r}_3, \tag{ii}$$

to find the λ_i 's and $\mathbf{r}_i$'s we solve the eigenvalue problem for $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$.

It follows from (i) and $F_{ij} = \partial y_i / \partial x_j$ that the deformation gradient tensor is

$$\mathbf{F} = \mathbf{I} + k\mathbf{e}_1 \otimes \mathbf{e}_2, \tag{iii}$$

and therefore from $\mathbf{C} = \mathbf{F}^T\mathbf{F}$,

$$\mathbf{C} = (\mathbf{I} + k\mathbf{e}_2 \otimes \mathbf{e}_1)(\mathbf{I} + k\mathbf{e}_1 \otimes \mathbf{e}_2) = \mathbf{e}_1 \otimes \mathbf{e}_1 + k(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + (1+k^2)\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{vi}$$

The eigenvalues of $\mathbf{C}$ are the roots λ of the equation $\det[\mathbf{C} - \lambda^2\mathbf{I}] = 0$:

$$\det[\mathbf{C} - \lambda^2\mathbf{I}] \;=\; \det \begin{pmatrix} 1-\lambda^2 & k & 0 \\ k & 1+k^2-\lambda^2 & 0 \\ 0 & 0 & 1-\lambda^2 \end{pmatrix} = (1-\lambda^2)(\lambda^4 - (2+k^2)\lambda^2 + 1) = 0. \tag{vii}$$

The roots of this equation are

$$\lambda_1^2 = \frac{2+k^2+k\sqrt{k^2+4}}{2}\ (\geq 1), \qquad \lambda_2^2 = \frac{2+k^2-k\sqrt{k^2+4}}{2}\ (\leq 1), \qquad \lambda_3^2 = 1. \tag{viii}$$

The eigenvector $\mathbf{r} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3$ corresponding to the eigenvalue λ^2 is given by $(\mathbf{C} - \lambda^2\mathbf{I})\mathbf{r} = \mathbf{o}$:

$$\begin{pmatrix} 1-\lambda^2 & k & 0 \\ k & 1+k^2-\lambda^2 & 0 \\ 0 & 0 & 1-\lambda^2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{ix}$$

For each $\lambda = \lambda_i$ this can be solved for r_1, r_2, r_3 thus leading to the eigenvectors

$$\mathbf{r}_1 = \ \cos\theta_r\ \mathbf{e}_1 + \sin\theta_r\ \mathbf{e}_2, \qquad \mathbf{r}_2 = \ -\sin\theta_r\ \mathbf{e}_1 + \cos\theta_r\ \mathbf{e}_2, \qquad \mathbf{r}_3 = \mathbf{e}_3, \tag{x}$$

where we have set

$$\tan 2\theta_r = -\frac{2}{k}, \qquad \frac{\pi}{4} \leq \theta_r < \frac{\pi}{2}. \tag{xii}$$

The angle θ_r is depicted in Figure 2.14. For the reasons given in the footnote on page 153, we anticipated $\lambda_3 = 1$ and $\mathbf{r}_3 = \mathbf{e}_3$. To find the principal stretches we simply take the square roots of $(viii)$:

$$\lambda_1 = \frac{\sqrt{k^2+4}+k}{2}\ (\geq 1), \qquad \lambda_2 = \frac{\sqrt{k^2+4}-k}{2}\ (\leq 1), \qquad \lambda_3 = 1. \qquad \square \tag{xiii}$$

We may now visualize the simple shear deformation $\mathbf{y} = \mathbf{F}\mathbf{x} = \mathbf{R}(\mathbf{U}\mathbf{x})$ in two steps as follows: First, the deformation $\mathbf{x} \mapsto \mathbf{U}\mathbf{x}$ stretches the square $OABC$ in Figure 2.15 by the amounts λ_1, λ_2 in the principal directions $\mathbf{r}_1, \mathbf{r}_2$ leading to the region $OA'B'C'$. This is then followed by the deformation $\mathbf{U}\mathbf{x} \mapsto \mathbf{R}(\mathbf{U}\mathbf{x})$ which rigidly rotates $OA'B'C'$ into the region OA_*B_*C which is the region occupied by the deformed body.

[15]Since this simple shear is a planar deformation in the x_1, x_2-plane with no stretch in the x_3-direction, one of the principal directions, say $\mathbf{r}_3$, will be $\mathbf{e}_3$ and the corresponding principal stretch $\lambda_3 = 1$. Moreover, since a simple shear is isochoric, $\det \mathbf{U} = \lambda_1\lambda_2\lambda_3 = \lambda_1\lambda_2 = 1$. Therefore one knows a priori that $\mathbf{U} = \lambda_1\mathbf{r}_1 \otimes \mathbf{r}_1 + \lambda_1^{-1}\mathbf{r}_2 \otimes \mathbf{r}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3$.

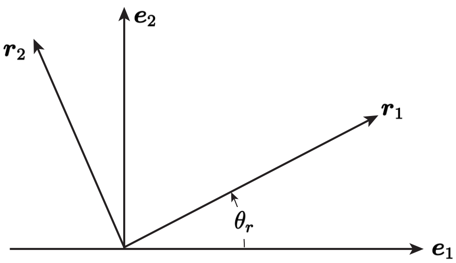


Figure 2.14: Principal directions $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ of the (right) Lagrangian stretch tensor $\mathbf{U}$. When $k \to 0$, the angle $\theta_r \to \pi/4$.

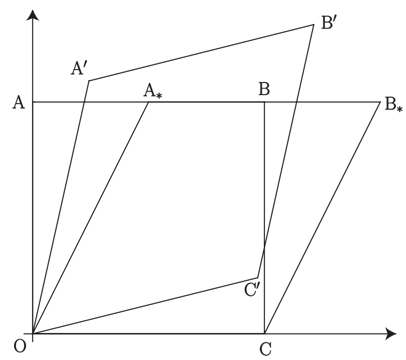


Figure 2.15: Simple shear deformation viewed in two steps: $\mathbf{y} = \mathbf{F}\mathbf{x} = \mathbf{R}(\mathbf{U}\mathbf{x})$. The pure stretch $\mathbf{x} \mapsto \mathbf{U}\mathbf{x}$ takes the region $OABC \mapsto O'A'B'C'$ and the subsequent rotation $\mathbf{U}\mathbf{x} \mapsto \mathbf{R}(\mathbf{U}\mathbf{x})$ takes $OA'B'C' \mapsto OA_*B_*C$.

Remark: The tensor $\mathbf{U}$ can be expressed with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by substituting (x) into $(ii)_1$ which leads to

$$\mathbf{U} = \frac{1}{\sqrt{4+k^2}}\Big(2\mathbf{e}_1 \otimes \mathbf{e}_1 + k(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + (2+k^2)\,\mathbf{e}_2 \otimes \mathbf{e}_2\Big) + \mathbf{e}_3 \otimes \mathbf{e}_3. \qquad (xvi)$$

The rotation tensor $\mathbf{R}$ can now be calculated using $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ which leads to

$$\mathbf{R} = \frac{1}{\sqrt{4+k^2}}\Big(2\mathbf{e}_1 \otimes \mathbf{e}_1 + k\mathbf{e}_1 \otimes \mathbf{e}_2 - k\mathbf{e}_2 \otimes \mathbf{e}_1 + 2\,\mathbf{e}_2 \otimes \mathbf{e}_2\Big) + \mathbf{e}_3 \otimes \mathbf{e}_3. \qquad (xix)$$

Problem 2.5.3. The region $\mathcal{R}_{\mathrm{R}}$ occupied by a body in a reference configuration is the unit cube

$$\mathcal{R}_{\mathrm{R}} = \{(x_1, x_2, x_3) \,:\, -1/2 \le x_1 \le 1/2,\ -1/2 \le x_2 \le 1/2,\ -1/2 \le x_3 \le 1/2\}.$$

The basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are parallel to the edges of the cube and all components are taken with respect to this basis. The body is subjected to the homogeneous deformation

$$\{y\} = [F]\{x\} \qquad \text{where} \qquad [F] = \begin{pmatrix} -a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a > 0,\ b > 0, \qquad (i)$$

that takes the particle at $(x_1, x_2, x_3) \in \mathcal{R}_{\rm R}$ to $(y_1, y_2, y_3) \in \mathcal{R}$. The negative signs are not typos.

(a) Calculate the components of the stretch tensors $\mathbf{U}, \mathbf{V}$ and the rotation tensor $\mathbf{R}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(b) Determine the principal stretches and principal directions of $\mathbf{U}$.

(c) Sketch the region $\mathcal{R}$ occupied by the body in the deformed configuration.

Solution: Observe that $[F] = [F]^T$ and therefore $[V]^2 = [B] = [F][F]^T = [F]^2$ and likewise $[U]^2 = [C] = [F]^T[F] = [F]^2$. Thus

$$[U]^2 = [V]^2 = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad [U] = [V] = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \square \tag{ii}$$

having made use of $a > 0$ and $b > 0$. Observe that $[U]$ and $[V]$ are symmetric and positive definite. The principal stretches are $a > 0, b > 0$ and 1 and the corresponding principal directions (of both $\mathbf{U}$ and $\mathbf{V}$) are $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$. $\square$

The matrix of components of the rotation tensor is

$$[R] = [F][U]^{-1} = \begin{pmatrix} -a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad \square \tag{iii}$$

The matrix $[R]$ is proper orthogonal $(\det[R] = +1)$. It represents a rotation through an angle π with the axis of rotation being $\mathbf{e}_3$ as shown in Figure 2.16.

In Figure 2.16, the points A, B, C and D in the reference configuration have the respective coordinates $(-1/2, -1/2, 0), (1/2, -1/2, 0), (1/2, 1/2, 0)$ and $(-1/2, 1/2, 0)$. The deformation maps them into the points A′, B′, C′ and D′ in the deformed configuration with coordinates $(a/2, b/2, 0), (-a/2, b/2, 0), (-a/2, -b/2, 0)$ and $(a/2, -b/2, 0)$. Observe the stretches by a and b in the $\mathbf{e}_1$- and $\mathbf{e}_2$-directions and the rotation through π about $\mathbf{e}_3$.

Problem 2.5.4. (See Chapter 3.3 for a continuation of this problem.) Bending of a block.

A body occupies the rectangular parallelepiped region $\mathcal{R}_{\rm R} = \{(x_1, x_2, x_3) \mid -A \le x_1 \le A, -B \le x_2 \le B, -C \le x_3 \le C\}$ in a reference configuration. The left-hand figure in Figure 2.17 shows a side view of this block looking down the x_3-axis. The block is subjected to a bending deformation in the x_1, x_2-plane that carries a generic particle from (x_1, x_2, x_3) to (y_1, y_2, y_3) and the region $\mathcal{R}_{\rm R} \mapsto \mathcal{R}$ as shown in Figure 2.17. Specifically, the deformation has the following characteristics:

- The body stretches uniformly in the x_3-direction in the sense that $y_3 = \Lambda x_3$ for some positive constant Λ.

- Every plane $x_3 = \mathsf{constant}$ in $\mathcal{R}_{\rm R}$ deforms identically.

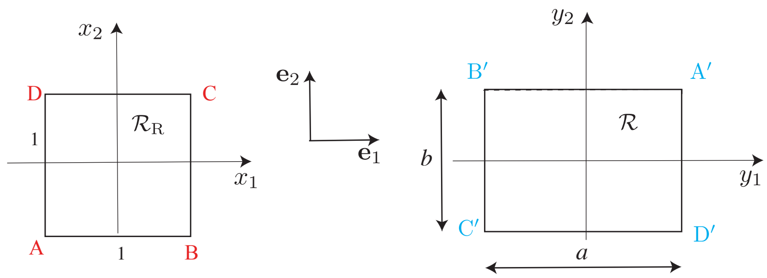


Figure 2.16: Figure for Problem 2.5.3: Mid-plane of the body looking down the x_3-axis. The body has been stretched by a, b and 1 in the $\mathbf{e}_1$-, $\mathbf{e}_2$- and $\mathbf{e}_3$-directions and rotated through an angle π about the $\mathbf{e}_3$-direction.

- Each horizontal straight line $x_2 = \mathsf{constant}$ is carried into a straight line in the deformed configuration, e.g. MN $\mapsto$ M′N′. Moreover (on each plane $x_3 = \mathsf{constant}$) the family of such straight lines corresponding to the various values of x_2 all pass through the same point $(y_1, y_2) = (0, 0)$ as depicted in Figure 2.17.
- Each vertical straight line $x_1 = \mathsf{constant}$ is deformed into a circular arc centered at $(0, 0)$ as shown in Figure 2.17, e.g. PQ $\mapsto$ P′Q′.
- The deformation is symmetric with respect to the x_1, x_3-plane.

(a) Given the shapes of the regions $\mathcal{R}$ and $\mathcal{R}_R$, and the nature of the bending deformation described, it is natural to use rectangular cartesian coordinates (x_1, x_2, x_3) and cylindrical polar coordinates (r, θ, z) to describe the undeformed and deformed configurations respectively, with associated basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. Thus express the deformation as

$$\left.\begin{aligned} y_1 &= r(x_1, x_2, x_3)\cos\theta(x_1, x_2, x_3), \\ y_2 &= r(x_1, x_2, x_3)\sin\theta(x_1, x_2, x_3), \\ y_3 &= \Lambda x_3, \end{aligned}\right\} \qquad \Lambda > 0, \qquad (i)$$

and determine the form of the functions $r(x_1, x_2, x_3)$ and $\theta(x_1, x_2, x_3)$ to the extent possible.

(b) Calculate the deformation gradient tensor $\mathbf{F}$. (Since $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are the natural bases to use when characterizing the reference and deformed configurations, you will find that the natural representation for $\mathbf{F}$ is with respect to the tensor basis $\mathbf{e}_r \otimes \mathbf{e}_1, \mathbf{e}_r \otimes \mathbf{e}_2, \ldots$.)

(c) By factoring $\mathbf{F}$ appropriately determine the stretch tensors $\mathbf{U}$ and $\mathbf{V}$ and the rotation tensor $\mathbf{R}$.

(d) What are the principal stretches and the principal Eulerian and Lagrangian directions of stretch?

Solution:

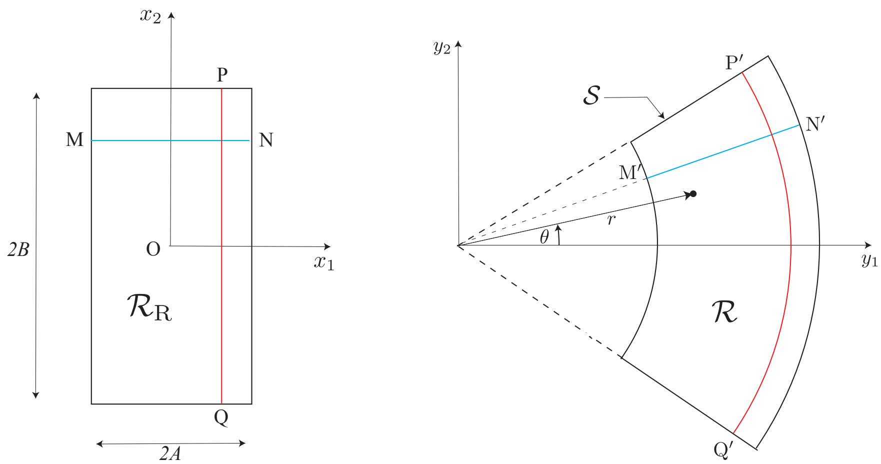


Figure 2.17: Left: In a reference configuration the body occupies the rectangular parallelepiped region $\mathcal{R}_{\mathrm{R}} = \{(x_1, x_2, x_3) : -A \le x_1 \le A, -B \le x_2 \le B, -C \le x_3 \le C\}$. Right: The region $\mathcal{R}$ in the deformed configuration. A vertical straight line in the reference configuration, e.g. PQ, is mapped into a circular arc in the deformed configuration, e.g. P′Q′. A horizontal straight line in the reference configuration, e.g. MN, is carried into a radial straight line, e.g. M′N′.

(a) Since every plane $x_3 =$ constant deforms identically, it follows that $r(x_1, x_2, x_3)$ and $\theta(x_1, x_2, x_3)$ must be independent of x_3 and so we can write the deformation as

$$y_1 = r(x_1, x_2)\cos\theta(x_1, x_2), \quad y_2 = r(x_1, x_2)\sin\theta(x_1, x_2), \quad y_3 = \Lambda x_3. \tag{ii}$$

– As one moves from M towards N along the horizontal straight line MN, the coordinate x_2 remains constant while x_1 increases. Thus considering its image M′N′, (the particle label) x_1 increases as one moves from M′ towards N′. However the orientation of M′N′, i.e. the angle $\theta(x_1, x_2)$, does not change. It follows that $\theta(x_1, x_2)$ cannot depend on x_1 and so

$$\theta = \theta(x_2). \tag{iii}$$

In view of symmetry,

$$\theta(x_2) = -\theta(-x_2). \tag{iv}$$

– Similarly, as one moves along the vertical straight line PQ, the coordinate x_1 remains constant while x_2 varies. Thus considering its image P′Q′, (the particle label) x_2 varies along it while its radius $r(x_1, x_2)$ does not change. It follows that $r(x_1, x_2)$ must be independent of x_2 and so

$$r = r(x_1). \tag{v}$$

Thus the bending deformation described in the problem is characterized by

$$y_1 = r(x_1)\cos\theta(x_2), \qquad y_2 = r(x_1)\sin\theta(x_2), \qquad y_3 = \Lambda x_3, \qquad \square \tag{vi}$$

where $r(x_1) > 0$ and $\theta(x_2) \in [-\pi, \pi]$ are arbitrary functions (with θ being an odd function).

(b) *Method 1:* The components of $\mathbf{F}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ can be calculated by differentiating (vi) with respect to x_1, x_2 and x_3 and using $F_{ij} = \partial y_i/\partial x_j$. This yields

$$\mathbf{F} = r' \cos\theta\, \mathbf{e}_1 \otimes \mathbf{e}_1 - r\theta' \sin\theta\, \mathbf{e}_1 \otimes \mathbf{e}_2 + r' \sin\theta\, \mathbf{e}_2 \otimes \mathbf{e}_1 + r\theta' \cos\theta\, \mathbf{e}_2 \otimes \mathbf{e}_2 + \Lambda\, \mathbf{e}_3 \otimes \mathbf{e}_3, \qquad \square \qquad (vii)$$

This representation of $\mathbf{F}$ does not provide much insight.

Method 2: Since it is natural to use the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ when describing the reference configuration and the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ for the deformed configuration, see Figure 2.17, we now calculate the components of the deformation gradient tensor with respect to the mixed basis $\mathbf{e}_r \otimes \mathbf{e}_1, \mathbf{e}_r \otimes \mathbf{e}_2, \ldots, \mathbf{e}_z \otimes \mathbf{e}_3$. We shall do this using the basic relation $d\mathbf{y} = \mathbf{F}d\mathbf{x}$.

First note that the basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are related to the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by

$$\mathbf{e}_r(\theta) = \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2, \qquad \mathbf{e}_\theta(\theta) = -\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2, \qquad \mathbf{e}_z = \mathbf{e}_3,$$

and therefore that

$$\frac{\partial \mathbf{e}_r}{\partial\theta} = \mathbf{e}_\theta, \qquad \frac{\partial \mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r. \qquad (viii)$$

Since the position vector of a particle in the reference configuration is

$$\mathbf{x} = \mathbf{x}(x_1, x_2, x_3) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3,$$

we have

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial x_1}\, dx_1 + \frac{\partial \mathbf{x}}{\partial x_2}\, dx_2 + \frac{\partial \mathbf{x}}{\partial x_3}\, dx_3 = \mathbf{e}_1\, dx_1 + \mathbf{e}_2\, dx_2 + \mathbf{e}_3\, dx_3. \qquad (ix)$$

We can write the position vector $\mathbf{y} = r\mathbf{e}_r + z\mathbf{e}_z$ of a particle in the deformed configuration in a little more detail as

$$\mathbf{y} = \mathbf{y}(x_1, x_2, x_3) = r(x_1)\mathbf{e}_r(\theta(x_2)) + \Lambda x_3 \mathbf{e}_z. \qquad (x)$$

Therefore

$$d\mathbf{y} = \frac{\partial \mathbf{y}}{\partial x_1}\, dx_1 + \frac{\partial \mathbf{y}}{\partial x_2}\, dx_2 + \frac{\partial \mathbf{y}}{\partial x_3}\, dx_3 \overset{(x)}{=} r'\, \mathbf{e}_r\, dx_1 + r\theta'\, \mathbf{e}_\theta\, dx_2 + \Lambda\, \mathbf{e}_z\, dx_3, \qquad (xi)$$

where we have used $(viii)_1$ and let a prime denote differentiation with respect to the argument. We finally replace dx_i in (xi) by $d\mathbf{x} \cdot \mathbf{e}_i$ which is a consequence of (ix). This leads to

$$\begin{aligned} d\mathbf{y} &= r'(d\mathbf{x} \cdot \mathbf{e}_1)\, \mathbf{e}_r + r\theta'(d\mathbf{x} \cdot \mathbf{e}_2)\, \mathbf{e}_\theta + \Lambda(d\mathbf{x} \cdot \mathbf{e}_3)\, \mathbf{e}_z = \\ &= r'(\mathbf{e}_r \otimes \mathbf{e}_1)\, d\mathbf{x} + r\theta'(\mathbf{e}_\theta \otimes \mathbf{e}_2)\, d\mathbf{x} + \Lambda(\mathbf{e}_z \otimes \mathbf{e}_3) d\mathbf{x} \\ &= \Big[r'(\mathbf{e}_r \otimes \mathbf{e}_1) + r\theta'(\mathbf{e}_\theta \otimes \mathbf{e}_2) + \Lambda(\mathbf{e}_z \otimes \mathbf{e}_3) \Big] d\mathbf{x} \\ &= \mathbf{F}\, d\mathbf{x}, \end{aligned}$$

where

$$\mathbf{F} = r'(x_1)\, (\mathbf{e}_r \otimes \mathbf{e}_1) + r\theta'(x_2)\, (\mathbf{e}_\theta \otimes \mathbf{e}_2) + \Lambda\, (\mathbf{e}_z \otimes \mathbf{e}_13). \qquad \square \qquad (xii)$$

Remark: Observe that we can write (vii) as

$$\mathbf{F} = r'(\mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta) \otimes \mathbf{e}_1 + r\theta'(-\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta) \otimes \mathbf{e}_2 + \Lambda\, \mathbf{e}_3 \otimes \mathbf{e}_3.$$

Substituting the expressions in the line above $(viii)$ into this gives (xii).

(c) The expression (xii) for $\mathbf{F}$ can be factored in two ways[16]:

$$\mathbf{F} = (\mathbf{e}_r \otimes \mathbf{e}_1 + \mathbf{e}_\theta \otimes \mathbf{e}_2 + \mathbf{e}_z \otimes \mathbf{e}_3)(r'\mathbf{e}_1 \otimes \mathbf{e}_1 + r\theta'\mathbf{e}_2 \otimes \mathbf{e}_2 + \Lambda\mathbf{e}_3 \otimes \mathbf{e}_3), \tag{xiii}$$

$$\mathbf{F} = (r'\mathbf{e}_r \otimes \mathbf{e}_r + r\theta'\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \Lambda\mathbf{e}_z \otimes \mathbf{e}_z)(\mathbf{e}_r \otimes \mathbf{e}_1 + \mathbf{e}_\theta \otimes \mathbf{e}_2 + \mathbf{e}_z \otimes \mathbf{e}_3). \tag{xiv}$$

Since $\mathbf{e}_r \otimes \mathbf{e}_1 + \mathbf{e}_\theta \otimes \mathbf{e}_2 + \mathbf{e}_z \otimes \mathbf{e}_3$ is a proper orthogonal tensor and the tensors multiplying them are symmetric and positive definite, we recognize $(xiii)$ and (xiv) as the respective polar decompositions $\mathbf{F} = \mathbf{RU}$ and $\mathbf{F} = \mathbf{VR}$. Therefore the rotation tensor $\mathbf{R}$ in the polar decomposition is

$$\mathbf{R} = \mathbf{e}_r \otimes \mathbf{e}_1 + \mathbf{e}_\theta \otimes \mathbf{e}_2 + \mathbf{e}_z \otimes \mathbf{e}_3, \qquad \square \tag{xv}$$

which is in fact the rotation that takes $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ into $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. Then, from $\mathbf{F} = \mathbf{RU}$ and $(xiii)$ we find

$$\mathbf{U} = r'\mathbf{e}_1 \otimes \mathbf{e}_1 + r\theta'\mathbf{e}_2 \otimes \mathbf{e}_2 + \Lambda\mathbf{e}_3 \otimes \mathbf{e}_3, \qquad \square \tag{xvi}$$

and similarly from $\mathbf{F} = \mathbf{VR}$ and (xiv) we obtain

$$\mathbf{V} = r'\mathbf{e}_r \otimes \mathbf{e}_r + r\theta'\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \Lambda\mathbf{e}_z \otimes \mathbf{e}_z. \qquad \square \tag{xvii}$$

Remark: Since the principal stretches are positive, we have assumed $r' > 0$ and $\theta' > 0$. What would $\mathbf{U}$ and $\mathbf{V}$ be if one or both of r' and θ' was negative?

(d) It follows from (xvi) and $(xvii)$ that the principal stretches are

$$\lambda_1 = r'(x_1), \qquad \lambda_2 = r(x_1)\theta'(x_2), \qquad \lambda_3 = \Lambda. \qquad \square \tag{xviii}$$

Moreover (xvi) shows that the principal directions of $\mathbf{U}$ are $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, while $(xvii)$ gives the principal directions of $\mathbf{V}$ to be $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. $\square$

Remark: Observe that the principal stretches depend on *both* x_1 and x_2 and therefore vary in both the radial and circumferential directions.

2.6 Strain.

It is clear that $\mathbf{U}$ (or $\mathbf{V}$) is the essential ingredient that characterizes the non-rigid part of the deformation gradient. “Strain” is simply an alternative measure of this part of the deformation, the only essential distinction between strain and stretch being that (by convention) the strain vanishes in a rigid deformation whereas the stretch equals the identity $\mathbf{I}$. Thus for *example* we might say that $\mathbf{U} - \mathbf{I}$ is the strain where $\mathbf{U}$ is the stretch.

[16] Alternatively one can calculate $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ from (xii) and then calculate $\mathbf{U}$ and $\mathbf{V}$ from the results.

Various measures of strain are used in the literature, examples of which we shall describe below. It should be pointed out that the continuum theory does not prefer[17] one strain measure over another; each is a one-to-one function of the stretch tensor and so all strain measures are equivalent. In fact, one does not even have to introduce the notion of strain and the theory could be based entirely on the stretch tensors $\mathbf{U}$ and $\mathbf{V}$.

The various measures of **Lagrangian strain** used in the literature are all related to the Lagrangian stretch $\mathbf{U}$ in a one-to-one manner. Examples include the Green Saint-Venant strain tensor, the Biot strain tensor, the generalized Green Saint-Venant (Seth-Hill) strain tensor and the Hencky (or logarithmic) strain tensor, defined by the respective expressions

$$
\begin{aligned}
&\text{Green Saint-Venant:} && \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) = \sum_{i=1}^{3} \frac{1}{2}(\lambda_i^2 - 1)\, \mathbf{r}_i \otimes \mathbf{r}_i, \\
&\text{Biot :} && \mathbf{U} - \mathbf{I} = \sum_{i=1}^{3} (\lambda_i - 1)\, \mathbf{r}_i \otimes \mathbf{r}_i, \\
& && \frac{1}{m}(\mathbf{U}^m - \mathbf{I}) = \sum_{i=1}^{3} \frac{1}{m}(\lambda_i^m - 1)\, \mathbf{r}_i \otimes \mathbf{r}_i, \quad m \neq 0, \\
&\text{Hencky :} && \ln \mathbf{U} = \sum_{i=1}^{3} \ln\, \lambda_i\, \mathbf{r}_i \otimes \mathbf{r}_i,
\end{aligned}
\tag{2.64}
$$

where m is a non-zero (positive or negative) integer; the generalized Green Saint-Venant strain tensor can be extended to non-integer values of m by defining it to be the tensor on the right-hand side of $(2.64)_3$. Observe that all of these strain measures vanish in a rigid deformation, i.e. when $\mathbf{U} = \mathbf{I}$. They are all symmetric and their principal directions are the Lagrangian principal directions $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$; the associated **principal strains** are

$$
\tfrac{1}{2}(\lambda_i^2 - 1), \qquad \lambda_i - 1, \qquad \tfrac{1}{m}(\lambda_i^m - 1), \qquad \text{and} \qquad \ln \lambda_i, \tag{2.65}
$$

respectively. In Section 2.6.1 (see also Problem 2.42) we will calculate the components of the Green Saint-Venant strain tensor and specialize them to simple shear.

Similarly, various measures of **Eulerian strain** are used in the literature, all of them being related to the Eulerian stretch $\mathbf{V}$ in a one-to-one manner. Examples include the Almansi strain, the generalized Almansi strain and the logarithmic strain, defined by the

[17]It may happen that the constitutive relation for a particular material takes an especially simple form when one particular strain measure is used in its characterization, while a different strain measure might lead to a simple constitutive description for some other material. This might then lead to a preference for one strain measure over another for a particular material.

respective expressions

$$\begin{aligned}
\frac{1}{2}(\mathbf{I}-\mathbf{V}^{-2}) &= \sum_{i=1}^{3}\frac{1}{2}(1-\lambda_i^{-2})\,\boldsymbol{\ell}_i\otimes\boldsymbol{\ell}_i\\
\frac{1}{m}(\mathbf{V}^{m}-\mathbf{I}) &= \sum_{i=1}^{3}\frac{1}{m}(\lambda_i^{m}-1)\,\boldsymbol{\ell}_i\otimes\boldsymbol{\ell}_i, \qquad (2.66)\\
\ln\mathbf{V} &= \sum_{i=1}^{3}\ln\,\lambda_i\,\boldsymbol{\ell}_i\otimes\boldsymbol{\ell}_i,
\end{aligned}$$

where m is a non-zero integer. The principal directions of all of these symmetric strain tensors are the Eulerian principal directions $\{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\boldsymbol{\ell}_3\}$.

The preceding examples may be *unified and generalized* as follows: Let $e(\lambda)$ be an arbitrary (for the moment) scalar-valued function defined for $0<\lambda<\infty$, and consider defining the Lagrangian strain tensor $\mathbf{E}(\mathbf{U})$ to be the tensor with eigenvectors $\mathbf{r}_i$ and corresponding eigenvalues $e(\lambda_i)$, i.e.

$$\mathbf{E}(\mathbf{U}) = e(\lambda_1)\,\mathbf{r}_1\otimes\mathbf{r}_1 \;+\; e(\lambda_2)\,\mathbf{r}_2\otimes\mathbf{r}_2 \;+\; e(\lambda_3)\,\mathbf{r}_3\otimes\mathbf{r}_3. \qquad (2.67)$$

Note that the **principal strains** associated with this tensor are $e(\lambda_1), e(\lambda_2)$ and $e(\lambda_3)$ and the corresponding principal directions are $\mathbf{r}_1,\mathbf{r}_2$ and $\mathbf{r}_3$.

In the undeformed configuration the principal stretches are unity and we would like the strain to vanish there. Thus we require $e(1)=0$. Next consider a "small deformation" of the body in which λ is close to unity. Then Taylor expansion of the function $e(\lambda)$ about $\lambda=1$ leads to

$$e(\lambda) = e(1)+e'(1)(\lambda-1)+\ldots = e'(1)\left(\frac{ds_y}{ds_x}-1\right)+\ldots = e'(1)\;\frac{ds_y-ds_x}{ds_x}+\ldots. \qquad (2.68)$$

In order that this coincide with the familiar definition of normal strain in an infinitesimal deformation, i.e. in order that $e\approx(ds_y-ds_x)/ds_x$, we must have $e'(1)=1$. Finally we require the normal strain $e(\lambda)$ to increase monotonically as the stretch λ increases and so we impose $e'(\lambda)>0$. Note then that the principal strain $e(\lambda)$ is positive for extensions $(\lambda>1)$ and negative for contractions $(\lambda<1)$.

Thus we define a Lagrangian strain $\mathbf{E}(\mathbf{U})$ by (2.67) where the function $e(\lambda)$ is required to have the properties

$$\begin{aligned}
&\text{a)}\quad e(1) \;=\; 0,\\
&\text{b)}\quad e'(1) \;=\; 1, \qquad (2.69)\\
&\text{c)}\quad e'(\lambda) \;>\; 0 \quad \text{for all} \quad \lambda>0.
\end{aligned}$$

Exercise: Is $\frac{1}{2n}\left(\mathbf{U}^n - \mathbf{U}^{-n}\right), n \neq 0$, an acceptable Lagrangian strain measure? Note that it is not a special case of $(2.64)_3$.

Observe that all Lagrangian strain tensors defined by (2.67) are symmetric. Their diagonal components E_{11}, E_{22} and E_{33} are known as the *normal* components of strain, while the off-diagonal components E_{12}, E_{23} and E_{31} are the *shear* components of strain. In the *principal basis* $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$, the matrix $[E]$ of strain components is diagonal and so the shear strains vanish in this basis, the normal components being the *principal strains*.

A generalized Eulerian strain tensor can be defined analogously by

$$\mathcal{E}(\mathbf{V}) = e(\lambda_1)\, \boldsymbol{\ell}_1 \otimes \boldsymbol{\ell}_1 \;+\; e(\lambda_2)\, \boldsymbol{\ell}_2 \otimes \boldsymbol{\ell}_2 \;+\; e(\lambda_3)\, \boldsymbol{\ell}_3 \otimes \boldsymbol{\ell}_3, \tag{2.70}$$

where $\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3$ are the principal directions of the Eulerian stretch tensor $\mathbf{V}$ and $e(\lambda)$ obeys (2.69).

2.6.1 Remarks on the Green Saint-Venant strain tensor.

While, as noted already, the theory does not prefer one strain measure over another, the Green Saint-Venant strain tensor has been used frequently in the (especially older) literature. It is therefore worth devoting some attention to it. From (2.58) and (2.64) the Green Saint-Venant strain tensor is

$$\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}). \tag{2.71}$$

- First we wish to express $\mathbf{E}$ in terms of the displacement gradient tensor $\nabla\mathbf{u}$. Since $\mathbf{y} = \mathbf{x} + \mathbf{u}$ it follows that

  $$\mathbf{F} = \nabla\mathbf{y} = \nabla(\mathbf{x} + \mathbf{u}) = \mathbf{I} + \nabla\mathbf{u}.$$

  Substituting this into (2.71) yields

  $$\mathbf{E} = \; \frac{1}{2}\Big((\mathbf{I} + \nabla\mathbf{u})^T(\mathbf{I} + \nabla\mathbf{u}) - \mathbf{I}\Big) = \frac{1}{2}\Big(\nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T\nabla\mathbf{u}\Big). \tag{2.72}$$

  Since $\nabla\mathbf{u}$ is the tensor with cartesian components $\partial u_i/\partial x_j$, this can be written in terms of cartesian components as

  $$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right), \tag{2.73}$$

  where the summation convention is being used on the repeated index k.

- Second we wish to interpret the components of the Green Saint-Venant strain tensor in terms of changes in length and changes in angle. Recall that the stretch of a material fiber in the direction $\mathbf{m}_\mathrm{R}$ in the reference configuration is

  $$\lambda(\mathbf{m}_\mathrm{R}) \;\stackrel{(2.30)}{=}\; |\mathbf{F}\mathbf{m}_\mathrm{R}| = \sqrt{\mathbf{F}\mathbf{m}_\mathrm{R}\cdot\mathbf{F}\mathbf{m}_\mathrm{R}} = \sqrt{\mathbf{F}^T\mathbf{F}\mathbf{m}_\mathrm{R}\cdot\mathbf{m}_\mathrm{R}} \;\stackrel{(2.71)}{=}\; \sqrt{(2\mathbf{E}\mathbf{m}_\mathrm{R}\cdot\mathbf{m}_\mathrm{R} + 1)}. \qquad (i)$$

Now consider a fiber oriented in the direction $\mathbf{m}_\mathrm{R} = \mathbf{e}_1$. Its stretch is

$$\lambda(\mathbf{e}_1) = |\mathbf{F}\mathbf{e}_1| = \sqrt{(2\mathbf{E}\mathbf{e}_1 \cdot \mathbf{e}_1 + 1)} \overset{(1.128)}{=} \sqrt{(2E_{11}+1)}. \qquad (ii)$$

Since the stretch λ represents the ratio of the deformed and undeformed lengths of the fiber, i.e. $\lambda = ds_y/ds_x$, it now follows that

$$\frac{ds_y - ds_x}{ds_x} = \lambda(\mathbf{e}_1) - 1 = \sqrt{(2E_{11}+1)} - 1. \qquad (2.74)$$

Thus we conclude that the change in length relative to the original length of a fiber in the direction $\mathbf{e}_1$ depends only on the normal strain E_{11}. It is not equal to E_{11} but is fully determined by E_{11}. If the normal strain $|E_{11}| \ll 1$, this approximates to the familiar expression

$$\frac{ds_y - ds_x}{ds_x} \approx E_{11}. \qquad (iii)$$

Analogous calculations may be carried out for fibers in the directions $\mathbf{e}_2$ and $\mathbf{e}_3$.

Now consider the change in angle between two fibers in the directions $\mathbf{e}_1$ and $\mathbf{e}_2$. From (2.35),

$$\cos\theta_y = \frac{\mathbf{F}\mathbf{e}_1 \cdot \mathbf{F}\mathbf{e}_2}{|\mathbf{F}\mathbf{e}_1|\,|\mathbf{F}\mathbf{e}_2|} = \frac{\mathbf{F}^T\mathbf{F}\mathbf{e}_1 \cdot \mathbf{e}_2}{|\mathbf{F}\mathbf{e}_1|\,|\mathbf{F}\mathbf{e}_2|} \overset{(ii)}{=} \frac{\mathbf{F}^T\mathbf{F}\mathbf{e}_1 \cdot \mathbf{e}_2}{\sqrt{(2E_{11}+1)}\,\sqrt{(2E_{22}+1)}} \overset{(2.71)}{=} \frac{2\mathbf{E}\mathbf{e}_1 \cdot \mathbf{e}_2}{\sqrt{(2E_{11}+1)}\,\sqrt{(2E_{22}+1)}}, \qquad (2.75)$$

where θ_y is the angle between the two fibers in the deformed configuration. In view of (1.128) this yields

$$\cos\theta_y = \frac{2\,E_{12}}{\sqrt{(2E_{11}+1)}\,\sqrt{(2E_{22}+1)}}, \qquad (2.76)$$

which shows that the angle between these two fibers in the deformed configuration depends on the shear strain E_{12} *and* the normal strains E_{11} and E_{22}. If the strains are small, i.e. $|E_{11}| \ll 1, |E_{22}| \ll 1$ and $|E_{12}| \ll 1$, this approximates to leading order to the familiar expression

$$\theta_x - \theta_y \approx 2E_{12},$$

where $\theta_x = \pi/2$.

– It is illuminating to calculate the components of $\mathbf{E}$ in a simple shear deformation

$$y_1 = x_1 + kx_2, \qquad y_2 = x_2, \qquad y_3 = x_3.$$

Differentiating this with respect to x_j and using $F_{ij} = \partial y_i/\partial x_j$ leads to

$$\mathbf{F} = \mathbf{I} + k\mathbf{e}_1 \otimes \mathbf{e}_2, \qquad \mathbf{C} = \mathbf{I} + k(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + k^2\mathbf{e}_2 \otimes \mathbf{e}_2,$$

and so the components of the Green Saint-Venant strain tensor are

$$[E] = \frac{1}{2}\big([C] - [I]\big) = \begin{pmatrix} 0 & k/2 & 0 \\ k/2 & k^2/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Observe that the normal strain $E_{22} \neq 0$. This is related to the fact that the line OP in Figure 2.4 increases in length when it deforms into OP'. Note that $E_{22} = O(k^2)$ so that in a linearized theory with a small amount of shear, $|k| \ll 1$, this term would be neglected, leading to a strain tensor whose only nonzero components are the shear strains E_{12} and E_{21}.

2.7 Some other coordinate systems.

2.7.1 Cylindrical polar coordinates.

In this section we illustrate working in other coordinate systems by calculating expressions for the deformation gradient tensor and the left Cauchy-Green deformation tensor in cylindrical polar coordinates.

In rectangular cartesian coordinates, a deformation is characterized by the mapping

$$y_1 = \widehat{y}_1(x_1, x_2, x_3), \quad y_2 = \widehat{y}_2(x_1, x_2, x_3) \quad y_3 = \widehat{y}_3(x_1, x_2, x_3), \qquad (i)$$

that takes the particle with coordinates (x_1, x_2, x_3) in the reference configuration to the point (y_1, y_2, y_3) in the deformed configurations. Let (R, Θ, Z) be the cylindrical polar coordinates of this particle in the undeformed configuration so that (see Figure 2.18)

$$x_1 = R\cos\Theta, \quad x_2 = R\sin\Theta, \quad x_3 = Z, \qquad (ii)$$

and let (r, θ, z) be its cylindrical polar coordinates in the deformed configuration so that

$$y_1 = r\cos\theta, \quad y_2 = r\sin\theta, \quad y_3 = z. \qquad (iii)$$

By combining (i) with (ii) and (iii) we can characterize the deformation in the form,

$$r = \widehat{r}(R, \Theta, Z), \qquad \theta = \widehat{\theta}(R, \Theta, Z), \qquad z = \widehat{\theta}(R, \Theta, Z). \qquad (iv)$$

The basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ associated with the cylindrical polar coordinates (R, Θ, Z) is shown in Figure 2.18 and is related to the fixed cartesian basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by

$$\left.\begin{aligned} \mathbf{e}_R(\Theta) &= \cos\Theta\ \mathbf{e}_1 + \sin\Theta\ \mathbf{e}_2, \\ \mathbf{e}_\Theta(\Theta) &= -\sin\Theta\ \mathbf{e}_1 + \cos\Theta\ \mathbf{e}_2, \\ \mathbf{e}_Z &= \mathbf{e}_3, \end{aligned}\right\} \qquad (v)$$

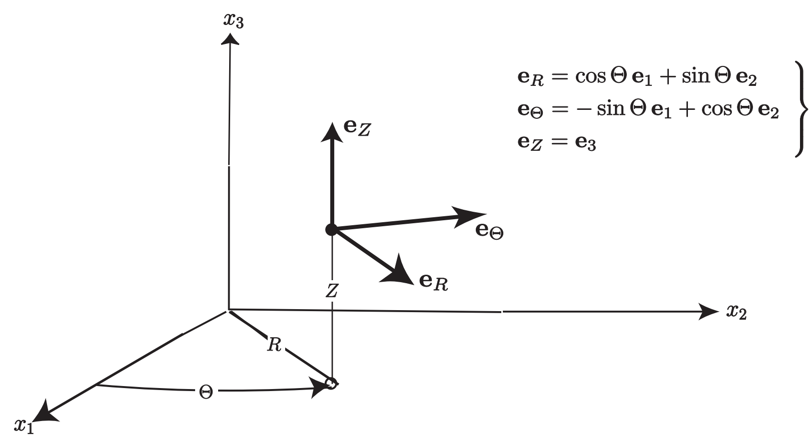


Figure 2.18: Cylindrical polar coordinates (R, Θ, Z) and the associated basis vectors $\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z$.

while the corresponding relation for the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ associated with the coordinates in the deformed configuration is

$$\left.\begin{aligned} \mathbf{e}_r(\theta) &= \cos\theta\ \mathbf{e}_1 + \sin\theta\ \mathbf{e}_2, \\ \mathbf{e}_\theta(\theta) &= -\sin\theta\ \mathbf{e}_1 + \cos\theta\ \mathbf{e}_2, \\ \mathbf{e}_z &= \mathbf{e}_3. \end{aligned}\right\} \qquad (vi)$$

Observe that in general, the basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ differs from the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$.

The position vectors of a particle in the undeformed and deformed configurations are, see Figure 2.18, $\mathbf{x} = R\,\mathbf{e}_R + Z\,\mathbf{e}_Z$ and $\mathbf{y} = r\mathbf{e}_r + z\mathbf{e}_z$, or in a little more detail,

$$\mathbf{x} = \widehat{\mathbf{x}}(R, \Theta, Z) = R\,\mathbf{e}_R + Z\,\mathbf{e}_Z, \qquad (vii)$$

$$\mathbf{y} = \widehat{\mathbf{y}}(R, \Theta, Z) = \Big(r\mathbf{e}_r + z\mathbf{e}_z\Big)\Big|_{r=\widehat{r},\ \theta=\widehat{\theta},\ z=\widehat{z}}\,, \qquad (viii)$$

respectively. We wish to calculate the deformation gradient tensor using $(iv), (vii)$ and $(viii)$. The general approach involves calculating the vectors $d\mathbf{y}$ and $d\mathbf{x}$ independently and then recognizing that they are related by $d\mathbf{y} = \mathbf{F}\,d\mathbf{x}$. A general treatment of orthogonal curvilinear coordinates can be found in Chapter 6 of Volume I.

We now proceed to calculate $d\mathbf{y}$. First, from $(viii)$ and the chain rule

$$d\mathbf{y} = \frac{\partial\widehat{\mathbf{y}}}{\partial R}dR + \frac{\partial\widehat{\mathbf{y}}}{\partial\Theta}d\Theta + \frac{\partial\widehat{\mathbf{y}}}{\partial Z}dZ. \qquad (ix)$$

Next, we calculate each term on the right-hand side of (ix):

$$\frac{\partial\widehat{\mathbf{y}}}{\partial R} \overset{(viii)}{=} \frac{\partial}{\partial R}(r\mathbf{e}_r + z\mathbf{e}_z) = \frac{\partial r}{\partial R}\mathbf{e}_r + r\frac{\partial\mathbf{e}_r}{\partial R} + \frac{\partial z}{\partial R}\mathbf{e}_z = \frac{\partial r}{\partial R}\mathbf{e}_r + r\frac{\partial\mathbf{e}_r}{\partial\theta}\frac{\partial\theta}{\partial R} + \frac{\partial z}{\partial R}\mathbf{e}_z,$$

where in getting to the second equality we used the fact that $\mathbf{e}_z$ does not depend on R, and in getting to the last equality we used the fact that according to (vi) the unit vector $\mathbf{e}_r$ depends on θ but not on r or z. Observe from (vi) that $\partial\mathbf{e}_r/\partial\theta = \mathbf{e}_\theta$. Thus we can write the preceding equation as

$$\frac{\partial\widehat{\mathbf{y}}}{\partial R} = \frac{\partial r}{\partial R}\mathbf{e}_r + r\frac{\partial\theta}{\partial R}\mathbf{e}_\theta + \frac{\partial z}{\partial R}\mathbf{e}_z. \tag{x}$$

The other terms in (ix) can be calculated similarly leading to

$$\frac{\partial\widehat{\mathbf{y}}}{\partial \Theta} = \frac{\partial r}{\partial \Theta}\mathbf{e}_r + r\frac{\partial\theta}{\partial \Theta}\mathbf{e}_\theta + \frac{\partial z}{\partial \Theta}\mathbf{e}_z, \tag{xi}$$

$$\frac{\partial\widehat{\mathbf{y}}}{\partial Z} = \frac{\partial r}{\partial Z}\mathbf{e}_r + r\frac{\partial\theta}{\partial Z}\mathbf{e}_\theta + \frac{\partial z}{\partial Z}\mathbf{e}_z. \tag{xii}$$

Substituting $(x), (xi), (xii)$ into (ix) now leads to

$$\begin{aligned} d\mathbf{y} \;\; &= \frac{\partial r}{\partial R}dR\,\mathbf{e}_r + r\frac{\partial\theta}{\partial R}dR\,\mathbf{e}_\theta + \frac{\partial z}{\partial R}dR\,\mathbf{e}_z + \\ &+ \frac{\partial r}{\partial \Theta}d\Theta\,\mathbf{e}_r + r\frac{\partial\theta}{\partial \Theta}d\Theta\,\mathbf{e}_\theta + \frac{\partial z}{\partial \Theta}d\Theta\,\mathbf{e}_z + \\ &+ \frac{\partial r}{\partial Z}dZ\,\mathbf{e}_r + r\frac{\partial\theta}{\partial Z}dZ\,\mathbf{e}_\theta + \frac{\partial z}{\partial Z}dZ\,\mathbf{e}_z. \end{aligned} \tag{$xiii$}$$

The next step is to express $dR, d\Theta$ and dZ in $(xiii)$ in terms of $d\mathbf{x}$. From (vii)

$$d\mathbf{x} = \frac{\partial\widehat{\mathbf{x}}}{\partial R}dR + \frac{\partial\widehat{\mathbf{x}}}{\partial \Theta}d\Theta + \frac{\partial\widehat{\mathbf{x}}}{\partial Z}dZ, \tag{xiv}$$

where

$$\begin{aligned} \frac{\partial\widehat{\mathbf{x}}}{\partial R} &= \frac{\partial}{\partial R}(R\mathbf{e}_R + Z\mathbf{e}_Z) \;\overset{(v)}{=}\; \frac{\partial R}{\partial R}\,\mathbf{e}_R = \mathbf{e}_R, \\ \frac{\partial\widehat{\mathbf{x}}}{\partial \Theta} &= \frac{\partial}{\partial \Theta}(R\mathbf{e}_R + Z\mathbf{e}_Z) \;\overset{(v)}{=}\; R\frac{\partial\mathbf{e}_R}{\partial\Theta} \;\overset{(v)}{=}\; R\mathbf{e}_\Theta, \\ \frac{\partial\widehat{\mathbf{x}}}{\partial Z} &= \frac{\partial}{\partial Z}(R\mathbf{e}_R + Z\mathbf{e}_Z) \;\overset{(v)}{=}\; \mathbf{e}_Z. \end{aligned}$$

Therefore (xiv) can be written as

$$d\mathbf{x} = dR\,\mathbf{e}_R + Rd\Theta\,\mathbf{e}_\Theta + dZ\,\mathbf{e}_Z, \tag{xv}$$

from which we obtain

$$dR = d\mathbf{x}\cdot\mathbf{e}_R, \qquad d\Theta = R^{-1}\,d\mathbf{x}\cdot\mathbf{e}_\Theta, \qquad dZ = d\mathbf{x}\cdot\mathbf{e}_Z. \tag{xvi}$$

Substituting (xvi) into $(xiii)$ gives

$$
\begin{aligned}
d\mathbf{y} \;&= \frac{\partial r}{\partial R}(d\mathbf{x}\cdot\mathbf{e}_R)\,\mathbf{e}_r + r\frac{\partial \theta}{\partial R}(d\mathbf{x}\cdot\mathbf{e}_R)\,\mathbf{e}_\theta + \frac{\partial z}{\partial R}(d\mathbf{x}\cdot\mathbf{e}_R)\,\mathbf{e}_z +\\
&+\frac{\partial r}{\partial \Theta}(R^{-1}\,d\mathbf{x}\cdot\mathbf{e}_\Theta)\,\mathbf{e}_r + r\frac{\partial \theta}{\partial \Theta}(R^{-1}\,d\mathbf{x}\cdot\mathbf{e}_\Theta)\,\mathbf{e}_\theta + \frac{\partial z}{\partial \Theta}(R^{-1}\,d\mathbf{x}\cdot\mathbf{e}_\Theta)\,\mathbf{e}_z\\
&+\frac{\partial r}{\partial Z}(d\mathbf{x}\cdot\mathbf{e}_Z)\,\mathbf{e}_r + r\frac{\partial \theta}{\partial Z}(d\mathbf{x}\cdot\mathbf{e}_Z)\,\mathbf{e}_\theta + \frac{\partial z}{\partial Z}(d\mathbf{x}\cdot\mathbf{e}_Z)\,\mathbf{e}_z,
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
d\mathbf{y} \;&= \frac{\partial r}{\partial R}(\mathbf{e}_r\otimes\mathbf{e}_R)\,d\mathbf{x} + r\frac{\partial \theta}{\partial R}(\mathbf{e}_\theta\otimes\mathbf{e}_R)\,d\mathbf{x} + \frac{\partial z}{\partial R}(\mathbf{e}_z\otimes\mathbf{e}_R)\,d\mathbf{x}+\\
&+\frac{1}{R}\frac{\partial r}{\partial \Theta}(\mathbf{e}_r\otimes\mathbf{e}_\Theta)\,d\mathbf{x} + \frac{r}{R}\frac{\partial \theta}{\partial \Theta}(\mathbf{e}_\theta\otimes\mathbf{e}_\Theta)\,d\mathbf{x} + \frac{1}{R}\frac{\partial z}{\partial \Theta}(\mathbf{e}_z\otimes\mathbf{e}_\Theta)\,d\mathbf{x}+\\
&+\frac{\partial r}{\partial Z}(\mathbf{e}_r\otimes\mathbf{e}_Z)\,d\mathbf{x} + r\frac{\partial \theta}{\partial Z}(\mathbf{e}_\theta\otimes\mathbf{e}_Z)\,d\mathbf{x} + \frac{\partial z}{\partial Z}(\mathbf{e}_z\otimes\mathbf{e}_Z)\,d\mathbf{x}.
\end{aligned}
$$

Since $d\mathbf{y} = \mathbf{F}d\mathbf{x}$ we can now read off the deformation gradient tensor $\mathbf{F}$ to be

$$
\begin{aligned}
\mathbf{F} \;&= \frac{\partial r}{\partial R}(\mathbf{e}_r\otimes\mathbf{e}_R) + \frac{1}{R}\frac{\partial r}{\partial \Theta}(\mathbf{e}_r\otimes\mathbf{e}_\Theta) + \frac{\partial r}{\partial Z}(\mathbf{e}_r\otimes\mathbf{e}_Z)+\\
&+r\frac{\partial \theta}{\partial R}(\mathbf{e}_\theta\otimes\mathbf{e}_R) + \frac{r}{R}\frac{\partial \theta}{\partial \Theta}(\mathbf{e}_\theta\otimes\mathbf{e}_\Theta) + r\frac{\partial \theta}{\partial Z}(\mathbf{e}_\theta\otimes\mathbf{e}_Z)+\\
&+\frac{\partial z}{\partial R}(\mathbf{e}_z\otimes\mathbf{e}_R) \;+\frac{1}{R}\frac{\partial z}{\partial \Theta}(\mathbf{e}_z\otimes\mathbf{e}_\Theta) \;+\frac{\partial z}{\partial Z}(\mathbf{e}_z\otimes\mathbf{e}_Z)\,.
\end{aligned}
\tag{2.77}
$$

Observe that the representation (2.77) involves both bases $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. The scalar coefficients are not the components of $\mathbf{F}$ in either basis. See Problem 1.6.5.

The left and right Cauchy Green deformation tensors $\mathbf{B}$ and $\mathbf{C}$ can now be readily calculated. We will see below that when (2.77) is used to calculate $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ the result comes out in terms of the components of $\mathbf{B}$ in the basis $\mathbf{e}_r\otimes\mathbf{e}_r, \mathbf{e}_r\otimes\mathbf{e}_\theta, \ldots \mathbf{e}_z\otimes\mathbf{e}_z$. On the other hand when $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is calculated the result emerges in terms of the components of $\mathbf{C}$ in the basis $\mathbf{e}_R\otimes\mathbf{e}_R, \mathbf{e}_R\otimes\mathbf{e}_\Theta, \ldots \mathbf{e}_Z\otimes\mathbf{e}_Z$. This is consistent with our previous reference to $\mathbf{B}$ as an Eulerian deformation tensor and $\mathbf{C}$ as a Lagrangian deformation tensor. Of course one can express either tensor in any basis of one's choice.

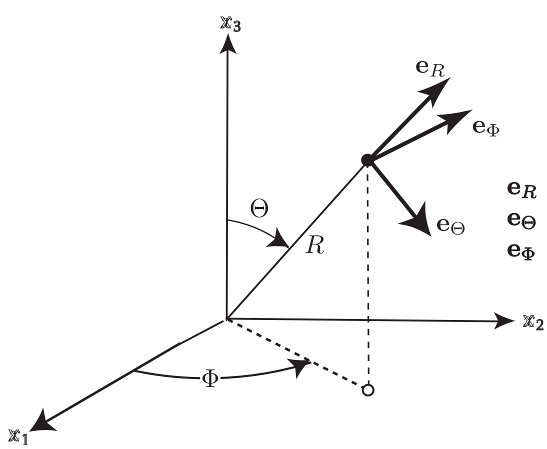
Turning to the left Cauchy-Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, it is straightforward to use (2.77) to show that as

$$
\begin{aligned}
\mathbf{B} = \;& B_{rr}\mathbf{e}_r\otimes\mathbf{e}_r \;+\; B_{\theta\theta}\mathbf{e}_\theta\otimes\mathbf{e}_\theta \;+\; B_{zz}\mathbf{e}_z\otimes\mathbf{e}_z \;+\\
&+B_{r\theta}(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r) \;+\; B_{rz}(\mathbf{e}_r\otimes\mathbf{e}_z + \mathbf{e}_z\otimes\mathbf{e}_r) \;+\\
&+B_{\theta z}(\mathbf{e}_z\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_z),
\end{aligned}
\tag{2.78}
$$

where

$$
\left.
\begin{aligned}
B_{rr} &= \left(\frac{\partial r}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial r}{\partial \Theta}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2, \\
B_{\theta\theta} &= r^2\left[\left(\frac{\partial \theta}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial \theta}{\partial \Theta}\right)^2 + \left(\frac{\partial \theta}{\partial Z}\right)^2\right], \\
B_{zz} &= \left(\frac{\partial z}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial z}{\partial \Theta}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2, \\
B_{r\theta} &= B_{\theta r} = r\left[\frac{\partial r}{\partial R}\frac{\partial \theta}{\partial R} + \frac{1}{R^2}\frac{\partial r}{\partial \Theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial r}{\partial Z}\frac{\partial \theta}{\partial Z}\right], \\
B_{rz} &= B_{zr} = \frac{\partial r}{\partial R}\frac{\partial z}{\partial R} + \frac{1}{R^2}\frac{\partial r}{\partial \Theta}\frac{\partial z}{\partial \Theta} + \frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z}, \\
B_{\theta z} &= B_{z\theta} = r\left[\frac{\partial \theta}{\partial R}\frac{\partial z}{\partial R} + \frac{1}{R^2}\frac{\partial z}{\partial \Theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \theta}{\partial Z}\frac{\partial z}{\partial Z}\right].
\end{aligned}
\right\} \tag{2.79}
$$

2.7.2 Spherical polar coordinates.



$$
\left.
\begin{aligned}
\mathbf{e}_R &= (\sin\Theta\cos\Phi)\,\mathbf{e}_1 + (\sin\Theta\sin\Phi)\,\mathbf{e}_2 + \cos\Theta\,\mathbf{e}_3 \\
\mathbf{e}_\Theta &= (\cos\Theta\cos\Phi)\,\mathbf{e}_1 + (\cos\Theta\sin\Phi)\,\mathbf{e}_2 - \sin\Theta\,\mathbf{e}_3 \\
\mathbf{e}_\Phi &= -\sin\Phi\,\mathbf{e}_1 + \cos\Phi\,\mathbf{e}_2
\end{aligned}
\right\}
$$

Figure 2.19: Spherical polar coordinates (R, Θ, Φ) and the associated basis vectors $\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi$.

Let (x_1, x_2, x_3) denote the rectangular cartesian coordinates of a particle in the reference configuration, and let (R, Θ, Φ) be its spherical polar coordinates; (see Figure 2.19) Then

$$
x_1 = R\sin\Theta\cos\Phi, \qquad x_2 = R\sin\Theta\sin\Phi, \qquad x_3 = R\cos\Theta. \tag{2.80}
$$

The associated basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi\}$ are related by

$$\left.\begin{array}{lcrcrcr} \mathbf{e}_R &=& (\sin\Theta\cos\Phi)\ \mathbf{e}_1 &+& (\sin\Theta\sin\Phi)\ \mathbf{e}_2 &+& \cos\Theta\ \mathbf{e}_3, \\ \mathbf{e}_\Theta &=& (\cos\Theta\cos\Phi)\ \mathbf{e}_1 &+& (\cos\Theta\sin\Phi)\ \mathbf{e}_2 &-& \sin\Theta\ \mathbf{e}_3, \\ \mathbf{e}_\Phi &=& -\sin\Phi\ \mathbf{e}_1 &+& \cos\Phi\ \mathbf{e}_2. && \end{array}\right\} \tag{2.81}$$

Let (y_1, y_2, y_3) denote the rectangular cartesian coordinates of a particle in the deformed configuration, and let (r, ϑ, φ) be its spherical polar coordinates. Then

$$y_1 = r\sin\vartheta\cos\varphi, \qquad y_2 = r\sin\vartheta\sin\varphi, \qquad y_3 = r\cos\vartheta. \tag{2.82}$$

The associated basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\varphi\}$ are related by

$$\left.\begin{array}{lcrcrcr} \mathbf{e}_r &=& (\sin\vartheta\cos\varphi)\ \mathbf{e}_1 &+& (\sin\vartheta\sin\varphi)\ \mathbf{e}_2 &+& \cos\vartheta\ \mathbf{e}_3, \\ \mathbf{e}_\vartheta &=& (\cos\vartheta\cos\varphi)\ \mathbf{e}_1 &+& (\cos\vartheta\sin\varphi)\ \mathbf{e}_2 &-& \sin\vartheta\ \mathbf{e}_3, \\ \mathbf{e}_\varphi &=& -\sin\varphi\ \mathbf{e}_1 &+& \cos\varphi\ \mathbf{e}_2. && \end{array}\right\} \tag{2.83}$$

The deformation can be characterized by

$$r = r(R, \Theta, \Phi), \qquad \vartheta = \vartheta(R, \Theta, \Phi), \qquad \varphi = \varphi(R, \Theta, \Phi), \tag{2.84}$$

A calculation similar to that in the preceding section gives the deformation gradient tensor with respect to the mixed bases $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$ to be

$$\begin{aligned} \mathbf{F} \ &= \frac{\partial r}{\partial R}(\mathbf{e}_r \otimes \mathbf{e}_R) + \frac{1}{R}\frac{\partial r}{\partial \Theta}(\mathbf{e}_r \otimes \mathbf{e}_\Theta) + \frac{1}{R\sin\Theta}\frac{\partial r}{\partial \Phi}(\mathbf{e}_r \otimes \mathbf{e}_\Phi) + \\ &+ r\frac{\partial \theta}{\partial R}(\mathbf{e}_\theta \otimes \mathbf{e}_R) + \frac{r}{R}\frac{\partial \theta}{\partial \Theta}(\mathbf{e}_\theta \otimes \mathbf{e}_\Theta) + \frac{r}{R\sin\Theta}\frac{\partial \theta}{\partial \Phi}(\mathbf{e}_\theta \otimes \mathbf{e}_\Phi) + \\ &+ r\sin\theta\frac{\partial \varphi}{\partial R}(\mathbf{e}_\varphi \otimes \mathbf{e}_R) \ + \frac{r\sin\theta}{R}\frac{\partial \varphi}{\partial \Theta}(\mathbf{e}_\varphi \otimes \mathbf{e}_\Theta) \ + \frac{r\sin\theta}{R\sin\Theta}\frac{\partial \varphi}{\partial \Phi}(\mathbf{e}_\varphi \otimes \mathbf{e}_\Phi)\,. \end{aligned} \tag{2.85}$$

The corresponding representation for the left Cauchy-Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is

$$\begin{aligned} \mathbf{B} = \ & B_{rr}\mathbf{e}_r \otimes \mathbf{e}_r \ + \ B_{\vartheta\vartheta}\mathbf{e}_\vartheta \otimes \mathbf{e}_\vartheta \ + \ B_{\varphi\varphi}\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi \ + \\ & + B_{r\vartheta}(\mathbf{e}_r \otimes \mathbf{e}_\vartheta + \mathbf{e}_\vartheta \otimes \mathbf{e}_r) \ + \ B_{r\varphi}(\mathbf{e}_r \otimes \mathbf{e}_\varphi + \mathbf{e}_\varphi \otimes \mathbf{e}_r) \ + \\ & + B_{\vartheta\varphi}(\mathbf{e}_\varphi \otimes \mathbf{e}_\vartheta + \mathbf{e}_\vartheta \otimes \mathbf{e}_\varphi), \end{aligned} \tag{2.86}$$

where

$$
\left.\begin{aligned}
B_{rr} &= \left(\frac{\partial r}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial r}{\partial \Theta}\right)^2 + \frac{1}{R^2 \sin^2\Theta}\left(\frac{\partial r}{\partial \Phi}\right)^2, \\
B_{\vartheta\vartheta} &= r^2\left[\left(\frac{\partial \vartheta}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial \vartheta}{\partial \Theta}\right)^2 + \frac{1}{R^2 \sin^2\Theta}\left(\frac{\partial \vartheta}{\partial \Phi}\right)^2\right], \\
B_{\varphi\varphi} &= r^2\,\sin^2\,\vartheta\left[\left(\frac{\partial \varphi}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial \varphi}{\partial \Theta}\right)^2 + \frac{1}{R^2 \sin^2\Theta}\left(\frac{\partial \varphi}{\partial \Phi}\right)^2\right] \\
B_{r\vartheta} &= B_{\vartheta r} = r\left[\frac{\partial r}{\partial R}\frac{\partial \vartheta}{\partial R} + \frac{1}{R^2}\frac{\partial r}{\partial \Theta}\frac{\partial \vartheta}{\partial \Theta} + \frac{1}{R^2 \sin^2\Theta}\frac{\partial r}{\partial \Phi}\frac{\partial \vartheta}{\partial \Phi}\right], \\
B_{r\varphi} &= B_{\varphi r} = r\,\sin\,\vartheta\left[\frac{\partial r}{\partial R}\frac{\partial \varphi}{\partial R} + \frac{1}{R^2}\frac{\partial r}{\partial \Theta}\frac{\partial \varphi}{\partial \Theta} + \frac{1}{R^2 \sin^2\Theta}\frac{\partial r}{\partial \Phi}\frac{\partial \varphi}{\partial \Phi}\right], \\
B_{\vartheta\varphi} &= B_{\varphi\vartheta} = r^2\,\sin\,\vartheta\left[\frac{\partial \vartheta}{\partial R}\frac{\partial \varphi}{\partial R} + \frac{1}{R^2}\frac{\partial \varphi}{\partial \Theta}\frac{\partial \vartheta}{\partial \Theta} + \frac{1}{R^2 \sin^2\Theta}\frac{\partial \vartheta}{\partial \Phi}\frac{\partial \varphi}{\partial \Phi}\right].
\end{aligned}\right\} \tag{2.87}
$$

2.7.3 Worked examples.

Problem 2.8.1. (Ogden) *(Extension and torsion of a solid circular cylinder.)* Let (R, Θ, Z) and (r, θ, z) be cylindrical polar coordinates of a particle in the reference and deformed configurations respectively with associated bases $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. The region $\mathcal{R}_R$ occupied by the body in a reference configuration is a circular cylinder of radius A and length L.

The body is uniformly stretched axially to a length $\ell = \Lambda L$ (with an accompanying transverse contraction), and the stretched cylinder in then subjected to a torsional deformation. Thus the cross section at Z in the reference configuration, displaces to ΛZ and then rotates through an angle $\alpha\Lambda Z$. In particular, one end of the cylinder is held fixed while the other rotates through an angle $\alpha\ell$. This deformation has the form

$$r = r(R), \qquad \theta = \Theta + \alpha\Lambda Z, \qquad z = \Lambda Z. \tag{i}$$

(a) Calculate the deformation gradient tensor.

(b) Determine $r(R)$ assuming the material to be incompressible,

(c) By factoring the deformation gradient tensor, show that locally, at each point of the body, the deformation is comprised of a rigid rotation, followed by a pure stretch, followed be a simple shear with shearing direction $\mathbf{e}_\theta$ and glide plane normal $\mathbf{e}_z$.

(d) Calculate the right Cauchy Green deformation tensor $\mathbf{C}$.

e) Calculate the principal stretches and the principal Lagrangian stretch directions.

Solution:

(a) The deformation gradient tensor is found by specializing (2.77) to the deformation (i) which leads to

$$\mathbf{F} = r'(R)(\mathbf{e}_r \otimes \mathbf{e}_R) + \frac{r}{R}(\mathbf{e}_\theta \otimes \mathbf{e}_\Theta) + \alpha\Lambda r(\mathbf{e}_\theta \otimes \mathbf{e}_Z) + \Lambda(\mathbf{e}_z \otimes \mathbf{e}_Z)\,. \qquad \Box$$

(b) When the material is incompressible

$$\det \mathbf{F} = 1 \qquad \Rightarrow \quad \Lambda \frac{r}{R}\frac{dr}{dR} = 1 \qquad \Rightarrow \quad r(R) = \sqrt{c + R^2/\Lambda},$$

where c is a constant of integration. However, since particles on the axis of the cylinder undergo no radial displacement it is necessary that $r(0) = 0$. This implies that $c = 0$ and therefore

$$r(R) = \Lambda^{-1/2} R. \qquad \Box \tag{ii}$$

It is convenient to set

$$\lambda := \Lambda^{-1/2}$$

so that $r'(R) = r(R)/R = \lambda$ and so

$$\mathbf{F} = \lambda(\mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta) + \alpha\Lambda r(\mathbf{e}_\theta \otimes \mathbf{e}_Z) + \Lambda(\mathbf{e}_z \otimes \mathbf{e}_Z)\,. \tag{iii}$$

(c) It can be readily verified that we can factor the deformation gradient tensor (iii) and write it as

$$\mathbf{F} = \mathbf{F}_1\mathbf{F}_2\mathbf{F}_3, \tag{iv}$$

where

$$\mathbf{F}_3 = \mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_z \otimes \mathbf{e}_Z, \tag{v}$$

$$\mathbf{F}_2 = \lambda\mathbf{e}_r \otimes \mathbf{e}_r + \lambda\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \Lambda\mathbf{e}_z \otimes \mathbf{e}_z, \tag{vi}$$

$$\mathbf{F}_1 = \mathbf{I} + \alpha r\, \mathbf{e}_\theta \otimes \mathbf{e}_z. \tag{vii}$$

Comparing (v) with (1.160) shows that $\mathbf{F}_3$ describes the rigid rotation that takes $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ into $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. Comparing (vi) with (2.7) shows that $\mathbf{F}_2$ describes a pure stretch with extension Λ in the z-direction and lateral contraction λ in the plane normal to it. Comparing (vii) with (2.15) shows that $\mathbf{F}_1$ describes a simple shear of amount αr in the shearing direction $\mathbf{e}_\theta$ with glide plane normal $\mathbf{e}_z$.

(d) The left Cauchy-Green tensor is

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \overset{(ii)}{=} \lambda^2(\mathbf{e}_R \otimes \mathbf{e}_R) + \lambda^2(\mathbf{e}_\Theta \otimes \mathbf{e}_\Theta) + \alpha\lambda\Lambda r(\mathbf{e}_\Theta \otimes \mathbf{e}_Z + \mathbf{e}_Z \otimes \mathbf{e}_\Theta) + \Lambda^2(1 + \alpha^2 r^2)(\mathbf{e}_Z \otimes \mathbf{e}_Z). \tag{viii}$$

(e) To find the principal stretches and principal directions of Lagrangian stretch, we can solve the eigenvalue problem for $\mathbf{C}$. However because of the special nature of the deformation under consideration, one can find the eigenvalues more easily. From the form of $\mathbf{C}$ in $(viii)$ (specifically since there are no shear components associated with $\mathbf{e}_R$) we conclude immediately that one principal value and principal direction are

$$\lambda_1 = \lambda = \Lambda^{-1/2}, \qquad \mathbf{r}_1 = \mathbf{e}_R. \tag{ix}$$

In a principal basis,

$$\mathbf{C} = \lambda_1^2(\mathbf{r}_1 \otimes \mathbf{r}_1) + \lambda_2^2(\mathbf{r}_2 \otimes \mathbf{r}_2) + \lambda_3^2(\mathbf{r}_3 \otimes \mathbf{r}_3). \qquad (x)$$

Calculating $\operatorname{tr}\mathbf{C}$ from $(viii)$ and (x) and equating the results gives

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 2\lambda^2 + \Lambda^2(1+\alpha^2 r^2) \qquad \Rightarrow \qquad \lambda_2^2 + \lambda_3^2 = \lambda^2 + \Lambda^2(1+\alpha^2 r^2). \qquad (xi)$$

Next calculating $\det\mathbf{F}$ from (iii) and equating the result to $\lambda_1\lambda_2\lambda_3$ yields

$$\lambda_1\lambda_2\lambda_3 = \lambda^2\Lambda \qquad \Rightarrow \qquad \lambda_2\lambda_3 = \lambda\Lambda. \qquad (xii)$$

Equations (xi) and (xii) can be solved for the principal stretches λ_2 and λ_3. (Note that in general $\lambda_2 \neq \lambda$ and $\lambda_3 \neq \Lambda$.)

Let the corresponding principal directions be

$$\mathbf{r}_2 = \cos\psi\,\mathbf{e}_\Theta + \sin\psi\,\mathbf{e}_Z, \qquad \mathbf{r}_3 = -\sin\psi\,\mathbf{e}_\Theta + \cos\psi\,\mathbf{e}_Z, \qquad (xiii)$$

for some to-be-determined angle ψ. Substituting $(ix)_2$ and $(xiii)$ into (x) and comparing the resulting expression with $(viii)$ leads to

$$\lambda_2^2\cos^2\psi + \lambda_3^2\sin^2\psi = \lambda^2, \qquad \lambda_2^2\sin^2\psi + \lambda_3^2\cos^2\psi = \Lambda^2(1+\alpha^2r^2), \qquad (\lambda_2^2 - \lambda_3^2)\sin\psi\cos\psi = \alpha\lambda\Lambda r.$$

These are three equations for λ_2, λ_3 and ψ. We do not need to solve them for the two principal stretches since we found them above in (xi) and (xii). By eliminating λ_2^2 and λ_3^2 from the three preceding equations we find

$$\tan 2\psi = \frac{2\alpha\lambda\Lambda r}{\lambda^2 - \Lambda^2(1+\alpha^2r^2)}. \qquad (xiv)$$

Thus the other two principal directions of $\mathbf{C}$ are given by $(xiii), (xiv)$.

2.8 Spatial and referential descriptions of a field.

Consider a scalar field $\phi(\mathbf{y})$ defined on the region $\mathcal{R}$ occupied by the body in the deformed configuration. For example it might represent the temperature field in the deformed body, with $\phi(\mathbf{y})$ being the temperature at the particle whose position in the deformed configuration is $\mathbf{y}$. This function $\phi(\mathbf{y})$, defined on $\mathcal{R}$, can be expressed as a function of $\mathbf{x}$ defined on $\mathcal{R}_{\rm R}$ by "changing variables" from $\mathbf{y} \to \mathbf{x}$:

$$\widehat{\phi}(\mathbf{x}) := \phi(\mathbf{y})\Big|_{\mathbf{y}=\widehat{\mathbf{y}}(\mathbf{x})} \qquad \text{for all } \mathbf{x} \in \mathcal{R}_{\rm R}, \qquad (2.88)$$

where $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x})$ is the deformation. *Note that $\widehat{\phi}(\mathbf{x})$ is not the temperature of the undeformed body.* The functions $\phi(\mathbf{y})$ and $\widehat{\phi}(\mathbf{x})$ both give the temperature at the same particle p in the deformed body, the particle having been identified in two different ways – in the former by its location $\mathbf{y}$ in the deformed configuration and in the latter by its position $\mathbf{x}$ in the reference configuration. One refers to the representation $\widehat{\phi}(\mathbf{x})$ as the **referential or material description** of the field under consideration, and $\phi(\mathbf{y})$ as its **spatial description**.

Since the deformation $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x})$ is one-to-one, there exists an inverse deformation

$$\mathbf{x} = \overline{\mathbf{x}}(\mathbf{y}) \tag{2.89}$$

that carries $\mathcal{R} \to \mathcal{R}_{\mathrm{R}}$. Therefore any function $\widehat{\phi}(\mathbf{x})$ defined on $\mathcal{R}_{\mathrm{R}}$ can be written as a function $\phi(\mathbf{y})$ defined on $\mathcal{R}$ by "changing variables" from $\mathbf{x} \to \mathbf{y}$:

$$\phi(\mathbf{y}) = \widehat{\phi}(\mathbf{x})\Big|_{\mathbf{x}=\overline{\mathbf{x}}(\mathbf{y})} \qquad \text{for all } \mathbf{y} \in \mathcal{R}. \tag{2.90}$$

If one takes the gradient of $\widehat{\phi}(\mathbf{x})$ with respect to $\mathbf{x}$ one gets a certain gradient vector that we will denote by $\mathrm{Grad}\,\phi$. It has cartesian components $\partial\widehat{\phi}/\partial x_i$. On the other hand if one takes the gradient of $\phi(\mathbf{y})$ with respect to $\mathbf{y}$ one gets a different gradient vector denoted by $\mathrm{grad}\,\phi$ with cartesian components $\partial\phi/\partial y_i$. Since we use two different symbols "Grad" and "grad" to denote these two gradients, it is not necessary to write, for example, $\mathrm{Grad}\,\widehat{\phi}$, it being understood that "Grad" applies on the referential field (and "grad" on the spatial field). The relation between $\mathrm{Grad}\,\phi$ and $\mathrm{grad}\,\phi$ can be determined by differentiating both sides of (2.88) with respect to $\mathbf{x}$, or both sides of (2.90) with respect to $\mathbf{y}$, and using the chain rule. (Problem 2.8.1)

In general, for arbitrary scalar fields $\phi(\mathbf{y})$ and $\psi(\mathbf{x})$, $\mathrm{grad}\,\phi$ and $\mathrm{Grad}\,\psi$ denote the respective vector fields with Cartesian components $\partial\phi/\partial y_i$ and $\partial\psi/\partial x_i$:

$$\mathrm{grad}\,\phi = \frac{\partial\phi}{\partial y_i}\,\mathbf{e}_i, \qquad \mathrm{Grad}\,\psi = \frac{\partial\psi}{\partial x_i}\,\mathbf{e}_i. \tag{2.91}$$

For arbitrary vector fields $\mathbf{a}(\mathbf{y})$ and $\mathbf{b}(\mathbf{x})$, $\mathrm{grad}\,\mathbf{a}$ and $\mathrm{Grad}\,\mathbf{b}$ denote the respective tensor fields with Cartesian components $\partial a_i/\partial y_j$ and $\partial b_i/\partial x_j$:

$$\mathrm{grad}\,\mathbf{a} = \frac{\partial a_i}{\partial y_j}\,\mathbf{e}_i \otimes \mathbf{e}_j, \qquad \mathrm{Grad}\,\mathbf{b} = \frac{\partial b_i}{\partial x_j}\,\mathbf{e}_i \otimes \mathbf{e}_j, \tag{2.92}$$

and

$$\mathrm{div}\,\mathbf{a} = \frac{\partial a_i}{\partial y_i}, \qquad \mathrm{Div}\,\mathbf{b} = \frac{\partial b_i}{\partial x_i}. \tag{2.93}$$

For arbitrary tensor fields $\mathbf{A}(\mathbf{y})$ and $\mathbf{B}(\mathbf{x})$, $\mathrm{div}\,\mathbf{A}$ and $\mathrm{Div}\,\mathbf{B}$ denote the respective vector fields with Cartesian components $\partial A_{ij}/\partial y_j$ and $\partial B_{ij}/\partial x_j$:

$$\mathrm{div}\,\mathbf{A} = \frac{\partial A_{ij}}{\partial y_j}\,\mathbf{e}_i, \qquad \mathrm{Div}\,\mathbf{B} = \frac{\partial B_{ij}}{\partial x_j}\,\mathbf{e}_i. \tag{2.94}$$

Let $\mathbf{u}(\mathbf{y})$ and $\widehat{\mathbf{u}}(\mathbf{x})$ be spatial and material descriptions of a vector-valued field:

$$\widehat{\mathbf{u}}(\mathbf{x}) = \mathbf{u}(\mathbf{y})\Big|_{\mathbf{y}=\widehat{\mathbf{y}}(\mathbf{x})}, \qquad \mathbf{u}(\mathbf{y}) = \widehat{\mathbf{u}}(\mathbf{x})\Big|_{\mathbf{x}=\overline{\mathbf{x}}(\mathbf{y})}. \tag{i}$$

Suppose that $\mathbf{u}(\mathbf{y})$ satisfies the differential equation

$$\operatorname{div} \mathbf{u}(\mathbf{y}) = 0 \qquad \text{at each } \mathbf{y} \in \mathcal{R}. \tag{ii}$$

Then by mapping $\mathbf{y} \to \mathbf{x}$ ("changing variables from $\mathbf{y}$ to $\mathbf{x}$") we have

$$\operatorname{div} \mathbf{u} = \frac{\partial u_i}{\partial y_i} = \frac{\partial \widehat{u}_i}{\partial x_j}\frac{\partial x_j}{\partial y_i} = \frac{\partial \widehat{u}_i}{\partial x_j} F^{-1}_{ji} = \frac{\partial \widehat{u}_i}{\partial x_j} F^{-T}_{ij} = \operatorname{Grad} \widehat{\mathbf{u}} \cdot \mathbf{F}^{-T}, \tag{iii}$$

where the dot in the last expression denotes the scalar product between the tensors Grad $\widehat{\mathbf{u}}$ and $\mathbf{F}^{-T}$. Thus the referential version of the differential equation (ii) is

$$\operatorname{Grad} \widehat{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{F}^{-T}(\mathbf{x}) = 0 \quad \text{at each } \mathbf{x} \in \mathcal{R}_{\mathrm{R}}. \tag{iv}$$

Observe that the differential equation (iv) *holds on* $\mathcal{R}_{\mathrm{R}}$.

2.8.1 Worked examples.

Problem 2.8.1. Let $\phi(\mathbf{y})$ and $\mathbf{v}(\mathbf{y})$ be a scalar and vector field defined on $\mathcal{R}$ and let $\widehat{\phi}(\mathbf{x})$ and $\widehat{\mathbf{v}}(\mathbf{x})$ be the corresponding scalar and vector fields defined on $\mathcal{R}_{\mathrm{R}}$ through the deformation $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x})$:

$$\widehat{\phi}(\mathbf{x}) = \phi(\widehat{\mathbf{y}}(\mathbf{x})), \qquad \widehat{\mathbf{v}}(\mathbf{x}) = \mathbf{v}(\widehat{\mathbf{y}}(\mathbf{x})).$$

Show that

$$\operatorname{Grad} \widehat{\phi} = \mathbf{F}^T \operatorname{grad} \phi, \qquad \operatorname{Grad} \widehat{\mathbf{v}} = \operatorname{grad} \mathbf{v}\ \mathbf{F}, \tag{2.95}$$

Here, in cartesian coordinates,

$$\Big(\operatorname{Grad} \widehat{\phi}\Big)_i = \frac{\partial \widehat{\phi}}{\partial x_i}, \qquad \Big(\operatorname{grad} \phi\Big)_i = \frac{\partial \phi}{\partial y_i}, \qquad \Big(\operatorname{Grad} \widehat{\mathbf{v}}\Big)_{ij} = \frac{\partial \widehat{v}_i}{\partial x_j}, \qquad \Big(\operatorname{grad} \mathbf{v}\Big)_{ij} = \frac{\partial v_i}{\partial y_j}.$$

Remark: It is not necessary to include the "hats" in (2.95) and the line below it since we use two different symbols, Grad and grad, for the two gradients. It is understood that ϕ is expressed referentially in Grad ϕ and spatially in grad ϕ.

Problem 2.8.2. Let $\mathbf{a}(\mathbf{y})$ and $\mathbf{b}(\mathbf{x})$ be spatial and referential representations of two vector fields. Suppose they are related by

$$\mathbf{b} = J\mathbf{F}^{-1}\mathbf{a}, \tag{i}$$

where $\mathbf{F} = \nabla\mathbf{y}$ is the deformation gradient tensor and $J = \det\mathbf{F}$ is the Jacobian determinant. Show that

$$\operatorname{Div}\mathbf{b} = J\operatorname{div}\mathbf{a}. \tag{ii}$$

Solution: The result follows from

$$\begin{aligned}\operatorname{Div}\mathbf{b} &= \frac{\partial b_i}{\partial x_i} \overset{(i)}{=} \frac{\partial}{\partial x_i}(JF^{-1}_{ij}a_j) = \frac{\partial}{\partial x_i}(JF^{-T}_{ji}a_j) = \frac{\partial}{\partial x_i}(JF^{-T}_{ji})a_j + JF^{-T}_{ji}\frac{\partial a_j}{\partial x_i} = \\ &\overset{(2.123)}{=} JF^{-T}_{ji}\frac{\partial a_j}{\partial x_i} = JF^{-1}_{ij}\frac{\partial a_j}{\partial y_k}\frac{\partial y_k}{\partial x_i} = JF^{-1}_{ij}\frac{\partial a_j}{\partial y_k}F_{ki} = J\delta_{kj}\frac{\partial a_j}{\partial y_k} = J\frac{\partial a_j}{\partial y_j} = J\operatorname{div}\mathbf{a}\end{aligned}$$

where we used the Piola identity (2.123) (page 217) in getting to the second line.

Problem 2.8.3. Let $\mathbf{A}(\mathbf{y})$ and $\mathbf{B}(\mathbf{x})$ be spatial and referential representations of two tensor fields. Suppose they are related by

$$\mathbf{B} = J\mathbf{A}\mathbf{F}^{-T}. \tag{i}$$

Show that

$$\operatorname{Div}\mathbf{B} = J\operatorname{div}\mathbf{A}, \tag{ii}$$

where the divergence of a tensor field is given in (2.94).

Solution: The result follows from

$$\begin{aligned}(\operatorname{Div}\mathbf{B})_k &= \frac{\partial B_{ki}}{\partial x_i} \overset{(i)}{=} \frac{\partial}{\partial x_i}(JA_{kj}F^{-T}_{ji}) = \frac{\partial}{\partial x_i}(JF^{-T}_{ji}A_{kj}) = \frac{\partial}{\partial x_i}(JF^{-T}_{ji})A_{kj} + JF^{-T}_{ji}\frac{\partial A_{kj}}{\partial x_i} = \\ &\overset{(2.123)}{=} JF^{-T}_{ji}\frac{\partial A_{kj}}{\partial x_i} = JF^{-1}_{ij}\frac{\partial A_{kj}}{\partial y_p}\frac{\partial y_p}{\partial x_i} = JF^{-1}_{ij}\frac{\partial A_{kj}}{\partial y_p}F_{pi} = J\delta_{pj}\frac{\partial A_{kj}}{\partial y_p} = J\frac{\partial A_{kj}}{\partial y_j} = J\operatorname{div}\mathbf{A}\end{aligned}$$

where we used the Piola identity (2.123) (page 217) in getting to the second line.

Problem 2.8.4. A rigid motion of a body is described by

$$\mathbf{y}(\mathbf{x}, t) = \mathbf{d}(t) + \mathbf{Q}(t)\mathbf{x}. \tag{2.96}$$

where the time-dependent vector $\mathbf{d}(t)$ describes the translation and the time-dependent proper orthogonal tensor $\mathbf{Q}(t)$ describes the rotation. Show that the velocity field, in spatial form, associated with a rigid motion can be expressed in the form

$$\mathbf{v}(\mathbf{y}, t) = \mathbf{c}(t) + \boldsymbol{\omega}(t)\times\mathbf{y}, \tag{2.97}$$

for some vectors $\mathbf{c}(t)$ and $\boldsymbol{\omega}(t)$.

Solution: The velocity of a particle $\mathbf{x}$ during the rigid motion (2.96) is

$$\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{d}}(t) + \dot{\mathbf{Q}}(t)\mathbf{x}, \tag{i}$$

where the superior dot denotes differentiation with respect to time. This velocity field can be described in spatial (Eulerian) form by using the inverse of the motion, $\mathbf{x} = \mathbf{Q}^T(\mathbf{y} - \mathbf{d})$, to swap $\mathbf{x}$ for $\mathbf{y}$:

$$\mathbf{v}(\mathbf{y}, t) = \dot{\mathbf{d}} + \dot{\mathbf{Q}}\,\mathbf{Q}^T(\mathbf{y} - \mathbf{d}) = \mathbf{c}(t) + \mathbf{W}(t)\mathbf{y}, \tag{ii}$$

where we have set

$$\mathbf{W}(t) := \dot{\mathbf{Q}}(t)\mathbf{Q}^T(t), \qquad \mathbf{c}(t) := \dot{\mathbf{d}}(t) - \dot{\mathbf{Q}}(t)\mathbf{Q}^T(t)\mathbf{d}(t). \tag{iii}$$

Since $\mathbf{Q}(t)$ is proper orthogonal at each instant we have $\mathbf{Q}(t)\mathbf{Q}^T(t) = \mathbf{I}$. Differentiating this with respect to time gives

$$\dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{W}(t) = -\mathbf{W}^T(t). \tag{iv}$$

Therefore $\mathbf{W}(t)$ is skew-symmetric and so we can write the velocity field as

$$\mathbf{v}(\mathbf{y}, t) = \mathbf{c}(t) + \boldsymbol{\omega}(t) \times \mathbf{y}$$

where $\boldsymbol{\omega}(t)$ is the axial vector associated with $\mathbf{W}(t)$; see (1.83).

2.9 Linearization.

The *displacement field* $\mathbf{u}(\mathbf{x})$ is related to the deformation by

$$\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x}. \tag{2.98}$$

The associated *displacement gradient tensor*

$$\mathbf{H} := \nabla \mathbf{u}, \tag{2.99}$$

has cartesian components

$$H_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{2.100}$$

Since $\mathbf{F} = \nabla \mathbf{y}$, it follows from (2.98) and (2.99) that the displacement gradient tensor $\mathbf{H}$ and the deformation gradient tensor $\mathbf{F}$ are related by

$$\mathbf{F} = \mathbf{I} + \mathbf{H}. \tag{2.101}$$

The various kinematic quantities encountered previously, such as the stretches $\mathbf{U}, \mathbf{V}$, the rotation $\mathbf{R}$ and the strain $\mathbf{E}$, were all expressed in terms of the deformation gradient tensor

$\mathbf{F}$, and so they can all be represented instead in terms of the displacement gradient tensor $\mathbf{H}$.

In the reference configuration we have $\mathbf{F} = \mathbf{I}$ and so $\mathbf{H} = \mathbf{0}$ by (2.101). If, in some sense (to be made precise), the body is deformed by a "small" amount, then $\mathbf{F}$ is close to $\mathbf{I}$ and $\mathbf{H}$ is close to $\mathbf{0}$. This is indeed the case in many physical circumstances and our goal in this section is to derive approximations for $\mathbf{U}, \mathbf{V}, \mathbf{R}, \mathbf{E}$ etc. in this particular setting. Note that when $\mathbf{F}$ is close to $\mathbf{I}$, both the stretch $\mathbf{U}$ *and* rotation $\mathbf{R}$ are close to $\mathbf{I}$, and so the setting we are considering involves small amounts of both stretching and rotation.

Before proceeding to derive suitable approximations, we first recall what we mean by the magnitude of a tensor and what it means when that magnitude is small. We note three preliminary algebraic results:

– First, recall from (1.121) that the norm (or magnitude) of a tensor $\mathbf{A}$ is defined as

$$|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}} \;=\; \sqrt{\mathrm{tr}\,(\mathbf{A}^T\mathbf{A})}. \tag{2.102}$$

In terms of the components A_{ij} this reads

$$|\mathbf{A}| = (A_{11}^2 + A_{12}^2 + A_{13}^2 + A_{21}^2 + \cdots + A_{33}^2)^{1/2}. \tag{2.103}$$

Observe that $|\mathbf{A}| > 0$ for all $\mathbf{A} \neq \mathbf{0}$. Moreover if $|\mathbf{A}| \to 0$ then *each* component $A_{ij} \to 0$ as well.

– Second, let $\mathbf{Z}(\mathbf{H})$ be a function that is defined for all 2-tensors $\mathbf{H}$ and whose values are also 2-tensors. We say that $\mathbf{Z}(\mathbf{H}) = \mathrm{O}(|\mathbf{H}|^n)$ as $|\mathbf{H}| \to 0$ if there exists a number $\alpha > 0$ such that $|\mathbf{Z}(\mathbf{H})| < \alpha|\mathbf{H}|^n$ as $|\mathbf{H}| \to 0$.

– And third, for a symmetric tensor $\mathbf{A}$ and real number m,

$$(\mathbf{I} + \mathbf{A})^m = \mathbf{I} + m\mathbf{A} + \mathrm{O}(|\mathbf{A}|^2) \quad \text{as} \quad |\mathbf{A}| \to 0 \tag{2.104}$$

which can be readily established in a principal basis of $\mathbf{A}$.

We are now in a position to linearize our preceding kinematic quantities in the special case of an **infinitesimal deformation**, defined as a deformation in which $\mathbf{H} = \nabla\mathbf{u} = \mathbf{F} - \mathbf{I}$ is small. To this end we set

$$\epsilon = |\mathbf{H}| \tag{2.105}$$

and conclude that as $\epsilon \to 0$,

$$\begin{aligned}
\mathbf{C} = \mathbf{U}^2 &= \mathbf{F}^T\mathbf{F} &\overset{(2.101)}{=}& \quad \mathbf{I} + \mathbf{H} + \mathbf{H}^T + \mathrm{O}(\epsilon^2),\\
\mathbf{B} = \mathbf{V}^2 &= \mathbf{F}\mathbf{F}^T &\overset{(2.101)}{=}& \quad \mathbf{I} + \mathbf{H} + \mathbf{H}^T + \mathrm{O}(\epsilon^2),\\
\mathbf{U} = \sqrt{\mathbf{U}^2} &= \sqrt{\mathbf{C}} &\overset{(2.104)}{=}& \quad \mathbf{I} + \tfrac{1}{2}(\boldsymbol{H} + \mathbf{H}^T) + \mathrm{O}(\epsilon^2),\\
\mathbf{V} = \sqrt{\mathbf{V}^2} &= \sqrt{\mathbf{B}} &\overset{(2.104)}{=}& \quad \mathbf{I} + \tfrac{1}{2}(\boldsymbol{H} + \mathbf{H}^T) + \mathrm{O}(\epsilon^2),\\
\mathbf{R} &= \mathbf{F}\mathbf{U}^{-1} &\overset{(2.101),(2.104)}{=}& \quad \mathbf{I} + \tfrac{1}{2}(\mathbf{H} - \mathbf{H}^T) + \mathrm{O}(\epsilon^2),
\end{aligned} \tag{2.106}$$

where the first two equations follow immediately by using $\mathbf{F} = \mathbf{I} + \mathbf{H}$, and we have used (2.104) in deriving the last three equations. By $(2.106)_1$ the Green Saint-Venant strain tensor can be written as

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) + \mathrm{O}(\epsilon^2). \tag{2.107}$$

More generally the Lagrangian strain tensor (2.67) can be linearized by Taylor expanding $e(\lambda)$ about $\lambda = 1$ giving

$$\mathbf{E}(\mathbf{U}) = \sum_{i=1}^{3} e(\lambda_i)\mathbf{r}_i \otimes \mathbf{r}_i = \sum_{i=1}^{3}(\lambda_i - 1)\mathbf{r}_i \otimes \boldsymbol{r}_i + \mathrm{O}\left(\epsilon^2\right) = \mathbf{U} - \mathbf{I} + \mathrm{O}\left(\epsilon^2\right) = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) + \mathrm{O}(\epsilon^2), \tag{2.108}$$

where we have used the properties $e(1) = 0, e'(1) = 1$.

One can also show that

$$J = \det \mathbf{F} = 1 + \operatorname{tr}\mathbf{H} + \mathrm{O}\left(\epsilon^2\right) = 1 + \operatorname{Div}\mathbf{u} + \mathrm{O}\left(\epsilon^2\right), \tag{2.109}$$

where in terms of its cartesian components, $\operatorname{Div}\mathbf{u} = \operatorname{tr}(\nabla \mathbf{u}) = \partial u_i/\partial x_i$.

Observe from $(2.106)_{3,4}$ and (2.108) that the stretch and strain tensors depend on $\mathbf{H}$ through only its symmetric part. Therefore we define a tensor $\boldsymbol{\varepsilon}$ by

$$\boldsymbol{\varepsilon} \;:=\; \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^T), \tag{2.110}$$

with cartesian components

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.111}$$

Note that the stretch tensors $\mathbf{U}$ and $\mathbf{V}$ and a general Lagrangian strain tensor $\mathbf{E}$ can be approximated as

$$\mathbf{U} = \mathbf{I} + \boldsymbol{\varepsilon} + \mathrm{O}\left(\epsilon^2\right), \qquad \mathbf{V} = \mathbf{I} + \boldsymbol{\varepsilon} + \mathrm{O}\left(\epsilon^2\right), \qquad \mathbf{E} = \boldsymbol{\varepsilon} + \mathrm{O}\left(\epsilon^2\right). \tag{2.112}$$

The symmetric tensor $\boldsymbol{\varepsilon}$ is known as the **infinitesimal strain tensor** and plays a central role in the theory of solids undergoing infinitesimal deformations. It is worth comparing the expression (2.111) for the components of the infinitesimal strain tensor with the corresponding expression (2.73) for those of the Green Saint-Venant strain tensor.

Remark: In Problem 2.2 it is shown that $\boldsymbol{\varepsilon}$ does *not* vanish in a rigid rotation $\mathbf{y} = \mathbf{Q}\mathbf{x}$ and therefore is *not* a suitable measure of strain in a finite deformation. However it is shown there (and below) that $\boldsymbol{\varepsilon}$ does vanish to leading order in an infinitesimal rigid rotation and therefore *is* appropriate in the study of infinitesimal deformations.

Remark: An eigenvalue ε_i of $\boldsymbol{\varepsilon}$ is related to the corresponding principal stretch λ_i by

$$\lambda_i = 1 + \varepsilon_i + \mathrm{O}(\epsilon^2). \tag{2.113}$$

Remark: Upon linearization, equations (2.74) and (2.76) read

$$\varepsilon_{11} \approx \frac{ds_y - ds_x}{ds_x}, \qquad \varepsilon_{12} \approx \frac{1}{2}\cos\theta_y \approx \frac{1}{2}(\pi/2 - \theta_y). \tag{2.114}$$

It follows from this that when the deformation is infinitesimal, the normal strain component ε_{11} represents the change in length per reference length of a fiber that was in the x_1-direction in the reference configuration; and that the shear strain component ε_{12} represents one half the decrease in angle between two fibers that were in the x_1- and x_2-directions in the reference configuration.

Remark: The stretch of a material fiber in the direction $\mathbf{m}$ is

$$\lambda = \big(\mathbf{F}\mathbf{m}\cdot\mathbf{F}\mathbf{m}\big)^{1/2} = \big((\mathbf{I}+\mathbf{H})\mathbf{m}\cdot(\mathbf{I}+\mathbf{H})\mathbf{m}\big)^{1/2} \doteq 1 + \frac{1}{2}(\mathbf{H}+\mathbf{H}^T)\mathbf{m}\cdot\mathbf{m} = 1 + \boldsymbol{\varepsilon}\mathbf{m}\cdot\mathbf{m}.$$

If the material is **inextensible** in the fiber direction $\mathbf{m}$, the only (infinitesimal) deformations it can undergo are those with

$$\boldsymbol{\varepsilon}\mathbf{m}\cdot\mathbf{m} = \varepsilon_{ij}m_im_j = 0. \tag{2.115}$$

Remark: Note from (2.109), (2.118) and (2.36) that

$$\frac{dV_y - dV_x}{dV_x} = J - 1 = \mathrm{Div}\,\mathbf{u} + \mathrm{O}(\epsilon^2) = \mathrm{tr}\,(\nabla\mathbf{u}) + \mathrm{O}(\epsilon^2) = \frac{\partial u_i}{\partial x_i} + \mathrm{O}(\epsilon^2) = \mathrm{tr}\,\boldsymbol{\varepsilon} + \mathrm{O}(\epsilon^2). \tag{2.116}$$

Thus the volumetric strain is measured by $\mathrm{tr}\,\boldsymbol{\varepsilon}$ in the infinitesimal deformation theory. If the material is **incompressible**, the only (infinitesimal) deformations it can undergo are those with

$$\mathrm{tr}\,\boldsymbol{\varepsilon} = \varepsilon_{kk} = 0. \tag{2.117}$$

Next, observe from $(2.106)_5$ that the rotation tensor $\mathbf{R}$ depends only on the skew-symmetric part of $\mathbf{H}$. Therefore we define a 2-tensor $\boldsymbol{\omega}$ by

$$\boldsymbol{\omega} \;:=\; \tfrac{1}{2}(\mathbf{H} - \mathbf{H}^T), \tag{2.118}$$

whose cartesian components are

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{2.119}$$

Observe that the rotation tensor $\mathbf{R}$ can be approximated as

$$\mathbf{R} = \mathbf{I} + \boldsymbol{\omega} + \mathrm{O}\left(\epsilon^2\right). \tag{2.120}$$

The tensor $\boldsymbol{\omega}$ is known as the **infinitesimal rotation tensor** and it is skew-symmetric (not orthogonal!)

Consider the particular displacement field

$$\mathbf{u}(\mathbf{x}) = \mathbf{W}\mathbf{x} + \mathbf{b} \tag{2.121}$$

where $\mathbf{W}$ is a constant skew-symmetric tensor and $\mathbf{b}$ is a constant vector. It is readily seen by substituting this into (2.110) that the associated infinitesimal strain field vanishes. Thus (2.121) describes an **infinitesimal rigid displacement**.

Remark: It is useful to observe from (2.21), (2.101) and (2.118) that a fiber $d\mathbf{x}$ in the reference configuration and its image $d\mathbf{y}$ in the deformed configuration are related by

$$d\mathbf{y} = d\mathbf{x} + \boldsymbol{\varepsilon}\, d\mathbf{x} + \boldsymbol{\omega}\, d\mathbf{x} + \mathrm{O}(\epsilon^2), \tag{2.122}$$

which shows that in the linearized theory the local deformation can be *additively* decomposed into a strain and a rotation. This is in contrast to the multiplicative decomposition $\mathrm{d}\mathbf{y} = \mathbf{R}\mathbf{U}\mathrm{d}\mathbf{x}$ for a finite deformation.

Remark: As noted previously, when $|\mathbf{H}|$ is small, *both* the strain and rotation are small. There are certain physical circumstances in which one wants to carry out a different linearization, i.e., linearization based on the smallness of some quantity other than $\nabla\mathbf{u}$. For example, consider rolling up a sheet of paper. If the rolled-up configuration is the deformed configuration and the flat one the reference configuration, in this situation one has large rotations $\mathbf{R}$ but small strains $\mathbf{U}-\mathbf{I}$. Thus one might wish to linearize based on the assumption that $|\mathbf{U}-\mathbf{I}|$ is small (but leave $\mathbf{R}$ arbitrary). Note that under these conditions $|\mathbf{H}|$ will not be small.

2.10 Exercises.

Problem 2.1. Bending of a block.

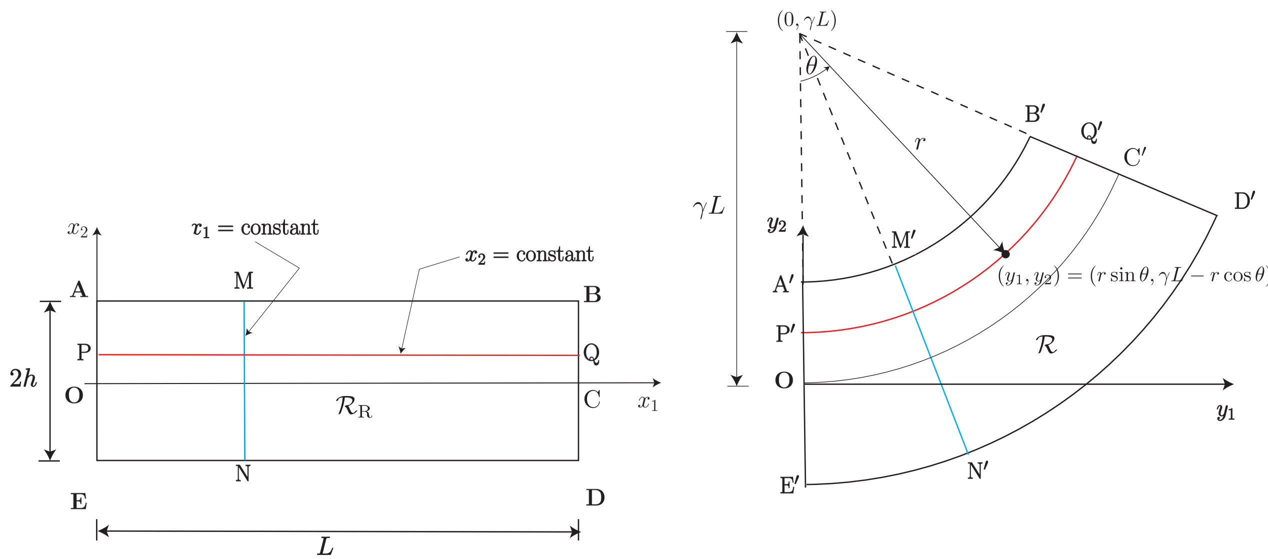


Figure 2.20: Left: Region $\mathcal{R}_\mathrm{R} = \{(x_1, x_2, x_3) : 0 \le x_1 \le L, -h \le x_2 \le h, -b \le x_3 \le b\}$ occupied by a body in a reference configuration. Right: The region $\mathcal{R}$ occupied by the body in the deformed configuration. The points P′, Q′, M′, N′, etc. are the images in the deformed configuration of the points P, Q, M, N, etc. in the reference configuration. Vertical straight lines, e.g. MN, in the reference configuration are mapped into straight lines, e.g. M′N′, that pass through the point $(0, \gamma L)$. Horizontal straight lines in the reference configuration, e.g. PQ, are carried into circular arcs, e.g. P′Q′. (Figure for Problem 2.1)

A body occupies the region $\mathcal{R}_\mathrm{R} = \{(x_1, x_2, x_3) : 0 \le x_1 \le L, -h \le x_2 \le h, -b \le x_3 \le b\}$ in a reference configuration. Figure 2.20 shows a side view of this block looking down the x_3-axis. The body is subjected to (some loading that leads to) a bending deformation that carries $\mathcal{R}_\mathrm{R}$ into the region $\mathcal{R}$ shown. The deformation has the features described in the figure caption and is therefore analogous to that in Problem 2.5.4. An analysis like that in Problem 2.5.4 shows that the deformation must have the form (Exercise)

$$y_1 = r(x_2)\sin\theta(x_1), \qquad y_2 = \gamma L - r(x_2)\cos\theta(x_1), \qquad y_3 = x_3, \tag{i}$$

where the functions $r(x_2)$ and $\theta(x_1)$ are subject to

$$r(0) = \gamma L, \qquad \theta(0) = 0, \tag{ii}$$

and

$$r(x_2) > 0, \qquad r'(x_2) < 0, \qquad \theta'(x_1) > 0. \tag{iii}$$

(a) Consider the (undeformed) material fiber $d\mathbf{x} = ds_x\,\mathbf{e}_1$ at an arbitrary point $(x_1, x_2, 0)$ in the body. Calculate its stretch $\lambda_1(x_1, x_2)$. Similarly consider the undeformed material fiber $d\mathbf{x} = ds_x\,\mathbf{e}_2$ and calculate its stretch $\lambda_2(x_1, x_2)$. Caution: we do not know that these are the *principal* stretches so the use of the symbols λ_1, λ_2 is probably not ideal.

(b) Suppose that a material fiber that lies on the x_2-axis in the undeformed configuration remains unstretched by the deformation. What additional information can you infer about $r(x_2)$ and $\theta(x_1)$?

(c) Specialize the stretch $\lambda_1(x_1, x_2)$ to the case where the material is incompressible. Show that $\lambda_1(x_1, x_2)$ varies nonlinearly with x_2. In addition, show that $\lambda_1(x_1, 0) = 1$, $\lambda_1(x_1, x_2) < 1$ for $x_2 > 0$ and $\lambda_1(x_1, x_2) > 1$ for $x_2 < 0$. Do **not** assume the material to be incompressible from here on.

(d) Calculate the components of the Green Saint-Venant and infinitesimal strain tensors $\mathbf{E}$ and $\boldsymbol{\varepsilon}$. Comment on the distinction.

(e) Under what conditions is the deformation infinitesimal? Specialize your preceding expressions for $\boldsymbol{\varepsilon}(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$ to this case. Make use of the fact that $\lambda_1 = \lambda_2 = 1$ in the reference configuration and therefore that in an infinitesimal deformation $|\lambda_1 - 1| \ll 1$ and $|\lambda_2 - 1| \ll 1$.

Solution:

(a) First calculate the deformation gradient tensor. Differentiating (i) with respect to x_j and using $F_{ij} = \partial y_i/\partial x_j$ gives the components of the deformation gradient tensor:

$$[F] = \begin{pmatrix} r\theta'\cos\theta & r'\sin\theta & 0 \\ r\theta'\sin\theta & -r'\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (iv)$$

where a prime denotes differentiation with respect to the argument. Tensorially,

$$\mathbf{F} = r\theta'\cos\theta\ \mathbf{e}_1\otimes\mathbf{e}_1 + r'\sin\theta\ \mathbf{e}_1\otimes\mathbf{e}_2 + r\theta'\sin\theta\ \mathbf{e}_2\otimes\mathbf{e}_1 - r'\cos\theta\ \mathbf{e}_2\otimes\mathbf{e}_2 + \mathbf{e}_3\otimes\mathbf{e}_3. \qquad (v)$$

Remark: On calculating $J = \det\mathbf{F}$, one obtains $J = \det\mathbf{F} = -rr'\theta'$. Thus $J > 0$ requires $r'\theta' < 0$.

Consider the undeformed fiber $d\mathbf{x} = ds_x\,\mathbf{e}_1$. Substituting this into $d\mathbf{y} = \mathbf{F}d\mathbf{x}$ and using (v) leads to

$$d\mathbf{y} = (r\theta'\cos\theta\, ds_x)\mathbf{e}_1 + (r\theta'\sin\theta\, ds_x)\mathbf{e}_2.$$

Therefore

$$ds_y = |d\mathbf{y}| = \sqrt{(r\theta'\cos\theta\, ds_x)^2 + (r\theta'\sin\theta\, ds_x)^2} = |r\theta'|\, ds_x \overset{(iii)}{=} r\theta'\, ds_x,$$

and so the stretch $\lambda_1 = ds_y/ds_x$ of such a fiber is

$$\lambda_1(x_1, x_2) = r(x_2)\theta'(x_1). \qquad \square \qquad (vi)$$

A similar calculation for an undeformed fiber $d\mathbf{x} = ds_x\,\mathbf{e}_2$ leads to the stretch

$$\lambda_2(x_1, x_2) = |r'(x_2)| = -r'(x_2). \qquad \square \qquad (vii)$$

(b) We are told that fibers lying on the x_1-axis remain unstretched. Thus from (vi)

$$\lambda_1(x_1, 0) = r(0)\theta'(x_1) = 1 \qquad \overset{(ii)_1}{\Rightarrow} \qquad \theta'(x_1) = 1/(\gamma L).$$

Integrating this equation and using $(ii)_2$ leads to

$$\theta(x_1) = \frac{x_1}{\gamma L}. \qquad \square \tag{viii}$$

(c) : If the material is incompressible we must have $\det \mathbf{F} = 1$. Calculating $\det \mathbf{F}$ from (iv) gives

$$\det \mathbf{F} = -r(x_2)r'(x_2)\theta'(x_1) = 1.$$

Since $\theta(x_1) = x_1/(\gamma L)$ this simplifies to

$$r(x_2)r'(x_2) = -\gamma L.$$

Integrating this and using $(ii)_1$ yields

$$r(x_2) = \gamma L\sqrt{1 - \frac{2x_2}{\gamma L}}. \tag{ix}$$

The stretch λ_1 is therefore given by $(vi), (viii), (ix)$ to be

$$\lambda_1(x_1, x_2) = \sqrt{1 - \frac{2x_2}{\gamma L}}, \qquad \square \tag{x}$$

from which we see that $\lambda_1(x_1, x_2)$ varies nonlinearly with x_2. Moreover we see that $\lambda_1(x_1, 0) = 1$, $\lambda_1(x_1, x_2) < 1$ for $x_2 > 0$ and $\lambda_1(x_1, x_2) > 1$ for $x_2 < 0$.

(d) The right Cauchy-Green tensor is

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \overset{(v)}{=} (r\theta')^2\ \mathbf{e}_1 \otimes \mathbf{e}_1 + (r')^2\ \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{xi}$$

We see that $\mathbf{C}$ is diagonal in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and therefore the three scalar factors appearing in (xi) are the squares of the principal stretches. It now follows from $(vi), (vii)$ and (xi) that the quantities λ_1 and λ_2 that we found earlier are in fact the principal stretches. We can write $\mathbf{C}$ as

$$\mathbf{C} = \lambda_1^2\ \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2^2\ \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{xii}$$

From (xi) we find that the Green Saint-Venant strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ has components

$$[E] = \begin{pmatrix} \frac{1}{2}(\lambda_1^2 - 1) & 0 & 0 \\ 0 & \frac{1}{2}(\lambda_2^2 - 1) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \square \tag{xiii}$$

where λ_1, λ_2 are given by $(vi), (vii)$. Observe that $\mathbf{E}$ is diagonal in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. On the other hand the matrix of components of the displacement gradient tensor $\mathbf{H} = \mathbf{F} - \mathbf{I}$, using by $(iv), (vi), (vii)$, is

$$[H] = \begin{pmatrix} \lambda_1 \cos\theta - 1 & -\lambda_2 \sin\theta & 0 \\ \lambda_1 \sin\theta & \lambda_2 \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{xiv}$$

Therefore the infinitesimal strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T)$ has components

$$[\varepsilon] = \begin{pmatrix} \lambda_1 \cos\theta - 1 & \frac{1}{2}(\lambda_1 - \lambda_2)\sin\theta & 0 \\ \frac{1}{2}(\lambda_1 - \lambda_2)\sin\theta & \lambda_2\cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad \square \tag{xv}$$

Observe that the shear strain component $E_{12} = 0$ but $\varepsilon_{12} \neq 0$. Observe from Figure 2.20 that the angle between two lines such as PQ and MN does not change due to the deformation. This shows that $\boldsymbol{\varepsilon}$ is not an appropriate strain measure in general (for large deformations).

(e) : Substituting $(viii)$ into (xiv) gives

$$[H] = \begin{pmatrix} \lambda_1 \cos\frac{x_1}{\gamma L} - 1 & -\lambda_2 \sin\frac{x_1}{\gamma L} & 0 \\ \lambda_1 \sin\frac{x_1}{\gamma L} & \lambda_2\cos\frac{x_1}{\gamma L} - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{xvi}$$

All elements of this matrix must be small for the deformation to be infinitesimal. First consider the requirement $|H_{12}| \ll 1$. Since $\lambda_2 \approx 1$ for infinitesimal deformations we must have $\sin x_1/(\gamma L) \ll 1$ which requires $x_1/(\gamma L) \ll 1$ which in turn requires $\gamma \gg x_1/L$ for all x_1 whence

$$\gamma \gg 1. \tag{$xvii$}$$

This says that the deformation is infinitesimal when the radius γL of the centerline in the deformed configuration is much larger than the length L of the undeformed beam (which is what we would expect). Thus with $1/\gamma$ as a small parameter, we want to approximate each term H_{ij} for small $1/\gamma$ dropping any terms that are quadratic or smaller. With this enforced $H_{11} \approx \lambda_1 - 1, H_{12} \approx -\lambda_2 x_1/(\gamma L), H_{21} \approx \lambda_2 x_1/(\gamma L)$ and $H_{22} \approx \lambda_2 - 1$ as $\gamma \to \infty$. Thus

$$[H] \approx \begin{pmatrix} \lambda_1 - 1 & -\lambda_2 \frac{x_1}{\gamma L} & 0 \\ \lambda_1 \frac{x_1}{\gamma L} & \lambda_2 - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \approx \begin{pmatrix} \lambda_1 - 1 & -\frac{x_1}{\gamma L} & 0 \\ \frac{x_1}{\gamma L} & \lambda_2 - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{$xviii$}$$

where in getting to the second expression we used the fact that $x_1/(\gamma L)$ is small and therefore replaced the stretches λ_1 and λ_2 in the two off-diagonal terms by their leading order approximations, i.e. unity. On using $(xviii)$,

$$[\varepsilon] = \frac{1}{2}([H] + [H]^T) = \begin{pmatrix} \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_2 - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad \square \tag{xix}$$

Observe that now $\varepsilon_{12} = 0$.

Remark: One can linearize the expression $(xiii)$ for the Green Saint Venant strain tensor directly. Note that for small deformations one has $|\lambda_i - 1| \ll 1$ and so

$$\frac{1}{2}(\lambda_i^2 - 1) = \frac{1}{2}(\lambda_i + 1)(\lambda_i - 1) \approx \frac{1}{2}(2)(\lambda_i - 1) = \lambda_i - 1.$$

Therefore based on $(xiii)$, the infinitesimal strain tensor must have terms $\lambda_1 - 1$ and $\lambda_2 - 1$ on the diagonal and zero off the diagonal, which is precisely (xix).

Exercise: Linearize the stretch λ_1 given by (x) and derive a familiar equation for the strain $\lambda_1 - 1$.

Problem 2.2. A rigid rotation of a body is described by the deformation $\mathbf{y} = \mathbf{Q}\mathbf{x}$ where $\mathbf{Q}$ is proper orthogonal. Consider the particular rigid rotation

$$\mathbf{Q} = \cos\theta(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \sin\theta(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) + \mathbf{e}_3 \otimes \mathbf{e}_3 \qquad (i)$$

that describes a rotation through an angle θ about the $\mathbf{e}_3$-axis; see Problem 1.4.14. The deformation $\mathbf{y} = \mathbf{Q}\mathbf{x}$ when written out explicitly in component form reads

$$\left.\begin{aligned} y_1 &= x_1\,\cos\theta + x_2\,\sin\theta, \\ y_2 &= -x_1\,\sin\theta + x_2\,\cos\theta, \\ y_3 &= x_3. \end{aligned}\right\} \qquad (ii)$$

Use (2.111) to calculate the components of the *infinitesimal strain tensor* $\boldsymbol{\varepsilon}$ associated with the deformation (ii). Explain why this strain tensor does not vanish even though the deformation is rigid.

Suppose that the deformation is infinitesimal in the sense that $|\theta| \ll 1$. Show that $\boldsymbol{\varepsilon}$ does vanish to leading order in this case. Moreover, show that the deformation (ii) reduces to the form (2.121) of an infinitesimal rigid rotation. *This shows that the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is not a good measure of strain for finite deformations but is appropriate for the study of infinitesimal deformations.*

Homogeneous deformations

Problem 2.3. (Based on Chadwick) (This problem will be revisited in Chapter 6 when we discuss the constitutive relation of an anisotropic material.) An *incompressible* body is reinforced by embedding two families of straight *inextensible* fibers in it as depicted in Figure 2.21. The fiber directions $\mathbf{m}_{\rm R}^{\pm}$ in the reference configuration are

$$\mathbf{m}_{\rm R}^{\pm} = \cos\Theta\,\mathbf{e}_1 \pm \sin\Theta\,\mathbf{e}_2, \qquad 0 < \Theta < \pi/2.$$

The body is subjected to the homogeneous deformation

$$y_1 = \lambda_1 x_1, \qquad y_2 = \lambda_2 x_2, \qquad y_3 = \lambda_3 x_3, \qquad (o)$$

where $\lambda_i > 0, i = 1, 2, 3$.

(a) Show that in view of the kinematic constraints used in modeling the material, i.e. inextensibility and incompressibility, the only deformations (of the above form) that this body can sustain are those that obey

$$\lambda_1^2\cos^2\Theta + \lambda_2^2\sin^2\Theta = 1, \qquad \lambda_1\lambda_2\lambda_3 = 1.$$

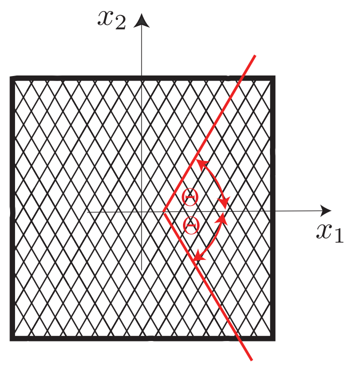


Figure 2.21: A cubic block reinforced with two families of inextensible fibers making angles $\pm\Theta$ with the x_1-axis in the reference configuration. (Figure for Problem 2.3)

(b) Show that the value of the stretch λ_1 is restricted to the range $0 < \lambda_1 < 1/\cos\Theta$. Why do you think λ_1 cannot be increased beyond a certain value?

(c) Let $\pm\theta$ be the angles that the fibers make with the y_1-axis in the deformed configuration. Analyze how θ varies as a function of λ_1. What value does θ approach when $\lambda_1 \to 1/\cos\Theta$? Now explain why the value of λ_1 cannot be increased beyond $1/\cos\Theta$.

(d) Analyze the variation of λ_2 as a function of λ_1. In particular, show that λ_2 decreases monotonically as λ_1 increases and that $\lambda_2 \to 0$ when $\lambda_1 \to 1/\cos\Theta$.

(e) Analyze the variation of λ_3 as a function of λ_1. In particular, show that as λ_1 increases, the body first contracts in the x_3-direction until λ_3 reaches the value $\sin 2\Theta$ and expands thereafter with $\lambda_3 \to \infty$ when $\lambda_1 \to 1/\cos\Theta$.

(f) Calculate the value of the angle between the two families of fibers in the deformed configuration when the body has contracted to its minimum value $\lambda_3 = \sin 2\Theta$ (corresponding to $\lambda_1 = 1/(\sqrt{2}\cos\Theta)$).

(g) Calculate the "Poisson's ratios" $-d\lambda_2/d\lambda_1$ at $\lambda_1 = 1$ and $-d\lambda_3/d\lambda_1$ at $\lambda_1 = 1$. Under what conditions (if any) do they take the value $1/2$?

Solution:

(a) The deformation gradient tensor associated with the given deformation (o) is

$$\mathbf{F} = \lambda_1\, \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\, \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3\, \mathbf{e}_3 \otimes \mathbf{e}_3.$$

In the undeformed configuration the fibers are oriented in the directions $\mathbf{m}_{\mathrm{R}}^{\pm} = \cos\Theta\,\mathbf{e}_1 \pm \sin\Theta\,\mathbf{e}_2$ and therefore the deformed images of $\mathbf{m}_{\mathrm{R}}^{\pm}$ are

$$\mathbf{F}\mathbf{m}_{\mathrm{R}}^{\pm} = \lambda_1 \cos\Theta \mathbf{e}_1 \pm \lambda_2 \sin\Theta \mathbf{e}_2. \qquad (i)$$

Since the fibers are inextensible we must have $|\mathbf{F}\mathbf{m}_{\mathrm{R}}^{\pm}| = |\mathbf{m}_{\mathrm{R}}^{\pm}| = 1$, i.e.

$$\lambda_1^2 \cos^2\Theta + \lambda_2^2 \sin^2\Theta = 1. \qquad \square \qquad (ii)$$

In addition, since the material is incompressible we must have $\det \mathbf{F} = 1$:

$$\lambda_1\lambda_2\lambda_3 = 1. \qquad \square \tag{iii}$$

(b) On rearranging (ii) we have $1 - \lambda_1^2\cos^2\Theta = \lambda_2^2\sin^2\Theta > 0$ where the inequality follows from $\lambda_2 > 0$ and $\sin\Theta > 0$. Thus

$$1 - \lambda_1^2\cos^2\Theta > 0 \qquad \Rightarrow \qquad 1 > \lambda_1\cos\Theta \qquad \Rightarrow \qquad 0 < \lambda_1 < 1/\cos\Theta, \qquad \square$$

having used the fact that λ_1 and $\cos\Theta$ are positive.

The two equations (ii) and (iii) constrain the values of the stretches. In particular, one can solve them for the stretches λ_2 and λ_3 in terms of λ_1 leading to

$$\lambda_2 = \frac{(1 - \lambda_1^2\cos^2\Theta)^{1/2}}{\sin\Theta}, \qquad 0 < \lambda_1 < 1/\cos\Theta, \tag{iv}$$

$$\lambda_3 = \frac{\sin\Theta}{\lambda_1(1 - \lambda_1^2\cos^2\Theta)^{1/2}}, \qquad 0 < \lambda_1 < 1/\cos\Theta, \tag{v}$$

where we have taken the positive square roots since $\lambda_1, \lambda_2, \lambda_3$ and $\sin\Theta$ are all positive. Thus the values of λ_2 and λ_3 are fully determined by λ_1 (and Θ).

(c) Consider a material fiber $d\mathbf{x} = ds(\cos\Theta\,\mathbf{e}_1 + \sin\Theta\,\mathbf{e}_2)$ in the reference configuration. Since in the deformed configuration it makes an angle θ with the y_1-direction (and its length remains ds), the deformation maps it into $d\mathbf{y} = ds(\cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2)$. Substituting $d\mathbf{x}$ and $d\mathbf{y}$ into $d\mathbf{y} = \mathbf{F}d\mathbf{x}$ and simplifying gives

$$\cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 \quad = (\lambda_1\mathbf{e}_1\otimes\mathbf{e}_1 + \lambda_2\mathbf{e}_2\otimes\mathbf{e}_2 + \lambda_3\mathbf{e}_3\otimes\mathbf{e}_3)(\cos\Theta\,\mathbf{e}_1 + \sin\Theta\,\mathbf{e}_2) = \lambda_1\cos\Theta\,\mathbf{e}_1 + \lambda_2\sin\Theta\,\mathbf{e}_2,$$

and so

$$\cos\theta = \lambda_1\cos\Theta, \qquad \square \tag{vii}$$

(and $\sin\theta = \lambda_2\sin\Theta$). This gives the angle in the deformed configuration as a function of the stretch λ_1 (and the angle Θ). Figure 2.22 shows the variation of θ with λ_1 according to (vii). Observe that the fiber angle θ decreases monotonically as the stretch λ_1 increases. Moreover,

$$\theta \to 0 \quad \text{as} \quad \lambda_1 \to 1/\cos\Theta,$$

(and $\theta \to \pi/2$ as $\lambda_1 \to 0$). Thus the fibers are aligned with the y_1-axis when $\lambda_1 \to 1/\cos\Theta$ and (since they are inextensible) the block cannot be stretched any further.

(d) Based on (iv), λ_2 decreases monotonically as λ_1 increases. Figure 2.23 shows the variation of λ_2 with λ_1 according to (iv). In particular $\lambda_2 \to 0$ as $\lambda_1 \to 1/\cos\Theta$. (Note: Since $\lambda_2 \to 0$ in this limit, the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$ requires that $\lambda_3 \to \infty$ and therefore in particular, λ_3 must increase with λ_1, at least for large values of stretch!)

(e) Differentiating (v) with respect to λ_1 yields

$$\frac{d\lambda_3}{d\lambda_1} = -\frac{\sin\Theta(1 - 2\lambda_1^2\cos^2\Theta)}{\lambda_1^2(1 - \lambda_1^2\cos^2\Theta)^{3/2}}, \tag{viii}$$

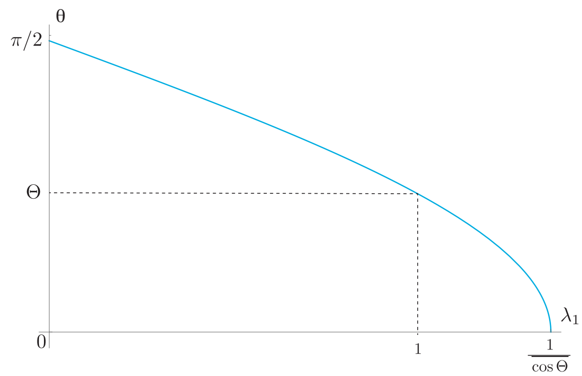


Figure 2.22: Angle θ in the deformed configuration versus stretch λ_1. In the reference configuration: $\lambda_1 = 1, \theta = \Theta$.

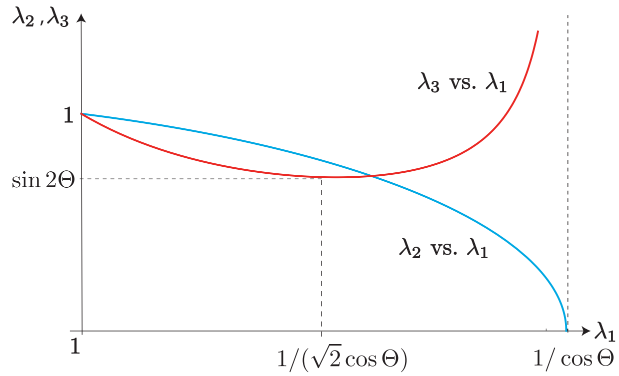


Figure 2.23: Variation of λ_2 and λ_3 versus λ_1 according to (iv) and (v). The figure has been drawn for $\Theta = 3\pi/8$.

from which we see that

$$\frac{d\lambda_3}{d\lambda_1}\begin{cases} < 0 & \text{for} \quad 0 < \lambda_1 < 1/(\sqrt{2}\cos\Theta), \\ = 0 & \text{for} \quad \lambda_1 = 1/(\sqrt{2}\cos\Theta), \\ > 0 & \text{for} \quad 1/(\sqrt{2}\cos\Theta) < \lambda_1 < 1/\cos\Theta. \end{cases}$$

Moreover

$$\lambda_3 = \sin 2\Theta \qquad \text{when} \qquad \lambda_1 = 1/(\sqrt{2}\cos\Theta). \tag{ix}$$

Therefore as λ_1 increases from 0, the value of λ_3 first decreases until it reaches its minimum value $\sin 2\Theta$ when $\lambda_1 = 1/(\sqrt{2}\cos\Theta)$ and increases thereafter. Therefore

$$\lambda_3 \geq \sin 2\Theta. \tag{x}$$

Observe that $\lambda_3 \to \infty$ as $\lambda_1 \to 1/\cos\Theta$. Figure 2.23 shows the variation of λ_3 with λ_1 according to (v). (The figure has been drawn for $\Theta = 3\pi/8$.)

Note that the value of λ_1 at which the minimum occurs, i.e. $\lambda_1 = 1/(\sqrt{2}\cos\Theta)$, is < 1 when $\Theta < \pi/4$ and > 1 when $\Theta > \pi/4$. Imagine increasing λ_1 from $\lambda_1 = 1$. If $\Theta < \pi/4$ then λ_3 increases monotonically as λ_1 increases. However if $\Theta > \pi/4$ then λ_3 first decreases until it reaches its minimum value and then increases.

(f) Observe from (vii) that

$$\theta = \pi/4 \quad \text{when} \quad \lambda_1 = \frac{1}{\sqrt{2}\cos\Theta}.$$

When $\theta = \pi/4$ (i.e. when $\lambda_1 = 1/(\sqrt{2}\cos\Theta)$ or $\lambda_3 = \sin 2\Theta$) the two families of fibers are orthogonal in the deformed configuration. This can be seen in Figure 2.22.

(g) Differentiating (iv) respect to λ_1 and evaluating the result at $\lambda_1 = 1$, and also evaluating ($viii$) at $\lambda_1 = 1$, gives

$$-\left.\frac{d\lambda_2}{d\lambda_1}\right|_{\lambda_1=1} = \cot^2\Theta, \qquad -\left.\frac{d\lambda_3}{d\lambda_1}\right|_{\lambda_1=1} = \frac{1-2\cos^2\Theta}{\sin^2\Theta}.$$

If the block was in a state of uniaxial stress in the x_1-direction (of course we are told nothing about stress here) then we would call the two quantities above the Poisson's ratios at infinitesimal deformations with respect to the two transverse directions x_2 and x_3. Note that (a) the values of the "Poisson ratios" depend on the angle Θ, (b) the "Poisson ratio" with respect to the x_2- direction is different to that with respect to the x_3-direction, and (c) the value associated with the x_3-direction will be either positive or negative depending on the angle Θ. It is readily shown from the preceding equation that if

$$-\left.\frac{d\lambda_2}{d\lambda_1}\right|_{\lambda_1=1} = -\left.\frac{d\lambda_3}{d\lambda_1}\right|_{\lambda_1=1} = \frac{1}{2},$$

the fiber direction in the reference configuration must have the special value $\approx 54.74^o$ corresponding to $\tan\Theta = \sqrt{2}$.

Problem 2.4. A body occupies a unit cube in a reference configuration:

$$\mathcal{R}_{\rm R} = \{(x_1, x_2, x_3) \,:\, -1/2 < x_1 < 1/2,\ -1/2 < x_2 < 1/2,\ -1/2 < x_3 < 1/2\}.$$

It is subjected to the deformation

$$y_1 = ax_1 + bx_2, \qquad y_2 = bx_1 + ax_2, \qquad y_3 = cx_3, \tag{i}$$

where a, b and c are constants with $c > 0$ and $a > b > 0$.

(a) Consider a plane $\mathcal{S}_{\rm R}$ defined by $x_1 + x_2 = \mathsf{constant}$ in the reference configuration. Under what conditions on a, b, c does the area of a patch on this surface not change due to the deformation (i)?

(b) Under what conditions on a, b, c does *every* material fiber on the plane $\mathcal{S}_{\rm R}$ remain *unstretched* by the deformation?

Solution: From (i) and $F_{ij} = \partial y_i/\partial x_j$ the matrix of components of the deformation gradient tensor is

$$[F] = \begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix}. \tag{ii}$$

Therefore

$$J = \det[F] = c(a^2 - b^2) > 0, \qquad (iii)$$

since $c > 0, a > b > 0$ (given).

(a) By Nanson's formula (with $J > 0$) the area transforms according to $dA_y\,\mathbf{n} = dA_x\,J\mathbf{F}^{-T}\mathbf{n}_R$ whence $dA_y = dA_x\,J|\mathbf{F}^{-T}\mathbf{n}_R|$. When the area does not change, $dA_x = dA_y$ and so

$$J|\mathbf{F}^{-T}\mathbf{n}_R| = 1. \qquad (iv)$$

A unit vector normal to the surface $\mathcal{S}_\mathrm{R}$ is

$$\mathbf{n}_R = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2. \qquad (v)$$

One readily finds from (ii) that

$$[F]^{-1} = (a^2 - b^2)^{-1}\begin{pmatrix} a & -b & 0 \\ -b & a & 0 \\ 0 & 0 & (a^2 - b^2)/c \end{pmatrix}. \qquad (vi)$$

Substituting $(iii), (v)$ and (vi) into (iv) and simplifying (and using $c > 0, a > b$) leads to

$$c^2(a-b)^2 = 1 \qquad \Rightarrow \qquad c(a-b) = 1. \qquad (vii)$$

(b) Let $\boldsymbol{\ell} = \ell_i\mathbf{e}_i$ be an arbitrary vector. If it lies in the plane $\mathcal{S}_\mathrm{R}$ it is perpendicular to $\mathbf{n}_R$ and so $\mathbf{n}_R \cdot \boldsymbol{\ell} = 0$:

$$\mathbf{n}_R \cdot \boldsymbol{\ell} = \left(\frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2\right) \cdot \ell_i\mathbf{e}_i = \frac{\ell_1}{\sqrt{2}} + \frac{\ell_2}{\sqrt{2}} = 0 \qquad \Rightarrow \qquad \ell_2 = -\ell_1,$$

and so such a vector has the form

$$\boldsymbol{\ell} = \ell_1\mathbf{e}_1 - \ell_1\mathbf{e}_2 + \ell_3\mathbf{e}_3, \qquad (viii)$$

where ℓ_1 and ℓ_3 are arbitrary. The deformed image of $\boldsymbol{\ell}$ is $\mathbf{F}\boldsymbol{\ell}$ and so every such fiber does not stretch if

$$|\mathbf{F}\boldsymbol{\ell}| = |\boldsymbol{\ell}|, \qquad (ix)$$

for all vectors $\boldsymbol{\ell}$ of the form $(viii)$. Substituting (ii) and $(viii)$ into (ix) and simplifying leads to

$$2\big[(a-b)^2 - 1\big]\ell_1^2 + \big[c^2 - 1\big]\ell_3^2 = 0 \qquad \text{for all } \ell_1, \ell_3.$$

This requires $(a-b)^2 - 1 = 0$ and $c^2 - 1 = 0$:

$$a - b = 1, \qquad c = 1, \qquad (x)$$

where we have used $c > 0, a > b > 0$.

Remark: As one would expect, (x) implies (vii) but the converse is not true.

Problem 2.5. (P. Rosakis) In this problem you are to show that if you know the deformation of any *three* linearly independent material fibers, then you can calculate $\mathbf{F}$ and therefore determine the deformation of *all* material fibers.

Consider three distinct non-coplanar material fibers identified by the three linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. (These fibers need not be perpendicular to each other and need not have the same lengths.) The body is subjected to a homogeneous deformation that carries these fibers into $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ respectively. You are given $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_3$. Derive a formula for the deformation gradient tensor in terms of these six vectors (alone) which then establishes the desired result.

Solution: We want to find a tensor $\mathbf{F}$ such that $\mathbf{F}\mathbf{a}_i = \mathbf{b}_i, i = 1, 2, 3$. If we can find 3 vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ such that $\mathbf{c}_i \cdot \mathbf{a}_j = \delta_{ij}$ (keep in mind that in this problem $\mathbf{a}_i \cdot \mathbf{a}_j \neq \delta_{ij}$) then, since $(\mathbf{b}_j \otimes \mathbf{c}_j)\mathbf{a}_i = (\mathbf{c}_j \cdot \mathbf{a}_i)\mathbf{b}_j = \delta_{ij}\mathbf{b}_j = \mathbf{b}_i$ it would follow that $\mathbf{F} = \mathbf{b}_i \otimes \mathbf{c}_i$. Thus our task is to determine three such vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$.

Since $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly independent, it follows from Problem 1.7 that $\mathbf{a}_i \times \mathbf{a}_j \neq \mathbf{o}$ and $(\mathbf{a}_i \times \mathbf{a}_j) \cdot \mathbf{a}_k \neq 0$ for distinct i, j, k. Therefore we can define three non-zero vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ related to $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ by[18]

$$\mathbf{c}_1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{(\mathbf{a}_2 \times \mathbf{a}_3) \cdot \mathbf{a}_1}, \qquad \mathbf{c}_2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{(\mathbf{a}_3 \times \mathbf{a}_1) \cdot \mathbf{a}_2}, \qquad \mathbf{c}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{(\mathbf{a}_1 \times \mathbf{a}_2) \cdot \mathbf{a}_3}. \tag{i}$$

Observe that the vector $\mathbf{c}_1$ is perpendicular to the vectors $\mathbf{a}_2$ and $\mathbf{a}_3$ and its length is such that $\mathbf{c}_1 \cdot \mathbf{a}_1 = 1$, i.e.

$$\mathbf{c}_1 \cdot \mathbf{a}_1 = 1, \qquad \mathbf{c}_1 \cdot \mathbf{a}_2 = \mathbf{c}_1 \cdot \mathbf{a}_3 = 0. \tag{ii}$$

The vectors $\mathbf{c}_2$ and $\mathbf{c}_3$ are related analogously to $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Therefore

$$\mathbf{c}_i \cdot \mathbf{a}_j = \delta_{ij}. \tag{iii}$$

It now follows from the remarks in the first paragraph that

$$\mathbf{F} = \mathbf{b}_j \otimes \mathbf{c}_j. \qquad \square$$

Remark: In the special case where each vector of the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is perpendicular to the other two, one sees from (i) that $\mathbf{c}_i = \mathbf{a}_i/|\mathbf{a}_i|^2$ and therefore that $\mathbf{c}_i$ is parallel to $\mathbf{a}_i$ and its length is $1/|\mathbf{a}_i|$. If in addition, $\mathbf{a}_i$ is a unit vector, then $\mathbf{c}_i = \mathbf{a}_i$.

Problem 2.6. (Ogden) A body undergoes a simple shear deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$, $\mathbf{F} = \mathbf{I} + k\,\mathbf{a} \otimes \mathbf{b}$, where $\mathbf{a}$ is the shearing direction and $\mathbf{b}$ is the glide plane normal. Consider a plane $\mathcal{S}_{\mathrm{R}}$ in the reference configuration that is perpendicular to the unit vector $\mathbf{n}_{\mathrm{R}} = \cos\theta\,\mathbf{a} + \sin\theta\,\mathbf{b}$. Calculate the ratio $(\Delta A_y/\Delta A_x)^2$ where ΔA_x is the area of a surface element on $\mathcal{S}_{\mathrm{R}}$ and ΔA_y is the area of its image in the deformed configuration; express your answer in terms of $\sin 2\theta$ and $\cos 2\theta$. Considering all such planes $\mathcal{S}_{\mathrm{R}}$, on which do the maximum and minimum values of $(\Delta A_y/\Delta A_x)^2$ occur? Calculate those values.

[18]The vectors $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ are said to be **reciprocal** to the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

Solution: The deformation gradient tensor associated with the simple shear is $\mathbf{F} = \mathbf{I} + k\mathbf{a} \otimes \mathbf{b}$, its determinant is $J = \det \mathbf{F} = 1$ and its inverse is

$$\mathbf{F}^{-1} = \mathbf{I} - k\mathbf{a} \otimes \mathbf{b}. \qquad (i)$$

Therefore $\mathbf{F}^{-T} = \mathbf{I} - k\mathbf{b} \otimes \mathbf{a}$. We are interested in a plane perpendicular to

$$\mathbf{n}_R = \cos\theta\,\mathbf{a} + \sin\theta\,\mathbf{b}. \qquad (ii)$$

Therefore from (2.39),

$$\begin{aligned}\left(\frac{\Delta A_y}{\Delta A_x}\right)^2 &= J^2|\mathbf{F}^{-T}\mathbf{n}_R|^2 = \big|(\mathbf{I} - k\mathbf{b} \otimes \mathbf{a})(\cos\theta\mathbf{a} + \sin\theta\mathbf{b})\big|^2 = \big|\cos\theta\mathbf{a} + (\sin\theta - k\cos\theta)\mathbf{b}\big|^2 = \\ &= (\cos\theta)^2 + (\sin\theta - k\cos\theta)^2 = 1 + \tfrac{k^2}{2}(1 + \cos 2\theta) - k\sin 2\theta = \\ &= 1 + \tfrac{k^2}{2} + k\sqrt{k^2/4 + 1}\,\big[\cos\theta_*\cos 2\theta + \sin\theta_*\sin 2\theta\big] = \\ &= 1 + \tfrac{k^2}{2} + k\sqrt{k^2/4 + 1}\,\cos(2\theta - \theta_*) \qquad (iii)\end{aligned}$$

where

$$\tan\theta_* = -2/k, \qquad \theta_* \in (\pi/4, \pi/2). \qquad (iv)$$

Therefore from (iii) we see that the maximum value of $(\Delta A_y/\Delta A_x)^2$ is

$$\left(\frac{\Delta A_y}{\Delta A_x}\right)^2 = 1 + \frac{k^2}{2} + k\sqrt{k^2/4 + 1}$$

which is associated with the plane defined by

$$2\theta = \theta_*. \qquad (v)$$

Likewise from (iii) we see that the minimum value of $(\Delta A_y/\Delta A_x)^2$ is

$$\left(\frac{\Delta A_y}{\Delta A_x}\right)^2 = 1 + \frac{k^2}{2} - k\sqrt{k^2/4 + 1}$$

which is associated with the plane defined by

$$2\theta = \pi + \theta_*. \qquad (vi)$$

The plane defined by (vi) is perpendicular to that defined by (v).

Problem 2.7. Let $\mathbf{y} = \mathbf{F}_1\mathbf{x}$ and $\mathbf{y} = \mathbf{F}_2\mathbf{x}$ be two arbitrary homogeneous deformations. Suppose that the deformation $\mathbf{y} = \mathbf{F}_1\mathbf{F}_2\mathbf{x}$ is a simple shear. Is the deformation $\mathbf{y} = \mathbf{F}_2\mathbf{F}_1\mathbf{x}$ also a simple shear? If it is, (either in general or under special circumstances), what is the associated amount of shear, glide plane normal and direction of shear?

Solution: Since $\mathbf{F}_1\mathbf{F}_2$ describes a simple shear it can be expressed as

$$\mathbf{F}_1\mathbf{F}_2 = \mathbf{I} + k\,\mathbf{m} \otimes \mathbf{n} \qquad \text{where} \quad |\mathbf{m}| = |\mathbf{n}| = 1, \quad \mathbf{m} \cdot \mathbf{n} = 0. \qquad (i)$$

Post-multiplying (i) by $\mathbf{F}_2^{-1}$ gives

$$\mathbf{F}_1 = \mathbf{F}_2^{-1} + k(\mathbf{m}\otimes\mathbf{n})\mathbf{F}_2^{-1} \overset{(1.78)}{=} \mathbf{F}_2^{-1} + k\big[\mathbf{m}\otimes(\mathbf{F}_2^{-T}\mathbf{n})\big], \qquad (ii)$$

and pre-multiplying (ii) by $\mathbf{F}_2$ leads to

$$\mathbf{F}_2\mathbf{F}_1 = \mathbf{I} + k\,\mathbf{F}_2\big[\mathbf{m}\otimes(\mathbf{F}_2^{-T}\mathbf{n})\big] \overset{(1.78)}{=} \mathbf{I} + k\big[(\mathbf{F}_2\mathbf{m})\otimes(\mathbf{F}_2^{-T}\mathbf{n})\big]. \qquad (iii)$$

Therefore the deformation corresponding to $\mathbf{F}_2\mathbf{F}_1$ is a simple shear provided $\mathbf{F}_2\mathbf{m}$ is perpendicular to $\mathbf{F}_2^{-T}\mathbf{n}$. That this is true can be verified by

$$(\mathbf{F}_2\mathbf{m})\cdot(\mathbf{F}_2^{-T}\mathbf{n}) \overset{(1.74)}{=} \mathbf{m}\cdot\mathbf{F}_2^T(\mathbf{F}_2^{-T}\mathbf{n}) = \mathbf{m}\cdot\mathbf{n} \overset{(i)}{=} 0. \qquad (iv)$$

Therefore $\mathbf{F}_2\mathbf{m}$ is perpendicular to $\mathbf{F}_2^{-T}\mathbf{n}$ and so $\mathbf{F}_2\mathbf{F}_1$ describes a simple shear. We can write (iii) in standard form as $\mathbf{I}+\kappa\mathbf{a}\otimes\mathbf{b}$ with $\mathbf{a}$ and $\mathbf{b}$ being unit vectors:

$$\mathbf{F}_2\mathbf{F}_1 = \mathbf{I}+\kappa\mathbf{a}\otimes\mathbf{b} \qquad \text{where} \quad \mathbf{a} = \frac{\mathbf{F}_2\mathbf{m}}{|\mathbf{F}_2\mathbf{m}|}, \quad \mathbf{b} = \frac{\mathbf{F}_2^{-T}\mathbf{n}}{|\mathbf{F}_2^{-T}\mathbf{n}|}, \quad \kappa = k\,|\mathbf{F}_2\mathbf{m}|\,|\mathbf{F}_2^{-T}\mathbf{n}|. \qquad \square$$

The shearing direction is $\mathbf{b}$, the normal to the glide plane is $\mathbf{a}$ and the amount of shear is κ.

Problem 2.8. (Ogden) Show that a simple shear with amount of shear k_1, shear direction $\mathbf{m}_1$ and glide plane normal $\mathbf{n}_1$ is commutative with a simple shear with amount of shear k_2, shear direction $\mathbf{m}_2$ and glide plane normal $\mathbf{n}_2$ if and only if

$$\text{either} \quad (a)\ \mathbf{m}_1 = \pm\mathbf{m}_2 \quad \text{or} \quad (b)\ \mathbf{n}_1 = \pm\mathbf{n}_2. \qquad (i)$$

In case (a) show that the composite deformation is a simple shear with shear direction $\mathbf{m}_1$, glide plane normal $k_1\mathbf{n}_1 \pm k_2\mathbf{n}_2$ and amount of shear $(k_1^2+k_2^2 \pm 2k_1k_2\mathbf{n}_1\cdot\mathbf{n}_2)^{1/2}$. In case (b) show that the composite deformation is a simple shear with shear direction $k_1\mathbf{m}_1 \pm k_2\mathbf{m}_2$, glide plane normal $\mathbf{n}$ and amount of shear $(k_1^2+k_2^2 \pm 2k_1k_2\mathbf{m}_1\cdot\mathbf{m}_2)^{1/2}$.

Solution: We are asked to show that $\mathbf{F}_1\mathbf{F}_2 = \mathbf{F}_2\mathbf{F}_1$ for the two deformation gradient tensors

$$\mathbf{F}_1 = \mathbf{I}+k_1\mathbf{m}_1\otimes\mathbf{n}_1, \qquad \mathbf{F}_2 = \mathbf{I}+k_2\mathbf{m}_2\otimes\mathbf{n}_2, \qquad (ii)$$

if and only if (i) holds.

From (ii) we find

$$\mathbf{F}_1\mathbf{F}_2 = \mathbf{I}+k_1\mathbf{m}_1\otimes\mathbf{n}_1+k_2\mathbf{m}_2\otimes\mathbf{n}_2+k_1k_2(\mathbf{m}_2\cdot\mathbf{n}_1)\mathbf{m}_1\otimes\mathbf{n}_2,$$

$$\mathbf{F}_2\mathbf{F}_1 = \mathbf{I}+k_1\mathbf{m}_1\otimes\mathbf{n}_1+k_2\mathbf{m}_2\otimes\mathbf{n}_2+k_1k_2(\mathbf{m}_1\cdot\mathbf{n}_2)\mathbf{m}_2\otimes\mathbf{n}_1,$$

and so

$$\mathbf{F}_1\mathbf{F}_2 = \mathbf{F}_2\mathbf{F}_1 \qquad (iii)$$

if and only if

$$(\mathbf{m}_2\cdot\mathbf{n}_1)\mathbf{m}_1\otimes\mathbf{n}_2 = (\mathbf{m}_1\cdot\mathbf{n}_2)\mathbf{m}_2\otimes\mathbf{n}_1. \qquad (iv)$$

Operating (iv) on the unit vector $\mathbf{n}_1$ yields

$$(\mathbf{m}_2 \cdot \mathbf{n}_1)(\mathbf{n}_2 \cdot \mathbf{n}_1)\mathbf{m}_1 = (\mathbf{m}_1 \cdot \mathbf{n}_2)\mathbf{m}_2. \tag{v}$$

Operating (iv) on the unit vector $\mathbf{n}_2$ gives

$$(\mathbf{m}_2 \cdot \mathbf{n}_1)\mathbf{m}_1 = (\mathbf{m}_1 \cdot \mathbf{n}_2)(\mathbf{n}_1 \cdot \mathbf{n}_2)\mathbf{m}_2. \tag{vi}$$

Operating (iv) on $\mathbf{m}_1$ and using $\mathbf{m}_1 \cdot \mathbf{n}_1 = 0$ leads to

$$(\mathbf{m}_2 \cdot \mathbf{n}_1)(\mathbf{n}_2 \cdot \mathbf{m}_1)\mathbf{m}_1 = 0. \tag{vii}$$

Since $|\mathbf{m}_1| = 1$ we know $\mathbf{m}_1 \neq \mathbf{o}$ and so $(vii) \Rightarrow$

$$(\mathbf{m}_2 \cdot \mathbf{n}_1)(\mathbf{n}_2 \cdot \mathbf{m}_1) = 0. \tag{$viii$}$$

Therefore either $\mathbf{m}_2 \cdot \mathbf{n}_1 = 0$ or $\mathbf{n}_2 \cdot \mathbf{m}_1 = 0$ or both are zero. Suppose $\mathbf{m}_2 \cdot \mathbf{n}_1 = 0$. Then (v) yields $(\mathbf{m}_1 \cdot \mathbf{n}_2)\mathbf{m}_2 = 0$ and since $\mathbf{m}_2 \neq \mathbf{o}$ it follows that

$$\mathbf{m}_1 \cdot \mathbf{n}_2 = 0. \tag{ix}$$

Suppose instead that $\mathbf{n}_2 \cdot \mathbf{m}_1 = 0$. Then (vi) yields $(\mathbf{m}_2 \cdot \mathbf{n}_1)\mathbf{m}_1 = 0$ and since $\mathbf{m}_1 \neq \mathbf{o}$ it follows that

$$\mathbf{m}_2 \cdot \mathbf{n}_1 = 0. \tag{x}$$

Thus both cases imply that necessarily

$$\mathbf{m}_1 \cdot \mathbf{n}_2 = 0 \quad \text{and} \quad \mathbf{m}_2 \cdot \mathbf{n}_1 = 0. \tag{xi}$$

Conversely, if (xi) holds we see that (iv) holds. Thus (xi) is necessary and sufficient in order that $\mathbf{F}_1\mathbf{F}_2 = \mathbf{F}_2\mathbf{F}_1$.

Finally, from the result in Problem 1.3.4 it follows that (xi) holds if and only if

$$\text{either} \quad \mathbf{m}_1 = \pm\mathbf{m}_2 \quad \text{or} \quad \mathbf{n}_1 = \pm\mathbf{n}_2.$$

Problem 2.9.

(a) Under what conditions does the direction of a material fiber remain unchanged (invariant) in a given deformation?

(b) The region $\mathcal{R}_{\mathrm{R}}$ occupied by a body in a reference configuration is a unit cube. Consider the following *isochoric* homogeneous deformation:

$$\mathbf{y} = \big(\lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3\big)\big(\mathbf{I} + k\,\mathbf{e}_1 \otimes \mathbf{e}_2\big)\mathbf{x}, \qquad \lambda_1 \neq 1, \lambda_2 \neq 1, k \neq 0, \tag{i}$$

where the orthonormal basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are aligned with the edges of the cube. Describe the physical nature of this deformation and list as many invariant directions as you can based on your intuition.

(c) Now show mathematically that there are exactly three directions that remain invariant in this deformation and determine them.

Solution:

(a) Consider a referential material fiber in the direction $\mathbf{m}_{\mathrm{R}}$. The direction of this fiber in the deformed configuration is parallel to $\mathbf{F}\mathbf{m}_{\mathrm{R}}$. Thus if the direction of this fiber remains unchanged, $\mathbf{m}_{\mathrm{R}}$ will be parallel to $\mathbf{F}\mathbf{m}_{\mathrm{R}}$ and so for some scalar μ we must have

$$\mathbf{F}\mathbf{m}_{\mathrm{R}} = \mu\, \mathbf{m}_{\mathrm{R}}.$$

Remark: This states that $\mathbf{m}_{\mathrm{R}}$ is an eigenvector of $\mathbf{F}$. Since $\mathbf{F}$ is not symmetric in general, it may not have a full complement of real eigenvalues and eigenvectors. In a three-dimensional vector space, $\mathbf{F}$ has three eigenvalues. If one of them is complex, its complex conjugate is also an eigenvalue, and so complex eigenvalues occur in pairs. Therefore in three dimensions, $\mathbf{F}$ has either one or three real eigenvalues. Thus an arbitrary deformation gradient tensor $\mathbf{F}$ will have (in general) either one or three directions that remain invariant. There maybe more if the real eigenvalues are repeated.

(b) The given deformation has the form $\mathbf{y} = \mathbf{F}_2\mathbf{F}_1\mathbf{x}$ and so it can be viewed in two steps. First $\mathbf{x} \to \mathbf{F}_1\mathbf{x}$ and then $\mathbf{F}_1\mathbf{x} \to \mathbf{F}_2(\mathbf{F}_1\mathbf{x})$. The tensor $\mathbf{F}_1 = \big(\mathbf{I} + k\,\mathbf{e}_1 \otimes \mathbf{e}_2\big)$ represents a *simple shear* in the x_1, x_2-plane with the direction of shearing being $\mathbf{e}_1$. The tensor $\mathbf{F}_2 = \big(\lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3\big)$ represents a *biaxial stretching* in the $\mathbf{e}_1$- and $\mathbf{e}_2$-directions. (Note that $\lambda_3 = 1$.) Thus the given deformation is the composition of these two deformations both of which are entirely in the $\mathbf{e}_1, \mathbf{e}_2$-plane.

Since particles have zero displacement in the $\mathbf{e}_3$-direction, we see geometrically that any referential material fiber in the $\mathbf{e}_3$-direction will remain in the $\mathbf{e}_3$-direction in the deformed configuration. Thus $\mathbf{e}_3$ is an invariant direction.

Next consider a referential fiber in the $\mathbf{e}_1$-direction. The simple shear will simply slide this fiber in the $\mathbf{e}_1$-direction. The biaxial stretch will stretch and translate this fiber without rotation. Thus any referential material fiber in the $\mathbf{e}_1$-direction will remain in the $\mathbf{e}_1$-direction in the deformed configuration. Thus $\mathbf{e}_1$ is also an invariant direction.

Since $\mathbf{e}_1$ and $\mathbf{e}_3$ are two distinct invariant directions (eigenvectors) of $\mathbf{F}$, with two corresponding real eigenvalues, it follows that $\mathbf{F}$ necessarily has a third real eigenvalue. The corresponding eigenvector will be a third invariant direction. It is not easy to determine this direction intuitively.

(c) We now proceed to calculate the invariant directions of $\mathbf{F}$ mathematically. First we note by differentiating (i) that

$$\mathbf{F} = \lambda_1\,\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\,\mathbf{e}_2 \otimes \mathbf{e}_2 + \,\mathbf{e}_3 \otimes \mathbf{e}_3 + k\lambda_1\mathbf{e}_1 \otimes \mathbf{e}_2.$$

We are told that this deformation is volume preserving whence $\det \mathbf{F} = 1$. Since $\det \mathbf{F} = \det \mathbf{F}_2 \det \mathbf{F}_1 = (\lambda_1\lambda_2)(1)$, this implies $\lambda_1\lambda_2 = 1$. It is convenient to set $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^{-1}$ whence

$$\mathbf{F} = \lambda\,\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda^{-1}\,\mathbf{e}_2 \otimes \mathbf{e}_2 + \,\mathbf{e}_3 \otimes \mathbf{e}_3 + k\lambda\mathbf{e}_1 \otimes \mathbf{e}_2.$$

In order to find the invariant directions we find the eigenvalues of $\mathbf{F}$ by solving $\det(\mathbf{F} - \mu\mathbf{I}) = 0$ for μ, and then finding the associated eigenvectors $\mathbf{m}_{\mathrm{R}}$ from $\mathbf{F}\mathbf{m}_{\mathrm{R}} = \mu\mathbf{m}_{\mathrm{R}}$. Thus we first solve

$$\det(\mathbf{F} - \mu\mathbf{I}) = \det \begin{pmatrix} \lambda - \mu & k\lambda & 0 \\ 0 & \lambda^{-1} - \mu & 0 \\ 0 & 0 & 1 - \mu \end{pmatrix} \; = \; (1-\mu)(\lambda - \mu)(\lambda^{-1} - \mu) = 0,$$

which yields the three eigenvalues $\mu_1 = 1$, $\mu_2 = \lambda$ and $\mu_3 = \lambda^{-1}$. The corresponding eigenvectors are then found from $\mathbf{F}\mathbf{m}_{\mathrm{R}}^{(i)} = \mu_i \mathbf{m}_{\mathrm{R}}^{(i)}$, $i = 1, 2, 3$, leading to

$$\mathbf{m}_{\mathrm{R}}^{(1)} = \mathbf{e}_3, \qquad \mathbf{m}_{\mathrm{R}}^{(2)} = \mathbf{e}_1, \qquad \mathbf{m}_{\mathrm{R}}^{(3)} = -k\lambda\mathbf{e}_1 + (\lambda - \lambda^{-1})\mathbf{e}_2.$$

The previous geometric discussion had already told us that the fiber directions $\mathbf{m}_{\mathrm{R}}^{(1)}$ and $\mathbf{m}_{\mathrm{R}}^{(2)}$ are invariant. We now know that fibers in the direction $\mathbf{m}_{\mathrm{R}}^{(3)}$ also preserve their direction.

Problem 2.10. (Ogden) Consider a planar, pure stretch

$$\mathbf{y} = \mathbf{F}\mathbf{x} \qquad \text{where} \quad \mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{i}$$

Let

$$\mathbf{m}_{\mathrm{R}}^{(1)} = \cos\Phi\,\mathbf{e}_1 + \sin\Phi\,\mathbf{e}_2, \qquad \mathbf{m}_{\mathrm{R}}^{(2)} = -\sin\Phi\,\mathbf{e}_1 + \cos\Phi\,\mathbf{e}_2, \qquad 0 < \Phi < \pi/2, \tag{ii}$$

be the directions of two (mutually orthogonal) material fibers in the reference configuration.

(a) Calculate the shear $\gamma \coloneqq \theta_x - \theta_y \in [-\pi/2, \pi/2]$ associated with this pair of fibers (Section 2.4.2).

(b) Show that $\gamma > 0$ for $\lambda_1 < \lambda_2$ and $\gamma < 0$ for $\lambda_1 > \lambda_2$.

(c) Show that the maximum absolute value of the shear $|\gamma|$ from among all such pairs of fibers is

$$\sin^{-1}\left(\frac{|\lambda_1^2 - \lambda_2^2|}{\lambda_1^2 + \lambda_2^2}\right). \tag{iii}$$

Solution:

(a) Using the notation in Section 2.4.2, let $\theta_x = \pi/2$ and θ_y denote the angles between this pair of fibers in the reference and deformed configurations respectively. The associated shear $\gamma \in [-\pi/2, \pi/2]$ is defined as $\gamma \coloneqq \theta_x - \theta_y = \pi/2 - \theta_y$. Therefore

$$\sin\gamma = \cos\theta_y. \tag{iv}$$

From (i) and (ii),

$$\mathbf{F}\mathbf{m}_{\mathrm{R}}^{(1)} = \lambda_1\cos\Phi\,\mathbf{e}_1 + \lambda_2\sin\Phi\,\mathbf{e}_2, \qquad \mathbf{F}\mathbf{m}_{\mathrm{R}}^{(2)} = -\lambda_1\sin\Phi\,\mathbf{e}_1 + \lambda_2\cos\Phi\,\mathbf{e}_2. \tag{v}$$

Substituting (v) into (2.35) gives

$$\cos\theta_y = \frac{-(\lambda_1^2 - \lambda_2^2)\sin\Phi\cos\Phi}{\left[\lambda_1^2\cos^2\Phi + \lambda_2^2\sin^2\Phi\right]^{1/2}\left[\lambda_1^2\sin^2\Phi + \lambda_2^2\cos^2\Phi\right]^{1/2}},$$

which after simplification and using (iv) yields

$$\sin\gamma = \frac{-(\lambda_1^2 - \lambda_2^2)\sin 2\Phi}{\left[4\lambda_1^2\lambda_2^2 + (\lambda_1^2 - \lambda_2^2)^2\sin^2 2\Phi\right]^{1/2}}. \tag{vi}$$

Thus the shear is

$$\gamma = \sin^{-1}\left[\frac{-(\lambda_1^2 - \lambda_2^2)\sin 2\Phi}{\left[4\lambda_1^2\lambda_2^2 + (\lambda_1^2 - \lambda_2^2)^2\sin^2 2\Phi\right]^{1/2}}\right]. \qquad \square \tag{vii}$$

(b) We are told that $0 < \Phi < \pi/2$ whence $\sin 2\Phi > 0$. Therefore from (vi) we see that $\sin\gamma > 0$ for $\lambda_1 < \lambda_2$ and $\sin\gamma < 0$ for $\lambda_1 > \lambda_2$ from which the desired result follows.

(c) With $\gamma \in [-\pi/2, \pi/2]$ we can write $\sin|\gamma| = |\sin\gamma|$, so that on dividing the numerator and denominator of the right-hand side of (vi) by $\sin 2\Phi$ we get

$$\sin|\gamma| = \frac{|(\lambda_1^2 - \lambda_2^2)|}{\left[4\lambda_1^2\lambda_2^2/\sin^2 2\Phi + (\lambda_1^2 - \lambda_2^2)^2\right]^{1/2}}. \qquad (viii)$$

Thus the largest value of $|\gamma|$ corresponds to the smallest value of the denominator of ($viii$) which in turn corresponds to the largest value of $\sin^2 2\Phi$ which is unity (at $\Phi = \pi/4$). Thus setting $\Phi = \pi/4$ in ($viii$) and simplifying gives

$$|\gamma|_{\max} = \sin^{-1}\left[\frac{|\lambda_1^2 - \lambda_2^2|}{\lambda_1^2 + \lambda_2^2}\right]. \qquad \square$$

Problem 2.11. Calculate the components of the Lagrangian logarithmic strain tensor $\mathbf{E} = \ln\mathbf{U}$ associated with a simple shear deformation

$$y_1 = x_1 + kx_2, \qquad y_2 = x_2, \qquad y_3 = x_3.$$

Solution: In Problem 2.5.2 we worked out the details of the polar decomposition of the deformation gradient tensor $\mathbf{F}$ for a simple shear. From those results, the eigenvalues of the right stretch tensor $\mathbf{U}$ were

$$\lambda_1 = \frac{\sqrt{k^2+4}+k}{2}, \qquad \lambda_2 = \frac{\sqrt{k^2+4}-k}{2}, \qquad \lambda_3 = 1, \qquad (i)$$

and the corresponding eigenvectors were

$$\mathbf{r}_1 = \cos\theta_r\,\mathbf{e}_1 + \sin\theta_r\,\mathbf{e}_2, \qquad \mathbf{r}_2 = -\sin\theta_r\,\mathbf{e}_1 + \cos\theta_r\,\mathbf{e}_2, \qquad \mathbf{r}_3 = \mathbf{e}_3, \qquad (ii)$$

where

$$\cos\theta_r = \frac{1}{\sqrt{1+\lambda_1^2}}, \qquad \sin\theta_r = \frac{\lambda_1}{\sqrt{1+\lambda_1^2}}. \qquad (iii)$$

Since the Lagrangian logarithmic strain tensor is given by

$$\ln\mathbf{U} = \ln\lambda_1\,\mathbf{r}_1\otimes\mathbf{r}_1 + \ln\lambda_2\,\mathbf{r}_2\otimes\mathbf{r}_2 + \ln\lambda_3\,\mathbf{r}_3\otimes\mathbf{r}_3,$$

we substitute (ii) into this and expand the result to get

$$\begin{aligned}\ln\mathbf{U} \;=\;& (\cos^2\theta_r\ln\lambda_1 + \sin^2\theta_r\ln\lambda_2)\mathbf{e}_1\otimes\mathbf{e}_1 + (\ln\lambda_1 - \ln\lambda_2)\sin\theta_r\cos\theta_r\,(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)\\ &+(\sin^2\theta_r\ln\lambda_1 + \cos^2\theta_r\ln\lambda_2)\mathbf{e}_2\otimes\mathbf{e}_2.\end{aligned}$$

The coefficient of $\mathbf{e}_i\otimes\mathbf{e}_j$ in this equation is the i,j-component of the tensor $\ln\mathbf{U}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Expressions for the λ's and θ_r in terms of the amount of shear k are given above in (i) and (iii).

Problem 2.12. (Chadwick) Consider an isochoric homogeneous deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$. Let $\mathbf{n}_{\mathrm{R}}$ be a unit vector (in the reference configuration) such that (a) the area of the surface normal to $\mathbf{n}_{\mathrm{R}}$ does not change and (b) a material fiber in the direction $\mathbf{n}_{\mathrm{R}}$ remains unstretched. Determine the requirements on $\mathbf{F}$ and $\mathbf{n}_{\mathrm{R}}$ for this to be possible. For mathematical simplicity restrict attention to the case where the principal stretches are distinct:

$$\lambda_1 > \lambda_2 > \lambda_3 > 0. \qquad (i)$$

Solution: Since both the surface and material fiber of interest are given in the reference configuration, we know that the area and length constraints can be written in terms of the Lagrangian stretch tensor $\mathbf{U}$ alone, or possibly more simply in terms of the Lagrangian Cauchy-Green tensor $\mathbf{C} = \mathbf{U}^2$.

Since the area of the surface normal to $\mathbf{n}_{\mathrm{R}}$ does not change, it follows from Nanson's formula with $\det \mathbf{F} = 1$ and $dA_y = dA_x$ that

$$|\mathbf{F}^{-T}\mathbf{n}_{\mathrm{R}}| = 1 \quad \Rightarrow \quad \mathbf{F}^{-T}\mathbf{n}_{\mathrm{R}} \cdot \mathbf{F}^{-T}\mathbf{n}_{\mathrm{R}} = 1 \quad \Rightarrow \quad \mathbf{F}^{-1}\mathbf{F}^{-T}\mathbf{n}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}} = 1 \quad \Rightarrow \quad \mathbf{C}^{-1}\mathbf{n}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}} = 1, \qquad (ii)$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. We are told that material fibers in the direction $\mathbf{n}_{\mathrm{R}}$ remain unstretched whence

$$|\mathbf{F}\mathbf{n}_{\mathrm{R}}| = 1 \quad \Rightarrow \quad \mathbf{F}\mathbf{n}_{\mathrm{R}} \cdot \mathbf{F}\mathbf{n}_{\mathrm{R}} = 1 \quad \Rightarrow \quad \mathbf{F}^T\mathbf{F}\mathbf{n}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}} = 1 \quad \Rightarrow \quad \mathbf{C}\mathbf{n}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}} = 1. \qquad (iii)$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the principal stretches and let n_1, n_2, n_3 be the components of $\mathbf{n}_{\mathrm{R}}$ in the associated principal basis (of the Lagrangian stretch). The eigenvalues of $\mathbf{C}$ are $\lambda_1^2, \lambda_2^2, \lambda_3^2$ and so

$$\mathbf{C}\mathbf{n}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}} = 1 \qquad \Rightarrow \quad \lambda_1^2 n_1^2 + \lambda_2^2 n_2^2 + \lambda_3^2 n_3^2 = 1, \qquad (iv)$$

$$\mathbf{C}^{-1}\mathbf{n}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}} = 1 \qquad \Rightarrow \quad \lambda_1^{-2} n_1^2 + \lambda_2^{-2} n_2^2 + \lambda_3^{-2} n_3^2 = 1. \qquad (v)$$

Since $\mathbf{n}_{\mathrm{R}}$ is a unit vector,

$$n_1^2 + n_2^2 + n_3^2 = 1. \qquad (vi)$$

Solving the three algebraic equations $(iv), (v), (vi)$ for the components of the normal vector leads to

$$n_1^2 = \frac{(1-\lambda_3^2)(1-\lambda_2^2)\lambda_1^2}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}, \qquad n_2^2 = \frac{(1-\lambda_3^2)(\lambda_1^2-1)\lambda_2^2}{(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)} \qquad n_3^2 = \frac{(\lambda_1^2-1)(\lambda_2^2-1)\lambda_3^2}{(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)}. \qquad (vii)$$

We now analyze the three expressions in (vii) keeping in mind that $\lambda_1\lambda_2\lambda_3 = 1$. We will show that the value of at least one of the λs must equal 1, and so we start by assuming that no λ equals 1. It then follows from (vii) that $n_i \neq 0$ for each $i = 1, 2, 3$ and therefore that $n_i^2 > 0$ for each i.

First consider consider n_1^2 and note that in view of (i) the denominator in $(vii)_1$ is positive. Thus in order to ensure that $n_1^2 > 0$ one must have either $\lambda_3 > 1, \lambda_2 > 1$ or $\lambda_3 < 1, \lambda_2 < 1$. However, since $\lambda_1\lambda_2\lambda_3 = 1$, in the former case we will have $\lambda_1 < 1$ in which event $\lambda_1 < 1 < \lambda_3$ contradicting (i). Thus only the second possibility can hold:

$$\lambda_3 < 1, \qquad \lambda_2 < 1. \qquad (viii)$$

Second consider n_2^2 and note that in view of (i) the denominator in $(vii)_2$ is positive. Thus $n_2^2 > 0$ requires that either $\lambda_1 > 1, \lambda_3 < 1$ or $\lambda_1 < 1, \lambda_3 > 1$. In the latter case we have $\lambda_1 < 1 < \lambda_3$ contradicting (i). Thus only the first possibility can hold:

$$\lambda_1 > 1, \qquad \lambda_3 < 1. \qquad (ix)$$

Finally consider n_3^2 and note again that the denominator in $(vii)_3$ is positive in view of (i). Thus $n_3^2 > 0$ requires that either $\lambda_1 < 1, \lambda_2 < 1$ or $\lambda_1 > 1, \lambda_2 > 1$. Since $\lambda_1\lambda_2\lambda_3 = 1$, in the former case we must have $\lambda_3 > 1$ and so again $\lambda_1 < 1 < \lambda_3$ contradicting (i). Thus only the second possibility can hold:

$$\lambda_1 > 1, \qquad \lambda_2 > 1. \tag{x}$$

All inequalities in $(viii), (ix), (x)$ must all hold. However $(viii)_2$ contradicts $(x)_2$. Therefore we conclude that the assumption that none of the λs has the value 1 cannot hold.

Two (or three) of the λs cannot equal 1 because of (i), and so one need only consider the possibility that one of the λs equals 1. However, if $\lambda_1 = 1$ it follows from (i) that $\lambda_3 < \lambda_2 < 1$ and so $\lambda_1\lambda_2\lambda_3 \neq 1$. Similarly if $\lambda_3 = 1$ it follows from (i) that $\lambda_1 > \lambda_2 > 1$ and so again, $\lambda_1\lambda_2\lambda_3 \neq 1$. Thus the only possibility is that the middle principal stretch $\lambda_2 = 1$.

In this case it is readily seen from (vii) that $\lambda_2 = 1, n_2 = \pm 1, n_1 = 0, n_3 = 0$. Note that equations $(iv), (v), (vi)$ are satisfied by this solution.

Summary: We have shown that (in the case where (i) holds), for there to exist a direction that both does not elongate and the area on the surface normal to it remains unchanged, it is necessary and sufficient that the middle principal stretch have the value 1 and that the direction of interest be the corresponding eigenvector.

Problem 2.13. (*This generalizes Problem 2.4(c). It is also related to Problem 2.32.*) A body occupies a region $\mathcal{R}_{\rm R}$ in a reference configuration and is subjected to a homogeneous deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$. Show that necessary and sufficient for there to exist a plane $\mathcal{S}_{\rm R}$ in $\mathcal{R}_{\rm R}$ such that all material fibers on $\mathcal{S}_{\rm R}$ remain unstretched is that $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ have the representation

$$\mathbf{C} = \mathbf{I} + \mathbf{a} \otimes \mathbf{n}_{\rm R} + \mathbf{n}_{\rm R} \otimes \mathbf{a} \tag{i}$$

for an arbitrary vector $\mathbf{a}$ with $\mathbf{n}_{\rm R}$ being a unit vector normal to $\mathcal{S}_{\rm R}$.

Solution: We first show sufficiency. Suppose that (i) holds. Then for any vector $\mathbf{v}$ on $\mathcal{S}_{\rm R}$

$$\begin{aligned} |\mathbf{F}\mathbf{v}|^2 \quad &= \quad \mathbf{F}\mathbf{v}\cdot\mathbf{F}\mathbf{v} = \mathbf{F}^T\mathbf{F}\mathbf{v}\cdot\mathbf{v} = \mathbf{C}\mathbf{v}\cdot\mathbf{v} = \\ &\overset{(i)}{=} \quad (\mathbf{I} + \mathbf{a}\otimes\mathbf{n}_{\rm R} + \mathbf{n}_{\rm R}\otimes\mathbf{a})\mathbf{v}\cdot\mathbf{v} = \Big[\mathbf{v} + (\mathbf{n}_{\rm R}\cdot\mathbf{v})\mathbf{a} + (\mathbf{a}\cdot\mathbf{v})\mathbf{n}_{\rm R}\Big]\cdot\mathbf{v} = \\ &= \quad \mathbf{v}\cdot\mathbf{v} + (\mathbf{n}_{\rm R}\cdot\mathbf{v})(\mathbf{a}\cdot\mathbf{v}) + (\mathbf{a}\cdot\mathbf{v})(\mathbf{n}_{\rm R}\cdot\mathbf{v}) = \mathbf{v}\cdot\mathbf{v} \end{aligned}$$

having used $\mathbf{n}_{\rm R}\cdot\mathbf{v} = 0$ in the last step. This establishes sufficiency.

Turning to necessity, we first establish a preliminary result. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in the plane $\mathcal{S}_{\rm R}$. Then $\mathbf{u} - \mathbf{v}$ also lies on $\mathcal{S}_{\rm R}$. Since these three vectors are unstretched by $\mathbf{F}$,

$$|\mathbf{F}(\mathbf{u}-\mathbf{v})|^2 = |\mathbf{u}-\mathbf{v}|^2, \qquad |\mathbf{F}\mathbf{u}|^2 = |\mathbf{u}|^2, \qquad |\mathbf{F}\mathbf{v}|^2 = |\mathbf{v}|^2. \tag{ii}$$

From this and

$$|\mathbf{F}(\mathbf{u}-\mathbf{v})|^2 = \mathbf{F}(\mathbf{u}-\mathbf{v})\cdot\mathbf{F}(\mathbf{u}-\mathbf{v}) = \mathbf{F}\mathbf{u}\cdot\mathbf{F}\mathbf{u} + \mathbf{F}\mathbf{v}\cdot\mathbf{F}\mathbf{v} - 2\mathbf{F}\mathbf{u}\cdot\mathbf{F}\mathbf{v} = |\mathbf{F}\mathbf{u}|^2 + |\mathbf{F}\mathbf{v}|^2 - 2\mathbf{F}\mathbf{u}\cdot\mathbf{F}\mathbf{v}$$

we conclude that

$$\mathbf{Fu} \cdot \mathbf{Fv} = \mathbf{u} \cdot \mathbf{v}. \tag{iii}$$

Now pick an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_3 = \mathbf{n}_\mathrm{R}$. Thus $\mathbf{e}_1$ and $\mathbf{e}_2$ lie in the plane perpendicular to $\mathbf{n}_\mathrm{R}$, i.e. on $\mathcal{S}_\mathrm{R}$, and so it follows from (iii) that

$$|\mathbf{Fe}_1| = 1, \qquad |\mathbf{Fe}_2| = 1, \qquad \mathbf{Fe}_1 \cdot \mathbf{Fe}_2 = 0;$$

and since $\mathbf{Fu} \cdot \mathbf{Fv} = \mathbf{F}^T\mathbf{Fu} \cdot \mathbf{v} = \mathbf{Cu} \cdot \mathbf{v}$ it tells us further that

$$\mathbf{Ce}_1 \cdot \mathbf{e}_1 = 1, \qquad \mathbf{Ce}_2 \cdot \mathbf{e}_2 = 1, \qquad \mathbf{Ce}_1 \cdot \mathbf{e}_2 = \mathbf{Ce}_2 \cdot \mathbf{e}_1 = 0. \tag{iv}$$

Let C_{ij} denote the components of (the symmetric tensor) $\mathbf{C}$ in this basis, i.e.

$$\mathbf{C} = C_{ij}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{v}$$

It now follows from (iv) that $C_{11} = C_{22} = 1, C_{12} = C_{21} = 0$. Thus (v) simplifies to

$$\mathbf{C} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + C_{13}(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1) + C_{23}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) + C_{33}\mathbf{e}_3 \otimes \mathbf{e}_3,$$

which we can rewrite as

$$\mathbf{C} = \mathbf{I} + \Big[C_{13}\mathbf{e}_1 + C_{23}\mathbf{e}_2 + \frac{1}{2}(C_{33}-1)\mathbf{e}_3\Big] \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \Big[C_{13}\mathbf{e}_1 + C_{23}\mathbf{e}_2) + \frac{1}{2}(C_{33}-1)\mathbf{e}_3\Big] \tag{vi}$$

having used $\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 = \mathbf{I}$.

Let $\mathbf{a}$ denote the vector

$$\mathbf{a} := C_{13}\mathbf{e}_1 + C_{23}\mathbf{e}_2 + \frac{1}{2}(C_{33}-1)\mathbf{e}_3. \tag{vii}$$

Then we can write (vi) as $\mathbf{C} = \mathbf{I} + \mathbf{a} \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{a}$, or, since $\mathbf{e}_3 = \mathbf{n}_\mathrm{R}$,

$$\mathbf{C} = \mathbf{I} + \mathbf{a} \otimes \mathbf{n}_\mathrm{R} + \mathbf{n}_\mathrm{R} \otimes \mathbf{a}. \tag{$viii$}$$

Since C_{13}, C_{23}, C_{33} are arbitrary, so is the vector $\mathbf{a}$. This shows that the representation $(viii)$ is necessary as well.

Cylindrical and spherical bodies.

Problem 2.14. *(Deformation of a hollow circular tube)* A body occupies a hollow circular cylindrical region $\mathcal{R}_\mathrm{R}$ in a reference configuration with inner radius A, outer radius B and length L:

$$\mathcal{R}_\mathrm{R} = \{(x_1, x_2, x_3) : A < (x_1^2 + x_2^2)^{1/2} < B,\ 0 < x_3 < L\}.$$

All components of vectors and tensors are taken with respect to the right-handed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ shown in Figure 2.24. A particle located at (x_1, x_2, x_3) in the reference configuration is carried to (y_1, y_2, y_3) by the deformation

$$\left.\begin{aligned} y_1 &= f(R)\Big[x_1\cos\phi(x_3) - x_2\sin\phi(x_3)\Big], \\ y_2 &= f(R)\Big[x_2\cos\phi(x_3) + x_1\sin\phi(x_3)\Big], \\ y_3 &= \Lambda x_3, \end{aligned}\right\}, \qquad R = (x_1^2 + x_2^2)^{1/2}. \tag{i}$$

Here $f(R)$ and $\phi(x_3)$ are smooth functions defined for $A \le R \le B$ and $0 \le x_3 \le L$ respectively and $\Lambda > 0$ is a constant. Describe the physical nature of this deformation: in particular, consider the particles that, in the undeformed configuration, lie on a circle $x_1^2 + x_2^2 = c^2$ (on the cross section at some fixed x_3). Determine (and describe) the curve on which these particles lie in the deformed configuration. Do the same for the particles on a radial straight line $x_2 = cx_1$. Determine the region $\mathcal{R}$ occupied by the body in the deformed configuration.

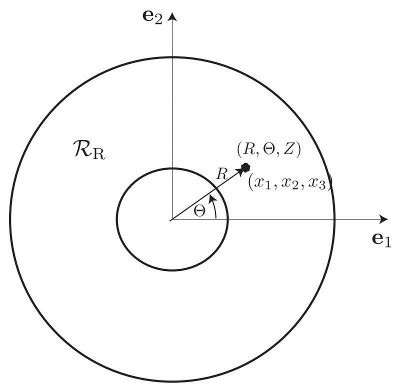


Figure 2.24: Cross-section of the region $\mathcal{R}_{\mathrm{R}}$ occupied by the body in a reference configuration: a hollow circular cylinder of inner radius A, outer radius B (and length L). (Figure for Problem 2.14)

Solution: First, from $(i)_3$ it is clear that this deformation involves a uniform stretching of the cylinder in the x_3-direction. Thus, in particular, the deformed length of the cylinder is ΛL.

The cylindrical polar coordinates (R, Θ, Z) of a particle in the reference configuration are related to the rectangular cartesian coordinates (x_1, x_2, x_3) by

$$x_1 = R\cos\Theta, \qquad x_2 = R\sin\Theta, \qquad x_3 = Z. \tag{ii}$$

By substituting (ii) into (i) one finds

$$y_1 = Rf(R)\cos\big(\Theta + \phi(Z)\big), \qquad y_2 = Rf(R)\sin\big(\Theta + \phi(Z)\big), \qquad y_3 = \lambda Z. \tag{iii}$$

It follows from (iii) that

$$y_1^2 + y_2^2 = \big(Rf(R)\big)^2. \qquad \square$$

Therefore particles that lie on any circle $R = c$ in the reference configuration are carried by the deformation onto a circle of radius $cf(c)$ in the deformed configuration. Thus the cylinder undergoes a radial expansion (if $f(c) > 1$) or radial contraction (if $f(c) < 1$). The function $f(R)$ characterizes the radial deformation of the cylinder. Particles on the inner and outer surfaces $R = A$ and $R = B$ lie on cylindrical surfaces of radii $a = Af(A)$ and $b = Bf(B)$ in the deformed configuration.

Next observe from (iii) that

$$\frac{y_2}{y_1} = \tan\Big(\Theta + \phi(Z)\Big). \qquad \square \tag{iv}$$

Therefore the particles that lie on any radial straight line $\Theta = \mathsf{constant}$ (i.e. $x_2/x_1 = \mathsf{constant}$) on the cross section $Z =$ constant in the reference configuration are carried by the deformation onto the radial straight line defined by (iv). Therefore radial straight lines remain straight but are rotated about the x_3-axis by $\phi(Z)$. Thus cross-sections of the cylinder are twisted by this deformation. According to (iv), the cross-section at $Z =$ constant rotates by $\phi(Z)$.

Thus the deformation (i) describes an axial stretching in the x_3-direction, an expansion (or contraction) in the radial directions and a twisting about the x_3-axis. The region $\mathcal{R}$ occupied by the body in the deformed configuration is

$$\mathcal{R} = \{(y_1, y_2, y_3) : Af(A) < (y_1^2 + y_2^2)^{1/2} < Bf(B),\ 0 < x_3 < \Lambda L\}. \qquad \square$$

Problem 2.15. (Spencer) *(Deformation of a solid circular cylinder)* The region occupied by a body in a reference configuration is a solid circular cylinder of radius A. Coordinate axes are chosen such that the axis of the cylinder coincides with the x_3-axis. The body undergoes the following deformation $(x_1, x_2, x_3) \mapsto (y_1, y_2, y_3)$:

$$\left.\begin{aligned} y_1 &= \lambda[x_1 \cos(\alpha x_3) + x_2 \sin(\alpha x_3)], \\ y_2 &= \lambda[-x_1 \sin(\alpha x_3) + x_2 \cos(\alpha x_3)], \\ y_3 &= \Lambda x_3, \end{aligned}\right\} \qquad (i)$$

where α, λ and Λ are positive constants.

(a) Describe this deformation.

(b) Consider a material fiber on the outer surface of the cylinder that, in the *reference* configuration is parallel to the axis of the cylinder. Calculate the stretch of this material fiber due to the deformation. Observe from your answer that though this referential fiber lies in the x_3-direction, it is not Λ alone that contributes to its stretch. Can you derive your answer by "physical" (elementary geometric) arguments alone?

(c) Consider a material fiber on the outer surface of the cylinder that, in the *deformed* configuration, is parallel to the axis of the cylinder. Calculate the stretch of this fiber. You may find useful the result in Problem 1.32, regarding the inverse of a tensor expressed in a mixed basis. An alternative way in which to calculate $\mathbf{F}^{-1}$ is to realize that it is the deformation gradient tensor for the inverse deformation $R = R(r, \theta, z), \Theta = \Theta(r, \theta, z), Z = Z(r, \theta, z)$.

Solution:

(a) Observe that the deformation (i) is *not* a homogeneous deformation and so cannot be written as $\mathbf{y} = \mathbf{F}\mathbf{x}$ with $\mathbf{F} = \nabla\mathbf{y}$. It is in fact a special case of the deformation in Problem 2.14. Since it involves 3 parameters λ, Λ and α we expect there to be 3 "types" of deformation.

While one can analyze (i) using cartesian coordinates, we shall use cylindrical polar coordinates instead. Let (R, Θ, Z) and (r, θ, z) be the cylindrical polar coordinates of a particle in the undeformed and deformed configurations respectively:

$$x_1 = R\cos\Theta,\ x_2 = R\sin\Theta,\ x_3 = Z \qquad \text{and} \qquad y_1 = r\cos\theta,\ y_2 = r\sin\theta,\ y_3 = z. \tag{ii}$$

On using $(ii)_{1,2,3}$ we can rewrite (i) as

$$y_1 = \lambda R\cos(\Theta - \alpha Z), \qquad y_2 = \lambda R\sin(\Theta - \alpha Z), \qquad y_3 = \Lambda Z,$$

which because of $(ii)_{4,5,6}$ can be written as

$$r = \lambda R, \qquad \theta = \Theta - \alpha Z, \qquad z = \Lambda Z. \tag{iii}$$

Thus we see that the deformation involves a stretch Λ in the axial direction, a stretch λ in the radial direction and a twisting where each cross section $Z = \mathsf{constant}$ rotates through an angle $-\alpha Z$ about the axis of the cylinder.

The associated deformation gradient tensor is found by specializing (2.77) to the deformation (iii):

$$\mathbf{F} = \lambda(\mathbf{e}_r \otimes \mathbf{e}_R) + \lambda(\mathbf{e}_\theta \otimes \mathbf{e}_\Theta) - \alpha\lambda R(\mathbf{e}_\theta \otimes \mathbf{e}_Z) + \Lambda(\mathbf{e}_z \otimes \mathbf{e}_Z)\,. \tag{iv}$$

(b) The material fiber of interest is in the direction $\mathbf{e}_Z$ in the reference configuration, i.e. $d\mathbf{x} = ds_x\mathbf{e}_Z$. Therefore from (iv) we find its deformed image to be

$$\begin{aligned} d\mathbf{y} &= \mathbf{F}\,d\mathbf{x} = \mathbf{F}(ds_x\mathbf{e}_Z) = ds_x\mathbf{F}\mathbf{e}_Z = \\ &= ds_x\Big(\lambda(\mathbf{e}_r \otimes \mathbf{e}_R) + \lambda(\mathbf{e}_\theta \otimes \mathbf{e}_\Theta) - \alpha\lambda A(\mathbf{e}_\theta \otimes \mathbf{e}_Z) + \Lambda(\mathbf{e}_z \otimes \mathbf{e}_Z)\Big)\mathbf{e}_Z = \\ &= ds_x\Big(-\alpha\lambda A\,\mathbf{e}_\theta + \Lambda\,\mathbf{e}_z\Big), \end{aligned}$$

where we have also set $R = A$ since the fiber is on the outer surface. Therefore the length of this fiber in the deformed configuration is

$$ds_y = |d\mathbf{y}| = ds_x\sqrt{\alpha^2\lambda^2A^2 + \Lambda^2} \qquad \Rightarrow \qquad \frac{ds_y}{ds_x} = \sqrt{\alpha^2\lambda^2A^2 + \Lambda^2}. \qquad \square$$

Observe that the stretch is not equal to Λ, the additional term being the stretch due to the torsion. If there was no torsional deformation, i.e. if $\alpha = 0$, this gives $ds_y/ds_x = \Lambda$.

(c) We now consider a fiber whose direction in the deformed configuration is $\mathbf{e}_z$, i.e. $d\mathbf{y} = ds_y\mathbf{e}_z$. Its image in the reference configuration is

$$d\mathbf{x} = \mathbf{F}^{-1}d\mathbf{y} = ds_y\,\mathbf{F}^{-1}\mathbf{e}_z. \tag{vi}$$

The inverse of the deformation gradient tensor $\mathbf{F}$ given in (iv), using the result in Problem 1.32, is

$$\mathbf{F}^{-1} = \lambda^{-1}(\mathbf{e}_R \otimes \mathbf{e}_r) + \lambda^{-1}(\mathbf{e}_\Theta \otimes \mathbf{e}_\theta) + \alpha\Lambda^{-1}R(\mathbf{e}_\Theta \otimes \mathbf{e}_z) + \Lambda^{-1}(\mathbf{e}_Z \otimes \mathbf{e}_z)\,. \tag{vii}$$

Alternatively, from (iii), the inverse mapping from $(R,\Theta,Z) \mapsto (r,\theta,z)$ is

$$R = \lambda^{-1}r, \qquad \Theta = \theta + \alpha\Lambda^{-1}z, \qquad Z = \Lambda^{-1}z. \tag{viii}$$

We can now use the formula (2.77) to calculate $\mathbf{F}^{-1}$ provided we swap (R, Θ, Z) for (r, θ, z) throughout the formula:

$$\mathbf{F}^{-1} = \frac{\partial R}{\partial r}\mathbf{e}_R \otimes \mathbf{e}_r + \frac{1}{r}\frac{\partial R}{\partial \theta}\mathbf{e}_R \otimes \mathbf{e}_\theta + \ldots + \frac{R}{r}\frac{\partial \Theta}{\partial \theta}\mathbf{e}_\Theta \otimes \mathbf{e}_\theta + R\frac{\partial \Theta}{\partial z}\mathbf{e}_\Theta \otimes \mathbf{e}_z + \ldots + \frac{\partial Z}{\partial z}\mathbf{e}_Z \otimes \mathbf{e}_z. \qquad (ix)$$

Substituting $(viii)$ into (ix) gives (vii).

We can now substitute (vii) and $R = A$ into (vi) which leads to

$$d\mathbf{x} = ds_y\Big(\alpha\Lambda^{-1}A\,\mathbf{e}_\Theta + \Lambda^{-1}\,\mathbf{e}_Z\Big),$$

and thus the length of the fiber in the reference configuration is

$$ds_x = |d\mathbf{x}| = ds_y\Lambda^{-1}\sqrt{\alpha^2A^2+1}.$$

Thus its stretch is

$$\frac{ds_y}{ds_x} = \frac{\Lambda}{\sqrt{1+A^2\alpha^2}}. \qquad \square$$

Observe that the stretch of this fiber is different to that of the fiber in part (b). For small amounts of torsion, $|\alpha| \ll 1$, both stretches are $ds_y/ds_x \approx \Lambda$.

Problem 2.16. *(Inflation and extension of a hollow circular tube)* The region $\mathcal{R}_\mathrm{R}$ occupied by a body in a reference configuration is a hollow circular cylinder of inner radius A, outer radius B and length L. It is subjected to the radially symmetric deformation

$$r = r(R), \qquad \theta = \Theta, \qquad z = \Lambda\,Z, \qquad (i)$$

where (R, Θ, Z) and (r, θ, z) are the respective cylindrical polar coordinates of a particle in the reference and deformed configurations, where $\Lambda > 0$ is a constant and $r(R) > 0$.

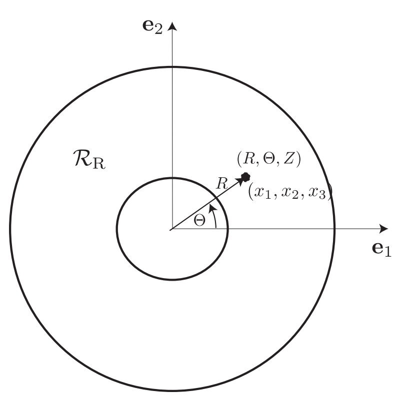


Figure 2.25: The region $\mathcal{R}_\mathrm{R}$ occupied by a body in a reference configuration is a hollow circular cylinder of inner radius A and outer radius B. (Figure for Problem 2.16)

(a) Calculate the principal stretches.

(b) Determine $r(R)$ (to the extent possible) if the material is incompressible. Assume that $r'(R) > 0$.

(c) Denote the stretch in the circumferential direction by $\lambda(R)$:

$$\lambda(R) = r(R)/R. \tag{ii}$$

Show that

$$A^2(\lambda_a^2\Lambda - 1) = B^2(\lambda_b^2\Lambda - 1) = R^2(\lambda^2(R)\Lambda - 1), \tag{iii}$$

where $\lambda_a = \lambda(A), \lambda_b = \Lambda(B)$.

Solution:

(a) The deformation gradient tensor is found by specializing (2.77) to the deformation (i):

$$\mathbf{F} = r'(R)(\mathbf{e}_r \otimes \mathbf{e}_R) + \frac{r(R)}{R}(\mathbf{e}_\theta \otimes \mathbf{e}_\Theta) + \Lambda(\mathbf{e}_z \otimes \mathbf{e}_Z)\,. \tag{iv}$$

In view of the cylindrical symmetry of the deformation, the basis vectors $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ coincide. Clearly, the polar decomposition of (iv) is $\mathbf{R} = \mathbf{I}$ and

$$\mathbf{U} = |r'(R)|(\mathbf{e}_R \otimes \mathbf{e}_R) + \frac{r(R)}{R}(\mathbf{e}_\Theta \otimes \mathbf{e}_\Theta) + \Lambda(\mathbf{e}_Z \otimes \mathbf{e}_Z),$$

having using the given facts that $r(R) > 0$ and $\Lambda > 0$. The principal stretches are therefore

$$\lambda_R = |r'(R)|, \qquad \lambda_\Theta = \frac{r(R)}{R}, \qquad \lambda_Z = \Lambda. \qquad \square \tag{v}$$

(b) For an incompressible material

$$\lambda_R\lambda_\Theta\lambda_Z = 1 \qquad \Rightarrow \qquad \Lambda r(R)r'(R) = R, \tag{vi}$$

having used the fact that $r'(R) > 0$ (additional information given to us). Writing this as

$$\frac{1}{2}\frac{d}{dR}\Big(r^2(R)\Big) = R/\Lambda,$$

allows us to integrate to get

$$r(R) = \sqrt{c + R^2/\Lambda}, \qquad \square \tag{vii}$$

where c is a constant.

(c) Using (ii) and (vii) gives

$$\lambda(R) = \frac{\sqrt{c + R^2/\Lambda}}{R}. \tag{viii}$$

Since $\lambda_a = \lambda(A)$ and $\lambda_b = \lambda(B)$ we have

$$\lambda_a = \frac{\sqrt{c + A^2/\Lambda}}{A}, \qquad \lambda_b = \frac{\sqrt{c + B^2/\Lambda}}{B}. \tag{ix}$$

Solving ($viii$) and each of (ix) for c gives

$$c = R^2\lambda^2(R) - R^2/\Lambda = A^2\lambda_a^2 - A^2/\Lambda = B^2\lambda_b^2 - B^2/\Lambda\ ,$$

which shows that

$$R^2(\Lambda\lambda^2(R) - 1) = A^2(\Lambda\lambda_a^2 - 1) = B^2(\Lambda\lambda_b^2 - 1). \qquad \square$$

Problem 2.17. (Ogden) *(Combined axial and azimuthal shear of a tube)* Let (R, Θ, Z) and (r, θ, z) be cylindrical polar coordinates of a particle in the reference and deformed configurations respectively with associated bases $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. The region $\mathcal{R}_R$ occupied by the body in a reference configuration is a hollow circular cylinder of inner radius A, outer radius B and length L.

Consider a deformation of the form

$$r = R, \qquad \theta = \Theta + \phi(R), \qquad z = Z + w(R). \tag{i}$$

In the special case $\phi \equiv 0$ this describes an "axial (telescopic) shearing" of the tube while the special case $w \equiv 0$ describes an "azimuthal shearing". Note that neither of these particular deformations, nor (i), is a torsional deformation.

We will study the stress in this tube in Problem 3.10.1 of Chapter 3.

(a) Calculate the deformation gradient tensor $\mathbf{F}$.

(b) By factoring $\mathbf{F}$ into the product of three tensors, show that locally, at each point of the body, the deformation is comprised of a rigid rotation, followed by a simple shear with glide plane normal $\mathbf{e}_r$ and shear direction $\mathbf{e}_\theta$, followed be a simple shear with glide plane normal $\mathbf{e}_r$ and shear direction $\mathbf{e}_z$. Determine the associated amounts of shear.

(c) If the material is incompressible, what does this tell you (if anything) about $\phi(R)$ and $w(R)$?

(d) Show that the composition of the two simple shears in part (b) corresponds to a simple shear with shearing direction $\mathbf{a}$, glide plane normal $\mathbf{e}_r$ and amount of shear k where

$$\mathbf{a} = \sin\beta\,\mathbf{e}_\theta + \cos\beta\,\mathbf{e}_z, \quad \tan\beta = \frac{k_1}{k_2}, \qquad k = k = \left[k_1^2 + k_2^2\right]^{1/2}, \quad k_1 = R\phi'(R), \quad k_2 = w'(R). \tag{ii}$$

Remark: Note that you now have the deformation gradient tensor factored as $\mathbf{F} = \mathbf{KQ}$ where $\mathbf{Q}$ is a rotation and $\mathbf{K}$ a simple shear. Keep in mind that this is *not* the polar decomposition of $\mathbf{F}$ and that $\mathbf{K} \neq \mathbf{V}$.

(e) Calculate the matrix of components of the left Cauchy Green deformation tensor $\mathbf{B}$ in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. (Express your answer in terms of k_1 and k_2.)

(f) Calculate the principal stretches and principal Eulerian stretch directions. Hint: You can make use the results of Problem 2.40. (Express your answer in terms of $k, \mathbf{a}, \mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_z$.)

Solution:

(a) From (2.77) and (i),

$$\mathbf{F} = \mathbf{e}_r \otimes \mathbf{e}_R + R\phi'\mathbf{e}_\theta \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + w'\mathbf{e}_z \otimes \mathbf{e}_R + \mathbf{e}_z \otimes \mathbf{e}_Z. \qquad \square \tag{iii}$$

(b) A simple shear with shearing direction $\mathbf{e}_\theta$ and glide plane normal $\mathbf{e}_r$ has the form

$$\mathbf{F}_1 = \mathbf{I} + k_1 \mathbf{e}_\theta \otimes \mathbf{e}_r; \qquad (iv)$$

a simple shear with shearing direction $\mathbf{e}_z$ and glide plane normal $\mathbf{e}_r$ has the form

$$\mathbf{F}_2 = \mathbf{I} + k_2 \mathbf{e}_z \otimes \mathbf{e}_r; \qquad (v)$$

and the rotation of the basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ into the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is described by the rotation tensor

$$\mathbf{Q} = \mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_z \otimes \mathbf{e}_Z. \qquad (vi)$$

On calculating $\mathbf{F}_2\mathbf{F}_1\mathbf{Q}$ we get

$$\begin{aligned} \mathbf{F}_2\mathbf{F}_1\mathbf{Q} &= (\mathbf{I} + k_2 \mathbf{e}_z \otimes \mathbf{e}_r)(\mathbf{I} + k_1 \mathbf{e}_\theta \otimes \mathbf{e}_r)(\mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_z \otimes \mathbf{e}_Z) = \\ &= (\mathbf{I} + k_2 \mathbf{e}_z \otimes \mathbf{e}_r)(\mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_z \otimes \mathbf{e}_Z + k_1 \mathbf{e}_\theta \otimes \mathbf{e}_R) = \\ &= \mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_z \otimes \mathbf{e}_Z + k_1 \mathbf{e}_\theta \otimes \mathbf{e}_R + k_2 \mathbf{e}_z \otimes \mathbf{e}_R. \qquad (vii) \end{aligned}$$

Equation (vii) is identical to (iii) provided

$$k_1 = R\phi'(R), \qquad k_2 = w'(R). \qquad (viii)$$

Therefore the deformation can be decomposed into the orthogonal $\mathbf{Q}$ that rotates $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ into $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$; followed by the simple shear $\mathbf{F}_1$ with shearing direction $\mathbf{e}_\theta$, glide plane normal $\mathbf{e}_r$ and amount of shear $k_1 = R\phi'(R)$; followed by the simple shear $\mathbf{F}_2$ with shearing direction $\mathbf{e}_z$, glide plane normal $\mathbf{e}_r$ and amount of shear $k_2 = w'(R)$. $\Box$

(c) If the material is incompressible we must have det $\mathbf{F} = 1$. Since a rotation and a simple shear are each isochoric, the composite deformation will be automatically isochoric. This can be confirmed from

$$\det \mathbf{F} = (\det \mathbf{F}_2)(\det \mathbf{F}_1)(\det \mathbf{Q}) = (1 + k_2 \mathbf{e}_z \cdot \mathbf{e}_r)(1 + k_1 \mathbf{e}_\theta \cdot \mathbf{e}_r)(1) = 1.$$

Therefore incompressibility imposes no restrictions on the functions $\phi(R($ and $w(R)$.

(d) Calculating $\mathbf{F}_2\mathbf{F}_1$,

$$\mathbf{F}_2\mathbf{F}_1 = (\mathbf{I} + k_2 \mathbf{e}_z \otimes \mathbf{e}_r)(\mathbf{I} + k_1 \mathbf{e}_\theta \otimes \mathbf{e}_r) = \mathbf{I} + (k_1 \mathbf{e}_\theta + k_2 \mathbf{e}_z) \otimes \mathbf{e}_r, \qquad (ix)$$

which we can write as

$$\mathbf{F}_2\mathbf{F}_1 = \mathbf{I} + k\mathbf{a} \otimes \mathbf{e}_r, \qquad (x)$$

where

$$k = \sqrt{k_1^2 + k_2^2}, \qquad \mathbf{a} = \sin\beta\, \mathbf{e}_\theta + \cos\beta\, \mathbf{e}_z, \qquad \sin\beta = k_1/k, \qquad \cos\beta = k_2/k. \qquad (xi)$$

Note that $\mathbf{a}$ is a unit vector. Thus $\mathbf{F}_2\mathbf{F}_1$ is a simple shear with shearing direction $\mathbf{a}$, glide plane normal $\mathbf{e}_r$ and amount of shear k. $\Box$

(e) The left Cauchy Green tensor is given by

$$\begin{aligned}
\mathbf{B} &= \mathbf{F}\mathbf{F}^T = (\mathbf{F}_2\mathbf{F}_1\mathbf{Q})(\mathbf{F}_2\mathbf{F}_1\mathbf{Q})^T = (\mathbf{F}_2\mathbf{F}_1\mathbf{Q})(\mathbf{Q}^T\mathbf{F}_1^T\mathbf{F}_2^T) = \mathbf{F}_2\mathbf{F}_1\mathbf{F}_1^T\mathbf{F}_2^T = \\
&= (\mathbf{I} + k_2\mathbf{e}_z \otimes \mathbf{e}_r)(\mathbf{I} + k_1\mathbf{e}_\theta \otimes \mathbf{e}_r)(\mathbf{I} + k_1\mathbf{e}_r \otimes \mathbf{e}_\theta)(\mathbf{I} + k_2\mathbf{e}_r \otimes \mathbf{e}_z) = \\
&= \mathbf{I} + k_1(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + k_2(\mathbf{e}_z \otimes \mathbf{e}_r + \mathbf{e}_r \otimes \mathbf{e}_z) + \\
&\qquad + k_1k_2(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta) + k_1^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta + k_2^2\mathbf{e}_z \otimes \mathbf{e}_z
\end{aligned}$$

and so the matrix of components of $\mathbf{B}$ in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is

$$[B] = \begin{pmatrix} 1 & k_1 & k_2 \\ k_1 & 1 + k_1^2 & k_1k_2 \\ k_2 & k_1k_2 & 1 + k_2^2 \end{pmatrix}.$$

(f) Since the deformation is a simple shear $\mathbf{I} + k\mathbf{a} \otimes \mathbf{e}_r$ (plus a rotation), we can determine the principal stretches and principal directions of the Eulerian stretch directly from Problem 2.40. From equation (iii) of Problem 2.40 the principal stretches are

$$\lambda_1 = \lambda, \qquad \lambda_2 = \lambda^{-1}, \qquad \lambda_3 = 1 \qquad \text{where} \qquad \lambda = \frac{1}{2}\left[\sqrt{k^2+4} + k\right], \qquad \square$$

where k is given by $(xi)_1$ and ($viii$). By identifying $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in equation (i) of Problem 2.40 with $\mathbf{a}, \mathbf{e}_r$ and $\mathbf{a} \times \mathbf{e}_r$ here, we find the corresponding principal directions to be

$$\boldsymbol{\ell}_1 = \cos\theta_\ell\mathbf{a} + \sin\theta_\ell\mathbf{e}_r, \qquad \boldsymbol{\ell}_2 = -\sin\theta_\ell\mathbf{a} + \cos\theta_\ell\mathbf{e}_r, \qquad \boldsymbol{\ell}_3 = \mathbf{a} \times \mathbf{e}_r = \cos\beta\mathbf{e}_\theta - \sin\beta\mathbf{e}_z, \qquad \square$$

where

$$\tan 2\theta_\ell = 2/k. \qquad \square$$

Problem 2.18. (Ogden) *(Inflation of a hollow spherical shell)* Let (R, Θ, Φ) and (r, θ, φ) be spherical polar coordinates in the reference and deformed configurations respectively with associated bases $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$. The hollow spherical region $\mathcal{R}_R$ occupied by an incompressible body in a reference configuration is described by

$$A \le R \le B, \qquad 0 \le \Theta \le 2\pi, \qquad 0 \le \Phi \le 2\pi. \tag{i}$$

The body is subjected to a spherically symmetric deformation

$$r = r(R), \qquad \theta = \Theta, \qquad \varphi = \Phi, \tag{ii}$$

and its inner and outer radii in the deformed configuration are a and b respectively.

(a) Calculate the principal stretches.

(b) Let

$$\lambda(R) = r(R)/R, \qquad \lambda_a = r(A)/A, \qquad \lambda_b = r(B)/B; \tag{iii}$$

these are the stretches in the circumferential direction at $R, R = A$ and $R = B$ respectively. Show that

$$A^3(\lambda_a^3 - 1) = R^3(\lambda^3 - 1) = B^3(\lambda_b^3 - 1), \qquad (iv)$$

and hence show that either $\lambda_a \geq \lambda_b \geq 1$ or $\lambda_a \leq \lambda_b \leq 1$.

(c) Now suppose that the sphere is thin-walled in the sense that

$$\varepsilon := T/R \ll 1, \qquad (v)$$

where $T = B - A$ and $R = (A+B)/2$ are the wall-thickness and mean radius of the sphere respectively in the reference configuration. Let λ_a and λ_b be the stretch at the inner and outer wall as in (iii) above, and let $\lambda = r/R$ be the *mean* stretch where r is the *mean* radius of the deformed sphere. Derive approximate expressions for the stretches λ_a and λ_b keeping terms of order ε. Your results will be of the form

$$\lambda_a = \lambda + \Delta\lambda_a\, \varepsilon + \mathrm{O}(\varepsilon^2), \qquad \lambda_b = \lambda + \Delta\lambda_b\, \varepsilon + \mathrm{O}(\varepsilon^2). \qquad (vi)$$

Solution:

(a) (See Problem 2.4.3 for an alternative calculation of the principal stretches that doesn't rely on (2.87).) It is readily seen from (ii) and (2.87) that

$$B_{rr} = \left(r'(R)\right)^2, \qquad B_{\vartheta\vartheta} = B_{\varphi\varphi} = \frac{r^2}{R^2}, \qquad B_{r\vartheta} = B_{r\varphi} = B_{\vartheta\varphi} = 0. \qquad (vii)$$

Therefore (assuming $r'(R) > 0$) the principal stretches are

$$\lambda_1 = r'(R), \qquad \lambda_2 = \lambda_3 = \frac{r}{R}. \qquad (viii)$$

Since the material is incompressible,

$$\lambda_1\lambda_2\lambda_3 = \lambda_1\lambda_2^2 = 1 \qquad \Rightarrow \qquad \lambda_1 = \lambda_2^{-2}.$$

Therefore the principal stretches can be expressed as

$$\lambda_1 = \lambda^{-2}, \quad \lambda_2 = \lambda_3 = \lambda \qquad \text{where} \qquad \lambda = \frac{r}{R}. \qquad \square$$

(b) On substituting $(viii)$ into $\lambda_1\lambda_2\lambda_3 = 1$ we get the differential equation

$$r^2\frac{dr}{dR} = R^2 \qquad \Rightarrow \qquad r^3 = R^3 + c.$$

Enforcing $r(A) = a$ and $r(B) = b$ yields $c = a^3 - A^3 = b^3 - B^3$ and so we have

$$r^3 = R^3 + a^3 - A^3 = R^3 + b^3 - B^3. \qquad (ix)$$

Substituting

$$\lambda = \frac{r}{R}, \qquad \lambda_a = \frac{a}{A}, \qquad \lambda_b = \frac{b}{B},$$

into (ix) in order to eliminate r, a and b leads to

$$\lambda^3 R^3 = R^3 + \lambda_a^3 A^3 - A^3 = R^3 + \lambda_b^3 B^3 - B^3,$$

which can be rewritten as

$$A^3(\lambda_a^3 - 1) = B^3(\lambda_b^3 - 1) = R^3(\lambda^3 - 1). \qquad \square \tag{x}$$

Consider the case $\lambda_b \neq 1$. The first equation in (x) then leads to

$$\frac{\lambda_a^3 - 1}{\lambda_b^3 - 1} = \frac{B^3}{A^3} \qquad \Rightarrow \qquad \frac{\lambda_a^3 - 1}{\lambda_b^3 - 1} > 1, \tag{xi}$$

since $B > A$. Therefore if $\lambda_b^3 - 1 > 0$ equation (xi) yields

$$\lambda_a^3 - 1 > \lambda_b^3 - 1 \qquad \Rightarrow \qquad \lambda_a > \lambda_b > 1. \qquad \square$$

On the other hand if $\lambda_b^3 - 1 < 0$ equation (xi) yields

$$\lambda_a^3 - 1 < \lambda_b^3 - 1 \qquad \Rightarrow \qquad \lambda_a < \lambda_b < 1. \qquad \square$$

In the case $\lambda_b = 1$ we see from (x) that $\lambda_a = 1$.

(c) Let R and T denote the mean radius and wall-thickness of the undeformed shell. We now turn to the case when the body is *thin-walled* in the sense that

$$\varepsilon = T/R \qquad \text{where} \quad T = B - A, \ R = \frac{1}{2}(A + B).$$

We can write the inner and outer undeformed radii of the body as

$$A = R - T/2 = R(1 - \varepsilon/2), \qquad B = R + T/2 = R(1 + \varepsilon/2), \tag{xii}$$

First consider λ_a. From (iv) and (xii):

$$\lambda_a = \left[1 + \frac{R^3}{A^3}(\lambda^3 - 1)\right]^{1/3} = \left[1 + \frac{\lambda^3 - 1}{(1 - \varepsilon/2)^3}\right]^{1/3}. \tag{$xiii$}$$

We now approximate $(xiii)$ for small ε dropping terms of $O(\varepsilon^2)$:

$$\begin{aligned}\lambda_a &= \left[1 + \frac{\lambda^3 - 1}{(1 - \varepsilon/2)^3}\right]^{1/3} = \left[1 + (1 - \varepsilon/2)^{-3}(\lambda^3 - 1)\right]^{1/3} \approx \left[1 + (1 + 3\varepsilon/2)(\lambda^3 - 1)\right]^{1/3} = \\ &= \left[\lambda^3 + \frac{3\varepsilon}{2}(\lambda^3 - 1)\right]^{1/3} = \lambda\left[1 + \frac{3\varepsilon}{2}\frac{(\lambda^3 - 1)}{\lambda^3}\right]^{1/3} \approx \lambda\left[1 + \frac{\varepsilon}{2}\frac{(\lambda^3 - 1)}{\lambda^3}\right] = \lambda + \varepsilon\frac{(\lambda^3 - 1)}{2\lambda^2},\end{aligned} \qquad \square$$

where in two steps of the preceding calculation we have used the binomial expansion $(1+\varepsilon)^n = 1+n\varepsilon+O(\varepsilon^2)$. Similarly approximating λ_b gives

$$\lambda_b = \lambda - \varepsilon\,\frac{\lambda^3 - 1}{2\lambda^2} + \mathrm{O}(\varepsilon^2). \qquad \square$$

Problem 2.19. *(Eversion of a circular cylindrical tube)* (See Problem 5.15 for a complete analysis of this problem.) A hollow circular cylindrical tube has inner and outer radii A and B and length L in a reference configuration. Choose rectangular cartesian coordinates such that the region occupied by the body in this configuration is

$$\mathcal{R}_{\mathrm{R}} = \{(x_1, x_2, x_3) \ : A^2 < x_1^2 + x_2^2 < B^2, \ -L/2 < x_3 < L/2\}.$$

Consider "everting" the tube by turning it inside out – imagine a sock being turned inside out. In particular, this deformation maps the inner surface in the reference configuration into the outer surface in the deformed configuration and the outer surface in the reference configuration into the inner surface in the deformed configuration. Assume that the everted shape of the body is a hollow circular cylinder of inner and outer radii a and b and length ℓ. (In order to maintain the body in this particular deformed configuration one may have to apply a suitable loading on the tube. Otherwise the everted body may not be a hollow circular cylinder with flat ends.) If (r, θ, z) and (R, Θ, Z) denote the cylindrical polar coordinates of a particle in the deformed and reference configurations, take the deformation to have the form

$$r = r(R), \qquad \theta = \Theta, \qquad z = z(Z), \tag{i}$$

where

$$r(A) = b, \qquad r(B) = a, \qquad z(L/2) = -\ell/2, \qquad z(-L/2) = \ell/2. \tag{ii}$$

(a) Determine the deformation, i.e. $r(R)$ and $z(Z)$, assuming

 (a1) that radial and axial fibers do not stretch, and alternatively

 (a2) that the deformation is isochoric. Calculate $\mathbf{C}, \mathbf{U}$ and $\mathbf{R}$.

(b) Verify that if you repeat this deformation, i.e. you evert the deformed configuration, you recover the reference configuration.

Problem 2.20. Calculate explicit expressions for the deformation gradient tensor $\mathbf{F}$ and the left and right Cauchy-Green tensors $\mathbf{B}$ and $\mathbf{C}$ using spherical polar coordinates (R, Θ, Φ) in the undeformed configuration and (r, θ, φ) in the deformed configuration.

Problem 2.21. Consider a body that, in a reference configuration, is identified with the *surface* $\mathcal{S}_R$ depicted on the left-hand side of Figure 2.26. The cross-section of $\mathcal{S}_R$ at each Z is a circle of radius $R(Z)$, with the radius increasing monotonically from $R(Z_0)$ to $R(Z_1)$. Note that this is a two-dimensional body (a surface) in three-dimensional physical space. If (x_1, x_2, x_3) and (R, Θ, Z) are the rectangular cartesian and cylindrical polar coordinates of a point on $\mathcal{S}_R$, then

$$x_1 = R(Z)\cos\Theta, \quad x_2 = R(Z)\sin\Theta, \quad x_3 = Z, \qquad Z_0 \le Z \le Z_1, \quad 0 \le \Theta < 2\pi. \tag{i}$$

A particle on $\mathcal{S}_R$ is located at $\mathbf{x} = R(Z)\,\mathbf{e}_R + Z\,\mathbf{e}_Z$.

A deformation "flattens" the surface $\mathcal{S}_R$ into the (two-dimensional) planar annular region $\mathcal{S}$ shown on the right-hand side of Figure 2.26, taking each meridional curve on $\mathcal{S}_R$ into a radial line on $\mathcal{S}$. (A meridional curve on $\mathcal{S}_R$ is a curve at constant Θ as depicted in Figure 2.27.) If (y_1, y_2) and (r, θ) are the rectangular cartesian and polar coordinates of a particle in $\mathcal{S}$, then

$$y_1 = r\cos\theta, \quad y_2 = r\sin\theta, \qquad r_0 \le r \le r_1, \quad 0 \le \theta < 2\pi. \tag{ii}$$

The position vector of a particle on $\mathcal{S}$ can be written as $\mathbf{y} = r\mathbf{e}_r$.

The axially symmetric deformation that takes $\mathcal{S}_R \to \mathcal{S}$ can be described in the form $r = \widehat{r}(Z), \theta = \Theta$. It is more natural however to use coordinates on the body (i.e. on each surface) and therefore, for the reference configuration, to use arc length S along a meridional curve instead of the vertical coordinate Z. The corresponding orthonormal basis is comprised of the unit vectors $\mathbf{e}_S$ (see Figure 2.27) and $\mathbf{e}_\Theta$. Thus we express the deformation in the form

$$r = r(S), \qquad \theta = \Theta. \qquad (iii)$$

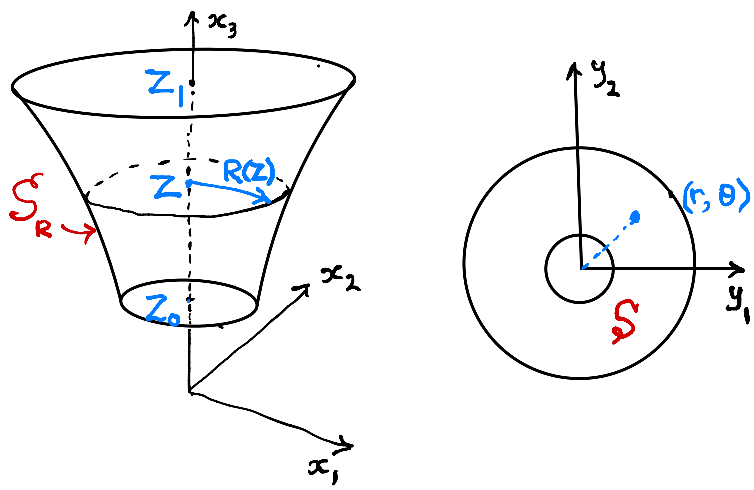


Figure 2.26: Reference configuration (left): A sheet $\mathcal{S}_\mathrm{R}$ with an axially varying circular cross-section of radius $R(Z)$ at a height Z. Deformed configuration (right): After flattening, planar circular annulus $\mathcal{S}$.

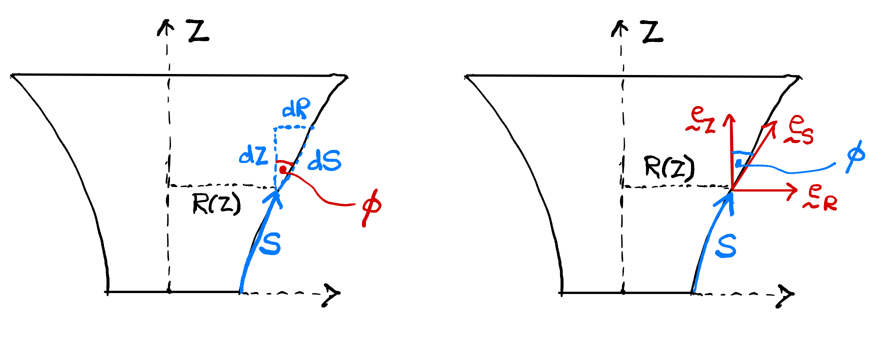


Figure 2.27: Arc length S and associated unit vector $\mathbf{e}_S$ along a meridional curve.

(a) Use $d\mathbf{y} = \mathbf{F}d\mathbf{x}$ to calculate the deformation gradient tensor $\mathbf{F}$ and express your answer in the mixed orthonormal bases $\{\mathbf{e}_S, \mathbf{e}_\Theta\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta\}$.

(b) Suppose that the deformation preserves area in the sense that the area of any part of $\mathcal{S}_R$ equals the area of its image on $\mathcal{S}$. Calculate the deformation $r(S)$. (Since the undeformed surface $\mathcal{S}_R$ is given, the function $R(S)$ is known.)

Solution:

(a) First consider the reference configuration: if the body had been three dimensional we would have used cylindrical polar coordinates (R, Θ, Z) with associated basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ and written the position vector of a particle in $\mathcal{R}_R$ as $\mathbf{x} = \mathbf{x}(R, \Theta, Z) = R\mathbf{e}_R(\Theta) + Z\mathbf{e}_Z$. Since the body here is two dimensional and $\mathbf{x} \in \mathcal{S}_R$, the R and Z coordinates are related by (the given function) $R = R(Z)$ and so we write the position vector of a point on $\mathcal{S}_R$ as

$$\mathbf{x} = \mathbf{x}(\Theta, Z) = R(Z)\mathbf{e}_R(\Theta) + Z\mathbf{e}_Z. \qquad (ii)$$

Thus

$$\begin{aligned} d\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial \Theta} d\Theta + \frac{\partial \mathbf{x}}{\partial Z} dZ = \\ &= \frac{\partial}{\partial \Theta}\Big(R(Z)\mathbf{e}_R(\Theta) + Z\mathbf{e}_Z\Big)d\Theta + \frac{\partial}{\partial Z}\Big(R(Z)\mathbf{e}_R(\Theta) + Z\mathbf{e}_Z\Big)dZ = \\ &= R\frac{\partial}{\partial \Theta}\big(\mathbf{e}_R(\Theta)\big)d\Theta + \left(\frac{dR}{dZ}\mathbf{e}_R + \mathbf{e}_Z\right) dZ = \\ &= R\mathbf{e}_\Theta d\Theta + \left(\frac{dR}{dZ}\mathbf{e}_R + \mathbf{e}_Z\right) dZ. \end{aligned} \qquad (iii)$$

In the expressions above we have displayed the argument of a function if it is to be differentiated, e.g. $R(Z)$ in the second line, but suppressed the argument otherwise, e.g. R in the third line. In getting to the fourth line we have used $\partial \mathbf{e}_R(\Theta)/\partial\Theta = \mathbf{e}_\Theta(\Theta)$ which follows from $\mathbf{e}_R(\Theta) = \cos\Theta\mathbf{e}_1 + \sin\Theta\mathbf{e}_2$ and $\mathbf{e}_\Theta(\Theta) = -\sin\Theta\mathbf{e}_1 + \cos\Theta\mathbf{e}_2$.

Even though a material fiber $d\mathbf{x}$ in the reference body lies in the surface $\mathcal{S}_R$, the particular representation (iii) involves three unit vectors, two of which don't lie in $\mathcal{S}_R$. As mentioned in the problem statement it is more natural to use orthogonal coordinates (S, Θ) *on* the surface $\mathcal{S}_R$ where S is arc length along a meridional curve; see Figure 2.27. The associated orthonormal basis is $\{\mathbf{e}_S, \mathbf{e}_\Theta\}$ where $\mathbf{e}_S$ is a unit tangent vector along a meridional curve in the direction of increasing arc length.

We see from Figure 2.27 that

$$dS\,\mathbf{e}_Z = dZ\,\mathbf{e}_Z + dR\,\mathbf{e}_R = \left(\mathbf{e}_Z + \frac{dR}{dZ}\,\mathbf{e}_R\right) dZ$$

and so we can write (iii) as

$$d\mathbf{x} = R\mathbf{e}_\Theta\,d\Theta + \mathbf{e}_S\,dS. \qquad (iv)$$

It follows from this that

$$dS = \mathbf{e}_S \cdot d\mathbf{x}, \qquad d\Theta = \frac{1}{R}\mathbf{e}_\Theta \cdot d\mathbf{x}. \qquad (v)$$

The following will be useful shortly: From Figure 2.27, the arc length as a function of height, $S(Z)$, is related to the radius as a function of height, $R(Z)$, by $dS^2 = dR^2 + dZ^2$, i.e.

$$\frac{dS}{dZ} = \sqrt{1 + \left(\frac{dR}{dZ}\right)^2}. \qquad (vi)$$

Integrating this gives

$$S(Z) = \int_{Z_0}^{Z} \sqrt{1 + \big(R'(\xi)\big)^2}\; d\xi, \qquad (vii)$$

where we have taken $S = 0$ at $Z = Z_0$, ξ is a dummy variable, and a prime denotes differentiation with respect to Z. This gives a relation between the functions $S(Z)$ and $R(Z)$. Since $R(Z)$ is given, we can consider $S(Z)$ to be known.

Second, consider the deformed configuration: The axially symmetric deformation that takes $\mathcal{S}_R \to \mathcal{S}$ is described by[19]

$$r = r(S), \qquad \theta = \Theta. \tag{viii}$$

The position vector $\mathbf{y} = r\,\mathbf{e}_r(\theta)$ of a particle on $\mathcal{S}$ can now be written in more detail as

$$\mathbf{y} = \mathbf{y}(\Theta, S) = r(S)\mathbf{e}_r(\Theta).$$

Therefore

$$\begin{aligned} d\mathbf{y} &= \frac{\partial \mathbf{y}}{\partial S}dS + \frac{\partial \mathbf{y}}{\partial \Theta}d\Theta = \frac{\partial}{\partial S}\Big(r(S)\mathbf{e}_r(\Theta)\Big)dS + \frac{\partial}{\partial \Theta}\Big(r(S)\mathbf{e}_r(\Theta)\Big)d\Theta = \\ &= \frac{dr}{dS}\,\mathbf{e}_r\,dS + r\frac{\partial}{\partial\Theta}\big(\mathbf{e}_r(\Theta)\big)\,d\Theta = \frac{dr}{dS}\,\mathbf{e}_r\,dS + r\mathbf{e}_\theta\,d\Theta. \end{aligned} \tag{ix}$$

Third, combine the preceding analyses by substituting (v) into (ix):

$$\begin{aligned} d\mathbf{y} &= \frac{dr}{dS}\,\mathbf{e}_r(\mathbf{e}_S\cdot d\mathbf{x}) + (r\mathbf{e}_\theta)\left(\frac{1}{R}\mathbf{e}_\Theta\cdot d\mathbf{x}\right) = \\ &= \frac{dr}{dS}(\mathbf{e}_r\otimes\mathbf{e}_S)d\mathbf{x} + \frac{r}{R}(\mathbf{e}_\theta\otimes\mathbf{e}_\Theta)d\mathbf{x} = \\ &= \left[\frac{dr}{dS}(\mathbf{e}_r\otimes\mathbf{e}_S) + \frac{r}{R}(\mathbf{e}_\theta\otimes\mathbf{e}_\Theta)\right]d\mathbf{x} \end{aligned}$$

and so we have

$$d\mathbf{y} = \mathbf{F}d\mathbf{x} \qquad \text{where} \quad \mathbf{F} = \frac{dr}{dS}(\mathbf{e}_r\otimes\mathbf{e}_S) + \frac{r}{R}(\mathbf{e}_\theta\otimes\mathbf{e}_\Theta). \qquad \square$$

(b) Consider a circular strip on $\mathcal{S}_R$ of radius R and width dS, its area being $2\pi R\,dS$. The corresponding annular region on $\mathcal{S}$ has radius r and width dr, its area being $2\pi r dr$. Thus the conservation of area requires $2\pi r\,dr = 2\pi R\,dS$ which can be written as the differential equation

$$r(S)\,\frac{dr}{dS}(S) = R(S); \tag{x}$$

recall that the function $R(S)$ is obtained from the given function $R(Z)$ by changing variables from Z to S using equation (vii). Observe that equation (xi) can alternatively be obtained from $\lambda_1\lambda_2 = 1$ (i.e. $\det\mathbf{F} = 1$) where $\mathbf{F} = \lambda_1(\mathbf{e}_r\otimes\mathbf{e}_S) + \lambda_2(\mathbf{e}_\theta\otimes\mathbf{e}_\Theta)$. Integrating (x) from the lower end $S = 0$ to some arbitrary S yields

$$\frac{1}{2}r^{\,2}(S) - \frac{1}{2}r^{\,2}(0) = \int_0^S R(\sigma)d\sigma \quad\Rightarrow\quad r(S) = \left[r^{\,2}(0) + 2\int_0^S R(\sigma)d\sigma\right]^{1/2}, \qquad \square \tag{xi}$$

where σ is a dummy variable.

[19] We can use the relation (vii) to convert a function of S into a function of Z and vice versa. We ought to use different symbols for such functions, e.g. $\widehat{r}(S)$ and $r(Z)$. For simplicity we shall not do this but we will avoid any confusion by displaying the arguments of the functions when important.

Alternatively, in terms of Z, we have from (x) and (vi),

$$r\,\frac{dr}{dZ}\frac{dZ}{dS} = R(Z) \qquad \text{and} \qquad \frac{dS}{dZ} = \sqrt{1+\big(R'(Z)\big)^2},$$

whence

$$r(Z)\,\frac{dr}{dZ}(Z) = R(Z)\sqrt{1+\big(R'(Z)\big)^2}.$$

Integrating this gives the deformation in the form $r = r(Z)$:

$$r(Z) = \left[r^2(Z_0) + 2\int_{Z_0}^{Z} R(\xi)\sqrt{1+\big(R'(\xi)\big)^2}\;d\xi\right]^{1/2},$$

where ξ is a dummy variable.

Some general considerations.

Problem 2.22. The stretch $\lambda(\mathbf{m}_\mathrm{R})$ of a fiber oriented (in the reference configuration) in the direction $\mathbf{m}_\mathrm{R}$ is

$$\lambda(\mathbf{m}_\mathrm{R}) = |\mathbf{F}\mathbf{m}_\mathrm{R}|. \qquad (i)$$

(a) Maximize $\lambda(\mathbf{m}_\mathrm{R})$ over all fiber directions $\mathbf{m}_\mathrm{R}$.

(b) Show that $\lambda(\mathbf{m}_\mathrm{R})$ can be written in terms of the principal stretches as

$$\lambda(\mathbf{m}_\mathrm{R}) = \sqrt{\lambda_1^2 m_1^2 + \lambda_2^2 m_2^2 + \lambda_3^2 m_3^2}, \qquad (ii)$$

where the m_k's are the components of $\mathbf{m}_\mathrm{R}$ in the principal basis of the right stretch tensor $\mathbf{U}$.

Solution: We write the square of the stretch as

$$\lambda^2(\mathbf{m}_\mathrm{R}) = \mathbf{F}\mathbf{m}_\mathrm{R}\cdot\mathbf{F}\mathbf{m}_\mathrm{R} = \mathbf{F}^T\mathbf{F}\mathbf{m}_\mathrm{R}\cdot\mathbf{m}_\mathrm{R} = \mathbf{C}\mathbf{m}_\mathrm{R}\cdot\mathbf{m}_\mathrm{R}. \qquad (iii)$$

When maximizing this over all unit vectors $\mathbf{m}_\mathrm{R}$, we must respect the constraint $\mathbf{m}_\mathrm{R}\cdot\mathbf{m}_\mathrm{R} = 1$. Thus we maximize the function

$$\phi(\mathbf{m}) = \mathbf{C}\mathbf{m}\cdot\mathbf{m} - \mu(\mathbf{m}\cdot\mathbf{m} - 1),$$

over *all* vectors $\mathbf{m}$ where μ is a Lagrange multiplier. This yields:

$$\frac{\partial\phi}{\partial m_j} = \frac{\partial}{\partial m_j}(C_{pq}m_q m_p - \mu m_p m_p + \mu) = C_{pq}\delta_{jq}m_p + C_{pq}m_q\delta_{jp} - 2\mu\delta_{jp}m_p = 2(C_{ij}m_j - \mu m_j) = 0$$

which tells us that a particular direction $\mathbf{m}_\mathrm{R}$ that maximizes $\lambda^2(\mathbf{m})$ obeys $\mathbf{C}\mathbf{m}_\mathrm{R} = \mu\mathbf{m}_\mathrm{R}$, i.e. it is an eigenvector of $\mathbf{C}$. Substituting $\mathbf{C}\mathbf{m}_\mathrm{R} = \mu\mathbf{m}_\mathrm{R}$ into (iii) yields $\lambda^2(\mathbf{m}_\mathrm{R}) = \mu$ and so the extrema of $\lambda^2(\mathbf{m}_\mathrm{R})$ are the eigenvalues of $\mathbf{C}$. The maximum value of $\lambda(\mathbf{m}_\mathrm{R})$ is therefore the largest of the principal stretches.

In a principal basis $\mathbf{C} = \lambda_1^2\mathbf{r}_1\otimes\mathbf{r}_1 + \lambda_2^2\mathbf{r}_2\otimes\mathbf{r}_2 + \lambda_3^2\mathbf{r}_3\otimes\mathbf{r}_3$. If we write $\mathbf{m}_\mathrm{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3$ then

$$\lambda^2(\mathbf{m}_\mathrm{R}) = \mathbf{C}\mathbf{m}_\mathrm{R}\cdot\mathbf{m}_\mathrm{R} = (\lambda_1^2\mathbf{r}_1\otimes\mathbf{r}_1 + \lambda_2^2\mathbf{r}_2\otimes\mathbf{r}_2 + \lambda_3^2\mathbf{r}_3\otimes\mathbf{r}_3)(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)\;\cdot\;(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)$$

which simplifies to (ii). Note that we can carry out the preceding maximization equivalently by maximizing $\lambda_1^2 m_1^2 + \lambda_2^2 m_2^2 + \lambda_3^2 m_3^2$ over all m_1, m_2, m_3 subject to the constraint $m_1^2 + m_2^2 + m_3^2 = 1$.

Problem 2.23. Show for any isochoric deformation that

$$I_1(\mathbf{C}) \geq 3, \qquad (i)$$

where $I_1(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ is the first principal invariant of $\mathbf{C}$. Moreover, show that $I_1 = 3$ if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

Likewise show for isochoric deformations that

$$I_2(\mathbf{C}) \geq 3, \qquad (ii)$$

where $I_2(\mathbf{C}) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$ is the second principal invariant of $\mathbf{C}$, and that $I_2 = 3$ if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

Solution: Since the deformations considered are isochoric,

$$\lambda_1\lambda_2\lambda_3 = 1. \qquad (iii)$$

Recall the inequality between the arithmetic and geometric means of a set of positive real numbers:

$$\frac{\alpha_1 + \alpha_2 + \ldots \alpha_n}{n} \geq \Big[\alpha_1\alpha_2\ldots\alpha_n\Big]^{1/n},$$

and further, that the two means are equal if and only if $\alpha_1 = \alpha_2 = \ldots = \alpha_n$. Apply this to the squares of the three principal stretches λ_i^2:

$$\frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{3} \geq \Big[\lambda_1^2\lambda_2^2\lambda_3^2\Big]^{1/3}.$$

Therefore after using (iii) we get

$$I_1(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \geq 3. \qquad \square$$

By the second part of the result quoted above, the inequality is strict unless $\lambda_1^2 = \lambda_2^2 = \lambda_3^2$ which by (iii) and the positivity of the the principal stretches shows that $I_1(\mathbf{C}) > 3$ unless $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

Alternatively, Rosakis has pointed out that one can establish this result by showing that the only extremum of the function $I_1(\lambda_1, \lambda_2) = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}$ in the first quadrant of the λ_1, λ_2-plane occurs at $(\lambda_1, \lambda_2) = (1, 1)$ and that it is a minimum.

By using (iii) we can write the second principal invariant $I_2(\mathbf{C}) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$ as

$$I_2(\mathbf{C}) = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2},$$

and the result follows by a calculation exactly as above.

Problem 2.24. (Piola identity) Show that

$$\int_{\mathcal{D}} \mathbf{n}\, dA_y = \mathbf{o}$$

where $\mathcal{D}$ is an arbitrary subregion of $\mathcal{R}$ and $\mathbf{n}$ is the outward pointing unit vector normal to its boundary $\partial\mathcal{D}$.

By using the relation $\mathbf{n} dA_y = J\mathbf{F}^{-T}\mathbf{n}_{\mathrm{R}} dA_x$ or otherwise, show that

$$\mathrm{Div}(J\mathbf{F}^{-T}) = \mathbf{o}, \tag{2.123}$$

where for any tensor field $\mathbf{A}(\mathbf{x})$, $\mathrm{Div}\,\mathbf{A}$ denotes the vector field with cartesian components $\partial A_{ij}/\partial x_j$.

Similarly show that

$$\mathrm{div}(J^{-1}\mathbf{F}^{T}) = \mathbf{o}, \tag{2.124}$$

where for any tensor field $\mathbf{A}(\mathbf{y})$, $\mathrm{div}\,\mathbf{A}$ denotes the vector field with cartesian components $\partial A_{ij}/\partial y_j$.

Solution: Recall that for any tensor field $\mathbf{A}(\mathbf{y})$ the divergence theorem states

$$\int_{\partial\mathcal{D}} \mathbf{A}\mathbf{n}\; dA = \int_{\mathcal{D}} \mathrm{div}\mathbf{A}\; dV, \tag{iii}$$

where $\mathcal{D}$ is an arbitrary subregion of $\mathcal{R}$ and $\mathbf{n}$ is the outward pointing unit vector normal to its boundary $\partial\mathcal{D}$. When applied to the choice $\mathbf{A} = \mathbf{I}$ we get

$$\int_{\partial\mathcal{D}} \mathbf{n}\; dA_y = 0. \qquad \square \tag{iv}$$

Let $\mathcal{D}_{\mathrm{R}}$ be the image of $\mathcal{D}$ in the reference configuration and let $\mathbf{n}_{\mathrm{R}}$ denote the outward pointing unit vector normal of its boundary $\partial\mathcal{D}_{\mathrm{R}}$. By using the relation $\mathbf{n}\; dA_y = J\mathbf{F}^{-T}\mathbf{n}_{\mathrm{R}}\; dA_x$

$$0 \overset{(iv)}{=} \int_{\partial\mathcal{D}} \mathbf{n}\; dA_y = \int_{\partial\mathcal{D}_{\mathrm{R}}} J\mathbf{F}^{-T}\mathbf{n}_{\mathrm{R}}\; dA_x = \int_{\partial\mathcal{D}_{\mathrm{R}}} \big(J\mathbf{F}^{-T}\big)\mathbf{n}_{\mathrm{R}}\; dA_x = \int_{\mathcal{D}_{\mathrm{R}}} \mathrm{Div}(J\mathbf{F}^{-T})\; dV_x$$

where in the last step we used the divergence theorem in the reference configuration. Since this holds for all choices of $\mathcal{D}_{\mathrm{R}}$, localization tells us that the integrand vanishes at each point in the body and so we get (2.123).

Likewise by using the relation $J^{-1}\mathbf{F}^T\mathbf{n}\; dA_y = \mathbf{n}_{\mathrm{R}}\; dA_x$ we have

$$0 = \int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{n}_{\mathrm{R}}\; dA_x = \int_{\partial\mathcal{D}} J^{-1}\mathbf{F}^T\mathbf{n}\; dA_y = \int_{\partial\mathcal{D}} (J^{-1}\mathbf{F}^T)\mathbf{n}\; dA_y = \int_{\mathcal{D}_{\mathrm{R}}} \mathrm{div}(J^{-1}\mathbf{F}^T)\; dV_x$$

which when localized yields (2.124).

Problem 2.25. Show that

$$\frac{\partial\lambda_i}{\partial\mathbf{C}} = \frac{1}{2\lambda_i}\mathbf{r}_i\otimes\mathbf{r}_i \qquad \text{and} \qquad \frac{\partial\lambda_i}{\partial\mathbf{F}} = \boldsymbol{\ell}_i\otimes\mathbf{r}_i, \tag{i}$$

where *the summation convention for repeated subscript is suspended.*

Solution: We shall follow the method for differentiation used in Problem 1.8.4. Consider a one-parameter family of right Cauchy-Green deformation tensors $\mathbf{C}(t)$ that depend smoothly on a parameter t.

Since $\mathbf{r}_i(t)$ is a unit vector,

$$\mathbf{r}_i(t) \cdot \mathbf{r}_i(t) = 1. \tag{ii}$$

Differentiating (ii) with respect to t gives

$$\dot{\mathbf{r}}_i \cdot \mathbf{r}_i = 0. \tag{iii}$$

Next, since λ_i^2 is an eigenvalue of $\mathbf{C}$ and $\mathbf{r}_i$ is the corresponding eigenvector,

$$\mathbf{C}\mathbf{r}_i = \lambda_i^2 \mathbf{r}_i. \tag{iv}$$

Differentiating this with respect to t yields

$$\dot{\mathbf{C}}\mathbf{r}_i + \mathbf{C}\dot{\mathbf{r}}_i = 2\lambda_i \left(\frac{\partial \lambda_i}{\partial \mathbf{C}} \cdot \dot{\mathbf{C}} \right) \mathbf{r}_i + \lambda_i^2 \dot{\mathbf{r}}_i.$$

Taking the scalar product of both sides of this equation with $\mathbf{r}_i$ and using (ii), (iii) leads to

$$\dot{\mathbf{C}}\mathbf{r}_i \cdot \mathbf{r}_i + \mathbf{C}\dot{\mathbf{r}}_i \cdot \mathbf{r}_i = 2\lambda_i \left(\frac{\partial \lambda_i}{\partial \mathbf{C}} \cdot \dot{\mathbf{C}} \right).$$

However $\mathbf{C}\dot{\mathbf{r}}_i \cdot \mathbf{r}_i = \dot{\mathbf{r}}_i \cdot \mathbf{C}\mathbf{r}_i \overset{(iv)}{=} \dot{\mathbf{r}}_i \cdot \lambda_i^2 \mathbf{r}_i \overset{(iii)}{=} 0$. Therefore the second term on the left-hand of the preceding equation drops out and we can write that equation as

$$\left(\mathbf{r}_i \otimes \mathbf{r}_i - 2\lambda_i \frac{\partial \lambda_i}{\partial \mathbf{C}} \right) \cdot \dot{\mathbf{C}} = 0.$$

Since this must hold for all $\dot{\mathbf{C}}$, and the terms in the parenthesis do not depend on $\dot{\mathbf{C}}$, by the argument used in Problem 1.8.4 we conclude that the terms inside the parenthesis must vanish:

$$\mathbf{r}_i \otimes \mathbf{r}_i - 2\lambda_i \frac{\partial \lambda_i}{\partial \mathbf{C}} = 0.$$

This leads to $(i)_1$.

By writing out in terms of components and using indicial notation it is readily seen that

$$\frac{\partial \lambda_i}{\partial \mathbf{F}} = 2\mathbf{F} \frac{\partial \lambda_i}{\partial \mathbf{C}}.$$

Thus the result $(i)_2$ follows upon using $\mathbf{F}(\mathbf{r}_i \otimes \mathbf{r}_i) = (\mathbf{R}\mathbf{U}\mathbf{r}_i) \otimes \mathbf{r}_i = \lambda_i(\mathbf{R}\mathbf{r}_i) \otimes \mathbf{r}_i = \lambda_i \boldsymbol{\ell}_i \otimes \mathbf{r}_i$.

Problem 2.26. Let $i_1(\mathbf{E}), i_2(\mathbf{E}), i_3(\mathbf{E})$ be the principal scalar invariants of the Green Saint-Venant strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$:

$$i_1 = \operatorname{tr} \mathbf{E}, \qquad i_2 = \frac{1}{2}(i_1^2 - \operatorname{tr} \mathbf{E}^2), \qquad i_3 = \det \mathbf{E}. \tag{i}$$

Show that a deformation is isochoric if

$$i_1 + 2i_2 + 4i_3 = 0. \tag{ii}$$

Problem 2.27. Show that the cartesian components $F_{ij}(\mathbf{x})$ of the deformation gradient tensor field must necessarily obey the following system of partial differential equations (**compatibility equations**)

$$\frac{\partial F_{ij}}{\partial x_k} = \frac{\partial F_{ik}}{\partial x_j}; \tag{2.125}$$

furthermore, show that this can be written as $\operatorname{Curl}\mathbf{F} = \mathbf{0}$.

Solution: Given the 3 components $y_i(\mathbf{x})$ of the deformation field, one can calculate the 9 components $F_{ij}(\mathbf{x})$ of the deformation gradient field by differentiating $y_i(\mathbf{x})$:

$$F_{ij}(\mathbf{x}) = \frac{\partial y_i}{\partial x_j}(\mathbf{x}).$$

Conversely, given the 9 components $F_{ij}(\mathbf{x})$ of the deformation gradient field, in order to determine the associated components $y_i(\mathbf{x})$ of the deformation field, one has the integrate the system of equations

$$\frac{\partial y_i}{\partial x_j}(\mathbf{x}) = F_{ij}(\mathbf{x}). \tag{i}$$

However, (i) is a system of 9 scalar differential equations for determining the 3 unknown scalar functions $y_i(\mathbf{x})$. It is therefore overdetermined and so can only be solved if the $F_{ij}(\mathbf{x})$'s satisfy suitable "integrability conditions". Differentiating (i) with respect to x_k gives

$$\frac{\partial^2 y_i}{\partial x_k \partial x_j}(\mathbf{x}) = \frac{\partial F_{ij}}{\partial x_k}(\mathbf{x}). \tag{ii}$$

On the other hand differentiating

$$\frac{\partial y_i}{\partial x_k}(\mathbf{x}) = F_{ik}(\mathbf{x}) \tag{iii}$$

with respect to x_j gives

$$\frac{\partial^2 y_i}{\partial x_j \partial x_k}(\mathbf{x}) = \frac{\partial F_{ik}}{\partial x_j}(\mathbf{x}). \tag{iv}$$

Changing the order of partial differentiation on the left-hand side of (iv) shows that the left-hand sides of (ii) and (iv) are the same. Thus we may equate their right-hand sides to get:

$$\frac{\partial F_{ij}}{\partial x_k} = \frac{\partial F_{ik}}{\partial x_j}. \qquad \square$$

This is a system of equations that the given fields $F_{ij}(\mathbf{x})$ must necessarily satisfy if there is to exist a corresponding set of $y_i(\mathbf{x})$'s.

We can write (2.125) as

$$0 = \frac{\partial F_{ij}}{\partial x_k} - \frac{\partial F_{ik}}{\partial x_j} = (\delta_{kp}\delta_{jq} - \delta_{kq}\delta_{jp})\frac{\partial F_{iq}}{\partial x_p} = e_{\ell kj} e_{\ell pq}\frac{\partial F_{iq}}{\partial x_p} \qquad \Rightarrow \qquad e_{\ell pq}\frac{\partial F_{iq}}{\partial x_p} = 0,$$

which by (1.169) is the cartesian representation of $\operatorname{Curl}\mathbf{F} = \mathbf{0}$.

Remark: A more challenging (and possibly more important) task is determining the compatibility equations to be satisfied by the components $E_{ij}(\mathbf{x})$ of a strain tensor field in order that there exist a corresponding deformation field.

Problem 2.28. Suppose that some Lagrangian strain tensor field vanishes at every point in the body: $\mathbf{E}(\mathbf{x}) = \mathbf{I}$ for all $\mathbf{x} \in \mathcal{R}_{\mathrm{R}}$. In this event, the Lagrangian stretch tensor field and the the right Cauchy-Green tensor field equal the identity at every point in the body: $\mathbf{U}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) = \mathbf{I}$ for all $\mathbf{x} \in \mathcal{R}_{\mathrm{R}}$. Show that the deformation gradient tensor field $\mathbf{F}(\mathbf{x})$ is orthogonal at each $\mathbf{x} \in \mathcal{R}_{\mathrm{R}}$. *Moreover*, show that $\mathbf{F}(\mathbf{x})$ is independent of $\mathbf{x}$ and therefore is a *constant* orthogonal tensor.

Solution: Since $\mathbf{C}(\mathbf{x}) = \mathbf{I}$ it follows that $\mathbf{U}(\mathbf{x}) = \mathbf{I}$ and therefore from the polar decomposition theorem that $\mathbf{F}(x)$ is orthogonal at each $\mathbf{x} \in \mathcal{R}_{\mathrm{R}}$.

Since $\mathbf{F}$ is orthogonal, $\mathbf{F}^T\mathbf{F} = \mathbf{I}$ or in cartesian components

$$F_{ki}F_{kj} = \delta_{ij}. \tag{i}$$

Since this holds at each $\mathbf{x}$ we may differentiate it respect to x_ℓ to get

$$\frac{\partial F_{ki}}{\partial x_\ell}F_{kj} + \frac{\partial F_{kj}}{\partial x_\ell}F_{ki} = 0. \tag{ii}$$

Equivalently, writing (i) as $F_{k\ell}F_{ki} = \delta_{\ell i}$ and differentiating with respect to x_j gives

$$\frac{\partial F_{k\ell}}{\partial x_j}F_{ki} + \frac{\partial F_{ki}}{\partial x_j}F_{k\ell} = 0, \tag{iii}$$

and likewise writing (i) as $F_{kj}F_{k\ell} = \delta_{j\ell}$ and differentiating with respect to x_i leads to

$$\frac{\partial F_{kj}}{\partial x_i}F_{k\ell} + \frac{\partial F_{k\ell}}{\partial x_i}F_{kj} = 0. \tag{iv}$$

The compatibility equations (2.125) tell us that $\partial F_{k\ell}/\partial x_j = \partial F_{kj}/\partial x_\ell$ and $\partial F_{ki}/\partial x_j = \partial F_{kj}/\partial x_i$ and so we can write (iii) as

$$\frac{\partial F_{kj}}{\partial x_\ell}F_{ki} + \frac{\partial F_{kj}}{\partial x_i}F_{k\ell} = 0. \tag{v}$$

Similarly since $\partial F_{k\ell}/\partial x_i = \partial F_{ki}/\partial x_\ell$ by the compatibility equations we can write (iv) as

$$\frac{\partial F_{kj}}{\partial x_i}F_{k\ell} + \frac{\partial F_{ki}}{\partial x_\ell}F_{kj} = 0. \tag{vi}$$

Adding (v) and (vi) and subtracting (ii) from the result yields

$$\frac{\partial F_{kj}}{\partial x_i}F_{k\ell} = 0.$$

Multiplying this by $F^{-1}_{\ell p}$ leads to

$$\frac{\partial F_{kj}}{\partial x_i} = 0.$$

Thus all partial derivatives of F_{ij} vanish, which therefore implies that $\mathbf{F}(\mathbf{x})$ is constant.

Problem 2.29. Suppose that the deformation gradient tensor field has the form $\mathbf{F}(\mathbf{x}) = \nabla\mathbf{y}(\mathbf{x}) = \phi(\mathbf{x})\mathbf{A}$ where $\phi(\mathbf{x})$ is a positive scalar-valued function defined for $\mathbf{x} \in \mathcal{R}_{\mathrm{R}}$ and $\mathbf{A}$ is a constant tensor with $\det \mathbf{A} > 0$. Show that necessarily $\phi(\mathbf{x})$ must be independent of $\mathbf{x}$ and therefore a constant.

Solution: We are told that $F_{ij}(\mathbf{x}) = \phi(\mathbf{x})\, A_{ij}$. Substituting this into the compatibility equations (2.125) (page 219) yields

$$\frac{\partial \phi}{\partial x_k}\, A_{ij} = \frac{\partial \phi}{\partial x_j}\, A_{ik}. \qquad (ii)$$

Since the result we want to prove involves ϕ and not $\mathbf{A}$, it is natural to first eliminate $\mathbf{A}$ which we can readily do since $\mathbf{A}$ is nonsingular and so its inverse exists. In order to simultaneously eliminate $\mathbf{A}$ from both sides of (ii) we multiply it by A^{-1}_{pi} to obtain

$$\frac{\partial \phi}{\partial x_k}\, A^{-1}_{pi} A_{ij} = \frac{\partial \phi}{\partial x_j}\, A^{-1}_{pi} A_{ik} \qquad \Rightarrow \qquad \frac{\partial \phi}{\partial x_k}\, \delta_{pj} = \frac{\partial \phi}{\partial x_j}\, \delta_{pk}.$$

This holds for all choices of the free indices p, j, k. Therefore it necessarily must hold with $p = j$:

$$\frac{\partial \phi}{\partial x_k}\, \delta_{jj} = \frac{\partial \phi}{\partial x_j}\, \delta_{jk} \qquad \Rightarrow \qquad 3\frac{\partial \phi}{\partial x_k} = \frac{\partial \phi}{\partial x_k},$$

where we have used $\delta_{jj} = 3$ and the substitution rule. Therefore

$$\frac{\partial \phi}{\partial x_k} = 0 \qquad \text{for } k = 1, 2, 3,$$

which says that all partial derivatives of $\phi(\mathbf{x})$ necessarily vanish and so $\phi(\mathbf{x})$ cannot depend on $\mathbf{x}$. Conversely, if $\phi(\mathbf{x})$ is independent of $\mathbf{x}$ the compatibility equations (ii) hold automatically.

Problem 2.30. Consider two deformations $\mathbf{y} = \mathbf{y}_1(\mathbf{x})$ and $\mathbf{y} = \mathbf{y}_2(\mathbf{x})$ related by a rigid deformation, i.e. the deformations are such that $\mathbf{y}_2(\mathbf{x}) = \mathbf{Q}\mathbf{y}_1(\mathbf{x}) + \mathbf{b}$ where $\mathbf{Q}$ is a constant rotation tensor and $\mathbf{b}$ is a constant vector. Show that the right Cauchy-Green tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ associated with these two deformations coincide: $\mathbf{C}_1 = \mathbf{C}_2$.

What is the corresponding relation between the left Cauchy-Green tensors $\mathbf{B}_1$ and $\mathbf{B}_2$?

Conversely, is it true that if $\mathbf{C}_1(\mathbf{x}) = \mathbf{C}_2(\mathbf{x})$ at each $\mathbf{x} \in \mathcal{R}_{\mathrm{R}}$ then the two deformations differ by a rigid deformation?

Problem 2.31. Among the various experiments on rubber that Rivlin and Saunders [7] carried out are some where they subjected a thin incompressible rubber sheet to a pure stretch

$$y_1 = \lambda_1 x_1, \qquad y_2 = \lambda_2 x_2, \qquad y_3 = \lambda_3 x_3. \qquad (i)$$

They varied the in-plane stretches λ_1, λ_2 keeping the value of the invariant $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ fixed[20]; see Section 4.6.1. Show that

$$\lambda_2^2 = \frac{1}{2}\Big[I_1 - \lambda_1^2 \pm \sqrt{(I_1 - \lambda_1^2)^2 - 4/\lambda_1^2}\Big], \qquad (ii)$$

and sketch the contours of the closed curves defined by (ii) in the λ_1, λ_2-plane corresponding to different fixed values of I_1.

[20]They also did experiments keeping I_2 fixed.

Solution: In view of incompressibility,

$$\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}. \tag{iii}$$

and therefore the principal scalar invariants can be written as

$$\left.\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \overset{(iii)}{=} \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}, \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \overset{(iii)}{=} \lambda_1^2\lambda_2^2 + \lambda_1^{-2} + \lambda_2^{-2}. \end{aligned}\right\} \tag{iv}$$

Equation $(iv)_1$ can be written as the bi-quadratic equation

$$\lambda_2^4 + (\lambda_1^2 - I_1)\lambda_2^2 + \lambda_1^{-2} = 0,$$

from which (ii) follows.

Problem 2.32. (See also Problem 2.33.) Consider a planar surface $\mathcal{S}_{\mathrm{R}}$ that passes through the region $\mathcal{R}_{\mathrm{R}}$ occupied by a body in the reference configuration. Let $\mathbf{n}_{\mathrm{R}}$ be a unit vector normal to $\mathcal{S}_{\mathrm{R}}$ and let $\mathcal{R}_{\mathrm{R}}^+$ denote the side into which $\mathbf{n}_{\mathrm{R}}$ points, $\mathcal{R}_{\mathrm{R}}^-$ the other side. Thus $\mathcal{S}_{\mathrm{R}}$ is a planar interface between two parts of the body.

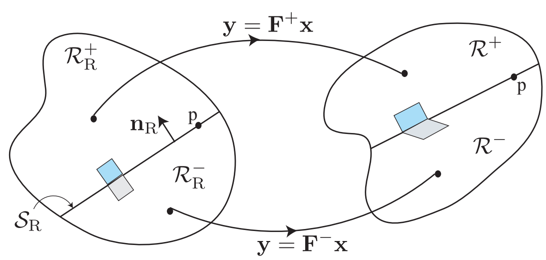


Figure 2.28: Problem 2.32: A piecewise homogeneous deformation. The deformation is continuous across $\mathcal{S}_{\mathrm{R}}$ but the deformation gradient tensor is not. This it depicted by the small grey and blue quadrilaterals.

Consider the piecewise homogeneous deformation

$$\mathbf{y} = \mathbf{y}(\mathbf{x}) = \begin{cases} \mathbf{F}^+\mathbf{x} & \text{for } \mathbf{x} \in \mathcal{R}_{\mathrm{R}}^+, \\ \mathbf{F}^-\mathbf{x} & \text{for } \mathbf{x} \in \mathcal{R}_{\mathrm{R}}^-, \end{cases} \tag{i}$$

where $\mathbf{F}^{\pm}$ are constant non-singular tensors. Show that this deformation $\mathbf{y}(\mathbf{x})$ is continuous across $\mathcal{S}_{\mathrm{R}}$ if and only if there is a constant vector $\mathbf{a}$ for which

$$\mathbf{F}^+ - \mathbf{F}^- = \mathbf{a} \otimes \mathbf{n}_{\mathrm{R}}. \tag{2.126}$$

This is known as the *Hadamard jump (compatibility) condition.* It plays an important role in studying interfaces between two material phases.

Interpret (2.126) geometrically. Specifically, with $\mathcal{S}$ being the image of $\mathcal{S}_{\mathrm{R}}$ in the deformed configuration, show that $\mathbf{F}^+$ differs from $\mathbf{F}^-$ by a simple shear with shearing direction $\mathbf{e}$ and glide plane normal $\mathbf{n}$ and a uniaxial extension in the direction $\mathbf{n}$. (Here the unit vectors $\mathbf{e}$ and $\mathbf{n}$ are in the plane $\mathcal{S}$ and normal to $\mathcal{S}$ respectively).

Solution: Since the deformation is continuous across $\mathcal{S}_R$, it follows from (i) that

$$\mathbf{F}^+\mathbf{x} = \mathbf{F}^-\mathbf{x} \qquad \text{for all vectors } \mathbf{x} \text{ in the plane } \mathcal{S}_R. \tag{ii}$$

Set

$$\mathbf{G} = \mathbf{F}^+ - \mathbf{F}^-, \tag{iii}$$

so that because of (ii)

$$\mathbf{G}\mathbf{x} = \mathbf{o} \qquad \text{for all vectors } \mathbf{x} \text{ in the plane } \mathcal{S}_R. \tag{iv}$$

Now pick an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_3 = \mathbf{n}_R$. Since $\mathbf{e}_3 = \mathbf{n}_R$ is perpendicular to the plane $\mathcal{S}_R$ it follows that the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ lie in the plane $\mathcal{S}_R$. Thus by taking $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{x} = \mathbf{e}_2$ in (iv),

$$\mathbf{G}\mathbf{e}_1 = \mathbf{o}, \qquad \mathbf{G}\mathbf{e}_2 = \mathbf{o}. \tag{v}$$

Solution 1: Let G_{ij} be the i,j component of $\mathbf{G}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ above:

$$\mathbf{G} = G_{ij}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{vi}$$

The equations in (v) imply that certain components of $\mathbf{G}$ vanish that we identify as follows: From (vi),

$$\mathbf{G}\mathbf{e}_1 = \big(G_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\big)\mathbf{e}_1 = G_{ij}(\mathbf{e}_j \cdot \mathbf{e}_1)\mathbf{e}_i = G_{ij}\delta_{1j}\mathbf{e}_i = G_{i1}\mathbf{e}_i.$$

Since $\mathbf{G}\mathbf{e}_1 = \mathbf{o}$ it follows that $G_{i1} = 0$, i.e. $G_{11} = G_{21} = G_{31} = 0$. Similarly from $\mathbf{G}\mathbf{e}_2 = \mathbf{o}$ we find that $G_{12} = G_{23} = G_{33} = 0$. Therefore the only non-zero components are G_{13}, G_{23} and G_{33} and so (vi) simplifies to

$$\mathbf{G} = G_{13}\mathbf{e}_1 \otimes \mathbf{e}_3 + G_{23}\mathbf{e}_2 \otimes \mathbf{e}_3 + G_{33}\mathbf{e}_3 \otimes \mathbf{e}_3 = \underbrace{(G_{13}\mathbf{e}_1 + G_{23}\mathbf{e}_2 + G_{33}\mathbf{e}_3)}_{\mathbf{a}} \otimes \mathbf{e}_3 = \mathbf{a} \otimes \mathbf{n}_R. \qquad \square$$

Solution 2:

Lemma: If $\mathbf{A}\mathbf{e}_i = \mathbf{o}$ for $i = 1, 2, 3$ where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis then $\mathbf{A} = \mathbf{0}$.

Now observe that

$$\Big[\mathbf{G} - \mathbf{G}(\mathbf{e}_3 \otimes \mathbf{e}_3)\Big]\mathbf{e}_\alpha = \mathbf{G}\mathbf{e}_\alpha - (\mathbf{e}_3 \cdot \mathbf{e}_\alpha)\mathbf{G}\mathbf{e}_3 = \mathbf{G}\mathbf{e}_\alpha \overset{(v)}{=} \mathbf{o} \qquad \text{for } \alpha = 1, 2,$$

$$\Big[\mathbf{G} - \mathbf{G}(\mathbf{e}_3 \otimes \mathbf{e}_3)\Big]\mathbf{e}_3 = \mathbf{G}\mathbf{e}_3 - (\mathbf{e}_3 \cdot \mathbf{e}_3)\mathbf{G}\mathbf{e}_3 = \mathbf{G}\mathbf{e}_3 - \mathbf{G}\mathbf{e}_3 = \mathbf{o}.$$

It therefore follows from the lemma above that $\mathbf{G} - \mathbf{G}(\mathbf{e}_3 \otimes \mathbf{e}_3) = \mathbf{0}$ and so

$$\mathbf{G} = \mathbf{G}(\mathbf{e}_3 \otimes \mathbf{e}_3) = \underbrace{(\mathbf{G}\mathbf{e}_3)}_{\mathbf{a}} \otimes \mathbf{e}_3 = \mathbf{a} \otimes \mathbf{n}_R. \qquad \square$$

To interpret the relation (2.126) between $\mathbf{F}^+$ and $\mathbf{F}^-$ we proceed as follows:

$$\begin{aligned}
\mathbf{F}^+ \; &= \mathbf{F}^- + \mathbf{a} \otimes \mathbf{n}_{\mathrm{R}} = \Big(\mathbf{I} + (\mathbf{a} \otimes \mathbf{n}_{\mathrm{R}})\, \bar{\mathbf{F}}^{\,-1}\Big)\mathbf{F}^- \overset{(1.78)}{=} \Big(\mathbf{I} + (\mathbf{a} \otimes \bar{\mathbf{F}}^{\,-T}\mathbf{n}_{\mathrm{R}})\Big)\mathbf{F}^- = \\
&\overset{(*)}{=} \Big(\mathbf{I} + \mathbf{b} \otimes \mathbf{n}\Big)\mathbf{F}^- = \\
&\overset{(**)}{=} \Big(\mathbf{I} + \big[(b_1\mathbf{e} + b_2\mathbf{n}) \otimes \mathbf{n}\big]\Big)\mathbf{F}^- = \Big(\mathbf{I} + b_1\mathbf{e} \otimes \mathbf{n} + b_2\mathbf{n} \otimes \mathbf{n}\Big)\mathbf{F}^- = \\
&= \Big(\mathbf{I} + \tfrac{b_1}{1+b_2}\mathbf{e} \otimes \mathbf{n}\Big)\Big(\mathbf{I} + b_2\mathbf{n} \otimes \mathbf{n}\Big)\mathbf{F}^-
\end{aligned}$$

In step $(*)$ we used $\mathbf{n} = \bar{\mathbf{F}}^{-T}\mathbf{n}_{\mathrm{R}}/|\,\bar{\mathbf{F}}^{-T}\mathbf{n}_{\mathrm{R}}|$ which follows from Nanson's formula and set $\mathbf{b} = |\,\bar{\mathbf{F}}^{-T}\mathbf{n}_{\mathrm{R}}|\,\mathbf{a}$; in step $(**)$ we wrote $\mathbf{b} = b_1\mathbf{e} + b_2\mathbf{n}$ where $b_1\mathbf{e}$ is the projection of $\mathbf{b}$ onto the plane $\mathcal{S}$ and $b_2\mathbf{n}$ is its projection onto the normal to $\mathcal{S}$ where $\mathbf{e}$ is a unit vector in the plane $\mathcal{S}$. Therefore the deformation on $\mathcal{R}_{\mathrm{R}}^+$ can be obtained by first subjecting it to the same deformation as that on $\mathcal{R}_{\mathrm{R}}^-$, followed by a uniaxial extension by b_1 in the direction of the normal $\mathbf{n}$, followed by a simple shear with glide plane $\mathcal{S}$, shearing direction $\mathbf{e}$ and amount of shear $b_1/(b_1+b_2)$.

Problem 2.33. (See also Problem 2.32.) Now consider the time-dependent version of Problem 2.32. Consider a planar surface $\mathcal{S}_t$ that passes through the region $\mathcal{R}_{\mathrm{R}}$ occupied by a body in the reference configuration. The surface propagates through the reference configuration with velocity $V_n\mathbf{n}_R$ where V_n is the constant propagation speed and $\mathbf{n}_R$ is the constant unit vector that is normal to $\mathcal{S}_t$ at all times. Note that since this surface propagates through the reference configuration, different particles lie on it at different times, and therefore it is not a material surface. (This is in contrast to the interface between two materials in a composite material.) Let $\mathcal{R}_{\mathrm{Rt}}^+$ denote the side into which $\mathbf{n}_{\mathrm{R}}$ points, $\mathcal{R}_{\mathrm{Rt}}^-$ the other side. Consider the piecewise homogeneous motion

$$\mathbf{y}(\mathbf{x},t) = \begin{cases} \mathbf{F}^+\mathbf{x} + \mathbf{v}^+t & \text{for } \mathbf{x} \in \mathcal{R}_{\mathrm{Rt}}^+,\ t > t_0, \\[2mm] \mathbf{F}^-\mathbf{x} + \mathbf{v}^-t & \text{for } \mathbf{x} \in \mathcal{R}_{\mathrm{Rt}}^-,\ t > t_0, \end{cases}$$

where $\mathbf{F}^{\pm}$ are constant non-singular tensors and $\mathbf{v}^{\pm}$ are constant vectors. Show that this motion is continuous across $\mathcal{S}_t$ at all times if and only if there is a constant vector $\mathbf{a}$ for which

$$\mathbf{F}^+ - \mathbf{F}^- = \mathbf{a} \otimes \mathbf{n}_{\mathrm{R}}, \tag{2.127}$$

and

$$\mathbf{v}^+ - \mathbf{v}^- = -V_n(\mathbf{F}^+ - \mathbf{F}^-)\mathbf{n}_R. \tag{2.128}$$

These *Hadamard jump (compatibility) conditions* generalize the special one in Problem 2.32. They play an important role in studying interfaces between two material phases.

Problem 2.34. This is a toy model of Problem 2.35 concerning a twinning deformation. Here we consider a 2-dimensional crystalline solid that in a reference configuration has a square lattice as shown in Figure 2.29.

A unit cell of the reference lattice is depicted by the grey square in Figure 2.30. The respective stretch tensors

$$\mathbf{U}_1 = \beta\mathbf{e}_1 \otimes \mathbf{e}_1 + \alpha\mathbf{e}_2 \otimes \mathbf{e}_2, \qquad \mathbf{U}_2 = \alpha\mathbf{e}_1 \otimes \mathbf{e}_1 + \beta\mathbf{e}_2 \otimes \mathbf{e}_2, \qquad \alpha > \beta > 0, \tag{i}$$

take the grey 1×1 unit cell into the blue $\beta \times \alpha$ rectangle and the pink $\alpha \times \beta$ rectangle as depicted in Figure 2.30. *Remark:* Note that one cannot rigidly rotate the pink rectangle into the blue rectangle in such a way that $a \mapsto a, b \mapsto b$ etc. (Imagine that "atoms" a, b, c, d sit at each vertex. One cannot rotate the pink rectangle into the blue rectangle in such a way that the positions of the atoms a, b, c and d align.) Therefore the blue and pink deformed configurations do not simply differ by a rigid rotation; they are called *variants* of each other.

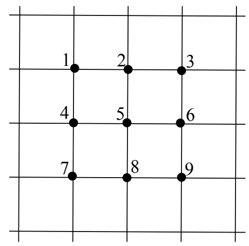


Figure 2.29: In a reference configuration the crystalline solid has a square lattice.

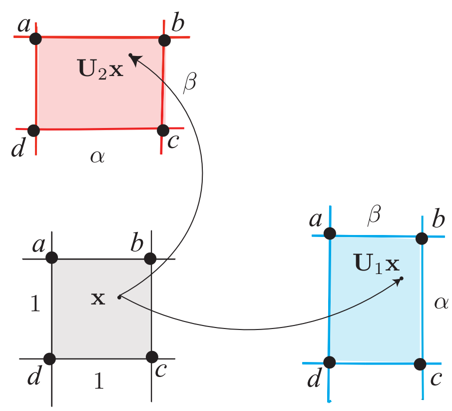


Figure 2.30: The grey square unit cell *abcd* is taken by the pure stretch $\mathbf{x} \mapsto \mathbf{U}_1\mathbf{x}$ into the blue rectangle *abcd*, while the pure stretch $\mathbf{x} \mapsto \mathbf{U}_2\mathbf{x}$ takes it into the pink rectangle *abcd*. Observe that one cannot rigidly rotate the pink rectangle into the blue rectangle in such a way that $a \mapsto a, b \mapsto b$ etc.

We are interested in the *co-existence of the two variants*, i.e. in the existence of a piecewise homogeneous, continuous, deformation that connects the two variants, specifically $\mathbf{U}_1$ to $\mathbf{QU}_2$, across a planar interface $\mathcal{S}_R$ (a straight line in two-dimensions); and if such a deformation does exist, to determine this interface and the rotation tensor $\mathbf{Q}$. Thus we consider the following piecewise homogeneous deformation, illustrated schematically in Figure 2.31:

$$\mathbf{y} = \begin{cases} \mathbf{F}_1\mathbf{x} = \mathbf{U}_1\mathbf{x} & \text{for } \mathbf{x} \in \mathcal{R}_1, \\ \\ \mathbf{F}_2\mathbf{x} = \mathbf{QU}_2\mathbf{x} & \text{for } \mathbf{x} \in \mathcal{R}_2. \end{cases} \qquad (ii)$$

(a) Show that the continuity of the piecewise homogeneous deformation (ii) requires $\mathbf{F}_2 - \mathbf{F}_1 = \mathbf{a} \otimes \mathbf{n}_R$ where the unit vector $\mathbf{n}_R$ is normal to $\mathcal{S}_R$ and $\mathbf{a}$ is an arbitrary vector; in Problem 2.32 you are asked to establish this result in three-dimensions. When $\mathbf{F}_1 = \mathbf{U}_1$ and $\mathbf{F}_2 = \mathbf{QU}_2$, the preceding requirement reads

$$\mathbf{QU}_2 - \mathbf{U}_1 = \mathbf{a} \otimes \mathbf{n}_R. \qquad (iii)$$

(b) Show by construction, i.e. by determining $\mathbf{Q}, \mathbf{a}$ and $\mathbf{n}_R$, that such a deformation does exist when $\mathbf{U}_1$ and $\mathbf{U}_2$ are given by (ii). Sketch a figure that interprets the deformation (ii).

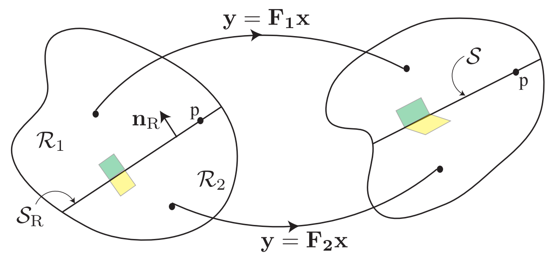


Figure 2.31: A piecewise homogeneous deformation. The deformation is continuous across $\mathcal{S}_R$ but the deformation gradient tensor is not. This is illustrated schematically by the yellow square that undergoes a simple shear and the green square that undergoes a pure stretch such that their deformed images fit continuously in the deformed configuration.

(c) Show that

$$\mathbf{Q}\mathbf{U}_2 = (\mathbf{I} + k\mathbf{b} \otimes \mathbf{n})\mathbf{U}_1, \qquad (iv)$$

for some scalar k and orthogonal unit vectors $\mathbf{b}$ and $\mathbf{n}$,

$$\mathbf{n} \cdot \mathbf{b} = 0, \qquad |\mathbf{n}| = 1, \qquad |\mathbf{b}| = 1. \qquad (v)$$

Note that the tensor $\mathbf{I} + k\mathbf{b} \otimes \mathbf{n}$ describes a simple shear. Sketch a figure that provides an alternative interpretation of the deformation (ii) in light of (iv).

Solution:

(a) Continuity of the deformation at points $\mathbf{x} \in \mathcal{S}_R$ requires

$$\mathbf{F}_2\mathbf{x} = \mathbf{F}_1\mathbf{x} \qquad \text{for all} \quad \mathbf{x} \in \mathcal{S}_\mathrm{R}.$$

Let

$$\mathbf{G} \coloneqq \mathbf{F}_2 - \mathbf{F}_1$$

so that

$$\mathbf{G}\mathbf{x} = \mathbf{o} \qquad \text{for all} \quad \mathbf{x} \in \mathcal{S}_\mathrm{R}.$$

Pick a basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ (for our two-dimensional vector space) where $\mathbf{f}_2 = \mathbf{n}_\mathrm{R}$ is normal to $\mathcal{S}_\mathrm{R}$. The unit vector $\mathbf{f}_1$ is then on the line $\mathcal{S}_\mathrm{R}$ and so the preceding equation is equivalent to

$$\mathbf{G}\mathbf{f}_1 = \mathbf{o}.$$

In terms of components in the basis $\{\mathbf{f}_1, \mathbf{f}_2\}$,

$$\mathbf{G}\mathbf{f}_1 = \mathbf{o} \quad \Rightarrow \quad G_{\xi\eta}\delta_{\eta 1} = 0 \quad \Rightarrow \quad G_{\xi 1} = 0 \quad \Rightarrow \quad G_{11} = G_{21} = 0,$$

where the subscripts ξ, η range over the values 1 and 2. Thus we can write the tensor $\mathbf{G}$ as

$$\mathbf{G} = \cancel{G_{11}}\mathbf{f}_1 \otimes \mathbf{f}_1 + G_{12}\mathbf{f}_1 \otimes \mathbf{f}_2 + \cancel{G_{21}}\mathbf{f}_2 \otimes \mathbf{f}_1 + G_{22}\mathbf{f}_2 \otimes \mathbf{f}_2 = \underbrace{(G_{12}\mathbf{f}_1 + G_{22}\mathbf{f}_2)}_{\mathbf{a}} \otimes \underbrace{\mathbf{f}_2}_{\mathbf{n}_R} = \mathbf{a} \otimes \mathbf{n}_\mathrm{R},$$

and thus

$$\mathbf{F}_2 - \mathbf{F}_1 = \mathbf{a} \otimes \mathbf{n}_{\mathrm{R}}. \qquad \square$$

(*b*) To determine $\mathbf{Q}, \mathbf{a}$ and $\mathbf{n}_R$ we express (*iii*) in terms of components in the cubic basis, i.e. the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ where the basis vectors are aligned with the edges of the square lattice:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} a_1 n_1 & a_1 n_2 \\ a_2 n_1 & a_2 n_2 \end{pmatrix}. \qquad (vi)$$

This gives

$$\left.\begin{aligned} \alpha\cos\theta - \beta &= a_1 n_1, \\ \beta\sin\theta &= a_1 n_2, \\ -\alpha\sin\theta &= a_2 n_1, \\ \beta\cos\theta - \alpha &= a_2 n_2, \end{aligned}\right\}$$

We can solve these, together with $n_1^2 + n_2^2 = 1$, to find n_1, n_2, θ, a_1 and a_2:

$$\mathbf{n}_R = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2, \qquad (vii)$$

$$\cos\theta = \frac{2\alpha\beta}{\alpha^2+\beta^2}, \qquad \sin\theta = \frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}, \qquad (viii)$$

$$\mathbf{a} = \sqrt{2}\left(\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\right)(\beta\mathbf{e}_1 - \alpha\mathbf{e}_2). \qquad (ix)$$

Observe from (*vii*) that the interface $\mathcal{S}_R$ in the reference configuration between $\mathcal{R}_1$ and $\mathcal{R}_2$ is inclined as depicted by the dashed line in Figure 2.32 (left-hand side). The lattice in the right-hand figure is the image of the reference lattice under the deformation (*ii*).

Observe that the deformation $\mathbf{x} \mapsto \mathbf{U}_1\mathbf{x}$ takes a unit cell such as 4587 in $\mathcal{R}_1$ into a blue rectangle 4587 in the deformed configuration; and likewise the deformation $\mathbf{x} \mapsto \mathbf{QU}_2\mathbf{x}$ takes a unit cell such as 2365 in $\mathcal{R}_2$ into a pink rectangle 2365 in the deformed configuration. On the other hand considering a unit cell such as 1254 that straddles the line $\mathcal{S}_R$, part of it is taken by $\mathbf{x} \mapsto \mathbf{U}_1\mathbf{x}$ into the triangle 154 and the other part is taken by $\mathbf{x} \mapsto \mathbf{QU}_2\mathbf{x}$ into the triangle 125. Details of this are shown in Figure 2.33.

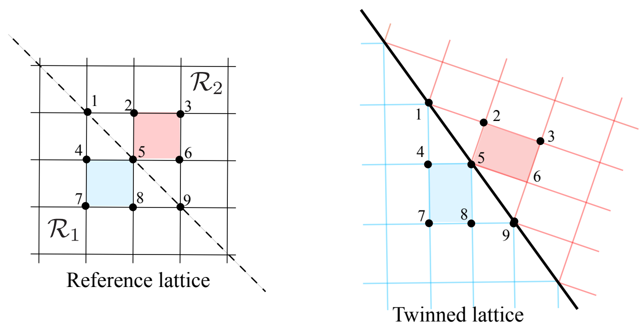


Figure 2.32: Left: Reference configuration with a square lattice. Right: Twinned configuration. The blue square 4587 has been deformed by $\mathbf{U}_1$; the pink square 2365 has been deformed by $\mathbf{U}_2$ and then rotated by $\mathbf{Q}$. The deformation is continuous.

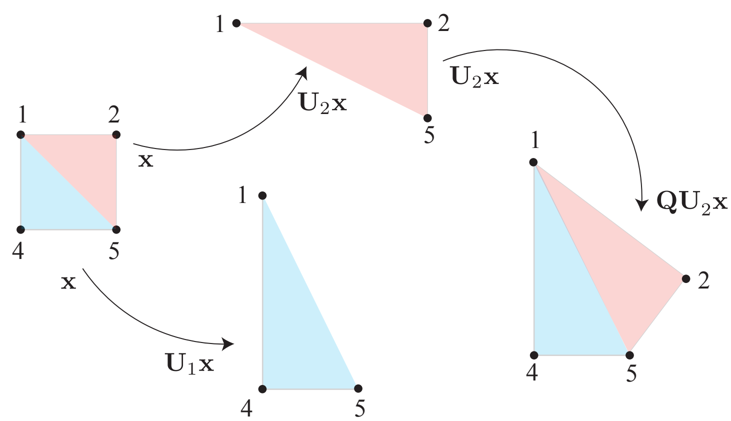


Figure 2.33: Geometric interpretation of equation (ii): The blue isosceles triangle 154 is taken by the pure stretch $\mathbf{x} \mapsto \mathbf{U}_1\mathbf{x}$ into the elongated blue triangle 154. The pink isosceles triangle is first taken by the pure stretch $\mathbf{x} \mapsto \mathbf{U}_2\mathbf{x}$ into the elongated pink triangle 125 (top) and then taken by the rigid rotation $\mathbf{U}_2\mathbf{x} \mapsto \mathbf{QU}_2\mathbf{x}$ into the rotated pink triangle (right).

Geometric interpretation of the deformation (ii)*:* This is described by Figure 2.33 as follows: the blue isosceles triangle 154 is taken by the pure stretch $\mathbf{x} \mapsto \mathbf{U}_1\mathbf{x}$ into the elongated blue triangle 154. The pink isosceles triangle is first taken by the pure stretch $\mathbf{x} \mapsto \mathbf{U}_2\mathbf{x}$ into the elongated pink triangle 125 (top) and then taken by the rigid rotation $\mathbf{U}_2\mathbf{x} \mapsto \mathbf{QU}_2\mathbf{x}$ into the rotated pink triangle (right).

We can determine the rotation tensor $\mathbf{Q}$ geometrically using Figure 2.34. As depicted there, let ϕ be the angle between a diagonal and the long side of either variant:

$$\tan\phi = \frac{\alpha}{\beta}, \qquad \sin\phi = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \qquad \cos\phi = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}. \tag{x}$$

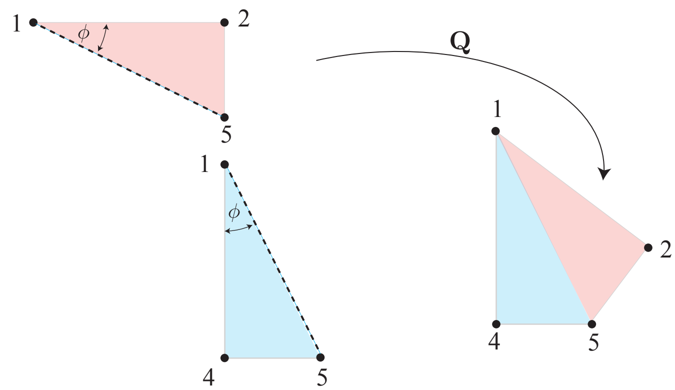


Figure 2.34: The rotation $\mathbf{Q}$: The angle between the dashed lines on the pink and blue triangles is $\pi/2-2\phi$, and so that is the angle by which the pink triangle must be rotated clockwise in order to make the dashed lines parallel.

By geometry, the angle by which the pink triangle on the left has to be rotated clockwise in order to make it similar to the pink triangle on the right is $\pi/2-2\phi$, and so the counter clockwise rotation is $\theta = -(\pi/2-2\phi)$. Thus

$$\cos\theta = \sin 2\phi = 2\sin\phi\cos\phi = \frac{2\alpha\beta}{\alpha^2+\beta^2}, \qquad \sin\theta = -\cos 2\phi = \sin^2\phi - \cos^2\phi = \frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}, \qquad (xi)$$

which coincides with $(viii)$.

(c) We write (iii) as

$$\mathbf{Q}\mathbf{U}_2 = \mathbf{U}_1 + \mathbf{a}\otimes\mathbf{n}_R = (\mathbf{I} + \mathbf{a}\otimes\mathbf{U}_1^{-1}\mathbf{n}_R)\mathbf{U}_1. \qquad (xii)$$

The tensor $\mathbf{I}+\mathbf{a}\otimes\mathbf{U}_1^{-1}\mathbf{n}_R$ represents a simple shear if (and only if) the vector $\mathbf{a}$ is perpendicular to the vector $\mathbf{U}_1^{-1}\mathbf{n}_R$. In order to investigate whether this is true or not we take the determinant of both sides of (xii) and use $\det\mathbf{Q} = 1$ and $\det(\mathbf{I}+\mathbf{a}\otimes\mathbf{b}) = 1+\mathbf{a}\cdot\mathbf{b}$. This gives

$$\det(\mathbf{Q}\mathbf{U}_2) = \det\mathbf{U}_2 = \det\left[(\mathbf{I}+\mathbf{a}\otimes\mathbf{U}_1^{-1}\mathbf{n}_R)\mathbf{U}_1\right] = (1+\mathbf{a}\cdot\mathbf{U}_1^{-1}\mathbf{n}_R)\det\mathbf{U}_1.$$

From (i) we know that $\det\mathbf{U}_1 = \det\mathbf{U}_2$ and so this yields

$$\mathbf{a}\cdot\mathbf{U}_1^{-1}\mathbf{n}_R = 0. \qquad (xiii)$$

Thus $\mathbf{U}_1^{-1}\mathbf{n}_R$ is orthogonal to $\mathbf{a}$ and so $\mathbf{I}+\mathbf{a}\otimes\mathbf{U}_1^{-1}\mathbf{n}_R$ is a simple shear. To write this in the standard form we divide $\mathbf{a}$ and $\mathbf{U}_1^{-1}\mathbf{n}_R$ by their respective magnitudes so that they can then be expressed as unit vectors. Thus we write (xii) as

$$\mathbf{Q}\mathbf{U}_2 = \left(\mathbf{I} + |\mathbf{a}|\,|\mathbf{U}_1^{-1}\mathbf{n}_R|\,\frac{\mathbf{a}}{|\mathbf{a}|}\otimes\frac{\mathbf{U}_1^{-1}\mathbf{n}_R}{|\mathbf{U}_1^{-1}\mathbf{n}_R|}\right)\mathbf{U}_1, \qquad (xiv)$$

and introduce

$$k = |\mathbf{a}|\,|\mathbf{U}_1^{-1}\mathbf{n}_R|, \qquad \mathbf{b} = \frac{\mathbf{a}}{|\mathbf{a}|}, \qquad \mathbf{n} = \frac{\mathbf{U}_1^{-1}\mathbf{n}_R}{|\mathbf{U}_1^{-1}\mathbf{n}_R|}. \qquad (xv)$$

and note that

$$\mathbf{n} \cdot \mathbf{b} = 0, \qquad |\mathbf{n}| = 1, \qquad |\mathbf{b}| = 1. \tag{xvi}$$

We can now write (xiv) as

$$\mathbf{Q}\mathbf{U}_2 = (\mathbf{I} + k\mathbf{b} \otimes \mathbf{n})\mathbf{U}_1. \tag{$xvii$}$$

Since $\mathbf{b}$ and $\mathbf{n}$ are orthogonal unit vectors it follows that $\mathbf{I} + k\mathbf{b} \otimes \mathbf{n}$ is a simple shear with glide plane normal $\mathbf{n}$, shearing direction $\mathbf{b}$ and amount of shear k.

To determine $k, \mathbf{n}$ and $\mathbf{b}$ (having previously found $\mathbf{Q}$) we simply substitute $(i), (vii), (ix)$ into (xv) to find

$$k = -\frac{\alpha^2 - \beta^2}{\alpha\beta}, \tag{$xiii$}$$

$$\mathbf{n} = \cos\phi\, \mathbf{e}_1 + \sin\phi\, \mathbf{e}_2, \qquad \mathbf{b} = -\sin\phi\, \mathbf{e}_1 + \cos\phi\, \mathbf{e}_2, \tag{xv}$$

where the angle ϕ is given by (x).

Alternative geometric interpretation of the deformation (ii)*:* According to the representation (ii) of the deformation, we deform the part $\mathcal{R}_1$ of the body by $\mathbf{U}_1$ while the part $\mathcal{R}_2$ is first deformed by $\mathbf{U}_2$ and then rotated by $\mathbf{Q}$. On the other hand by using (iv) we can represent this same deformation as

$$\mathbf{y} = \begin{cases} \mathbf{U}_1\mathbf{x} & \text{for } \mathbf{x} \in \mathcal{R}_1, \\ (\mathbf{I} + k\mathbf{b} \otimes \mathbf{n})\mathbf{U}_1\mathbf{x} & \text{for } \mathbf{x} \in \mathcal{R}_2. \end{cases} \tag{$\star$}$$

Thus an alternative way in which to view this same deformation is to deform the entire body $\mathcal{R}_1 \cup \mathcal{R}_2$ by $\mathbf{U}_1$ and to then subject the part $\mathcal{R}_2$ only to the simple shear $\mathbf{I} + k\mathbf{b} \otimes \mathbf{n}$. This is illustrated in Figure 2.35

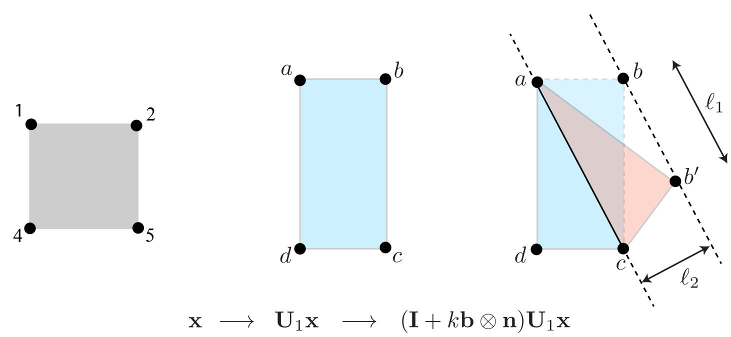


Figure 2.35: Geometric interpretation of equation $\star$. The deformation $\mathbf{x} \mapsto \mathbf{U}_1\mathbf{x}$ takes the grey square unit cell 1254 into the rectangle $abcd$. The triangle abc is now subjected to a simple shear as shown in the rightmost figure that takes abc into $ab'c$. The latter deformation is described by $\mathbf{z} \mapsto (\mathbf{I} + k\mathbf{b} \otimes \mathbf{n})\mathbf{z}$ where $\mathbf{z}$ is the position vector of a point in abc in the middle configuration.

Remark: One can calculate the various parameters involved in the deformation in Figure 2.35 geometrically. The vector $\mathbf{n}$ is perpendicular to the solid dark line in Figure 2.32 from which $(xv)_1$ follows. The vector $\mathbf{b}$

is perpendicular to $\mathbf{n}$. One can calculate geometrically from Figure 2.35 that the distance ℓ_1 that b moves during the shear, when $b \mapsto b'$, is

$$\ell_1 = |bb'| = \sqrt{\alpha^2+\beta^2} - 2\beta\sin\phi = \frac{\alpha^2-\beta^2}{\sqrt{\alpha^2+\beta^2}};$$

and the distance ℓ_2 from the line bb' on which b slides to the line ac is

$$\ell_2 = \beta\cos\phi = \frac{\alpha\beta}{\sqrt{\alpha^2+\beta^2}}.$$

Therefore the amount of shear is

$$\frac{\ell_1}{\ell_2} = \frac{\alpha^2-\beta^2}{\alpha\beta}.$$

which recovers $(xviii)$ except for the sign. (Question: why the sign difference?)

Problem 2.35. (See also Problems 2.34.) This problem arises when studying the microstructure of a certain two-phase material. In one phase, the crystallographic lattice underlying the material is cubic and this is called the austenite phase. In the other, the lattice is tetragonal and this phase is called the martensite phase. There are three variants of the martensite phase. Suppose that the reference configuration $\mathbf{F} = \mathbf{I}$ corresponds to the austenite phase. The three stretch tensors $\mathbf{U}_k$ given below in $(i), (ii)$ describe the deformation from the austenite phase into the three martensite variants.

The answer to question (a) below is yes and therefore an interface, oriented in a specific way, separating one martensite variant from another can exist. The answer to question (b) is no and therefore an interface separating one martensite variant from austenite cannot exist. *Reference:* K. Bhattacharya, *Microstructure of Martensite*, Oxford, 2003.

Let $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ be an orthonormal basis. Consider the 3 symmetric positive definite tensors $\mathbf{U}_1, \mathbf{U}_2$ and $\mathbf{U}_3$ defined by

$$\mathbf{U}_k = \alpha\mathbf{I} + (\beta-\alpha)\mathbf{r}_k \otimes \mathbf{r}_k, \qquad \alpha \neq 1,\ \beta \neq 1,\ \alpha \neq \beta,\ \alpha > 0,\ \beta > 0, \qquad k = 1,2,3. \qquad (i)$$

Here α and β are constant (lattice) parameters. The components of these three tensors in the basis $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ are

$$[U_1] = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \qquad [U_2] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \qquad [U_3] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}. \qquad (ii)$$

Therefore the deformation $\mathbf{y} = \mathbf{U}_1\mathbf{x}$ takes a $1 \times 1 \times 1$ cube and stretches it in the $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ directions by stretches β, α, α and maps the cube into a $\beta \times \alpha \times \alpha$ tetragon. The deformations $\mathbf{y} = \mathbf{U}_2\mathbf{x}$ and $\mathbf{y} = \mathbf{U}_3\mathbf{x}$ are similar with the stretch by β being in the $\mathbf{r}_2$ and $\mathbf{r}_3$ directions respectively. Consider the Hadamard compatibility condition (2.127):

$$\mathbf{F}^+ - \mathbf{F}^- = \mathbf{a} \otimes \mathbf{n}_{\mathrm{R}}. \qquad (iii)$$

(a) Here you want to study the possibility of a piecewise homogeneous (two-phase) deformation that involves two variants of martensite, one on each side of the interface. Accordingly take $\mathbf{F}^+ = \mathbf{R}^+\mathbf{U}_2$, $\mathbf{F}^- = \mathbf{U}_1$ in (iii). Do there exist a proper orthogonal tensor $\mathbf{R}^+$, a unit vector $\mathbf{n}_{\mathrm{R}}$ and a vector $\mathbf{a}$ such that (iii) holds? If yes, find $\mathbf{R}^+, \mathbf{n}_{\mathrm{R}}$ and $\mathbf{a}$.

(b) Here you want to study the possibility of a two-phase deformation that involves austenite on one side of the interface and one variant of martensite on the other. Accordingly take $\mathbf{F}^+ = \mathbf{R}^+\mathbf{U}_2,\ \mathbf{F}^- = \mathbf{I}$. Do there exist a proper orthogonal tensor $\mathbf{R}^+$, a unit vector $\mathbf{n}_\mathrm{R}$ and a vector $\mathbf{a}$ such that (*iii*) holds? If yes, find $\mathbf{R}^+, \mathbf{n}_\mathrm{R}$ and $\mathbf{a}$.

Problem 2.36. *(Average deformation gradient tensor.)* Define the average value of the deformation gradient tensor field in a body to be

$$\overline{\mathbf{F}} := \frac{1}{\mathsf{vol}} \int_{\mathcal{R}_\mathrm{R}} \mathbf{F}(\mathbf{x})\, dV_x,$$

where vol is the volume of the region $\mathcal{R}_\mathrm{R}$. Show that

$$\overline{\mathbf{F}} = \frac{1}{\mathsf{vol}} \int_{\partial\mathcal{R}_\mathrm{R}} \mathbf{y}(\mathbf{x}) \otimes \mathbf{n}_\mathrm{R}\, dA_x$$

and therefore that the average value of the deformation gradient tensor field in a body depends only on the deformation of the boundary $\partial\mathcal{R}_\mathrm{R}$. Show this

- first in the case where $\mathbf{F}(\mathbf{x})$ is continuous on $\mathcal{R}_\mathrm{R}$, and
- second in the case where $\mathbf{F}(\mathbf{x})$ is piecewise continuous on $\mathcal{R}_\mathrm{R}$. Specifically, suppose there is a surface $\mathcal{S}_\mathrm{R} \subset \mathcal{R}_\mathrm{R}$ with $\mathbf{F}(\mathbf{x})$ being continuous on either side of $\mathcal{S}_\mathrm{R}$ but discontinuous at $\mathcal{S}_\mathrm{R}$ but with the deformation $\mathbf{y}(\mathbf{x})$ being continuous on $\mathcal{R}_\mathrm{R}$ including on $\mathcal{S}_\mathrm{R}$.

Problem 2.37. *Decomposition of an arbitrary isochoric planar deformation gradient tensor:* Show that any planar isochoric deformation deformation gradient tensor $\mathbf{F}$ is equivalent to a suitable simple shear followed by a rotation, i.e. show that one can express such a tensor $\mathbf{F}$ as

$$\mathbf{F} = \overline{\mathbf{Q}}\,\mathbf{K},$$

where $\overline{\mathbf{Q}}$ is proper orthogonal and $\mathbf{K} = \mathbf{I} + k\mathbf{a} \otimes \mathbf{b}$ for some scalar k and mutually orthogonal unit vectors $\mathbf{a}$ and $\mathbf{b}$. Note: if the deformation is planar in the plane spanned by $\mathbf{r}_1$ and $\mathbf{r}_2$ then $\mathbf{a}$ and $\mathbf{b}$ are in that same plane and the rotation $\overline{\mathbf{Q}}$ is about $\mathbf{r}_3$.

Solution: Let $\mathbf{U}$ be the Lagrangian stretch tensor associated with an arbitrary planar isochoric deformation gradient tensor $\mathbf{F} = \mathbf{R}\mathbf{U}$. It can be expressed in spectral form as

$$\mathbf{U} = \lambda\, \mathbf{r}_1 \otimes \mathbf{r}_1 + \lambda^{-1}\mathbf{r}_2 \otimes \mathbf{r}_2, \qquad \lambda > 1, \tag{ii}$$

where without loss of generality we have taken $\lambda > 1$ and it is sufficient to work in two-dimensions. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be a second basis and let θ be the angle between $\mathbf{r}_1$ and $\mathbf{e}_1$:

$$\mathbf{r}_1 = \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2, \qquad \mathbf{r}_2 = -\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2. \tag{iii}$$

Motivated by the results of Problem 2.5.2 we take θ to be the angle defined by

$$\tan\theta = \lambda, \qquad \theta \in (\pi/4, \pi/2). \tag{iv}$$

Then

$$\sin\theta = \frac{\lambda}{\sqrt{1+\lambda^2}}, \qquad \cos\theta = \frac{1}{\sqrt{1+\lambda^2}}, \qquad \tan 2\theta = -\frac{2}{\lambda - \lambda^{-1}}. \tag{v}$$

Substituting (iii) and (v) into (ii) allows us to write $\mathbf{U}$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ as

$$\mathbf{U} = \frac{2\lambda}{\lambda^2+1}\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{\lambda^2-1}{\lambda^2+1}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \frac{\lambda(\lambda^2+\lambda^{-2})}{\lambda^2+1}\mathbf{e}_2 \otimes \mathbf{e}_2. \tag{vi}$$

Let $\mathbf{Q}$ be an orthogonal tensor:

$$\mathbf{Q} = \cos\psi\, \mathbf{e}_1 \otimes \mathbf{e}_1 - \sin\psi\, \mathbf{e}_1 \otimes \mathbf{e}_2 + \sin\psi\, \mathbf{e}_2 \otimes \mathbf{e}_1 + \cos\psi\, \mathbf{e}_2 \otimes \mathbf{e}_2, \tag{vii}$$

where the angle of rotation $\psi \in (0, \pi/2)$ is to be determined suitably. If we calculate $\mathbf{Q}^T\mathbf{U}$ and set the coefficient of $\mathbf{e}_2 \otimes \mathbf{e}_1$ equal to zero we get

$$\frac{\lambda^2-1}{\lambda^2+1}\cos\psi - \frac{2\lambda}{\lambda^2+1}\sin\psi = 0 \qquad \Rightarrow \qquad \tan\psi = \frac{\lambda - \lambda^{-1}}{2}, \tag{viii}$$

from which we get

$$\cos\psi = \frac{2}{\lambda + \lambda^{-1}}, \qquad \sin\psi = \frac{\lambda - \lambda^{-1}}{\lambda + \lambda^{-1}}. \tag{ix}$$

Taking this choice for ψ and calculating $\mathbf{Q}^T\mathbf{U}$ from (vi) and (vii) leads to

$$\mathbf{Q}^T\mathbf{U} = \mathbf{I} + k\mathbf{e}_1 \otimes \mathbf{e}_2 \qquad \text{where} \quad k := \lambda - \lambda^{-1}.$$

Thus letting $\mathbf{K}$ denote the simple shear

$$\mathbf{K} := \mathbf{I} + k\mathbf{e}_1 \otimes \mathbf{e}_2, \qquad k := \lambda - \lambda^{-1},$$

we have $\mathbf{Q}^T\mathbf{U} = \mathbf{K}$ and therefore

$$\mathbf{U} = \mathbf{QK} \qquad \Rightarrow \qquad \mathbf{F} = \mathbf{RU} = \mathbf{RQ}\,\mathbf{K} \qquad \square$$

where $\overline{\mathbf{Q}} = \mathbf{RQ}$ is proper orthogonal.

Problem 2.38. *Decomposition of an arbitrary isochoric deformation gradient tensor:* Show that an arbitrary isochoric homogeneous deformation can be viewed as a uniaxial extension with accompanying lateral contraction, a simple shear in the plane normal to the direction of extension, and a rigid rotation.

Solution: The deformation gradient tensor $\mathbf{F}$ can be written by the polar decomposition theorem as

$$\mathbf{F} = \mathbf{RU} = \mathbf{R}(\lambda_1\mathbf{r}_1 \otimes \mathbf{r}_1 + \lambda_2\mathbf{r}_2 \otimes \mathbf{r}_2 + \lambda_3\mathbf{r}_3 \otimes \mathbf{r}_3). \tag{i}$$

Since the deformation is isochoric,

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{ii}$$

The stretch tensor $\mathbf{U}$ can therefore be written as

$$\begin{aligned}
\mathbf{U} &= \lambda_1 \mathbf{r}_1 \otimes \mathbf{r}_1 + \lambda_2 \mathbf{r}_2 \otimes \mathbf{r}_2 + \lambda_3 \mathbf{r}_3 \otimes \mathbf{r}_3 = \\
&= \lambda_1 \lambda_3^{1/2} \lambda_3^{-1/2} \mathbf{r}_1 \otimes \mathbf{r}_1 + \lambda_2 \lambda_3^{1/2} \lambda_3^{-1/2} \mathbf{r}_2 \otimes \mathbf{r}_2 + \lambda_3 \mathbf{r}_3 \otimes \mathbf{r}_3 = \\
&\overset{(\star)}{=} \Lambda \lambda_3^{-1/2} \mathbf{r}_1 \otimes \mathbf{r}_1 + \Lambda^{-1} \lambda_3^{-1/2} \mathbf{r}_2 \otimes \mathbf{r}_2 + \lambda_3 \mathbf{r}_3 \otimes \mathbf{r}_3 = \\
&= (\Lambda \mathbf{r}_1 \otimes \mathbf{r}_1 + \Lambda^{-1} \mathbf{r}_2 \otimes \mathbf{r}_2 + \mathbf{r}_3 \otimes \mathbf{r}_3)(\lambda_3^{-1/2} \mathbf{r}_1 \otimes \mathbf{r}_1 + \lambda_3^{-1/2} \mathbf{r}_2 \otimes \mathbf{r}_2 + \lambda_3 \mathbf{r}_3 \otimes \mathbf{r}_3) = \\
&\overset{(\star\star)}{=} (\Lambda \mathbf{r}_1 \otimes \mathbf{r}_1 + \Lambda^{-1} \mathbf{r}_2 \otimes \mathbf{r}_2 + \mathbf{r}_3 \otimes \mathbf{r}_3)\, \mathbf{E} = \\
&= \mathbf{QKE}, \qquad \square
\end{aligned}$$

in step $(\star)$ we set

$$\Lambda = \lambda_1 \lambda_3^{1/2}, \qquad \Lambda^{-1} = \lambda_2 \lambda_3^{1/2},$$

in step $(\star\star)$ we set

$$\mathbf{E} := \lambda_3^{-1/2} \mathbf{r}_1 \otimes \mathbf{r}_1 + \lambda_3^{-1/2} \mathbf{r}_2 \otimes \mathbf{r}_2 + \lambda_3 \mathbf{r}_3 \otimes \mathbf{r}_3,$$

and in the last step we used the result of Problem 2.37 with $\mathbf{Q}$ being a rotation and $\mathbf{K}$ a simple shear in the plane spanned by $\mathbf{r}_1, \mathbf{r}_2$.

Observe that the tensor $\mathbf{E}$ describes an isochoric uniaxial extension in the direction $\mathbf{r}_3$ with accompanied lateral contraction.

Problem 2.39. (See also Problem 2.5.2 .) Calculate the principal stretches associated with the simple shear

$$y_1 = x_1 + k x_2, \qquad y_2 = x_2, \qquad y_3 = x_3, \qquad k > 0. \tag{i}$$

In Problem 2.5.2 we used a direct (but tedious) way by calculating $\mathbf{F}$, then $\mathbf{C} = \mathbf{F}^T \mathbf{F}$, determining its eigenvalues, and taking their square roots. Instead, carry out your calculations by getting two different expressions for the first invariant $I_1(\mathbf{C})$ and equating them, keeping in mind that the deformation is planar and isochoric.

Solution: Differentiating (i) and using $F_{ij} = \partial y_i / \partial x_j$ gives

$$\mathbf{F} = \mathbf{I} + k \mathbf{e}_1 \otimes \mathbf{e}_2. \tag{ii}$$

The deformation (i) is planar in the x_1, x_2-plane and so one principal stretch is unity with corresponding principal direction $\mathbf{e}_3$:

$$\lambda_3 = 1. \tag{iii}$$

The deformation (i) is isochoric since $\det \mathbf{F} = 1$ and therefore

$$\lambda_1 \lambda_2 \lambda_3 = 1 \qquad \overset{(iii)}{\Rightarrow} \qquad \lambda_2 = \lambda_1^{-1}. \tag{iv}$$

Next, the right Cauchy-Green deformation tensor is

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{I} + k^2 \mathbf{e}_2 \otimes \mathbf{e}_2 + k(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1),$$

and so its first invariant is

$$I_1 = \operatorname{tr}\mathbf{C} = 3 + k^2. \tag{v}$$

However the first invariant can be written in terms of the principal stretches as

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \overset{(iii),(iv)}{=} \lambda_1^2 + \lambda_1^{-2} + 1. \tag{vi}$$

Equating (v) and (vi) gives

$$\lambda_1^2 + \lambda_1^{-2} = 2 + k^2 \qquad \Rightarrow \qquad (\lambda_1 - \lambda_1^{-1})^2 = k^2.$$

We take the positive square root of this equation (letting $\lambda_1 \geq \lambda_2$ whence $\lambda_1 - \lambda_1^{-1} > 0$):

$$\lambda_1 - \lambda_1^{-1} = k \qquad \Rightarrow \qquad \lambda_1^2 - k\lambda_1 - 1 = 0 \qquad \Rightarrow \qquad \lambda_1 = \frac{k + \sqrt{k^2+4}}{2}, \qquad \square$$

where we took the positive sign of the square root since the negative sign gives a negative value for λ_1.

Problem 2.40. In Problem 2.5.2 we determined the rotation $\mathbf{R}$ and Lagrangian stretch tensor $\mathbf{U}$ associated with a simple shear $\mathbf{y} = \mathbf{F}\mathbf{x}$, $\mathbf{F} = \mathbf{I} + k\mathbf{e}_1 \otimes \mathbf{e}_2$, and then graphically interpreted that deformation viewed as $\mathbf{y} = \mathbf{R}(\mathbf{U}\mathbf{x})$. Carry out a corresponding graphical interpretation of a simple shear deformation represented as $\mathbf{y} = \mathbf{V}(\mathbf{R}\mathbf{x})$.

Solution: According to $\mathbf{y} = \mathbf{V}(\mathbf{R}\mathbf{x})$ we first rigidly rotate the square using the mapping $\mathbf{x} \to \mathbf{R}\mathbf{x}$ where the rotation tensor $\mathbf{R}$ is the same as that in the decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$. We then stretch the square by the amounts λ_1 and λ_2 in the directions $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ to get the region occupied by the deformed body. This is depicted in Figure 2.36. Since we have already found $\mathbf{R}$ and the λ_i's, it remains to find $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$, the third principal direction being $\boldsymbol{\ell}_3 = \mathbf{r}_3 = \mathbf{e}_3$.

The principal directions $\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3$ of the left stretch tensor $\mathbf{V}$ are related to the principal directions $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ of the right stretch tensor $\mathbf{U}$ by $\boldsymbol{\ell}_i = \mathbf{R}\mathbf{r}_i$. Since we found $\mathbf{R}$ and $\mathbf{r}_i$ in Problem 2.5.2, the principal directions $\boldsymbol{\ell}_i$ can be readily shown to be

$$\boldsymbol{\ell}_1 = \mathbf{R}\mathbf{r}_1 = \ \cos\theta_\ell \ \mathbf{e}_1 + \sin\theta_\ell \ \mathbf{e}_2, \qquad \boldsymbol{\ell}_2 = \mathbf{R}\mathbf{r}_2 = \ -\sin\theta_\ell \ \mathbf{e}_1 + \cos\theta_\ell \ \mathbf{e}_2, \qquad \boldsymbol{\ell}_3 = \mathbf{e}_3, \tag{i}$$

where

$$\tan 2\theta_\ell = 2/k. \tag{ii}$$

(Note that $\theta_r = \pi/2 - \theta_\ell$ where θ_r is the corresponding angle in the right polar decomposition; see Problem 2.5.2.) The corresponding principal stretches are

$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda^{-1}, \quad \lambda_3 = 1 \qquad \text{where} \quad \lambda = \frac{1}{2}\left[\sqrt{k^2+4} + k\right], \tag{iii}$$

which are the common eigenvalues of $\mathbf{V}$ and $\mathbf{U}$.

Remark: The tensor $\mathbf{V}$ can be readily expressed with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by using $\mathbf{V} = \mathbf{F}\mathbf{R}^T$ together with the results of Problem 2.5.2. This leads to

$$\mathbf{V} = \frac{1}{\sqrt{4+k^2}}\Big((2+k^2)\mathbf{e}_1 \otimes \mathbf{e}_1 + k(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + 2\,\mathbf{e}_2 \otimes \mathbf{e}_2\Big) + \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{ii}$$

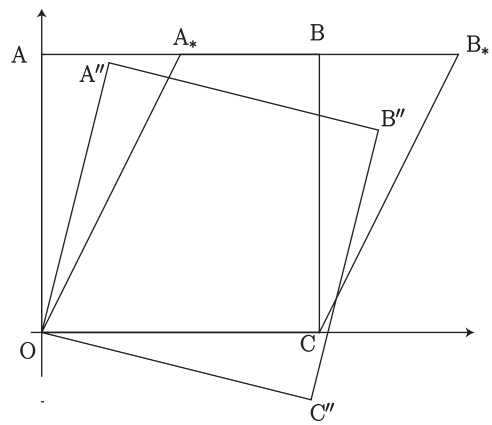


Figure 2.36: Problem 2.40: Simple shear deformation $\mathbf{y} = (\mathbf{I} + k\mathbf{e}_1 \otimes \mathbf{e}_2)\mathbf{x} = \mathbf{V}(\mathbf{R}\mathbf{x})$ viewed in two steps: The rotation $\mathbf{x} \to \mathbf{R}\mathbf{x}$ takes the region $OABC \to OA''B''C''$ and the pure stretch $\mathbf{R}\mathbf{x} \to \mathbf{V}(\mathbf{R}\mathbf{x})$ takes $OA''B''C'' \to OA_*B_*C$.

Problem 2.41. Two material fibers AB and AC in the reference configuration have equal length s_0 and are oriented in the respective directions $\mathbf{e}_1$ and $\mathbf{e}_2$. A homogeneous deformation

$$\mathbf{y} = \mathbf{F}\mathbf{x}, \qquad \mathbf{F} = \text{ constant},$$

maps these fibers into $A'B'$ and $A'C'$ that have lengths s_1, s_2 with the angle between them being $\pi/2 - \phi$. The quantities s_0, s_1, s_2 and ϕ have been measured.

Calculate the strain components E_{11}, E_{22} and E_{12} in terms of s_0, s_1, s_2 and ϕ where $\mathbf{E}$ is the Green Saint-Venant strain tensor. Linearize your answer to the case of an infinitesimal deformation.

Solution:

Set $ds_x = s_0$ and $ds_y = s_1$ in (2.74) to get

$$\frac{s_1 - s_0}{s_0} = \sqrt{1 + 2E_{11}} \qquad \Rightarrow \qquad E_{11} = \frac{1}{2}\left[\left(\frac{s_1}{s_0}\right)^2 - 1\right]. \tag{i}$$

Similarly setting $ds_x = s_0, ds_y = s_2$ in the analogous formula involving E_{22} yields

$$E_{22} = \frac{1}{2}\left[\left(\frac{s_2}{s_0}\right)^2 - 1\right]. \tag{ii}$$

Next, set $\theta_y = \pi/2 - \phi$ in (2.76) and use the values of E_{11} and E_{22} from above to get

$$\sin\phi = \frac{2E_{12}}{\sqrt{1 + 2E_{11}}\sqrt{1 + 2E_{22}}}$$

whence

$$E_{12} = \frac{1}{2}\,\frac{s_1}{s_0}\,\frac{s_2}{s_0}\,\sin\phi. \tag{iii}$$

For an infinitesimal deformation, we set $s_1 = s_0 + \Delta s_1, \Delta s_1 \ll s_0$, substitute this into (i) and approximate the result for small $\Delta s_1/s_0$. This leads to

$$E_{11} = \frac{s_1 - s_0}{s_0}$$

with the error being quadratic. Similarly one finds

$$E_{22} = \frac{s_2 - s_0}{s_0}$$

to linear accuracy. Substituting $s_1 = s_0 + \Delta s_1$ and $s_2 = s_0 + \Delta s_2$ into (iii) and approximating the result for small $\phi, \Delta s_1/s_0$ and $\Delta s_2/s_0$ leads to

$$E_{12} = \frac{1}{2}\phi$$

where the error is quadratic.

Problem 2.42. In Problem 2.5.2 we calculated the stretch tensor $\mathbf{U}$ and rotation tensor $\mathbf{R}$ associated with a simple shear. Linearize those results for small amounts of shear k and thus derive the specializations of $(2.112)_1$ and (2.120) to simple shear. Calculate also the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ and compare and contrast your result with the expression you derived for the Green Saint-Venant strain tensor in Section 2.6.1.

Problem 2.43. This problem involves a planar deformation and for convenience we shall display only the in-plane equations. As shown in Figure 2.37, the body occupies a rectangular strip of width W and height H in a reference configuration. Coordinate axes are chosen such that

$$\mathcal{R}_{\rm R} = \{(x_1, x_2) : \ 0 \le x_1 \le W, \ 0 \le x_2 \le H\}.$$

The deformation takes the point $(x_1, x_2) \to (y_1, y_2)$ and the region $\mathcal{R}_{\rm R} \to \mathcal{R}$. Let (r, θ) be the polar coordinates in the deformed configuration,

$$y_1 = r\cos\theta, \ \ y_2 = r\sin\theta, \qquad (i)$$

with associated basis vectors

$$\mathbf{e}_r(\theta) = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2, \qquad \mathbf{e}_\theta(\theta) = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2. \qquad (ii)$$

The deformation can be characterized by

$$r = r(x_1, x_2), \quad \theta = \theta(x_1, x_2), \qquad (x_1, x_2) \in \mathcal{R}_{\rm R}. \qquad (iii)$$

(a) Though the figure shows the region in the deformed configuration to be a circular annulus, in this part of the problem do *not* assume the deformation (iii) to possess any form of symmetry. (You would have to consider such non-symmetric deformations if, for example, your goal was to study the stability of the cylindrically symmetric one.) Calculate the deformation gradient tensor $\mathbf{F}$ and the left Cauchy-Green tensor

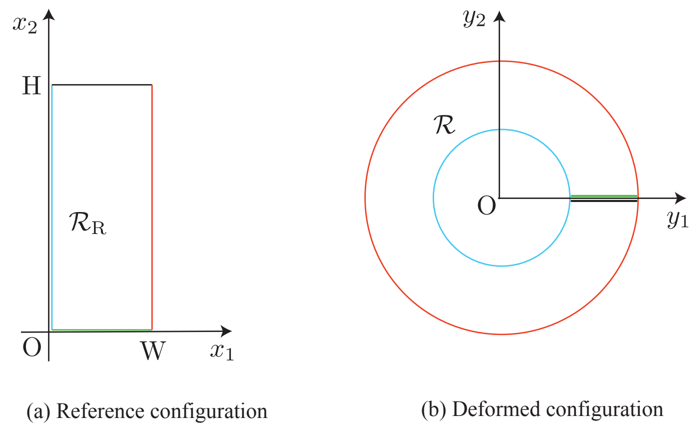


Figure 2.37: The body occupying a rectangular strip in the reference configuration, is rolled up into its deformed configuration. The deformation takes $(x_1, x_2) \to (y_1, y_2)$ and $\mathcal{R}_{\rm R} \to \mathcal{R}$. Despite the figure on the right, in part (a) of this problem do not assume the region $\mathcal{R}$ to be a circular annulus. Figure for Problem 2.43.

$\mathbf{B}$ using the bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ in the reference and deformed configurations respectively. Derive a condition on $r(x_1, x_2), \theta(x_1, x_2)$ and their partial derivatives if the material is incompressible.

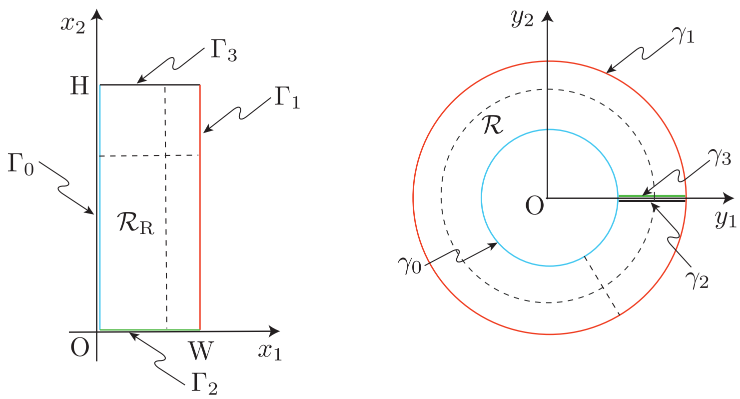
(b) Now consider the special case where the deformation carries each vertical line $x_1 =$ constant in $\mathcal{R}_{\rm R}$ into a circle $r =$ constant in $\mathcal{R}$, and each horizontal line $x_2 =$ constant in $\mathcal{R}_{\rm R}$ into a radial line $\theta =$ constant in $\mathcal{R}$. This is illustrated by the dashed curves in Figure 2.38. The left- and right-hand boundaries $x_1 = 0$ and $x_1 = W$ map into circles of radii r_0 and r_1 respectively. The bottom edge of the strip $x_2 = 0$ and the top edge of the strip $x_2 = H$ map into the respective radial lines $\theta = 0$ and $\theta = 2\pi$. What form do the functions $r(x_1, x_2)$ and $\theta(x_1, x_2)$ have in this case? Specialize your expressions for $\mathbf{F}$ and $\mathbf{B}$ from part (a) to this case. Moreover, use the incompressibility condition to find $r(x_1, x_2)$ and $\theta(x_1, x_2)$. What are the principal stretches λ_r and λ_θ?

(c) Finally, suppose that the boundary Γ_0 (see Figure 2.38) is not stretched by this deformation. Specialize the deformation and the principal stretches from part (b) to this case.

Solution:

(a) To calculate the deformation gradient tensor $\mathbf{F}$ we use

$$\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2, \qquad \mathbf{y} = r(x_1, x_2)\, \mathbf{e}_r(\theta(x_1, x_2)),$$



(a) Reference configuration (b) Deformed configuration

Figure 2.38: The body, occupying a rectangular strip in the reference configuration, is rolled up into a circular annulus in its deformed configuration. As shown by the dashed curves, every vertical line maps into a circle and every horizontal line maps into a radial line. The boundaries $\Gamma_i \to \gamma_i, i = 0, 1, 2, 3$.

and (ii) and proceed as follows using the chain rule:

$$
\begin{aligned}
d\mathbf{y} &= \frac{\partial \mathbf{y}}{\partial x_1} dx_1 + \frac{\partial \mathbf{y}}{\partial x_2} dx_2 = \\
&= \frac{\partial}{\partial x_1}\big[r(x_1,x_2)\,\mathbf{e}_r(\theta(x_1,x_2))\big]dx_1 + \frac{\partial}{\partial x_2}\big[r(x_1,x_2)\,\mathbf{e}_r(\theta(x_1,x_2))\big]dx_2 = \\
&= \frac{\partial r}{\partial x_1} dx_1\,\mathbf{e}_r + r\frac{\partial \theta}{\partial x_1} dx_1\,\mathbf{e}_\theta + \frac{\partial r}{\partial x_2} dx_2\,\mathbf{e}_r + r\frac{\partial \theta}{\partial x_2} dx_2\,\mathbf{e}_r,
\end{aligned}
$$

where we have used the fact that $\partial \mathbf{e}_r/\partial\theta = \mathbf{e}_\theta$. From $d\mathbf{x} = dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2$ we have

$$
dx_1 = d\mathbf{x}\cdot\mathbf{e}_1, \qquad dx_2 = d\mathbf{x}\cdot\mathbf{e}_2.
$$

Substituting this into the preceding equation gives

$$
\begin{aligned}
d\mathbf{y} &= \frac{\partial r}{\partial x_1}(d\mathbf{x}\cdot\mathbf{e}_1)\,\mathbf{e}_r + r\frac{\partial \theta}{\partial x_1}(d\mathbf{x}\cdot\mathbf{e}_1)\,\mathbf{e}_\theta + \frac{\partial r}{\partial x_2}(d\mathbf{x}\cdot\mathbf{e}_2)\,\mathbf{e}_r + r\frac{\partial \theta}{\partial x_2}(d\mathbf{x}\cdot\mathbf{e}_2)\,\mathbf{e}_r = \\
&= \frac{\partial r}{\partial x_1}(\mathbf{e}_r\otimes\mathbf{e}_1)\,d\mathbf{x} + r\frac{\partial \theta}{\partial x_1}(\mathbf{e}_\theta\otimes\mathbf{e}_1)\,d\mathbf{x} + \frac{\partial r}{\partial x_2}(\mathbf{e}_r\otimes\mathbf{e}_2)\,d\mathbf{x} + r\frac{\partial \theta}{\partial x_2}(\mathbf{e}_r\otimes\mathbf{e}_2)\,d\mathbf{x} \\
&= \mathbf{F}\,d\mathbf{x}
\end{aligned}
$$

where

$$
\mathbf{F} = \frac{\partial r}{\partial x_1}(\mathbf{e}_r\otimes\mathbf{e}_1) + r\frac{\partial \theta}{\partial x_1}(\mathbf{e}_\theta\otimes\mathbf{e}_1) + \frac{\partial r}{\partial x_2}(\mathbf{e}_r\otimes\mathbf{e}_2) + r\frac{\partial \theta}{\partial x_2}(\mathbf{e}_r\otimes\mathbf{e}_2). \qquad (iv)
$$

The associated left Cauchy-Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is

$$
\mathbf{B} = \mathbf{F}\mathbf{F}^T = B_{rr}\mathbf{e}_r\otimes\mathbf{e}_r + B_{r\theta}(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r) + B_{\theta\theta}\mathbf{e}_\theta\otimes\mathbf{e}_\theta, \qquad (v)
$$

where

$$B_{rr} = \left(\frac{\partial r}{\partial x_1}\right)^2 + \left(\frac{\partial r}{\partial x_2}\right)^2, \qquad B_{\theta\theta} = r^2\left(\frac{\partial \theta}{\partial x_1}\right)^2 + r^2\left(\frac{\partial \theta}{\partial x_2}\right)^2,$$
$$B_{r\theta} = B_{\theta r} = r\frac{\partial r}{\partial x_1}\frac{\partial \theta}{\partial x_1} + r\frac{\partial r}{\partial x_2}\frac{\partial \theta}{\partial x_2}, \tag{vi}$$

If the material is incompressible then the deformation has to be locally volume preserving and so $\det \mathbf{F} =$

$$r\left(\frac{\partial r}{\partial x_1}\frac{\partial \theta}{\partial x_2} - \frac{\partial r}{\partial x_2}\frac{\partial \theta}{\partial x_1}\right) = 1. \tag{vii}$$

(b) Since each vertical line $x_1 =$ constant maps into a circle of radius $r =$ constant it follows that $r(x_1, x_2)$ has to be independent of x_2. Furthermore, since the left- and right-hand boundaries $x_1 = 0$ and $x_1 = W$ map into circles of radius r_0 and r_1 respectively, one must have $r(0) = r_0, r(W) = r_1$. Similarly since each horizontal line $x_2 =$ constant maps into a radial line $\theta =$ constant it follows that $\theta(x_1, x_2)$ has to be independent of x_1. Since the bottom edge of the strip $x_2 = 0$ maps into the radial line $\theta = 0$ and the top edge of the strip $x_2 = H$ maps into the radial line $\theta = 2\pi$ we must have $\theta(0) = 0, \theta(H) = 2\pi$. Thus the deformation (iii) specializes to

$$r = r(x_1), \qquad \theta = \theta(x_2). \tag{viii}$$

with

$$r(0) = r_0, \quad r(W) = r_1, \qquad \theta(0) = 0, \quad \theta(H) = 2\pi. \tag{ix}$$

In this case the preceding results $(iv), (v)$ specialize to

$$\mathbf{F} = r'\mathbf{e}_r \otimes \mathbf{e}_1 + r\theta'\mathbf{e}_\theta \otimes \mathbf{e}_2, \tag{x}$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (r')^2\,\mathbf{e}_r \otimes \mathbf{e}_r + r^2\,(\theta')^2\,\mathbf{e}_\theta \otimes \mathbf{e}_\theta. \tag{xi}$$

Since $\mathbf{B} = \lambda_r^2 \mathbf{e}_r \otimes \mathbf{e}_r + \lambda_\theta^2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta$, we conclude that the principal stretches are

$$\lambda_r = r', \qquad \lambda_\theta = r\theta', \tag{xii}$$

where we have assumed $r' > 0, \theta' > 0$. Otherwise the principal stretches would be $\lambda_r = |r'|, \lambda_\theta = r|\theta'|$. Observe that

$$\mathbf{F} = \lambda_r \mathbf{e}_r \otimes \mathbf{e}_1 + \lambda_\theta \mathbf{e}_\theta \otimes \mathbf{e}_2, \qquad \mathbf{B} = \lambda_r^2 \mathbf{e}_r \otimes \mathbf{e}_r + \lambda_\theta^2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta.$$

The consequence (vii) of incompressibility now specializes to

$$\det \mathbf{F} = rr'\theta' = 1 \qquad \Rightarrow \qquad r(x_1)r'(x_1)\theta'(x_2) = 1$$

By separating variables in this equation we conclude that

$$r(x_1)r'(x_1) = \frac{1}{\theta'(x_2)} = c_1 \text{ (constant)},$$

which when integrated yields

$$r(x_1) = \sqrt{2c_1 x_1 + c_2}, \qquad \theta(x_2) = \frac{x_2}{c_1} + c_4,$$

where all the c's are constants. The boundary conditions $\theta(0) = 0, \theta(H) = 2\pi$ require $c_4 = 0, c_1 = H/(2\pi)$, while the boundary conditions $r(0) = r_0, r(W) = r_1$ yield $c_2 = r_0^2$, $r_1^2 = r_0^2 + HW/\pi$. Thus the deformation $(viii)$ specializes to

$$r(x_1) = \sqrt{Hx_1/\pi + r_0^2}, \qquad \theta(x_2) = \frac{2\pi x_2}{H}, \tag{$xiii$}$$

with the radius r_1 of the outer boundary being

$$r_1 = \sqrt{r_0^2 + HW/\pi}. \tag{xiv}$$

We could have written (xiv) immediately by equating the areas $\pi r_1^2 - \pi r_0^2 = HW$.

Observe from $(xii)_2$ and $(xiii)$ that $\lambda_\theta = 2\pi r/H$. Again we could have written this down directly since a vertical line of length H maps into a circle of radius r. By incompressibility, $\lambda_r = 1/\lambda_\theta$. Thus the principal stretches can be written as

$$\lambda_r = \frac{H}{2\pi r}, \qquad \lambda_\theta = \frac{2\pi r}{H}. \tag{xv}$$

(c) Since Γ_0 and γ_0 have the same lengths, $2\pi r_0 = H$ and so the inner radius of the annulus is

$$r_0 = H/2\pi. \tag{xvi}$$

Substituting (xvi) into $(xiii)$ and (xiv) allows the deformation to be written as

$$r(x_1) = r_0\sqrt{1 + 2x_1/(\pi r_0)}, \qquad \theta(x_2) = \frac{x_2}{r_0}, \tag{$xvii$}$$

and the radius r_1 of the outer boundary as

$$r_1 = r_0\sqrt{1 + 2W/r_0}. \tag{$xviii$}$$

From (xv) and (xvi) the principal stretches can be expressed as

$$\lambda_r = \frac{r_0}{r}, \qquad \lambda_\theta = \frac{r}{r_0}. \tag{xix}$$

Problem 2.44. (*Measures of volumetric and shape change.*)

(a) Multiplicatively decompose an arbitrary deformation gradient tensor $\mathbf{F}$ into the product of a tensor $\alpha\mathbf{I}$ that captures the entire volume change associated with $\mathbf{F}$ and a tensor $\overline{\mathbf{F}}$ that involves no volume change, i.e. given $\mathbf{F}$, find α and $\overline{\mathbf{F}}$ such that

$$\mathbf{F} = (\alpha\mathbf{I})\overline{\mathbf{F}} = \alpha\overline{\mathbf{F}} \qquad \text{where} \quad \det\overline{\mathbf{F}} = 1. \tag{i}$$

One speaks of the part $\alpha\mathbf{I}$ as the volumetric part of the deformation gradient tensor $\mathbf{F}$ while the part $\overline{\mathbf{F}}$ is the "shape change" (plus rotation) part of $\mathbf{F}$.

(b) Define the "modified left Cauchy-Green deformation tensor" $\overline{\mathbf{B}}$ and its principal scalar invariants $\overline{I}_1, \overline{I}_2, \overline{I}_3$ by

$$\overline{\mathbf{B}} := \overline{\mathbf{F}}\,\overline{\mathbf{F}}^T, \qquad \overline{I}_1 := \operatorname{tr}\overline{\mathbf{B}}, \qquad \overline{I}_2 := \frac{1}{2}\big[(\operatorname{tr}\overline{\mathbf{B}})^2 - \operatorname{tr}\overline{\mathbf{B}}^2\big], \qquad \overline{I}_3 := \det\overline{\mathbf{B}}. \tag{2.129}$$

Derive expressions for $\mathbf{B}$, I_1, I_2, J in terms of $\overline{\mathbf{B}}$, $\overline{I}_1, \overline{I}_2, \overline{I}_3$ where, as usual

$$\mathbf{B} = \mathbf{F}\,\mathbf{F}^T, \qquad I_1 = \mathrm{tr}\,\mathbf{B}, \qquad I_2 = \frac{1}{2}\big[(\mathrm{tr}\,\mathbf{B})^2 - \mathrm{tr}\,\mathbf{B}^2\big], \qquad J = \sqrt{I_3} = \sqrt{\det\mathbf{B}}, \tag{ii}$$

and show that there is a one-to-one relation between $\{I_1, I_2, J\}$ and $\{\overline{I}_1, \overline{I}_2, J\}$.

(c) Derive linearized expressions for the volumetric and shape change measures

$$\alpha\mathbf{I} - \mathbf{I} \qquad \text{and} \qquad \overline{\mathbf{E}} := \frac{1}{2}\left[\overline{\mathbf{F}}\,\overline{\mathbf{F}}^T - \mathbf{I}\right]$$

when the displacement gradient is small. Express your answers in terms of the infinitesimal strain and rotation tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$.

Solution:

(a) We are told that $\det\mathbf{F} = \det(\alpha\mathbf{I})$. It therefore follows from $J = \det\mathbf{F} = \det(\alpha\mathbf{I}) = \alpha^3$ that $\alpha = J^{1/3}$ whence from (i):

$$\overline{\mathbf{F}} = \mathrm{J}^{-1/3}\mathbf{F}, \qquad \mathbf{F} = J^{1/3}\overline{\mathbf{F}}. \tag{2.130}$$

Given $\mathbf{F}$, its factors $\mathrm{J}^{-1/3}\mathbf{I}$ and $\overline{\mathbf{F}}$ are given by $J = \det\mathbf{F}$ and $(2.130)_1$. Conversely, given J and $\overline{\mathbf{F}}$, $(2.130)_2$ gives $\mathbf{F}$. Thus there is a one-to-one relation between $\mathbf{F}$ and the pair $J, \overline{\mathbf{F}}$.

(b) Starting from $(ii)_1$,

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \overset{(2.130)_2}{=} (J^{1/3}\overline{\mathbf{F}})(J^{1/3}\overline{\mathbf{F}}^T) = J^{2/3}\overline{\mathbf{F}}\overline{\mathbf{F}}^T \overset{(2.129)_1}{=} J^{2/3}\overline{\mathbf{B}}. \qquad \square \tag{2.131}$$

From $(ii)_2$,

$$I_1 = \mathrm{tr}\,\mathbf{B} \overset{(2.131)}{=} J^{2/3}\mathrm{tr}\,\overline{\mathbf{B}} \overset{(2.129)_2}{=} J^{2/3}\overline{I}_1. \tag{iii}$$

Since

$$\mathrm{tr}\,\mathbf{B}^2 \overset{(2.131)}{=} \mathrm{tr}\,(J^{2/3}\overline{\mathbf{B}})^2 = J^{4/3}\mathrm{tr}\,\overline{\mathbf{B}}^2, \tag{iv}$$

it follows from $(2.129)_3$, $(ii)_3$, (iii) and (iv) that

$$I_2 = J^{4/3}\overline{I}_2. \tag{v}$$

Thus, given $\overline{I}_1, \overline{I}_2, J$ one can determine I_1, I_2, J from

$$I_1 = J^{2/3}\overline{I}_1, \qquad I_2 = J^{4/3}\overline{I}_2, \tag{2.132}$$

and conversely, given I_1, I_2, J one can determine $\overline{I}_1, \overline{I}_2, J$ from

$$\overline{I}_1 = J^{-2/3}I_1, \qquad \overline{I}_2 = J^{-4/3}I_2. \tag{2.133}$$

(c) The right Cauchy Green tensor can be expressed as

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = (J^{1/3}\overline{\mathbf{F}})^T(J^{1/3}\overline{\mathbf{F}}) = \mathrm{J}^{2/3}\overline{\mathbf{F}}^T\overline{\mathbf{F}}. \tag{v}$$

To linearize the preceding results we first note that

$$J = \det \mathbf{F} = \det(\mathbf{I}+\mathbf{H}) = 1 + \operatorname{tr}\mathbf{H} + O(|\mathbf{H}|^2),$$

where

$$\mathbf{H} = \nabla \mathbf{u}.$$

Therefore by the binomial expansion

$$J^n = [1 + \operatorname{tr}\mathbf{H} + \ldots]^n = 1 + n\operatorname{tr}\mathbf{H} + O(|\mathbf{H}|^2).$$

Therefore

$$\mathrm{J}^{1/3}\mathbf{I} - \mathbf{I} = \left[1 + \frac{1}{3}\operatorname{tr}\mathbf{H} + \ldots\right]\mathbf{I} - \mathbf{I} = \frac{1}{3}\left(\operatorname{tr}\mathbf{H}\right)\mathbf{I} + \ldots. \qquad (vi)$$

Similarly,

$$\overline{\mathbf{F}} = \mathrm{J}^{-1/3}\mathbf{F} = \left[1 - \frac{1}{3}\operatorname{tr}\mathbf{H} + \ldots\right](\mathbf{I}+\mathbf{H}) = \mathbf{I} + \left[\mathbf{H} - \frac{1}{3}\left(\operatorname{tr}\mathbf{H}\right)\mathbf{I}\right] + \ldots$$

and so

$$\overline{\mathbf{F}}\overline{\mathbf{F}}^T - \mathbf{I} = \mathbf{H} + \mathbf{H}^T - \frac{2}{3}\left(\operatorname{tr}\mathbf{H}\right) + \ldots \qquad (vii)$$

Recall that the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is defined by

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T). \qquad (viii)$$

It now follows from $(vi), (vii), (viii)$ that

$$J^{1/3}\mathbf{I} - \mathbf{I} = \frac{1}{3}\left(\operatorname{tr}\boldsymbol{\varepsilon}\right)\mathbf{I}, \qquad \square \qquad (ix)$$

$$\overline{\mathbf{E}} := \frac{1}{2}\left[\overline{\mathbf{F}}\,\overline{\mathbf{F}}^T - \mathbf{I}\right] = \boldsymbol{\varepsilon} - \frac{1}{3}\left(\operatorname{tr}\boldsymbol{\varepsilon}\right)\mathbf{I} \qquad \square \qquad (x)$$

The term on the right-hand side of (x) is known as the deviatoric (infinitesimal) strain; it is a measure of shape change. Observe from (x) that $\boldsymbol{\varepsilon}$ is the sum of a volumetric term and shape change term (whereas in the finite deformation theory we have a multiplicative decomposition).

Problem 2.45. *Rigid deformation.* A deformation $\mathbf{y}(\mathbf{x})$ is said to be rigid if it preserves the distance between all pairs of particles, i.e. if (2.16) holds. Show that a deformation is rigid if and only if it has the form

$$\mathbf{y}(\mathbf{x}) = \mathbf{Q}\mathbf{x} + \boldsymbol{b} \qquad (i)$$

where $\mathbf{Q}$ is a constant orthogonal tensor and $\boldsymbol{b}$ is a constant vector.

Solution: A deformation is rigid if the distance $|\mathbf{z} - \mathbf{x}|$ between *any* two particles $\mathbf{x}$ and $\mathbf{z}$ in the reference configuration equals the distance $|\mathbf{y}(\mathbf{z}) - \mathbf{y}(\mathbf{x})|$ between them in the deformed configuration:

$$|\mathbf{y}(\mathbf{z}) - \mathbf{y}(\mathbf{x})|^2 = |\mathbf{z} - \mathbf{x}|^2 \qquad \text{for all } \mathbf{x}, \mathbf{z} \in \mathcal{R}_{\mathrm{R}},$$

which can write in component form as

$$\Big[y_i(\mathbf{z}) - y_i(\mathbf{x})\Big]\Big[y_i(\mathbf{z}) - y_i(\mathbf{x})\Big] = (z_i - x_i)(z_i - x_i) \qquad \text{for all } \mathbf{x}, \mathbf{z} \in \mathcal{R}_{\rm R}. \tag{ii}$$

Since (ii) holds for all $\mathbf{x}$, we may take its derivative with respect to x_j to get

$$-2F_{ij}(\mathbf{x})\,(y_i(\mathbf{z}) - y_i(\mathbf{x})) = -2(z_j - x_j) \qquad \text{for all } \mathbf{x}, \mathbf{z} \in \mathcal{R}_{\rm R}, \tag{iii}$$

where $F_{ij}(\mathbf{x}) = \partial y_i(\mathbf{x})/\partial x_j$ are the components of the deformation gradient tensor. Since (iii) holds for all $\mathbf{z}$ we may take its derivative with respect to z_k to obtain $F_{ij}(\mathbf{x})F_{ik}(\mathbf{z}) = \delta_{jk}$, i.e.

$$\mathbf{F}^T(\mathbf{x})\mathbf{F}(\boldsymbol{z}) = \mathbf{1} \qquad \text{for all } \boldsymbol{x}, \mathbf{z} \in \mathcal{R}_{\rm R}. \tag{iv}$$

Finally, since (iv) holds for all $\mathbf{x}$ and all $\mathbf{z}$, we can take $\mathbf{x} = \mathbf{z}$ in (iv) to get

$$\mathbf{F}^T(\mathbf{x})\mathbf{F}(\mathbf{x}) = \mathbf{I} \qquad \text{for all } \mathbf{x} \in \mathcal{R}_{\rm R}.$$

Thus we conclude that $\mathbf{F}(\mathbf{x})$ is an orthogonal tensor at each $\mathbf{x}$. In fact, since $\det \boldsymbol{F} > 0$, it is proper orthogonal and therefore represents a rotation.

The (possible) dependence of $\mathbf{F}$ on $\mathbf{x}$ implies that $\mathbf{F}$ might be a different proper orthogonal tensor at different points $\mathbf{x}$ in the body. However, returning to (iv), multiplying both sides of it by $\mathbf{F}(\mathbf{x})$ and recalling that $\mathbf{F}$ is orthogonal gives

$$\mathbf{F}(\mathbf{z}) = \mathbf{F}(\mathbf{x}) \qquad \text{at all } \mathbf{x}, \mathbf{z} \in \mathcal{R}_{\rm R},$$

which implies that $\mathbf{F}(\mathbf{x})$ is a *constant* tensor.

Therefore (a) since we have shown that $\mathbf{F}(\mathbf{x})$ is a *constant* tensor, it follows by integrating $\nabla\mathbf{y} = \mathbf{F}$ that the deformation necessarily has the form $\mathbf{y}(\mathbf{x}) = \mathbf{F}\mathbf{x} + \mathbf{b}$ for a constant tensor $\mathbf{F}$ and a constant vector $\mathbf{b}$; and (b) since we have shown that $\mathbf{F}$ is proper orthogonal, this now leads to (i).

Conversely it is easy to verify that (i) satisfies (ii).

Problem 2.46. *"Orientation" preserving deformation.* A triplet of vectors $\{d\mathbf{x}^{(1)},\, d\mathbf{x}^{(2)}, d\mathbf{x}^{(3)}\}$ is right-handed if

$$(d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}) \cdot d\mathbf{x}^{(3)} > 0.$$

A deformation is said to preserve orientation if every right-handed linearly-independent triplet of material fibers $\{d\mathbf{x}^{(1)},\, d\mathbf{x}^{(2)}, d\mathbf{x}^{(3)}\}$ is carried into a right-handed triplet of fibers $\{d\mathbf{y}^{(1)}, d\mathbf{y}^{(2)}, d\mathbf{y}^{(3)}\}$. Show that a deformation $\mathbf{y}(\mathbf{x})$ is orientation preserving if and only if

$$\det\mathbf{F} > 0 \qquad \text{where } \mathbf{F} = \nabla\mathbf{y}. \tag{i}$$

Solution: Consider a triplet of linearly independent material fibers $\{d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}, d\mathbf{x}^{(3)}\}$. We are told it is right-handed and so

$$(d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}) \cdot d\mathbf{x}^{(3)} > 0. \tag{ii}$$

The deformation carries these fibers into the three fibers $d\mathbf{y}^{(1)} = \mathbf{F}d\mathbf{x}^{(1)}$, $d\mathbf{y}^{(2)} = \mathbf{F}d\mathbf{x}^{(2)}$, $d\mathbf{y}^{(3)} = \mathbf{F}d\mathbf{x}^{(3)}$. We what to find the condition under which this second triplet of vectors is also right-handed, i.e. $(d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)}) \cdot d\mathbf{y}^{(3)} > 0$. We proceed as follows:

$$\begin{aligned}(d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)}) \cdot d\mathbf{y}^{(3)} &= (\mathbf{F}d\mathbf{x}^{(1)} \times \mathbf{F}d\mathbf{x}^{(2)}) \cdot \mathbf{F}d\mathbf{x}^{(3)}, \\ &= \det\mathbf{F}\ (d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}) \cdot d\mathbf{x}^{(3)},\end{aligned}$$

where is the second step we have used the vector identity $(\mathbf{Ta} \times \mathbf{Tb}) \cdot \mathbf{Tc} = \det\mathbf{T}\ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ that holds for any three linearly independent vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and nonsingular tensor $\mathbf{T}$; see Problem 1.4.15. Therefore given (ii),

$$(d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)}) \cdot d\mathbf{y}^{(3)} > 0. \qquad (iii)$$

if and only if (i) holds. This establishes the result.

Problem 2.47. *(Change of Area).* Derive *Nanson's formula*, i.e. calculate the relationship between two material area elements $dA_x\mathbf{n}_\mathrm{R}$ and $dA_y\mathbf{n}$ in the reference and deformed configurations respectively; see Figure 2.10.

Solution: Consider the parallelogram in the reference configuration defined by the fibers $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ as shown in Figure 2.10. Let $\mathrm{d}A_x$ denote its area and let $\mathbf{n}_\mathrm{R}$ be a unit vector normal to this plane. Then, from the definition of the vector product between the vectors $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ we have

$$d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} = |d\mathbf{x}^{(1)}|\ |d\mathbf{x}^{(2)}|\ \sin\theta\ \mathbf{n}_\mathrm{R}$$

where θ is the angle between $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$. However it is readily seen by geometry that $|d\mathbf{x}^{(1)}|$ is the length of the base of the parallelogram and $|d\mathbf{x}^{(2)}|\ \sin\theta$ is its height. Therefore

$$|d\mathbf{x}^{(1)}|\ |d\mathbf{x}^{(2)}|\ \sin\theta = dA_x,$$

and so

$$d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} = \mathrm{d}A_x\ \mathbf{n}_\mathrm{R}. \qquad (i)$$

Similarly if $\mathrm{d}A_y$ and $\mathbf{n}$ are the area and unit normal vector, respectively, to the surface in the deformed configuration defined by $d\mathbf{y}^{(1)}$ and $d\mathbf{y}^{(2)}$, then

$$d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)} = \mathrm{d}A_y\ \mathbf{n}. \qquad (ii)$$

Note that the surfaces under consideration (shown shaded in Figure 2.10) are “material” surfaces in the sense that they composed of the same particles. The unit vectors $\mathbf{n}_\mathrm{R}$ and $\mathbf{n}$ are defined by the fact that they are normal to these material surface elements.

It now follows that

$$\begin{aligned}\mathrm{d}A_y\ \mathbf{n} &= d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)} = \mathbf{F}d\mathbf{x}^{(1)} \times \mathbf{F}d\mathbf{x}^{(2)} = \\ &= \det\mathbf{F}\ \mathbf{F}^{-T}\ (d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}) = \\ &\overset{(i)}{=} \det\mathbf{F}\ \mathbf{F}^{-T}\ (dA_x\mathbf{n}_\mathrm{R}) = dA_x\ \det\mathbf{F}\ \mathbf{F}^{-T}\mathbf{n}_\mathrm{R},\end{aligned}$$

where in getting to the second line we have used the vector identity $\mathbf{Ta}\times\mathbf{Tb} = \det\mathbf{T}\,\mathbf{T}^{-T}(\mathbf{a}\times\mathbf{b})$ that holds for any pair of linearly independent vectors $\mathbf{a},\mathbf{b}$ and nonsingular 2-tensor $\mathbf{T}$; see (1.194) on page 88. We are thus led to the desired result

$$\mathrm{d}A_y\,\mathbf{n} = \mathrm{d}A_x\,J\,\mathbf{F}^{-T}\,\mathbf{n}_\mathrm{R}\ . \tag{iii}$$

that relates the vector areas $\mathrm{d}A_y\mathbf{n}$ and $\mathrm{d}A_x\mathbf{n}_\mathrm{R}$. By taking the magnitude of this vector equation we find that the areas dA_y and dA_x are related by

$$\mathrm{d}A_y = \mathrm{d}A_x\,J\,|\mathbf{F}^{-T}\,\mathbf{n}_\mathrm{R}|; \tag{iv}$$

On substituting (iv) into (iii) we find that the unit normal vectors $\mathbf{n}_\mathrm{R}$ and $\mathbf{n}$ are related by

$$\mathbf{n} = \frac{\mathbf{F}^{-T}\mathbf{n}_\mathrm{R}}{|\mathbf{F}^{-T}\mathbf{n}_\mathrm{R}|}. \tag{v}$$

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2.11 Appendix

2.11.1 The material time derivative.

Next consider a time-dependent **motion** of the body on some time interval $[t_0, t_1]$. The motion takes the particle located at $\mathbf{x}$ in the reference configuration to the location $\widehat{\mathbf{y}}(\mathbf{x}, t)$ at time t:

$$\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}, t), \qquad \mathbf{x} \in \mathcal{R}_{\mathrm{R}}, \quad \mathbf{y} \in \mathcal{R}_t, \quad t \in [t_0, t_1], \tag{2.134}$$

$\mathcal{R}_t$ being the region of space occupied by the body at time t. Note that $\mathcal{R}_t$ evolves with time. Since there is a one-to-one relation between $\mathbf{x}$ and $\mathbf{y}$ at each time, there is an inverse mapping

$$\mathbf{x} = \overline{\mathbf{x}}(\mathbf{y}, t)$$

that takes $\mathcal{R}_t \mapsto \mathcal{R}_{\mathrm{R}}$.

Keep in mind that the location of a particle in the reference configuration serves as a convenient tag by which to identify the particle and so $\mathbf{x}$ serves as a proxy for a particle label. Thus when we want to consider the rate of change of some field at a fixed particle, we consider its rate of change at fixed $\mathbf{x}$.

Now consider a field $\phi(\mathbf{y}, t)$ defined on $\mathcal{R}_t$. Though this represents ϕ spatially, suppose we want to calculate its time rate of change at a fixed particle – the so-called **material time derivative** of ϕ. We shall use a superior dot to denote this time rate of change of ϕ by writing $\dot{\phi}$. In order to calculate $\dot{\phi}$ we first use the motion $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}, t)$ to map ϕ into the reference configuration thus obtaining its material representation $\widehat{\phi}(\mathbf{x}, t)$ where

$$\widehat{\phi}(\mathbf{x}, t) = \phi(\widehat{\mathbf{y}}(\mathbf{x}, t), t). \tag{2.135}$$

Then by $\dot{\phi}$ we mean

$$\dot{\phi} = \frac{\partial \widehat{\phi}}{\partial t}(\mathbf{x}, t).$$

In particular, the velocity of a particle at time t is the time rate of change of position at a fixed particle:

$$\widehat{\mathbf{v}}(\mathbf{x}, t) := \dot{\mathbf{y}} = \frac{\partial \widehat{\mathbf{y}}}{\partial t}(\mathbf{x}, t). \tag{2.136}$$

Following the discussion in Section 2.8, we can express the velocity field spatially in the form $\mathbf{v}(\mathbf{y}, t)$ or referentially in the form $\widehat{\mathbf{v}}(\mathbf{x}, t)$ where these two representations are related by

$$\mathbf{v}(\mathbf{y}, t) = \widehat{\mathbf{v}}(\overline{\mathbf{x}}(\mathbf{y}, t), t), \qquad \widehat{\mathbf{v}}(\mathbf{x}, t) = \mathbf{v}(\widehat{\mathbf{y}}(\mathbf{x}, t), t).$$

Returning to a generic function $\phi(\mathbf{y}, t)$, we can calculate its material time derivative by differentiating (2.135) with respect to time (keeping $\mathbf{x}$ fixed) and using the chain rule:

$$\dot{\phi} = \frac{\partial \widehat{\phi}}{\partial t}(\mathbf{x}, t) = \frac{\partial \phi}{\partial y_i}(\mathbf{y}, t)\frac{\partial \widehat{y}_i}{\partial t}(\mathbf{x}, t) + \frac{\partial \phi}{\partial t}(\mathbf{y}, t) = \frac{\partial \phi}{\partial y_i} v_i + \frac{\partial \phi}{\partial t}$$

which we can write as

$$\dot{\phi} = \mathbf{v} \cdot \mathrm{grad}\, \phi + \frac{\partial \phi}{\partial t}, \tag{2.137}$$

where $\mathrm{grad}\, \phi$ is the vector field with cartesian components $\partial\phi/\partial y_i$.

Exercise: Show that

$$\dot{J} = J\, \mathsf{div}\, \mathbf{v} \tag{2.138}$$

where $J = \det \mathbf{F}$. In cartesian components, $\mathsf{div}\, \mathbf{v} = \partial v_i/\partial y_i$.

Exercise: The *velocity gradient tensor* $\mathbf{L}$ is the tensor with cartesian components $\partial v_i/\partial y_j$:

$$\mathbf{L} := \mathrm{grad}\, \mathbf{v}, \qquad L_{ij} = \frac{\partial v_i}{\partial y_j}(\mathbf{y}, t). \tag{2.139}$$

Show that

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F} \qquad \text{and} \qquad \mathrm{div}\, \mathbf{v} = \mathrm{tr}\, \mathbf{L}. \tag{2.140}$$

2.11.2 A transport theorem.

In subsequent chapters we will need to calculate the rate of change of energy associated with some part of the body. Suppose that this part occupies a subregion $\mathcal{D}_t \subset \mathcal{R}_t$ at time t, keeping in mind that even though $\mathcal{D}_t$ moves through space, the same material particles are associated with it at all times. We will therefore have to evaluate a term of the form

$$\frac{d}{dt}\int_{\mathcal{D}_t} \phi(\mathbf{y}, t)\, dV_y.$$

If $\mathcal{D}_t$ did not depend on t we would simply take the derivative inside the integral but here we must pay attention to the fact that $\mathcal{D}_t$ is time-dependent. In order to get around the time dependency of $\mathcal{D}_t$, we map $\mathcal{D}_t$ into the (time-independent) region $\mathcal{D}_{\mathrm{R}}$ that it occupies in the reference configuration using the motion $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}, t)$. Under this mapping $\mathcal{D}_t \mapsto \mathcal{D}_{\mathrm{R}}$, $\phi(\mathbf{y}, t) \mapsto \widehat{\phi}(\mathbf{x}, t)$ and $dV_y \mapsto J dV_x$. Accordingly

$$\frac{d}{dt}\int_{\mathcal{D}_t} \phi(\mathbf{y}, t)\, dV_y = \frac{d}{dt}\int_{\mathcal{D}_{\mathrm{R}}} \widehat{\phi}(\mathbf{x}, t)\, J\, dV_x.$$

We can now take the derivative inside the integral since $\mathcal{D}_{\mathrm{R}}$ is time independent:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{D}_t} \phi(\mathbf{y}, t)\, dV_y &= \frac{d}{dt}\int_{\mathcal{D}_{\mathrm{R}}} \widehat{\phi}(\mathbf{x}, t)\, J(\mathbf{x}, t)\, dV_x = \int_{\mathcal{D}_{\mathrm{R}}} \frac{\partial}{\partial t}\Big(\widehat{\phi}(\mathbf{x}, t)\, J(\mathbf{x}, t)\Big)\, dV_x = \\
&= \int_{\mathcal{D}_{\mathrm{R}}} (\dot{\phi} J + \phi \dot{J})\, dV_x \overset{(2.138)}{=} \int_{\mathcal{D}_{\mathrm{R}}} (\dot{\phi} J + \phi J\, \mathsf{div}\, \mathbf{v})\, dV_x = \\
&= \int_{\mathcal{D}_{\mathrm{R}}} (\dot{\phi} + \phi\, \mathsf{div}\, \mathbf{v})\, J\, dV_x = \int_{\mathcal{D}_t} (\dot{\phi} + \phi\, \mathsf{div}\, \mathbf{v})\, dV_y,
\end{aligned}$$

where in getting to the very last expression we reverted from $\mathcal{D}_{\mathrm{R}} \mapsto \mathcal{D}_t$ and $J dV_x \mapsto dV_y$. Therefore we have the **transport formula**

$$\frac{d}{dt}\int_{\mathcal{D}_t} \phi \, dV_y = \int_{\mathcal{D}_t} (\dot{\phi} + \phi \, \mathsf{div}\,\mathbf{v}) \, dV_y \tag{2.141}$$

for the function $\phi(\mathbf{y}, t)$. Similar transport formulae can be written for vector and tensor fields, as well as for fields defined on a moving surface $\mathcal{S}_t$ or a moving curve $\mathcal{L}_t$; see Volume II.

Finally we note an illuminating alternative form of (2.141). First, we can rewrite (2.137) as

$$\dot{\phi} = \mathrm{grad}\,\phi \cdot \mathbf{v} + \frac{\partial \phi}{\partial t}(\mathbf{y}, t) = \mathrm{div}\,(\phi \mathbf{v}) - \phi \, \mathrm{div}\,\mathbf{v} + \frac{\partial \phi}{\partial t}(\mathbf{y}, t),$$

where $\mathrm{div}\,\mathbf{v}$ is the scalar field $\partial v_i / \partial y_i$. Substituting this into (2.141) yields

$$\frac{d}{dt}\int_{\mathcal{D}_t} \phi \, dV_y = \int_{\mathcal{D}_t} \left(\frac{\partial \phi}{\partial t} + \mathrm{div}\,(\phi \mathbf{v}) \right) dV_y. \tag{2.142}$$

Finally we use the divergence theorem (1.173) to rewrite the last term thus obtaining the following alternate form of the transport formula:

$$\frac{d}{dt}\int_{\mathcal{D}_t} \phi \, dV_y = \int_{\mathcal{D}_t} \frac{\partial \phi}{\partial t} \, dV_y + \int_{\partial \mathcal{D}_t} \phi \, \mathbf{v} \cdot \mathbf{n} \, dA_y. \tag{2.143}$$

In this representation, the last term characterizes the flux of ϕ across the boundary $\partial \mathcal{D}_t$.

2.11.3 Exercises.

Problem 2.11.1. Consider the particular (time-dependent) *motion* $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}, t)$:

$$y_1 = a(t)x_1 + b(t)x_2, \qquad y_2 = c(t)x_2, \qquad y_3 = d(t)x_3. \tag{i}$$

Calculate the particle velocity field and express it in both referential (material) form and spatial form.

Calculate the particle acceleration field and express it in spatial form.

Calculate the components of $\mathrm{Grad}\,\mathbf{v}$, the tensor with cartesian components $\partial v_i(\mathbf{x}, t)/\partial x_j$.

Calculate the components of the velocity gradient tensor $\mathbf{L} = \mathrm{grad}\,\mathbf{v}$ where $\mathrm{grad}\,\mathbf{v}$ is the tensor with cartesian components $\partial v_i(\mathbf{y}, t)/\partial y_j$.

Calculate also the *stretching tensor* field $\mathbf{D}(\mathbf{y}, t)$:

$$\mathbf{D} := \frac{1}{2}(\mathbf{L} + \mathbf{L}^T). \tag{2.144}$$

Solution: Differentiating (i) at a fixed particle, i.e. with $\mathbf{x}$ held fixed, gives the particle velocity field

$$v_1(\mathbf{x}, t) = \dot{a}x_1 + \dot{b}x_2, \qquad v_2(\mathbf{x}, t) = \dot{c}x_2, \qquad v_3(\mathbf{x}, t) = \dot{d}x_3. \tag{ii}$$

In order to express the velocity field spatially, i.e. as a function of $\mathbf{y}$ and t, we must first solve (i) to get $\mathbf{x} = \mathbf{x}(\mathbf{y}, t)$:

$$x_1 = \frac{1}{a}\left(y_1 - \frac{b}{c}y_2\right), \qquad x_2 = \frac{1}{c}y_2, \qquad x_3 = \frac{1}{d}y_3. \tag{iii}$$

We can now find the spatial representation $\mathbf{v}(\mathbf{y}, t)$ of the velocity by substituting (iii) into (ii):

$$v_1(\mathbf{y}, t) = \frac{\dot a}{a}\left(y_1 - \frac{b}{c}y_2\right) + \frac{\dot b}{c}y_2, \qquad v_2(\mathbf{y}, t) = \frac{\dot c}{c}y_2, \qquad v_3(\mathbf{y}, t) = \frac{\dot d}{d}y_3. \tag{iv}$$

The velocity gradient tensor $\mathbf{L}(\mathbf{y}, t)$ has components $L_{ij} = \partial v_i/\partial y_j$ which we find by differentiating (iv) with respect to the y_j's:

$$\mathbf{L}(\mathbf{y}, t) = \frac{\dot a}{a}\,\mathbf{e}_1\otimes\mathbf{e}_1 + \left(\frac{\dot b}{c} - \frac{b\dot a}{ac}\right)\mathbf{e}_1\otimes\mathbf{e}_2 + \frac{\dot c}{c}\,\mathbf{e}_2\otimes\mathbf{e}_2 + \frac{\dot d}{d}\,\mathbf{e}_3\otimes\mathbf{e}_3. \tag{v}$$

The rate of deformation tensor $\mathbf{D}$ is therefore

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \frac{\dot a}{a}\,\mathbf{e}_1\otimes\mathbf{e}_1 + \frac{1}{2}\left(\frac{\dot b}{c} - \frac{b\dot a}{ac}\right)(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1) + \frac{\dot c}{c}\,\mathbf{e}_2\otimes\mathbf{e}_2 + \frac{\dot d}{d}\,\mathbf{e}_3\otimes\mathbf{e}_3. \tag{vi}$$

Problem 2.11.2. A body undergoes a motion $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}, t)$ and occupies a region $\mathcal{R}_t$ at time t. Calculate the rate of change of the surface area of the outer boundary of the body:

$$\frac{d}{dt}\int_{\partial\mathcal{R}_t} dA_y. \tag{i}$$

Solution: The following preliminary results will be useful. Recall from (2.39) and (2.40) that

$$dA_y = J|\mathbf{F}^{-T}\mathbf{n}_R|\,dA_x, \qquad \mathbf{n} = \frac{\mathbf{F}^{-T}\mathbf{n}_R}{|\mathbf{F}^{-T}\mathbf{n}_R|}, \tag{ii}$$

from (2.140) that

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \tag{iii}$$

and from (1.207) that

$$\dot{\mathbf{F}}^{-1} = -\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{iv}$$

It follows from (iv) that

$$\dot{\mathbf{F}}^{-T} = -\mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{F}^{-T} \overset{(iii)}{=} \mathbf{L}^T\mathbf{F}^{-T}. \tag{v}$$

Next,

$$\frac{\partial}{\partial t}|\mathbf{F}^{-T}\mathbf{n}_R|^2 = 2|\mathbf{F}^{-T}\mathbf{n}_R|\;\frac{\partial}{\partial t}|\mathbf{F}^{-T}\mathbf{n}_R|, \tag{vi}$$

and alternatively

$$\begin{aligned}\frac{\partial}{\partial t}|\mathbf{F}^{-T}\mathbf{n}_R|^2 \quad &= \frac{\partial}{\partial t}(\mathbf{F}^{-T}\mathbf{n}_R\cdot\mathbf{F}^{-T}\mathbf{n}_R) = 2\dot{\mathbf{F}}^{-T}\mathbf{n}_R\cdot\mathbf{F}^{-T}\mathbf{n}_R = \\ &\overset{(ii)_2}{=} 2|\mathbf{F}^{-T}\mathbf{n}_R|\,\dot{\mathbf{F}}^{-T}\mathbf{n}_R\cdot\mathbf{n} \overset{(v)}{=} 2|\mathbf{F}^{-T}\mathbf{n}_R|\,\mathbf{L}^T\mathbf{F}^{-T}\mathbf{n}_R\cdot\mathbf{n} = \\ &\overset{(ii)_2}{=} 2|\mathbf{F}^{-T}\mathbf{n}_R|^2\,\mathbf{L}^T\mathbf{n}\cdot\mathbf{n} = 2|\mathbf{F}^{-T}\mathbf{n}_R|^2\,\mathbf{L}\mathbf{n}\cdot\mathbf{n}.\end{aligned} \tag{vii}$$

Combining (vi) and (vii)

$$\frac{\partial}{\partial t}|\mathbf{F}^{-T}\mathbf{n}_R| = |\mathbf{F}^{-T}\mathbf{n}_R|\ \mathbf{L}\mathbf{n}\cdot\mathbf{n}. \qquad (viii)$$

We can now calculate the rate of change of surface area as follows:

$$\begin{aligned}
\frac{d}{dt}\int_{\partial\mathcal{R}_t} dA_y \quad &\overset{(ii)_1}{=} \frac{d}{dt}\int_{\partial\mathcal{R}_R} J\,|\mathbf{F}^{-T}\mathbf{n}_R|\,dA_x = \int_{\partial\mathcal{R}_R}\left[\dot{J}\,|\mathbf{F}^{-T}\mathbf{n}_R| + J\,\frac{\partial}{\partial t}|\mathbf{F}^{-T}\mathbf{n}_R|\right] dA_x = \\
&\overset{(viii),(2.138)}{=} \int_{\partial\mathcal{R}_R}\Big[J\,\mathrm{div}\,\mathbf{v}\ |\mathbf{F}^{-T}\mathbf{n}_R| + J\ |\mathbf{F}^{-T}\mathbf{n}_R|\,\mathbf{L}\mathbf{n}\cdot\mathbf{n}\Big]\,dA_x = \\
&= \int_{\partial\mathcal{R}_R}[\mathrm{div}\,\mathbf{v}\ +\ \mathbf{L}\mathbf{n}\cdot\mathbf{n}]\,J\,|\mathbf{F}^{-T}\mathbf{n}_R|dA_x = \\
&\overset{(ii)_1}{=} \int_{\partial\mathcal{R}_t}[\mathrm{div}\,\mathbf{v} - \mathbf{L}\mathbf{n}\cdot\mathbf{n}]\,dA_y = \int_{\partial\mathcal{R}_t}[\mathrm{tr}\,\mathbf{L} - \mathbf{L}\mathbf{n}\cdot\mathbf{n}]\,dA_y = \\
&= \int_{\partial\mathcal{R}_t}\mathbf{L}\cdot[\mathbf{I} - \mathbf{n}\otimes\mathbf{n}]\,dA_y. \qquad \square
\end{aligned}$$

Problem 2.11.3. *(A transport theorem.)* A body undergoes a motion $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}, t)$ and occupies a region $\mathcal{R}_t$ at time t. Let $\mathcal{S}_t$ be an evolving *material surface* in the interior of $\mathcal{R}_t$ – by a material surface we mean that the same material particles lie on $\mathcal{S}_t$ at all times even though $\mathcal{S}_t$ moves through space. Let $\mathbf{g}(\mathbf{y}, t)$ be a smooth vector field defined for all $\mathbf{y} \in \mathcal{R}_t$ at each t. Show that

$$\frac{d}{dt}\int_{\mathcal{S}_t}\mathbf{g}\cdot\mathbf{n}\,dA_y = \int_{\mathcal{S}_t}(\dot{\mathbf{g}} + (\mathrm{tr}\,\mathbf{L})\,\mathbf{g} - \mathbf{L}\mathbf{g})\cdot\mathbf{n}\,dA_y, \qquad (2.145)$$

where $\dot{\mathbf{g}}$ is the material time derivative of $\mathbf{g}$ as defined just above (2.138) and $\mathbf{L} = \mathrm{grad}\,\mathbf{v}$ is the velocity gradient tensor defined in (2.139).

Chapter 3

Force, Equilibrium Principles and Stress

In this chapter we consider the equilibrium principles of force and moment balance and their consequences. The analysis holds no matter what the constitutive characteristics of the material, provided only that it can be modeled as a continuum. Our focus will be on purely mechanical issues. A more complete discussion (including inertial effects) can be found in the references listed at the end of this chapter

A roadmap of this chapter is as follows: in Section 3.1 we introduce the notion of force, more specifically body force and traction, and discuss their various attributes. The global balance laws for force and moment equilibrium are stated in Section 3.2, and from them we deduce the notion of stress and discuss it in Section 3.3. Section 3.4 is devoted to deriving the field equations associated with the balance laws. Principal stresses and principal directions are discussed in Section 3.5. The analysis and discussion up to this point are carried out entirely using the geometric characteristics of the deformed configuration without any mention of a reference configuration or the deformation. It is often useful however to work with an (equivalent) formulation with respect to a reference configuration. Accordingly in Section 3.7 we reformulate the *geometric aspects* of the preceding analysis to be those associated with a reference configuration and the Piola stress tensor is introduced. Section 3.8 considers the rate at which stress does work – the stress power – , and the notion of work-conjugate stress-strain pairs is discussed in Section 3.8.1. The preceding results are linearized in Section 3.9. Finally in Section 3.10 we examine the equilibrium equations in cylindrical and spherical polar coordinates.

All fields encountered in this chapter will be assumed to be smooth. That is, we assume them to be differentiable as many times as needed, and that these derivatives are continuous. This must be relaxed when, for example, we consider a two-phase material where the stress field will be discontinuous at an interface between two phases (Problem 3.28).

3.1 Force.

We are concerned with the *deformed configuration* of the body. In this configuration the body occupies a region $\mathcal{R}$, and an arbitrary part of the body[1] occupies a region $\mathcal{D}$ that is a subregion of $\mathcal{R}$. It is convenient to refer to $\mathcal{D}$ as a part of the body (rather than to use the more cumbersome but precise phrase "the region occupied by a part of the body"). In this configuration, a generic particle is located at $\mathbf{y} \in \mathcal{R}$. Inertial effects are not considered and when we refer to time t, we only use it to discuss a one-parameter family of configurations – a so-called "quasi-static motion".

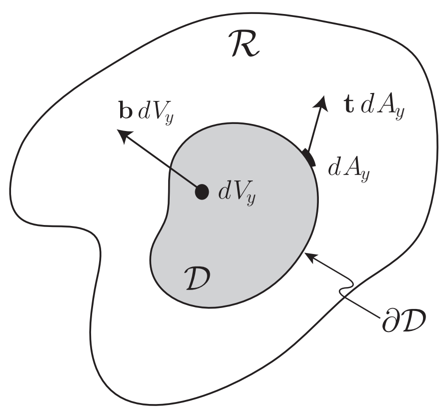

Figure 3.1: Forces acting on a part $\mathcal{D}$ of the body: the traction $\mathbf{t}$ is a force per unit area acting at points on the boundary $\partial\mathcal{D}$ due to contact between $\mathcal{D}$ and the rest of the body across the surface $\partial\mathcal{D}$. The body force density $\mathbf{b}$ is a force per unit volume acting at points in the interior of $\mathcal{D}$ applied by agents outside the body.

We now turn our attention to the forces acting *on* an arbitrary part $\mathcal{D}$ of the deformed body. As depicted in Figure 3.1 we assume there are two types of forces: *body forces* that act at each point in the interior of $\mathcal{D}$ and are applied by agents outside of the body, and *contact*

[1] A *part* of a body involves the same set of particles in all configurations. For a more careful discussion, see Volume II.

forces or **tractions**[2] that act at points on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ and represent forces due to contact between $\mathcal{D}$ and the rest of the body[3] across the surface $\partial\mathcal{D}$. The body force density $\mathbf{b}$ is a force *per unit (deformed) volume*[4], while the contact force density $\mathbf{t}$ is a force *per unit (deformed) surface area*; see Figure 3.1.

In order to characterize a force, we must specify how it contributes to (a) the resultant force, (b) the resultant moment about an arbitrary fixed (pivot) point, and (c) how it does work.

Since $\mathbf{b}$ is a force per unit volume distributed over $\mathcal{D}$, its resultant is its volume integral over $\mathcal{D}$. Similarly since $\mathbf{t}$ is a force per unit area distributed over the boundary $\partial\mathcal{D}$, its resultant is its surface integral over $\partial\mathcal{D}$. The *resultant external force* on the part $\mathcal{D}$ under consideration is thus taken to be

$$\int_{\mathcal{D}} \mathbf{b}\, \mathrm{d}V_y + \int_{\partial\mathcal{D}} \mathbf{t}\, \mathrm{d}A_y\,; \tag{3.1}$$

the *resultant moment* of the external forces acting on $\mathcal{D}$ about an arbitrary fixed point O is taken to be

$$\int_{\mathcal{D}} \mathbf{y}\times\mathbf{b}\, \mathrm{d}V_y + \int_{\partial\mathcal{D}} \mathbf{y}\times\mathbf{t}\, \mathrm{d}A_y \tag{3.2}$$

where $\mathbf{y}$ is position with respect to O; and the *rate of working* of the external forces acting on $\mathcal{D}$ is taken to be

$$\int_{\mathcal{D}} \mathbf{b}\cdot\mathbf{v}\, \mathrm{d}V_y + \int_{\partial\mathcal{D}} \mathbf{t}\cdot\mathbf{v}\, \mathrm{d}A_y\,, \tag{3.3}$$

where $\mathbf{v}$ is particle velocity. Note that $\mathbf{t}$ represents a force per unit *deformed* area and $\mathbf{b}$ a force per unit *deformed* volume.

In order for the formulae (3.1) - (3.3) to be useful, we must specify the variables on which $\mathbf{b}$ and $\mathbf{t}$ depend. We expect that the body force density may depend on position $\mathbf{y}$ and so we assume that

$$\mathbf{b} = \mathbf{b}(\mathbf{y}). \tag{3.4}$$

We now turn to the traction $\mathbf{t}$. It too will depend on the position $\mathbf{y}$ but it cannot depend *only* on $\mathbf{y}$. To see this consider Figure 3.2. The two figures there both show the same region $\mathcal{D}$; the point A in both is the same and its position vector is $\mathbf{y}_A$. In the left-hand figure, $\mathcal{D}_1$

[2] Some authors call this the *stress vector*.

[3] If part of $\partial\mathcal{D}$ coincides with a part of $\partial\mathcal{R}$, the contact force on that part of the surface is applied by an outside agent.

[4] One can alternatively characterize the body force as a force per unit mass.

and $\mathcal{D}_2$ are two parts of the body and A lies on the interface between them. The right-hand figure shows two different parts, $\mathcal{D}_3$ and $\mathcal{D}_4$, and A lies on the interface between these two parts as well. The interface between $\mathcal{D}_3$ and $\mathcal{D}_4$ is different to that between $\mathcal{D}_1$ and $\mathcal{D}_2$ though A lies on both interfaces. If the traction $\mathbf{t}$ depended *only* on $\mathbf{y}$, then the traction at A would be $\mathbf{t}(\mathbf{y}_A)$ and the force per unit area applied by[5] $\mathcal{D}_1$ on $\mathcal{D}_2$ at A and the force per unit area applied by $\mathcal{D}_3$ on $\mathcal{D}_4$ at A would both be $\mathbf{t}(\mathbf{y}_A)$. However we do not expect the force applied by $\mathcal{D}_1$ on $\mathcal{D}_2$ to be the same as that applied by $\mathcal{D}_3$ on $\mathcal{D}_4$.

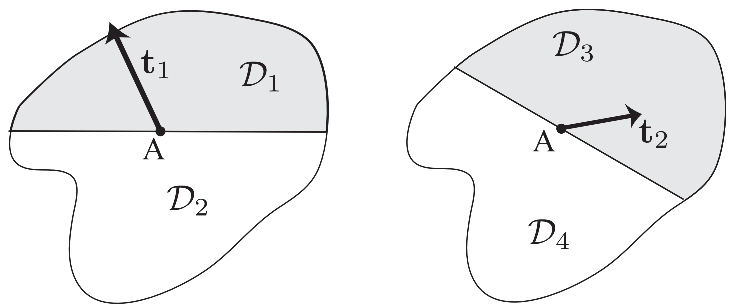

Figure 3.2: Traction depends on the surface on which it acts: Regions $\mathcal{D}_1$ and $\mathcal{D}_2$ are occupied by two parts of a body, while $\mathcal{D}_3$ and $\mathcal{D}_4$ are occupied by a different pair of parts. Planar interfaces separate these parts while the point A is common to both interfaces. The traction $\mathbf{t}_1$ in the figure on the left is applied at A by the material in $\mathcal{D}_1$ on that in $\mathcal{D}_2$. The traction $\mathbf{t}_2$ in the figure on the right is applied at A by the material in $\mathcal{D}_3$ on that in $\mathcal{D}_4$. Even though both tractions are associated with the same point $\mathbf{y}_A$ there is no reason to expect that $\mathbf{t}_1 = \mathbf{t}_2$.

Therefore the traction must depend on the specific *surface* at $\mathbf{y}$ on which it acts. To first order, a surface is defined by its unit normal vector $\mathbf{n}$, and so we shall assume that

$$\mathbf{t} = \mathbf{t}(\mathbf{y}, \mathbf{n}). \tag{3.5}$$

Of course $\mathbf{n}$ would vary along $\partial\mathcal{D}$ and so it too is a function of $\mathbf{y}$: $\mathbf{n} = \mathbf{n}(\mathbf{y})$. According to (3.5) the dependence of the traction on the surface is only through the normal vector and not, for example, the curvature of the surface. The ansatz (3.5) is known as *Cauchy's hypothesis*.

It is worth emphasizing that according to (3.3) the traction $\mathbf{t}(\mathbf{y}, \mathbf{n})$ denotes the force per unit area *applied by* the part outside $\mathcal{D}$ *on* the material inside $\mathcal{D}$. Now consider a (not-necessarily closed) surface $\mathcal{S}$ in the body and let $\mathbf{y}$ be a point on this surface and let $\mathbf{n}$ be a

[5]Question: Is this the force applied by $\mathcal{D}_2$ on $\mathcal{D}_1$ or by $\mathcal{D}_1$ on $\mathcal{D}_2$?

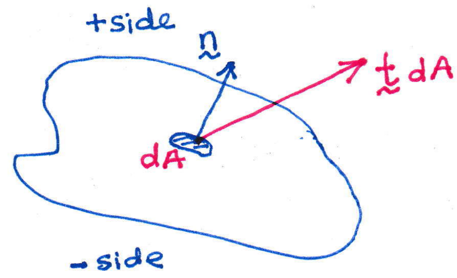

Figure 3.3: The force vector $\mathbf{t}(\mathbf{n})\, dA_y$ acting on an infinitesimal surface element. This force is applied by the material on the positive side of the surface on the material on the negative side. The positive side is the one into which the unit normal vector $\mathbf{n}$ points.

unit vector that is normal to $\mathcal{S}$ as shown in Figure 3.3. The side of $\mathcal{S}$ into which $\mathbf{n}$ points is referred to as the *positive side* of $\mathcal{S}$ and the other is the *negative side*. By convention, the traction vector $\mathbf{t}(\mathbf{y}, \mathbf{n})$ denotes the force per unit area applied *by the material on the positive side on the material on the negative side.*

Is this consistent with our earlier discussion of the traction on the closed surface $\partial\mathcal{D}$? If the unit normal vector $\mathbf{n}$ on $\partial\mathcal{D}$ is taken so *it points out* of $\mathcal{D}$, then the positive side of the surface is the outside of $\mathcal{D}$ and so $\mathbf{t}(\mathbf{y}, \mathbf{n})$ is the traction applied by the part outside of $\mathcal{D}$ on $\mathcal{D}$. This is exactly what we had earlier, the point being that the unit normal vector should be pointing outwards.

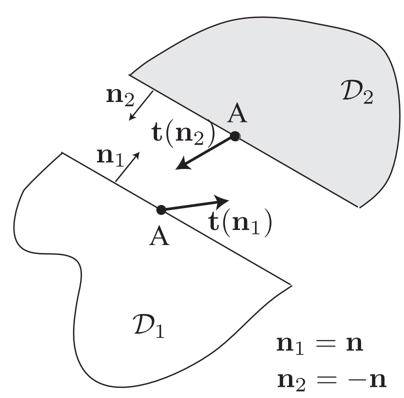

Figure 3.4: The unit *outward* normal vector to $\mathcal{D}_1$ is $\mathbf{n}_1$ and so the traction that is applied by $\mathcal{D}_2$ on $\mathcal{D}_1$ is $\mathbf{t}(\mathbf{n}_1)$. The unit *outward* normal vector to $\mathcal{D}_2$ is $\mathbf{n}_2$ and so the traction applied by $\mathcal{D}_1$ on $\mathcal{D}_2$ is $\mathbf{t}(\mathbf{n}_2)$. While $\mathbf{n}_1 = -\mathbf{n}_2$ we do not know (yet) whether $\mathbf{t}(\mathbf{n}_1) = -\mathbf{t}(\mathbf{n}_2)$.

Continuing to focus on the dependence of the traction on the normal vector, consider the body shown in Figure 3.4. In order to calculate the traction acting *on* $\mathcal{D}_1$ at A, we draw the unit normal vector to the interface that points outward from $\mathcal{D}_1$. This is denoted by $\mathbf{n}_1$ in the figure. Thus the traction acting on $\mathcal{D}_1$ is $\mathbf{t}(\mathbf{y}_A, \mathbf{n}_1)$; this is applied by $\mathcal{D}_2$. On the other hand if we want to calculate the traction acting *on* $\mathcal{D}_2$ at A, we draw the unit normal vector $\mathbf{n}_2$ that points outward from $\mathcal{D}_2$. Thus the traction acting *on* $\mathcal{D}_2$ is $\mathbf{t}(\mathbf{y}_A, \mathbf{n}_2)$; this is applied by $\mathcal{D}_1$. We do not (yet) know how $\mathbf{t}(\mathbf{y}_A, \mathbf{n}_1)$ relates to $\mathbf{t}(\mathbf{y}_A, \mathbf{n}_2)$ though $\mathbf{n}_1 = -\mathbf{n}_2$, i.e. how $\mathbf{t}(\mathbf{y}, \mathbf{n})$ relates to $\mathbf{t}(\mathbf{y}, -\mathbf{n})$.

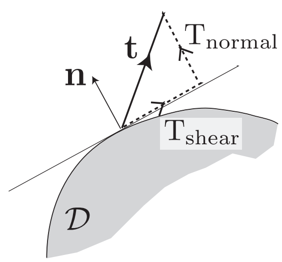

Figure 3.5: Components of the traction $\mathbf{t}$: normal stress T_{normal} and resultant shear stress T_{shear} .

Finally we emphasize that the traction acts in a direction that *need not be normal to the surface*; i.e., as depicted in Figure 3.5, $\mathbf{t}(\mathbf{n})$ is not in general parallel to $\mathbf{n}$ (where here and the rest of this paragraph we suppress the dependency on $\mathbf{y}$). The component of traction that is normal to the surface is called the *normal stress* and we denote it by T_{normal}:

$$T_{\text{normal}}(\mathbf{n}) := \mathbf{t}(\mathbf{n}) \cdot \mathbf{n}\,; \tag{3.6}$$

the associated normal traction vector is $T_{\text{normal}}\,\mathbf{n} = (\mathbf{t}\cdot\mathbf{n})\,\mathbf{n}$. When $T_{\text{normal}} > 0$ we say it is *tensile*, *compressive* when $T_{\text{normal}} < 0$. The resultant shear traction vector is $\mathbf{t} - (\mathbf{t}\cdot\mathbf{n})\mathbf{n} = (\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\mathbf{t}$. Its magnitude, the *resultant shear stress* T_{shear}, by the Pythagorean theorem is

$$T_{\text{shear}}(\mathbf{n}) := \sqrt{|\mathbf{t}|^2 - T^2_{\text{normal}}} = \sqrt{\big[\mathbf{t}(\mathbf{n})\cdot\mathbf{t}(\mathbf{n})\big] - \big[\mathbf{t}(\mathbf{n})\cdot\mathbf{n}\big]^2}. \tag{3.7}$$

A natural (and important) question to ask is: “from among all planes through a given point, on which is $T_{\text{normal}}(\mathbf{n})$ largest? And on which is it smallest?” This requires one to consider $T_{\text{normal}}(\mathbf{n})$ as a function of the unit vector $\mathbf{n}$ and to find the specific vector(s) $\mathbf{n}$ at which it has its extrema. One can ask a similar question for the shear stress $T_{\text{shear}}(\mathbf{n})$. We shall revisit these questions once we have more information on how $\mathbf{t}(\mathbf{n})$ depends on $\mathbf{n}$.

3.2 Force and moment equilibrium.

The *equilibrium principle for force balance* postulates that the resultant force on every part of the body vanishes:

$$\int_{\mathcal{D}} \mathbf{b}\, \mathrm{d}V_y + \int_{\partial\mathcal{D}} \mathbf{t}\, \mathrm{d}A_y = \mathbf{o} \qquad \text{for all } \mathcal{D} \subset \mathcal{R}. \tag{3.8}$$

Similarly, the *equilibrium principle of moment balance* postulates that the resultant moment (about a fixed point O) on every part of the body vanishes:

$$\int_{\mathcal{D}} \mathbf{y} \times \mathbf{b}\, \mathrm{d}V_y + \int_{\partial\mathcal{D}} \mathbf{y} \times \mathbf{t}\, \mathrm{d}A_y = \mathbf{o} \qquad \text{for all } \mathcal{D} \subset \mathcal{R}. \tag{3.9}$$

Both (3.8) and (3.9) must hold for *every part* of the body.

An equation that holds at each point $\mathbf{y}$ is said to be "local" while one that holds for each part $\mathcal{D}$ is said to be "global". Global statements such as (3.8) and (3.9) are convenient when formulating the basic balance principles. When solving a specific boundary-value problem however it is more useful to have a local version of that principle. The local statement corresponding to a balance law is said to be the associated **field equation**.

From the discussion in Section 3.1 we know that the integrand of the surface integral term in (3.8) depends on the unit normal vector $\mathbf{n}$. If this dependence is linear, and we do not yet know if this is true, then the integrand would have the form $\mathsf{A}\mathbf{n}$ where A is some 2-tensor. In this event we can use the divergence theorem to rewrite the surface integral as a volume integral, and the equation would have the form of a single volume integral over $\mathcal{D}$ that is to vanish. Since this balance law is to hold for all parts $\mathcal{D}$ of the body, then provided the integrand is continuous, we conclude by localization (Section 1.8.3) that the integrand itself must vanish at each point $\mathbf{y} \in \mathcal{R}$. This leads to the field equation associated with (3.8). We could simplify (3.9) similarly. This is what we shall carry out in Section 3.4 below, but before we do that we must show that the traction $\mathbf{t}(\mathbf{y}, \mathbf{n})$ depends linearly on $\mathbf{n}$.

Example: Show that force and moment balance, (3.8) and (3.9), hold if and only if the rate of working of the tractions and body forces vanishes in all steady rigid motions.

Solution The rate of working (power) of the tractions and body forces is given by (3.3) where $\mathbf{v}$ is particle velocity. In the special case of a steady rigid motion, the velocity is given by equation (2.97) of Problem 2.8.4:

$$\mathbf{v}(\mathbf{y}) = \mathbf{c} + \boldsymbol{\omega} \times \mathbf{y}, \tag{i}$$

where the constant vectors $\boldsymbol{\omega}$ and $\mathbf{c}$ represent the angular and translational velocities respectively. Substituting (i) into (3.3) gives the rate of working in a steady rigid motion to be

$$\mathbb{P}_{\text{rigid}} = \int_{\partial\mathcal{D}} \mathbf{t} \cdot (\mathbf{c} + \boldsymbol{\omega} \times \mathbf{y})\, dA_y + \int_{\mathcal{D}} \mathbf{b} \cdot (\mathbf{c} + \boldsymbol{\omega} \times \mathbf{y})\, dV_y,$$

which we can write as

$$\mathbb{P}_{\text{rigid}} = \mathbf{c} \cdot \left[\int_{\partial\mathcal{D}} \mathbf{t}\, dA_y + \int_{\mathcal{D}} \mathbf{b}\, dV_y \right] + \left[\int_{\partial\mathcal{D}} \mathbf{t} \cdot (\boldsymbol{\omega} \times \mathbf{y})\, dA_y + \int_{\mathcal{D}} \mathbf{b} \cdot (\boldsymbol{\omega} \times \mathbf{y})\, dV_y \right].$$

This can be rewritten using the vector identity $\mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) = \mathbf{q} \cdot (\mathbf{r} \times \mathbf{p})$ as

$$\mathbb{P}_{\text{rigid}} = \mathbf{c} \cdot \left[\int_{\partial\mathcal{D}} \mathbf{t}\, dA_y + \int_{\mathcal{D}} \mathbf{b}\, dV_y \right] + \left[\int_{\partial\mathcal{D}} \boldsymbol{\omega} \cdot (\mathbf{y} \times \mathbf{t})\, dA_y + \int_{\mathcal{D}} \boldsymbol{\omega} \cdot (\mathbf{y} \times \mathbf{b})\, dV_y \right],$$

from which we conclude that

$$\mathbb{P}_{\text{rigid}} = \mathbf{c} \cdot \left[\int_{\partial\mathcal{D}} \mathbf{t}\, dA_y + \int_{\mathcal{D}} \mathbf{b}\, dV_y \right] + \boldsymbol{\omega} \cdot \left[\int_{\partial\mathcal{D}} \mathbf{y} \times \mathbf{t}\, dA_y + \int_{\mathcal{D}} \mathbf{y} \times \mathbf{b}\, dV_y \right]. \qquad (ii)$$

Therefore when force and moment balance, (3.8) and (3.9), hold, it follows from (ii) that the rate of working vanishes: $\mathbb{P}_{\text{rigid}} = 0$. Conversely if the rate of working vanishes is every steady rigid motion, i.e. if $\mathbb{P}_{\text{rigid}} = 0$ for all vectors $\mathbf{c}$ and $\boldsymbol{\omega}$, it follows from (ii) that force and moment balance necessarily hold.

3.3 Consequences of force balance. Stress.

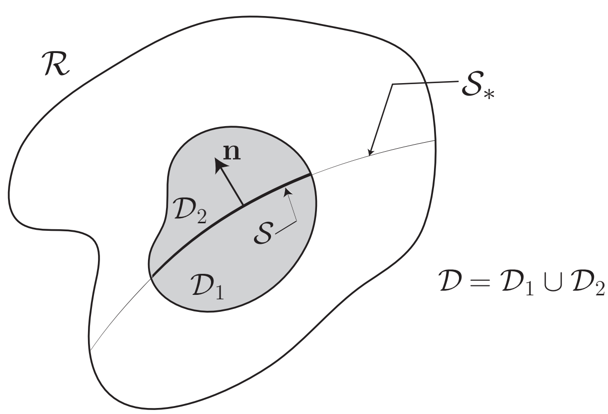

Figure 3.6: A surface $\mathcal{S}_*$ contained within $\mathcal{R}$ intersects the sub-region $\mathcal{D}$ and separates it into two parts $\mathcal{D}_1$ and $\mathcal{D}_2$.

We now explore several implications of force balance. The focus in this section is on how the traction vector $\mathbf{t}(\mathbf{y}, \mathbf{n})$ depends on the unit vector $\mathbf{n}$. The position $\mathbf{y}$ will play no central role in our discussion and so it will be convenient to suppress $\mathbf{y}$ and write $\mathbf{t}(\mathbf{n})$ instead of $\mathbf{t}(\mathbf{y}, \mathbf{n})$.

Consequence (1): The "equal and opposite" property $\mathbf{t}(\mathbf{n}) = -\mathbf{t}(-\mathbf{n})$.

Let $\mathcal{S}_*$ be an arbitrary surface contained within the region $\mathcal{R}$ occupied by the deformed body as shown in Figure 3.6. Pick a sub-region $\mathcal{D}$ that is intersected by $\mathcal{S}_*$ and is thus separated into regions $\mathcal{D}_1$ and $\mathcal{D}_2$: $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. $\mathcal{S}$ is the portion of $\mathcal{S}_*$ that is contained within $\mathcal{D}$ and is therefore the interface between $\mathcal{D}_1$ and $\mathcal{D}_2$. Note that the unit normal vector $\mathbf{n}$ on $\mathcal{S}$ shown in the figure is outward to $\mathcal{D}_1$ whereas $-\mathbf{n}$ is outward to $\mathcal{D}_2$. Thus when force balance (3.8) is applied to $\mathcal{D}_1$, the traction term will involve the integral of $\mathbf{t}(\mathbf{n})$ over $\mathcal{S}$, whereas when it is applied to $\mathcal{D}_2$, it will involve the integral of $\mathbf{t}(-\mathbf{n})$ over $\mathcal{S}$. We now apply (3.8) to each of the regions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}$ individually, and then subtract the first two of the resulting equations from the third. This leads to (Exercise)

$$\int_{\mathcal{S}} [\mathbf{t}(\mathbf{n}) + \mathbf{t}(-\mathbf{n})] \; dA_y \;=\; 0. \tag{3.10}$$

Since this must hold for arbitrary choices of $\mathcal{D}$, and therefore for arbitrary choices of $\mathcal{S}$, it follows by localization[6] that the integrand must vanish at each point on $\mathcal{S}$. Thus we conclude that

$$\mathbf{t}(-\mathbf{n}) = -\mathbf{t}(\mathbf{n}) \tag{3.11}$$

for all unit vectors $\mathbf{n}$.

Observe that this is the analog for a continuum of Newton's third law for particles. It says that the traction exerted on the positive side of a surface by the negative side, is equal in magnitude and opposite in direction to the traction exerted on the negative side by the positive side. While this appears to be a consequence of force balance and not a separate postulate, it is in fact implicitly buried within the assumption that the force on $\mathcal{D}$ is given by (3.1).

Consequence (2): The traction $\mathbf{t}(\mathbf{n})$ is a linear function of $\mathbf{n}$.

We now derive an expression for the traction on a plane oriented in an arbitrary direction $\mathbf{n}$ in terms of the tractions on three mutually orthogonal planes, e.g. planes normal to the basis vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$. This leads to a second, critically important, consequence of force balance, namely that the traction vector $\mathbf{t}(\mathbf{n})$ depends *linearly* on the normal vector $\mathbf{n}$. This is called Cauchy's Theorem.

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[6] See Section 1.8.3 for the volume integral version of localization.

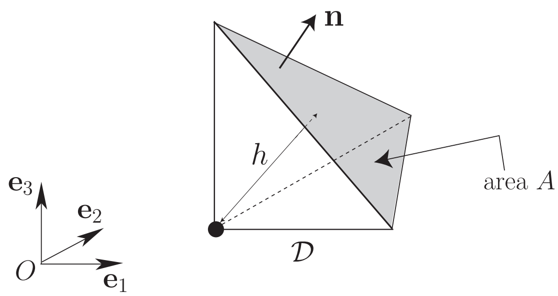


Figure 3.7: Tetrahedral subregion $\mathcal{D}$ of the body.

In order to establish this, consider the tetrahedral subregion $\mathcal{D}$ shown in Figure 3.7 with three of its faces parallel to the coordinate planes. Observe that the unit outward normal vectors to the four faces of $\mathcal{D}$ are $\mathbf{n}$, $-\mathbf{e}_1$, $-\mathbf{e}_2$ and $-\mathbf{e}_3$. Moreover, if the area of the shaded face is A, one can readily show from geometry that the area, A_k, of the face normal to $\mathbf{e}_k$ is $n_k A$. Next we apply force balance to this tetrahedron, and take the limit of the resulting equation as the height $h \to 0$ keeping the orientations of all faces fixed. In this limit the volumetric term (which involves the body force) approaches zero like h^3 whereas the area terms (which involve the traction) approach zero like h^2. Therefore the volumetric term goes to zero faster than the area terms and so only the area terms survive in this limit leading to

$$\lim_{h\to 0}\; A\mathbf{t}(\mathbf{n}) + A_1\mathbf{t}(-\mathbf{e}_1) + A_2\mathbf{t}(-\mathbf{e}_2) + A_3\mathbf{t}(-\mathbf{e}_3) = \mathbf{0}. \tag{3.12}$$

Because of (3.11) and $A_k = n_k A$, this leads to

$$\mathbf{t}(\mathbf{n}) = n_1\mathbf{t}(\mathbf{e}_1) + n_2\mathbf{t}(\mathbf{e}_2) + n_3\mathbf{t}(\mathbf{e}_3) = \mathbf{t}(\mathbf{e}_k)n_k. \tag{3.13}$$

Equation (3.13) tells us that *if we know the tractions on three mutually orthogonal planes*, for example the planes normal to the basis vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$, *we can calculate the traction on any other plane from that information alone.* Observe by writing (3.13) as

$$\mathbf{t}(n_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3) = n_1\mathbf{t}(\mathbf{e}_1)+n_2\mathbf{t}(\mathbf{e}_2)+n_3\mathbf{t}(\mathbf{e}_3) \qquad \text{for all } n_1, n_2, n_3 \text{ with } n_1^2+n_2^2+n_3^2 = 1,$$

that $\mathbf{t}(\mathbf{n})$ is a linear function of $\mathbf{n}$ on the set of all unit vectors.

Consequence (3): The stress tensor $\mathbf{T}$.

As observed above, in order to calculate the traction on an arbitrary plane we only need know the tractions $\mathbf{t}(\mathbf{e}_1), \mathbf{t}(\mathbf{e}_2), \mathbf{t}(\mathbf{e}_3)$ on the three coordinate planes. It is natural therefore to "label" the components of these three traction vectors. Since each traction vector has three components, we have a total of nine components to label.

Accordingly let $T_{ij}, i, j = 1, 2, 3$, be the *ith* component of the traction $\mathbf{t}(\mathbf{e}_j)$:

$$T_{ij} \coloneqq t_i(\mathbf{e}_j) = \mathbf{t}(\mathbf{e}_j) \cdot \mathbf{e}_i.$$

This is illustrated in Figure 3.8(a). Note that the second subscript of T_{ij} identifies the surface on which the traction acts and the first identifies the direction of that traction component. Thus each T_{ij} represents a force per unit deformed area acting on a particular coordinate plane in a particular direction. An equivalent way in which to write the preceding equation is $\mathbf{t}(\mathbf{e}_j) = T_{ij}\mathbf{e}_i$. Thus we have

$$\boxed{T_{ij} \coloneqq t_i(\mathbf{e}_j) = \mathbf{t}(\mathbf{e}_j) \cdot \mathbf{e}_i, \qquad\qquad \mathbf{t}(\mathbf{e}_j) = T_{ij}\mathbf{e}_i.} \tag{3.14}$$

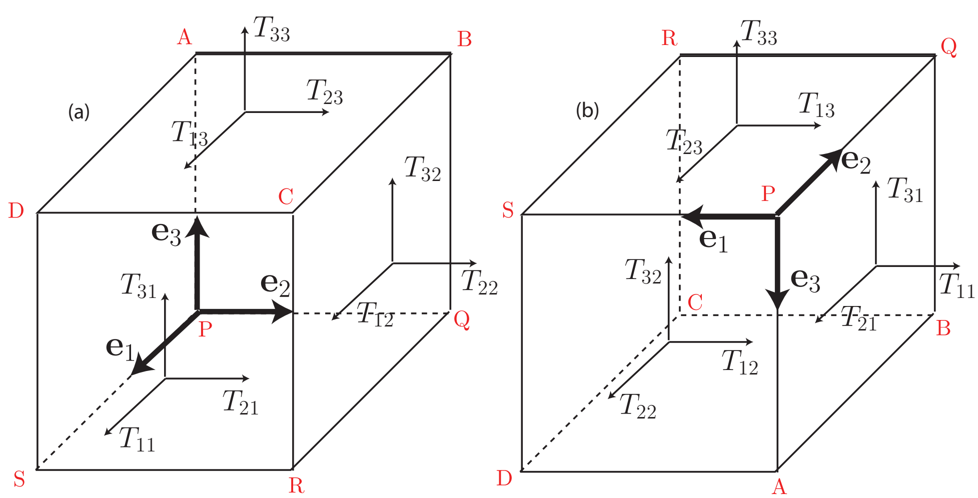


Figure 3.8: The figure shows two views of the same cubic region and the same basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Figure (a) shows the stress components T_{ij} acting on the three faces with outward normals $+\mathbf{e}_1$, $+\mathbf{e}_2$ and $+\mathbf{e}_3$. Observe how the figure is consistent with $T_{ij} = \mathbf{t}(\mathbf{e}_j) \cdot (\mathbf{e}_i)$. Figure (b) shows the stress components T_{ij} acting on the three faces with outward normals $-\mathbf{e}_1$, $-\mathbf{e}_2$ and $-\mathbf{e}_3$. Note in this case the consistency with $T_{ij} = \mathbf{t}(-\mathbf{e}_j) \cdot (-\mathbf{e}_i)$.

In order to determine the traction components on a face whose outward normal is in the *negative* j^{th}-direction we observe from (3.11), $(3.14)_2$ that

$$\mathbf{t}(-\mathbf{e}_j) = -\mathbf{t}(\mathbf{e}_j) = -T_{ij}\mathbf{e}_i = T_{ij}(-\mathbf{e}_i). \tag{3.15}$$

Therefore the force/area acting on a surface with unit normal $-\mathbf{e}_j$, in the direction $-\mathbf{e}_i$, is T_{ij}. This is illustrated in Figure 3.8(b).

The 9 elements T_{ij} may be assembled into a matrix $[T]$. The elements T_{11}, T_{22} and T_{33} on the diagonal of $[T]$ are known as the **normal stress** components; the off-diagonal terms $T_{ij}, i \neq j$, are the **shear stress** components.

We now return to the expression (3.13) for the traction on an arbitrary surface and substitute (3.14) and $n_j = \mathbf{e}_j \cdot \mathbf{n}$ into it:

$$\mathbf{t}(\mathbf{n}) \overset{(3.13)}{=} \mathbf{t}(\mathbf{e}_j)n_j \overset{(3.14)}{=} (T_{ij}\mathbf{e}_i)n_j = T_{ij}n_j\mathbf{e}_i = T_{ij}(\mathbf{e}_j \cdot \mathbf{n})\mathbf{e}_i = T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{n} = \Big(T_{ij}\,\mathbf{e}_i \otimes \mathbf{e}_j\Big)\mathbf{n}\,. \tag{3.16}$$

Let $\mathbf{T}$ be the second-order tensor whose components in the basis $\{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3\}$ are T_{ij}, i.e[7].

$$\boxed{\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j.} \tag{3.17}$$

It now follows that (3.16) can be written as $\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n}$, or by writing out all the arguments:

$$\boxed{\mathbf{t}(\mathbf{y}, \mathbf{n}) = \mathbf{T}(\mathbf{y})\mathbf{n}.} \tag{3.18}$$

In terms of components,

$$t_i(\mathbf{n}) = T_{ij}n_j, \qquad\qquad \{t\} = [T]\{n\}. \tag{3.19}$$

The tensor $\mathbf{T}(\mathbf{y})$ is known as the **Cauchy stress tensor**. Observe that $\mathbf{T}$ *does not depend on* the normal vector $\mathbf{n}$. Therefore we may speak of the stress *at a point*. In contrast, when speaking of traction, we must speak of the traction *on a surface through a point*. When $\mathbf{T}(\mathbf{y})$ is known, equation (3.18) can be used to calculate the traction $\mathbf{t}(\mathbf{y}, \mathbf{n})$ on *any* plane through $\mathbf{y}$. The equilibrium principle of moment balance will show that $\mathbf{T}$ is symmetric.

As noted earlier, the component T_{ij} of the stress tensor represents the i^{th} component of the force per unit area acting on a surface whose normal is in the j^{th} direction. It is worth reiterating that we have been concerned with the region occupied by the *deformed* body and therefore (a) the surface referenced above must be normal to the j^{th} direction in the *deformed* configuration, and (b) the area referenced above refers to area in the *deformed* configuration. The middle figure in Figure 3.16 illustrates this in a special case.

The normal stress and the magnitude of the resultant shear stress introduced in (3.6) and (3.7) can now be written in the following respective forms with the dependence on $\mathbf{n}$ made explicit:

$$T_{\text{normal}}(\mathbf{n}) = \mathbf{T}\mathbf{n} \cdot \mathbf{n}\,, \tag{3.20}$$

[7]See Section 1.4.3, in particular (1.200).

$$T_{\text{shear}}(\mathbf{n}) = \sqrt{\big(\mathbf{Tn}\cdot\mathbf{Tn}\big) - \big(\mathbf{Tn}\cdot\mathbf{n}\big)^2}. \tag{3.21}$$

In Section 3.5 we shall discuss the maximum values of these two quantities (over all unit vectors $\mathbf{n}$).

In cylindrical polar coordinates (for example) we have

$$\begin{aligned}\mathbf{T} \;&= T_{rr}\mathbf{e}_r\otimes\mathbf{e}_r + T_{r\theta}\mathbf{e}_r\otimes\mathbf{e}_\theta + T_{rz}\mathbf{e}_r\otimes\mathbf{e}_z +\\ &\quad + T_{\theta r}\,\mathbf{e}_\theta\otimes\mathbf{e}_r + T_{\theta\theta}\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta + T_{\theta z}\,\mathbf{e}_\theta\otimes\mathbf{e}_z +\\ &\quad + T_{zr}\,\mathbf{e}_z\otimes\mathbf{e}_r + T_{z\theta}\,\mathbf{e}_z\otimes\mathbf{e}_\theta + T_{zz}\,\mathbf{e}_z\otimes\mathbf{e}_z.\end{aligned} \tag{3.22}$$

Consider for example a circular cylindrical body, the outward unit normal vector on the curved surface being $\mathbf{e}_r$. The traction on this surface is

$$\mathbf{t}(\mathbf{e}_r) = \mathbf{Te}_r \overset{(3.22)}{=} T_{rr}\mathbf{e}_r + T_{\theta r}\,\mathbf{e}_\theta + T_{zr}\,\mathbf{e}_z,$$

and we see (again) that the first subscript tells us the direction of a traction component and the second indicates the surface on which it acts.

3.3.1 Some particular stress tensors.

Consider the stress tensor $\mathbf{T}(\mathbf{y})$ at a particular point $\mathbf{y}$. Let T_{ij} be the components of $\mathbf{T}$ in an orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$.

– *Uniaxial stress.* The particular case where the only nonzero component of stress is $T_{11} = T$, i.e.

$$\mathbf{T} = \mathrm{T}\,\mathbf{e}_1\otimes\mathbf{e}_1, \qquad [T] = \begin{pmatrix} T & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

describes a uniaxial stress in the $\mathbf{e}_1$-direction. Observe that the traction on an arbitrary plane is $\mathbf{t}(\mathbf{n}) = \mathbf{Tn} = T(\mathbf{e}_1\otimes\mathbf{e}_1)\mathbf{n} = T\,(\mathbf{n}\cdot\mathbf{e}_1)\,\mathbf{e}_1 = Tn_1\,\mathbf{e}_1$. Thus the traction on *every* plane acts in the $\mathbf{e}_1$-direction, though its value depends on the plane (through n_1). A uniaxial stress of magnitude T in some direction $\mathbf{m}$ is described by

$$\mathbf{T} = \mathrm{T}\,\mathbf{m}\otimes\mathbf{m}.$$

– *Hydrostatic stress.* The special case where $\mathbf{T}$ has the form

$$\mathbf{T} = T\,\mathbf{I}, \qquad [T] = \begin{pmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{pmatrix},$$

describes a hydrostatic stress (a pure pressure $-T$). Observe that the traction on an arbitrary plane is $\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n} = T\,\mathbf{n}$. Thus the traction on every plane acts in the direction normal to that plane and has magnitude T.

– *Pure shear.* Finally,

$$\mathbf{T} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \qquad [T] = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

describes a *pure shear* stress state with respect to the $\mathbf{e}_1$, $\mathbf{e}_2$ directions. A pure shear with respect to an arbitrary pair of orthogonal directions $\mathbf{a}$ and $\mathbf{b}$ is described by

$$\mathbf{T} = \tau(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}).$$

3.3.2 Worked examples.

Problem 3.3.1. The surface $\mathcal{S}$ in Figure 3.9 with unit outward normal vector $\mathbf{e}_2$ is traction-free. Therefore

$$\mathbf{t}(\mathbf{e}_2) = \mathbf{o} \quad \stackrel{(3.14)_1}{\Rightarrow} \quad T_{k2} = 0 \quad \Rightarrow \quad T_{12} = T_{22} = T_{32} = 0 \quad \text{on } \mathcal{S}.$$

Therefore the three components of stress T_{12}, T_{22}, T_{32} must vanish on $\mathcal{S}$. Note that *it is not necessary* that the remaining stress components vanish. The surface $\mathcal{S}$ is traction-free but that does not mean a point on the surface has to be stress-free. That is, $\mathbf{t}(\mathbf{n}) = \mathbf{o}$ for some $\mathbf{n}$ does not imply $\mathbf{T} = \mathbf{0}$.

Problem 3.3.2. The region $\mathcal{R}$ occupied by a body in the deformed configuration is a prismatic cylinder whose cross section is an equilateral triangle as shown in Figure 3.10. Determine the normal and shear traction components that must be applied (as shown in the left-hand figure) such that the Cauchy stress tensor is a pure shear

$$\mathbf{T} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \qquad (i)$$

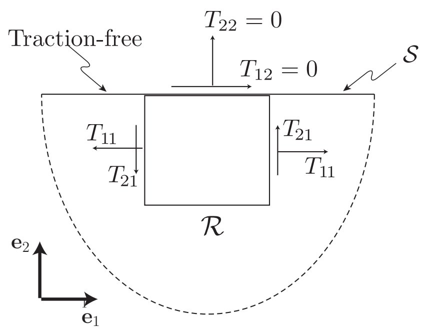


Figure 3.9: The surface $\mathcal{S}$, perpendicular to $\mathbf{e}_2$, is traction-free, and so the stress components $T_{12} = T_{22} = T_{32} = 0$ on $\mathcal{S}$. However the remaining stress components need not vanish on $\mathcal{S}$: the surface is traction-free but a point on the surface might not be stress-free.

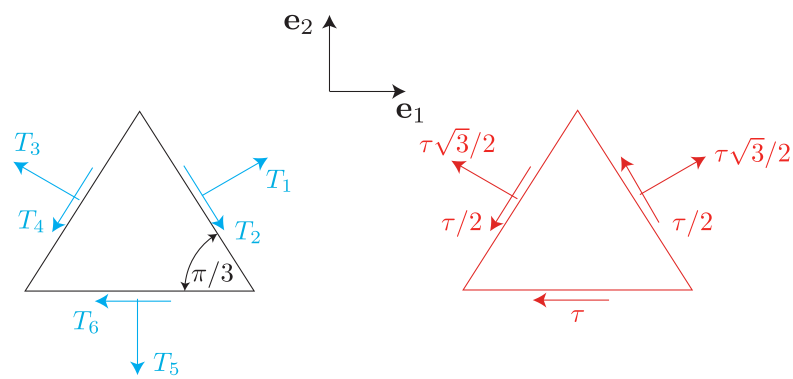


Figure 3.10: Left: Traction components to be determined.

Solution: First consider the bottom surface. The unit outward normal is $\mathbf{n} = -\mathbf{e}_2$ and so the traction vector on this surface is

$$\mathbf{t} = \mathbf{T}\mathbf{n} = \Big[\tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)\Big](-\mathbf{e}_2) = -\tau\mathbf{e}_1. \qquad (ii)$$

In the figure, this traction is shown as $\mathbf{t} = -T_6\mathbf{e}_1 - T_5\mathbf{e}_2$ which when compared with (ii) gives $T_5 = 0, T_6 = \tau$.

Next consider the upper right-hand surface. The unit outward normal vector is

$$\mathbf{n} = (\sqrt{3}/2)\mathbf{e}_1 + (1/2)\mathbf{e}_2, \qquad (iii)$$

and so the traction on this surface is

$$\mathbf{t} = \mathbf{T}\mathbf{n} \overset{(i),(iii)}{=} \Big[\tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)\Big]\left(\frac{\sqrt{3}}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2\right) = \frac{\tau}{2}\mathbf{e}_1 + \frac{\tau\sqrt{3}}{2}\mathbf{e}_2. \qquad (iv)$$

The traction component T_1 in the figure is in the direction $\mathbf{n}$ and so

$$T_1 = \mathbf{t}\cdot\mathbf{n} \overset{(iv),(iii)}{=} \left(\frac{\tau}{2}\mathbf{e}_1 + \frac{\tau\sqrt{3}}{2}\mathbf{e}_2\right)\Big((\sqrt{3}/2)\mathbf{e}_1 + (1/2)\mathbf{e}_2\Big) = \frac{\tau\sqrt{3}}{2}. \qquad (v)$$

The traction component T_2 in the figure is in the direction

$$(1/2)\mathbf{e}_1 - (\sqrt{3}/2)\mathbf{e}_2, \qquad (vi)$$

(that is perpendicular to $\mathbf{n}$) and so

$$T_2 = \mathbf{t}\cdot\left(\frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2\right) \stackrel{(iv),(vi)}{=} \left(\frac{\tau}{2}\mathbf{e}_1 + \frac{\tau\sqrt{3}}{2}\mathbf{e}_2\right)\cdot\left(\frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2\right) = -\frac{\tau}{2}.$$

A similar calculation gives

$$T_3 = -\tau\sqrt{3}/2, \qquad T_4 = \tau/2.$$

The right-hand figure in Figure 3.10 displays these results. As an exercise you may wish to confirm force and moment equilibrium.

Problem 3.3.3. The stress tensor $\mathbf{T}$ at a particular point in a certain body corresponds to a state of pure shear of magnitude τ with respect to the directions $\mathbf{e}_1', \mathbf{e}_2'$. As shown in Figure 3.11, the vectors $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ are obtained by rotating the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ through an angle $\pi/4$ about $\mathbf{e}_3$. Calculate the components of $\mathbf{T}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Solution: Since $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ is obtained by rotating $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ through an angle $\pi/4$ about $\mathbf{e}_3$,

$$\mathbf{e}_1' = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2, \qquad \mathbf{e}_2' = -\frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2, \qquad \mathbf{e}_3' = \mathbf{e}_3. \qquad (i)$$

We are told that

$$\mathbf{T} = \tau(\mathbf{e}_1'\otimes\mathbf{e}_2' + \mathbf{e}_2'\otimes\mathbf{e}_1'). \qquad (ii)$$

Substituting (i) into (ii) and simplifying, for example

$$\mathbf{e}_1'\otimes\mathbf{e}_2' = \left(\frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2\right)\otimes\left(-\frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2\right) = -\frac{1}{2}\mathbf{e}_1\otimes\mathbf{e}_1 - \frac{1}{2}\mathbf{e}_2\otimes\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_1\otimes\mathbf{e}_2 + \frac{1}{2}\mathbf{e}_2\otimes\mathbf{e}_2,$$

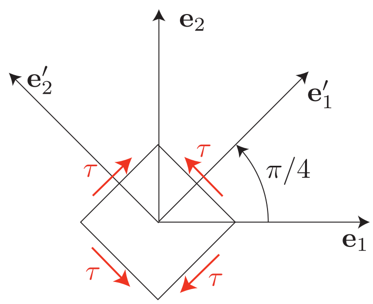
leads to

$$\mathbf{T} = -\tau\mathbf{e}_1\otimes\mathbf{e}_1 + \tau\mathbf{e}_1\otimes\mathbf{e}_1. \qquad (iii)$$

Thus the matrix of components of $\mathbf{T}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is

$$[T] = \begin{pmatrix} -\tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (iv)$$

Observe from (iv) that $\mathbf{T}$ can be viewed as the superposition of a uniaxial compressive stress τ in the $\mathbf{e}_1$-direction and a uniaxial tensile stress τ in the $\mathbf{e}_2$-direction (when $\tau > 0$). We also know that (this same stress tensor) $\mathbf{T}$ can be viewed as a pure shear with respect to $\mathbf{e}_1', \mathbf{e}_2'$. This is depicted in Figure 3.11. This example illustrates how the components of $\mathbf{T}$ depend on the basis.



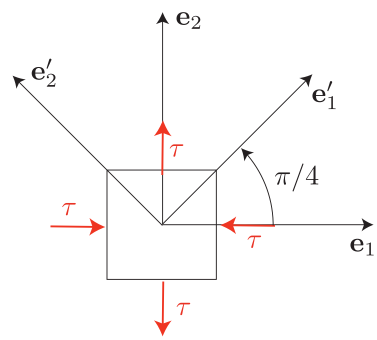


Figure 3.11: Left: The stress tensor $\mathbf{T}$ is a simple shear of magnitude τ with respect to the directions $\mathbf{e}_1', \mathbf{e}_2'$. Right: Equivalently it is the superposition of a uniaxial compressive stress τ in the $\mathbf{e}_1$-direction and a uniaxial tensile stress τ in the $\mathbf{e}_2$-direction. (Problem 3.3.3)

Problem 3.3.4. (Continued in Problem 3.24.) Consider a body that in the deformed configuration occupies the annular sector $a \le r \le b, -\beta \le \theta \le \beta, -1/2 \le z \le 1/2$ shown in Figure 3.12. We are using cylindrical polar coordinates (r, θ, z) in the deformed configuration with associated basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. Assume that the Cauchy stress tensor is symmetric and that the components $T_{zr} = T_{z\theta} = T_{zz} = 0$ so that

$$\mathbf{T} = T_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + T_{r\theta}(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + T_{\theta\theta}\, \mathbf{e}_\theta \otimes \mathbf{e}_\theta. \qquad (i)$$

Assume further that the remaining stress components depend on r and θ (but not z).

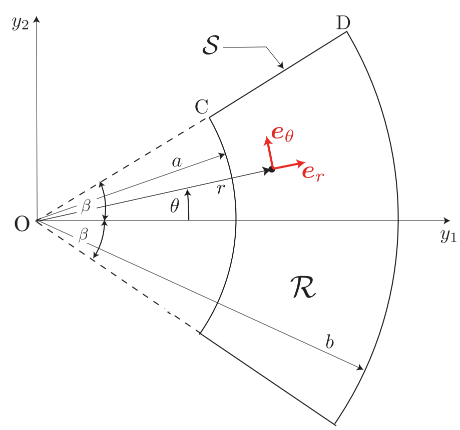


Figure 3.12: In the deformed configuration the body occupies the annular sector $\mathcal{R} = \{(r, \theta, z) : a \le r \le b, -\beta \le \theta \le \beta, -1/2 \le z \le 1/2\}$. (Figure for Problem 3.3.4 .)

Determine the restrictions on the stress components $T_{rr}(r, \theta), T_{r\theta}(r, \theta)$ and $T_{\theta\theta}(r, \theta)$ arising from the following requirements: (a) the outer curved boundary $r = b$ is traction-free; (b) the inner curved boundary $r = a$ is also traction-free; (c) the resultant force on the top inclined surface $\mathcal{S}$ vanishes; and (d) the resultant moment on $\mathcal{S}$ about O is $m\mathbf{e}_z$.

Later, in Problem 3.24, after we have developed the equilibrium equations, we will explore the consequences of equilibrium.

The kinematics of the bending of a rectangular block into a shape like the one shown in Figure 3.12 was examined previously in Problem 2.5.4. Here we are not told what the undeformed configuration of the body is.

Solution: First consider the outer curved surface $r = b$. Since the outward pointing unit normal vector to it is $\mathbf{e}_r$, the traction $\mathbf{T}\mathbf{e}_r$ acting on this surface has radial and circumferential components T_{rr} and $T_{\theta r}$ respectively. Since (each point on) this surface is traction-free, it follows that

$$T_{rr}(b,\theta) = T_{\theta r}(b,\theta) = 0 \qquad \text{for all} \quad -\beta \le \theta \le \beta. \tag{ii}$$

Likewise, at the traction-free inner curved surface $r = a$ we have

$$T_{rr}(a,\theta) = T_{\theta r}(a,\theta) = 0 \qquad \text{for all} \quad -\beta \le \theta \le \beta. \tag{iii}$$

Next consider the flat inclined surface $\mathcal{S}$ on which $\theta = \beta$. Keep in mind that, in general, $\mathbf{e}_r = \mathbf{e}_r(\theta), \mathbf{e}_\theta = \mathbf{e}_\theta(\theta)$. The outward pointing unit vector on it is $\mathbf{e}_\theta$, or more precisely $\mathbf{e}_\theta(\beta)$, and so from (i), the traction on $\mathcal{S}$ is

$$\mathbf{t} = \mathbf{T}\mathbf{e}_\theta\Big|_{\theta=\beta} = \big(T_{r\theta}\,\mathbf{e}_r + T_{\theta\theta}\,\mathbf{e}_\theta\big)\Big|_{\theta=\beta} = T_{r\theta}(r,\beta)\,\mathbf{e}_r(\beta) + T_{\theta\theta}(r,\beta)\,\mathbf{e}_\theta(\beta). \tag{iv}$$

The resultant force on this surface is therefore

$$\begin{aligned}\int_{\mathcal{S}} \mathbf{t}\,dA_y = \int_a^b\int_{-1/2}^{1/2} \mathbf{t}\,dzdr = \int_a^b \mathbf{t}\,dr \quad &= \int_a^b \Big[T_{r\theta}(r,\beta)\,\mathbf{e}_r(\beta) + T_{\theta\theta}(r,\beta)\,\mathbf{e}_\theta(\beta)\Big]\,dr = \\ &= \left[\int_a^b T_{r\theta}(r,\beta)\,dr\right]\mathbf{e}_r(\beta) + \left[\int_a^b T_{\theta\theta}(r,\beta)\,dr\right]\mathbf{e}_\theta(\beta).\end{aligned}$$

Since the resultant force on this surface vanishes it follows that

$$\int_a^b T_{r\theta}(r,\beta)\,dr = 0, \qquad \int_a^b T_{\theta\theta}(r,\beta)\,dr = 0. \tag{v}$$

The resultant moment on $\mathcal{S}$ about O is

$$\begin{aligned}\int_{\mathcal{S}} \mathbf{y}\times\mathbf{t}\,dA_y = \int_a^b\int_{-1/2}^{1/2} \mathbf{y}\times\mathbf{t}\,dzdr = \int_a^b \mathbf{y}\times\mathbf{t}\,dr \quad &= \int_a^b r\,\mathbf{e}_r(\beta)\times\Big[T_{r\theta}(r,\beta)\,\mathbf{e}_r(\beta) + T_{\theta\theta}(r,\beta)\,\mathbf{e}_\theta(\beta)\Big]\,dr = \\ &= \left[\int_a^b rT_{\theta\theta}(r,\beta)\,dr\right]\mathbf{e}_z, \qquad (vi)\end{aligned}$$

where we have used $\mathbf{e}_r\times\mathbf{e}_r = \mathbf{o}$ and $\mathbf{e}_r\times\mathbf{e}_\theta = \mathbf{e}_z$. (Please derive (vi) using physical arguments, without using the vector cross-product.) We are told that the resultant moment on this surface is $m\mathbf{e}_z$ and so

$$\left[\int_a^b rT_{\theta\theta}(r,\beta)\,dr\right] = m. \tag{vii}$$

3.4 Field equations.

We are now in a position to derive the local versions – the field equations – of the global equilibrium principles for force and moment balance (3.8) and (3.9).

Consequence (4): Equilibrium equations.

Consider *force balance* (3.8), which in component form reads

$$\int_{\partial\mathcal{D}} t_i \,\mathrm{d}A_y + \int_{\mathcal{D}} b_i \,\mathrm{d}V_y = 0 \ . \tag{3.23}$$

Considering the first term, we first trade traction for stress using (3.18), and then convert the surface integral into a volume integral by using the divergence theorem:

$$\int_{\partial\mathcal{D}} t_i \,\mathrm{d}A_y = \int_{\partial\mathcal{D}} T_{ij}n_j \,\mathrm{d}A_y = \int_{\mathcal{D}} \frac{\partial T_{ij}}{\partial y_j} \,dV_y, \tag{3.24}$$

and so (3.23) yields

$$\int_{\mathcal{D}} \left(\frac{\partial T_{ij}}{\partial y_j} + b_i \right) \mathrm{d}V_y = 0 \ . \tag{3.25}$$

Since this must hold for all parts $\mathcal{D}$ of the body, and assuming the integrand to be continuous, it follows by localization (Section 1.8.3) that the integrand itself must vanish at each point in $\mathcal{R}$. We thus conclude that

$$\frac{\partial T_{ij}}{\partial y_j} + b_i \ = \ 0 \qquad \text{at each } \mathbf{y} \in \mathcal{R}, \tag{3.26}$$

which can be written in basis-free form by using (1.168) (page 69) as

$$\mathrm{div}\,\mathbf{T} + \mathbf{b} = \mathbf{0} \qquad \text{at each } \mathbf{y} \in \mathcal{R}. \tag{3.27}$$

The *equilibrium equation* (3.27) is the field equations corresponding to force balance. It must hold at each point in the body.

Conversely, when the equilibrium equation (3.27) and the traction-stress relation (3.18) hold, then the global force balance law (3.8) holds. (Show this.)

See also Problem 10.4.1.

Consequence (5): Symmetry of the stress tensor[8].

[8]Problem 3.17 derives this result without using components in a basis.

We turn next to *moment balance* (3.9). Recall that for any two vectors $\mathbf{a}$ and $\mathbf{b}$, the i^{th} component of the vector $\mathbf{a}\times\mathbf{b}$ is $e_{ijk}a_jb_k$ where e_{ijk} is the Levi-Civita symbol introduced in (1.38). Thus we can write (3.9) in component form as

$$\int_{\partial\mathcal{D}}(\mathbf{y}\times\mathbf{t})_i\,\mathrm{d}A_y+\int_{\mathcal{D}}(\mathbf{y}\times\mathbf{b})_i\,\mathrm{d}V_y=\int_{\partial\mathcal{D}}e_{ijk}\,y_j\,t_k\,\mathrm{d}A_y+\int_{\mathcal{D}}e_{ijk}\,y_jb_k\,\mathrm{d}V_y=0\ . \tag{3.28}$$

The term involving the traction can be simplified by first using the traction-stress relation (3.19), then using the divergence theorem and finally expanding the result. This leads to

$$\begin{aligned}\int_{\partial\mathcal{D}}e_{ijk}\,y_j\,t_k\,\mathrm{d}A_y &= \int_{\partial\mathcal{D}}e_{ijk}\,y_j\,T_{km}n_m\,\mathrm{d}A_y=\int_{\mathcal{D}}e_{ijk}\,\frac{\partial}{\partial y_m}(y_j\,T_{km})\,\mathrm{d}V_y\\ &= \int_{\mathcal{D}}e_{ijk}\left(\delta_{jm}\,T_{km}+y_j\,\frac{\partial T_{km}}{\partial y_m}\right)\,\mathrm{d}V_y\ .\end{aligned} \tag{3.29}$$

Substituting (3.29) into (3.28) and making use of the equilibrium equation (3.26) now yields

$$\int_{\mathcal{D}}e_{ijk}\,T_{kj}\,\mathrm{d}V_y=0\ . \tag{3.30}$$

Since (3.30) must hold for all choices of D, it follows by localization that

$$e_{ijk}\,T_{kj}=0 \qquad \text{at each } \mathbf{y}\in\mathcal{R}. \tag{3.31}$$

One way in which to see what the three scalar equations (3.31) imply is to write them out explicitly. For example for $i=1$ we have $e_{1jk}\,T_{kj}=e_{123}\,T_{32}+e_{132}\,T_{23}$ because all of the other e_{ijk} terms have at least two repeated subscripts and thus vanish. Since $e_{123}=1$ and $e_{132}=-1$ it now follows that $e_{1jk}\,T_{kj}=T_{32}-T_{23}$ and therefore (3.31) implies that $T_{23}=T_{32}$. The cases $i=2$ and $i=3$ can be dealt with similarly. Thus we conclude that

$$T_{12}=T_{21},\qquad T_{23}=T_{32},\qquad T_{31}=T_{13},$$

which we can write as

$$T_{ij}=T_{ji} \qquad \text{at each } \mathbf{y}\in\mathcal{R}. \tag{3.32}$$

Thus the stress tensor $\mathbf{T}$ is *symmetric*:

$$\mathbf{T}=\mathbf{T}^T \qquad \text{at each } \mathbf{y}\in\mathcal{R}. \tag{3.33}$$

This is equivalent to (3.31) and is a local consequence of moment balance.

Exercise: To show the symmetry of the stress tensor without explicitly writing out the terms in (3.31) (as we did above) multiply (3.31) by e_{ipq} and use of the identity $e_{ijk}\,e_{ipq}=\delta_{jp}\delta_{kq}-\delta_{jq}\delta_{kp}$.

Conversely, when the symmetry condition (3.33), the equilibrium equation (3.27), and the traction-stress relation (3.18) all hold, then the global moment balance (3.9) holds. (Show this.)

3.4.1 Summary

In summary, the global equilibrium principles of force and moment balance hold if and only if the Cauchy stress tensor field $\mathbf{T}(\mathbf{y})$ obeys

$$\boxed{\begin{array}{rcl} \text{div}\,\mathbf{T}+\mathbf{b} &=& \mathbf{o}, \\ & & \qquad\qquad \text{at each } \mathbf{y}\in\mathcal{R}, \\ \mathbf{T} &=& \mathbf{T}^T, \end{array}} \tag{3.34}$$

with the traction on a surface related to the stress through

$$\boxed{\mathbf{t}(\mathbf{y},\mathbf{n}) = \mathbf{T}(\mathbf{y})\mathbf{n}.} \tag{3.35}$$

In cartesian components,

$$\frac{\partial T_{ij}}{\partial y_j} + b_i = 0, \qquad T_{ij} = T_{ji}, \qquad t_i = T_{ij}n_j. \tag{3.36}$$

3.5 Principal stresses.

Since the Cauchy stress tensor $\mathbf{T}$ is symmetric, it has three real eigenvalues, τ_1, τ_2, τ_3, and a set of three corresponding orthonormal eigenvectors, $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3$:

$$\mathbf{T}\mathfrak{t}_i = \tau_i\ \mathfrak{t}_i \qquad\qquad \text{(no sum on i).} \tag{3.37}$$

The eigenvalues τ_i are called the *principal stresses* and the eigenvectors $\mathfrak{t}_i$ define the *principal directions of Cauchy stress*. (Caution: Please note the distinction between the two lowercase boldface t's: $\mathbf{t}$ and $\mathfrak{t}_i$ denoting the traction and the principal stress directions respectively.) The triplet of vectors $\{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3\}$ defines an orthonormal basis referred to as a principal basis for stress. The matrix of stress components in this basis is diagonal and is given by

$$[T] = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}. \tag{3.38}$$

We can express $\mathbf{T}$ as

$$\mathbf{T} = \tau_1\, \mathfrak{t}_1 \otimes \mathfrak{t}_1 + \tau_2\, \mathfrak{t}_2 \otimes \mathfrak{t}_2 + \tau_3\, \mathfrak{t}_3 \otimes \mathfrak{t}_3 = \sum_{i=1}^{3} \tau_i\, \mathfrak{t}_i \otimes \mathfrak{t}_i. \tag{3.39}$$

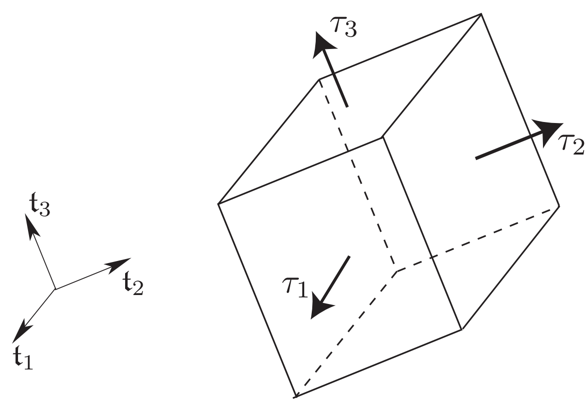


Figure 3.13: Principal stresses τ_1, τ_2, τ_3 and corresponding principal directions $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3$.

When the components of $\mathbf{T}$ and $\mathbf{n}$ are taken with respect to a principal basis for $\mathbf{T}$, one can show that the normal stress (3.6) and the magnitude of the resultant shear stress (3.7) can be written as (Problems 3.4 and 3.5)

$$T_{\text{normal}}(\mathbf{n}) = \mathbf{t}(\mathbf{n}) \cdot \mathbf{n} = \mathbf{Tn} \cdot \mathbf{n} = T_{ij} n_i n_j = \tau_1 \, n_1^2 + \tau_2 \, n_2^2 + \tau_3 \, n_3^2, \tag{3.40}$$

$$T_{\text{shear}}^2 = |\mathbf{t}(\mathbf{n})|^2 - T_{\text{normal}}^2 = \tau_1^2 n_1^2 + \tau_2^2 n_2^2 + \tau_3^2 n_3^2 - (\tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2)^2. \tag{3.41}$$

An important characteristic of the principal stresses and principal directions can be seen by asking the question "from among all planes passing through a given point, on which is $T_{\text{normal}}(\mathbf{n})$ largest? And on which is it smallest?" This requires one to consider $T_{\text{normal}}(\mathbf{n})$ as a function of the unit vector $\mathbf{n}$ and to find the specific vector(s) $\mathbf{n}$ at which it has its extrema. One can show (Problem 3.4) that the maximum value of $T_{\text{normal}}(\mathbf{n})$ over all unit vectors $\mathbf{n}$ is the largest of the principal stresses:

$$T_{\text{normal}}(\mathbf{n})\Big|_{\text{max over } \mathbf{n}} = \text{maximum of } \{\tau_1, \tau_2, \tau_3\}, \tag{3.42}$$

and that the smallest value of $T_{\text{normal}}(\mathbf{n})$ is the smallest principal stress.

The maximum value of $T_{\text{shear}}(\mathbf{n})$ over all unit vectors $\mathbf{n}$ is (Problem 3.5)

$$T_{\text{shear}}(\mathbf{n})\Big|_{\text{max over } \mathbf{n}} = \text{maximum of } \left\{ \frac{1}{2}|\tau_1 - \tau_2|, \quad \frac{1}{2}|\tau_2 - \tau_3|, \quad \frac{1}{2}|\tau_3 - \tau_1| \right\}. \tag{3.43}$$

One can also show that there is always a plane (through each point of a body) on which $T_{\text{shear}}(\mathbf{n})$ vanishes, but in general, there is no plane on which $T_{\text{normal}}(\mathbf{n})$ vanishes (though there might be in special cases) (Problem 3.6).

Finally it is worth emphasizing that the principal directions of the stress tensor $\mathbf{T}$ have no relationship, in general, to the principal directions of the stretch tensors $\mathbf{U}$ or $\mathbf{V}$. There may be a relationship between them for *particular materials*, but this depends on the constitutive law. In particular, we will find that for an isotropic elastic material, the principal directions $\{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3\}$ of $\mathbf{T}$ coincide with the principal directions $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ of the Eulerian stretch tensor $\mathbf{V}$. See also Problem 3.30.

3.6 Mean pressure and deviatoric stress.

It is sometimes convenient to decompose the stress additively into the sum of two parts, a hydrostatic part and a deviatoric part. By definition, the *mean pressure* is the (negative of the) average normal stress

$$= -\frac{1}{3}T_{kk} = -\frac{1}{3}\,\mathrm{tr}\,\mathbf{T},$$

and so the hydrostatic part of stress is $\frac{1}{3}\,(\mathrm{tr}\,\mathbf{T})\,\mathbf{I}$. The remaining part of the stress is called the *deviatoric part* which we denote by

$$\mathbf{T}^{(\mathrm{dev})} := \mathbf{T} - \frac{1}{3}(\mathrm{tr}\ \mathbf{T})\mathbf{I}, \qquad T_{ij}^{(\mathrm{dev})} = T_{ij} - \frac{1}{3}T_{kk}\delta_{ij}. \tag{3.44}$$

Note that the trace of the deviatoric stress vanishes. Thus we have the decomposition

$$\mathbf{T} = \mathbf{T}^{(\mathrm{dev})} + \frac{1}{3}(\mathrm{tr}\ \mathbf{T})\mathbf{I}. \tag{3.45}$$

3.7 Formulation of mechanical principles with respect to a reference configuration.

A few videos on some of the material in this section can be found here.

Thus far, our discussion of traction, stress, balance laws and field equations, did not allude to a reference configuration. Though not conceptually necessary, it is frequently convenient to refer the kinematic quantities entering the discussion of traction and stress, i.e. the area and surface normal vector, to the corresponding quantities in a reference configuration. Often, the reference configuration can be chosen in a convenient manner while the deformed configuration is not known a priori.

First consider an infinitesimal part of the body whose volume in the deformed configuration is dV_y. The body force on this part is $\mathbf{b}\,\mathrm{d}V_y$ where $\mathbf{b}$ is the body force per unit deformed volume. Let $\mathrm{d}V_x$ denote the volume of this part in a reference configuration. Then we can introduce the *body force per unit reference volume*, $\mathbf{b}_{\mathrm{R}}$, in terms of which the body force on this part can equivalently be expressed as $\mathbf{b}_{\mathrm{R}}\,\mathrm{d}V_x$. Therefore we have

$$\text{Body force on infinitesimal part} = \boxed{\mathbf{b}\,dV_y = \mathbf{b}_{\mathrm{R}}\,dV_x.} \tag{3.46}$$

Keep in mind that the force $\mathbf{b}_{\mathrm{R}}\,dV_x$ acts on the *deformed* body. We know from Section 2.4.3 that these volumes are related by $dV_y = J\,dV_x$ where $J = \det \mathbf{F}$. Thus

$$\boxed{\mathbf{b}_{\mathrm{R}} = J\,\mathbf{b}.} \tag{3.47}$$

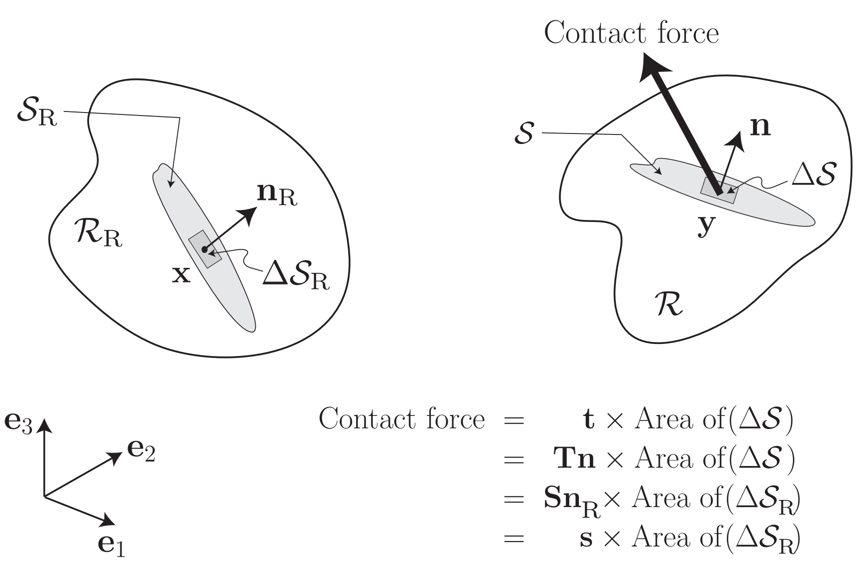


Figure 3.14: The surface $\mathcal{S}$ and a surface element $\Delta\mathcal{S}$ in the deformed configuration, and their images $\mathcal{S}_{\mathrm{R}}$ and $\Delta\mathcal{S}_{\mathrm{R}}$ in the reference configuration. The vectors $\mathbf{n}$ and $\mathbf{n}_{\mathrm{R}}$ are normal to these respective surfaces. Different (equivalent) ways for characterizing the contact force on $\Delta\mathcal{S}$ are noted in the figure. Keep in mind that the contact force acts on the deformed surface.

Next we turn to traction. Recall that the Cauchy traction $\mathbf{t}(\mathbf{n})$ represents the contact force per unit *deformed* area and that it acts on the surface whose normal in the *deformed* configuration is $\mathbf{n}$. Let $\Delta\mathcal{S}$ be an infinitesimal surface in $\mathcal{R}$ of area dA_y with unit normal $\mathbf{n}$. The contact force on this surface is therefore $\mathbf{t}(\mathbf{n})\,dA_y$. Let $\Delta\mathcal{S}_{\mathrm{R}}$ be the image of this surface in a reference configuration with area $\mathrm{d}A_x$ and unit normal $\mathbf{n}_{\mathrm{R}}$; see Figure 3.14. We now introduce the *contact force per unit reference area*, $\mathbf{s}(\mathbf{n}_{\rm R})$, in terms of which the contact force on $\Delta\mathcal{S}$ can be equivalently written as $\mathbf{s}(\mathbf{n}_{\rm R})\, dA_x$. Thus we have

$$\text{Contact force on } \Delta\mathcal{S} \;=\; \mathbf{t}(\mathbf{n})\, \mathrm{d}A_y \;=\; \mathbf{s}(\mathbf{n}_{\rm R})\, \mathrm{d}A_x. \tag{3.48}$$

Keep in mind that the force $\mathbf{s}(\mathbf{n}_{\rm R})\, \mathrm{d}A_x$ acts on the *deformed* surface as illustrated in the right-hand figure in Figure 3.14. We can rewrite this using $\mathbf{t} = \mathbf{T}\mathbf{n}$ and Nanson's formula $\mathrm{d}A_y\, \mathbf{n} = \mathrm{d}A_x\, J\, \mathbf{F}^{-T}\, \mathbf{n}_{\rm R}$ as

$$\begin{aligned}
\text{Contact force on } \Delta\mathcal{S} \;&=\; \mathbf{s}(\mathbf{n}_{\rm R})\, \mathrm{d}A_x = \\
&=\; \mathbf{t}(\mathbf{n})\, \mathrm{d}A_y = \mathbf{T}\mathbf{n}\, dA_y = \; \mathbf{T}\,(J\, \mathbf{F}^{-T}\, \mathbf{n}_{\rm R}\, \mathrm{d}A_x) = \\
&=\; (J\, \mathbf{T}\, \mathbf{F}^{-T})\, \mathbf{n}_{\rm R}\, \mathrm{d}A_x.
\end{aligned} \tag{3.49}$$

It is natural therefore to introduce a tensor $\mathbf{S}$ defined by

$$\boxed{\mathbf{S} = J\, \mathbf{T}\, \mathbf{F}^{-T}\ ,} \tag{3.50}$$

so that

$$\boxed{\mathbf{s} = \mathbf{S}\, \mathbf{n}_{\rm R}\ .} \tag{3.51}$$

The contact force on $\Delta\mathcal{S}$ can now be written in the equivalent forms

$$\text{Contact force on } \Delta\mathcal{S} \;= \boxed{\mathbf{t}\, \mathrm{d}A_y = \mathbf{s}\, \mathrm{d}A_x,} \tag{3.52}$$

and

$$\text{Contact force on } \Delta\mathcal{S} \;= \boxed{\mathbf{T}\mathbf{n}\, \mathrm{d}A_y = \mathbf{S}\mathbf{n}_{\rm R}\, \mathrm{d}A_x\ .} \tag{3.53}$$

This is described by the text in Figure 3.14. The contact force per unit referential area, $\mathbf{s}(\mathbf{n}_{\rm R})$, is called the **Piola traction vector** and the associated tensor $\mathbf{S}$ is the **Piola stress tensor**. These are sometimes referred to as[9] the first Piola-Kirchhoff traction and first Piola-Kirchhoff stress respectively.

The physical significance of the components of $\mathbf{S}$, can be deduced as follows. Let

$$\mathbf{S} = S_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j.$$

Consider a surface parallel to one of the coordinate planes, say $\mathbf{n}_{\rm R} = \mathbf{e}_j$. According to $\mathbf{s}(\mathbf{n}_{\rm R}) = \mathbf{S}\mathbf{n}_{\rm R}$, the traction on this surface is $\mathbf{s}(\mathbf{e}_j) = \mathbf{S}\mathbf{e}_j$. The $i^{\rm th}$-component of this traction is therefore $s_i(\mathbf{e}_j) = \mathbf{s}(\mathbf{e}_j)\cdot\mathbf{e}_i = \mathbf{S}\mathbf{e}_j\cdot\mathbf{e}_i = S_{ij}$. Thus

$$S_{ij} = s_i(\mathbf{e}_j). \tag{3.54}$$

[9] Our terminology is based on the historical notes in Section 210 of [8].

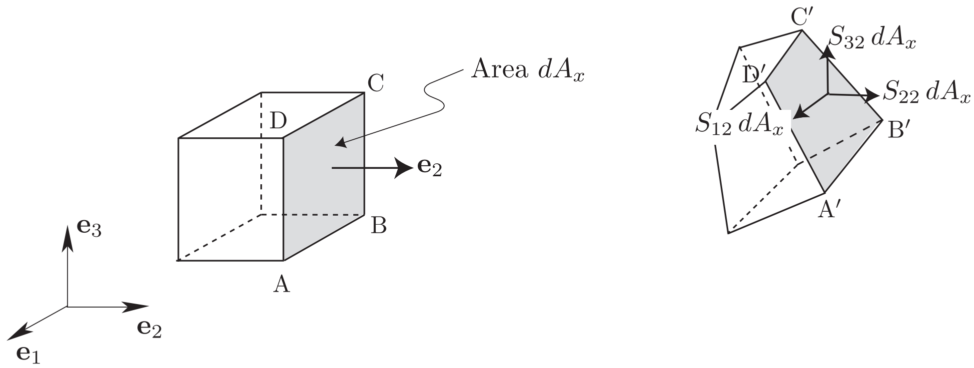


Figure 3.15: Physical significance of the components of the stress tensor $\mathbf{S}$: the shaded surface in the reference configuration is normal to $\mathbf{e}_2$ and has area dA_x. The i^{th} component of *force* acting on the image of this surface in the deformed configuration is $S_{i2} \times dA_x$.

Therefore S_{ij} is the i^{th} component of force per unit referential area acting on the surface that is normal to the j^{th} direction in the reference configuration.

For example consider a surface $\Delta\mathcal{S}$ that is normal to $\mathbf{e}_2$ in the reference configuration as shown in Figure 3.15. Then by taking $\mathbf{n}_{\mathrm{R}} = \mathbf{e}_2$ in (3.51), the contact force on $\Delta\mathcal{S}$ can be written as

$$\begin{aligned}\text{Contact force on } \Delta\mathcal{S} \ &= \mathbf{t}\ \mathrm{d}A_y = \mathbf{s}\ \mathrm{d}A_x = \mathbf{S}\,\mathbf{e}_2\ \mathrm{d}A_x = (S_{12}\,\mathbf{e}_1 + S_{22}\,\mathbf{e}_2 + S_{32}\,\mathbf{e}_3)\ \mathrm{d}A_x = \\ &= (S_{12}\ \mathrm{d}A_x)\,\mathbf{e}_1 + (S_{22}\ \mathrm{d}A_x)\,\mathbf{e}_2 + (S_{32}\ \mathrm{d}A_x)\,\mathbf{e}_3\,.\end{aligned} \tag{3.55}$$

This is illustrated in Figure 3.15.

To illustrate this further, consider a simple shear deformation of a block as shown in Figure 3.16. The leftmost figure shows the region $\mathcal{R}_{\mathrm{R}}$, while both the middle and rightmost figures show the region $\mathcal{R}$. The unit outward normal vector to the face $C'D'$ is $\mathbf{n}$ and therefore the

$$\text{Contact force on } C'D' = \mathbf{T}\mathbf{n} \times |C'D'| = [(T_{1j}n_j)\mathbf{e}_1 + (T_{2j}n_j)\mathbf{e}_2 + (T_{3j}n_j)\mathbf{e}_3] \times |C'D'|\ ; \tag{3.56}$$

this is illustrated in the middle figure. Since the face CD, which is the pre-image of $C'D'$, has a unit outward normal $\mathbf{e}_1$, we can *equivalently* write

$$\text{Contact force on } C'D' = \mathbf{S}\mathbf{n}_R \times |CD| = \mathbf{S}\mathbf{e}_1 \times |CD| = [S_{11}\mathbf{e}_1 + S_{21}\mathbf{e}_2 + S_{31}\mathbf{e}_3] \times |CD|; \tag{3.57}$$

this is illustrated in the rightmost figure. Similarly the unit outward normal vectors to the

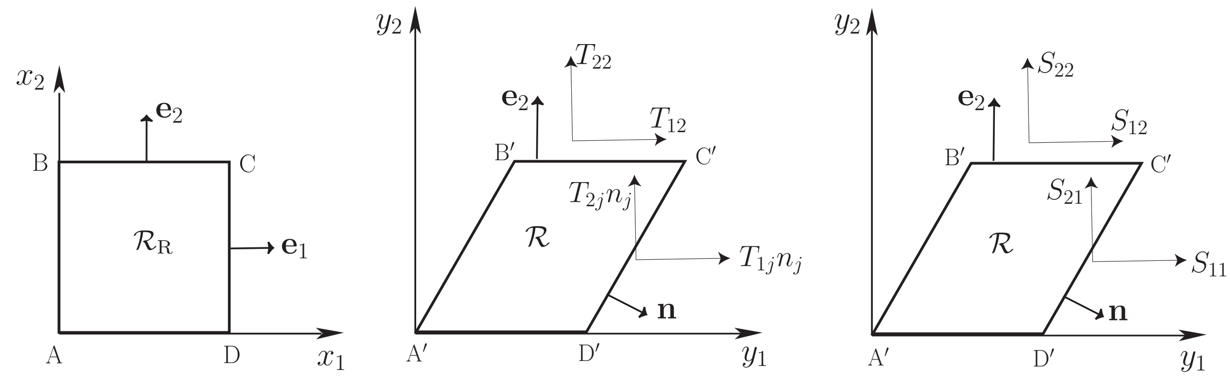


Figure 3.16: Simple shear. The middle and rightmost figures *both* show the region $\mathcal{R}$ occupied by the body in the deformed configuration. They show the tractions on the faces $B'C'$ and $C'D'$ in two different, but equivalent, ways: the middle figure describes the traction in terms of the components of the stress $\mathbf{T}$ while the rightmost figure describes them in terms of the components of $\mathbf{S}$. The corresponding forces are found by multiplying each traction by the area of the relevant surface in either the deformed or reference configuration as appropriate.

face $B'C'$ and its pre-image BC are both $\mathbf{e}_2$, and therefore we can write

$$\begin{aligned} \text{Contact force on } B'C' \quad &= \quad \mathbf{T}\mathbf{e}_2 \times |B'C'| = [T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3] \times |B'C'|, \\ &= \quad \mathbf{S}\mathbf{e}_2 \times |BC| = [S_{12}\mathbf{e}_1 + S_{22}\mathbf{e}_2 + S_{32}\mathbf{e}_3] \times |BC|; \end{aligned} \tag{3.58}$$

these are also displayed in Figure 3.16.

Suppose we use different bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ in the reference and deformed configurations respectively so that $\mathbf{y} = y_i\mathbf{e}'_i$ and $\mathbf{x} = x_i\mathbf{e}_i$, with the deformation described by $y_1 = y_1(x_1, x_2, x_3), y_2 = y_2(x_1, x_2, x_3), y_3 = y_3(x_1, x_2, x_3)$. The associated deformation gradient tensor can be expressed in the mixed bases as

$$\mathbf{F} = F_{ij}\, \mathbf{e}'_i \otimes \mathbf{e}_j. \tag{3.59}$$

If the basis vectors do not depend on the coordinates then $F_{ij} = \partial y_i/\partial x_j$, but even otherwise, $\mathbf{F}$ has the representation (3.59). It follows from (1.77) that

$$\mathbf{F}^T = F_{ij}\, \mathbf{e}_j \otimes \mathbf{e}'_i.$$

Next on using the result in Problem 1.32 we find

$$\mathbf{F}^{-T} = F^{-1}_{ij}\, \mathbf{e}'_j \otimes \mathbf{e}_i$$

where $[F]^{-1}$ is the matrix inverse of $[F]$. Thus with the Cauchy stress represented as

$$\mathbf{T} = T_{ij}\,\mathbf{e}'_i \otimes \mathbf{e}'_j,$$

we find

$$\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T} = J\left(T_{ij}\,\mathbf{e}'_i \otimes \mathbf{e}'_j\right)\left(F^{-1}_{k\ell}\,\mathbf{e}'_\ell \otimes \mathbf{e}_k\right) = S_{ij}\,\mathbf{e}'_i \otimes \mathbf{e}_j \quad \text{where} \quad [S] = J[T][F]^{-T}. \tag{3.60}$$

Thus $\mathbf{S}$, like $\mathbf{F}$, is most naturally expressed in the mixed bases.

We now turn to the equilibrium equations. Before proceeding further, you might wish to review the discussion in Section 2.8 of the material and spatial descriptions of a field. As mentioned there, any field defined on $\mathcal{R}$ and described spatially as a function of $\mathbf{y}$ can be converted to a field defined on $\mathcal{R}_\mathrm{R}$ that is described referentially as a function of the reference position $\mathbf{x}$. We do this by using the deformation $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x})$ to change $\mathbf{y} \to \mathbf{x}$. It turns out that it is convenient to express the Piola stress tensor field referentially as $\mathbf{S}(\mathbf{x})$, so that (3.50), (3.51) would read (with more detail)

$$\mathbf{S}(\mathbf{x}) = J(\mathbf{x})\,\mathbf{T}(\widehat{\mathbf{y}}(\mathbf{x}))\,\mathbf{F}^{-T}(\mathbf{x})\ , \tag{3.61}$$

$$\mathbf{s}(\mathbf{x}, \mathbf{n}_\mathrm{R}) = \mathbf{S}(\mathbf{x})\,\mathbf{n}_\mathrm{R}\ . \tag{3.62}$$

Substituting $\mathbf{T} = \mathrm{J}^{-1}\mathbf{S}\mathbf{F}^T$ into $\operatorname{div}\mathbf{T}$ and using (2.124) from page 217 shows that

$$\operatorname{div}\mathbf{T} = \mathrm{J}^{-1}\operatorname{Div}\mathbf{S},$$

where $\operatorname{div}\mathbf{T}$ and $\operatorname{Div}\mathbf{S}$ are the vector fields whose ith cartesian components are

$$\Big(\operatorname{div}\mathbf{T}\Big)_i = \frac{\partial T_{ij}}{\partial y_j}, \qquad \Big(\operatorname{Div}\mathbf{S}\Big)_i = \frac{\partial S_{ij}}{\partial x_j},$$

respectively[10]. Therefore $\operatorname{div}\mathbf{T} + \mathbf{b} = \mathrm{J}^{-1}\operatorname{Div}\mathbf{S} + \mathrm{J}^{-1}\,\mathbf{b}_\mathrm{R}$ and so the equilibrium equation $(3.34)_1$ can be written in the equivalent form

$$\boxed{\operatorname{Div}\mathbf{S} + \mathbf{b}_\mathrm{R} = \mathbf{0} \qquad \text{for all } \mathbf{x} \in \mathcal{R}_R.} \tag{3.63}$$

The moment balance equation $(3.34)_2$, in view of $\mathbf{T} = \mathrm{J}^{-1}\mathbf{S}\mathbf{F}^T$, yields

$$\boxed{\mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T \qquad \text{for all } \mathbf{x} \in \mathcal{R}_R.} \tag{3.64}$$

[10] Note the distinction between Div and div. For any tensor field $\mathbf{A}(\mathbf{x})$, the vector field $\operatorname{Div}\mathbf{A}$ has cartesian components $\partial A_{ij}/\partial x_j$:

$$\Big(\operatorname{Div}\mathbf{A}\Big)_i = \frac{\partial A_{ij}}{\partial x_j}, \qquad \operatorname{Div}\mathbf{A} = \frac{\partial A_{ij}}{\partial x_j}\,\mathbf{e}_i.$$

There is a parallel distinction between Grad/grad, and Curl/curl.

The field equations (3.63) and (3.64) hold at every point $\mathbf{x} \in \mathcal{R}_{\mathrm{R}}$. In cartesian components, equations (3.63), (3.64) and $\mathbf{s} = \mathbf{S}\mathbf{n}_{\mathrm{R}}$ read

$$\frac{\partial S_{ij}}{\partial x_j} + b_i^{\mathrm{R}} = 0, \qquad S_{ik}F_{jk} = F_{ik}S_{jk}, \qquad s_i = S_{ij}n_j^{\mathrm{R}}. \tag{3.65}$$

Note that the Piola stress tensor is *not* symmetric in general. This implies in particular that $\mathbf{S}$ may not have three real eigenvalues and so we will not (usually) speak of its principal values.

Before leaving this section it is instructive to express the various terms of the global balance laws for force and moment equilibrium in terms of these referential quantities. Let $\mathcal{D}_{\mathrm{R}}$ and $\mathcal{D}$ be the regions occupied by a part of the body in the reference and deformed configurations respectively. By integrating (3.52) over the body and using (3.51) we see that the resultant contact force on this part is

$$= \int_{\partial\mathcal{D}} \mathbf{t}\, dA_y \overset{(3.52)}{=} \int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{s}\, dA_x \overset{(3.51)}{=} \int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{S}\mathbf{n}_{\mathrm{R}}\, dA_x.$$

By integrating (3.46) over the body we see that the resultant body force on this part is

$$= \int_{\mathcal{D}} \mathbf{b}\, dV_y = \int_{\mathcal{D}_{\mathrm{R}}} \mathbf{b}_{\mathrm{R}}\, dV_x.$$

Consequently, the balance law (3.8) for force equilibrium can be written equivalently as

$$\int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{s}\, \mathrm{d}A_x + \int_{\mathcal{D}_{\mathrm{R}}} \mathbf{b}_{\mathrm{R}}\, \mathrm{d}V_x = \mathbf{0}, \tag{3.66}$$

which must hold for all $\mathcal{D}_{\mathrm{R}} \subset \mathcal{R}_{\mathrm{R}}$. An alternative derivation of the field equation $(3.64)_1$ involves using $\mathbf{s} = \mathbf{S}\mathbf{n}_{\mathrm{R}}$ and the divergence theorem on (3.66) and then localizing the result.

Similarly, the resultant moment of the contact force is given by

$$= \int_{\partial\mathcal{D}} \mathbf{y} \times \mathbf{t}\, dA_y \overset{(3.52)}{=} \int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{y}(\mathbf{x}) \times \mathbf{s}\, dA_x \overset{(3.51)}{=} \int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{y}(\mathbf{x}) \times \mathbf{S}\mathbf{n}_{\mathrm{R}}\, dA_x.$$

In this way one finds that the balance law for moment equilibrium (3.9) can be written equivalently as

$$\int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{y}(\mathbf{x}) \times \mathbf{S}\mathbf{n}_{\mathrm{R}}\, dA_x + \int_{\mathcal{D}_{\mathrm{R}}} \mathbf{y}(\mathbf{x}) \times \mathbf{b}_{\mathrm{R}}\, dV_x = \mathbf{0}\ . \tag{3.67}$$

It is worth emphasizing that it is *not* $\mathbf{x} \times \mathbf{S}\mathbf{n}_{\mathrm{R}}$ but rather $\mathbf{y}(\mathbf{x}) \times \mathbf{S}\mathbf{n}_{\mathrm{R}}$ that appears here. (Had it been $\mathbf{x} \times \mathbf{S}\mathbf{n}_{\mathrm{R}}$ this would have led to $\mathbf{S} = \mathbf{S}^T$.)

3.7.1 Worked examples.

Problem 3.7.1. Bending of a block.

Consider a block undergoing a bending deformation as depicted in Figure 3.17, the kinematics of which were analyzed previously in Problem 2.5.4. We found there that the deformation from (x_1, x_2, x_3) to (y_1, y_2, y_3) is characterized by

$$y_1 = r(x_1)\cos\theta(x_2), \qquad y_2 = r(x_1)\sin\theta(x_2), \qquad y_3 = \Lambda x_3, \tag{i}$$

where

$$r(x_1) > 0, \qquad r'(x_1) > 0, \qquad \theta'(x_2) > 0, \qquad \theta(x_2) = -\theta(-x_2), \qquad \Lambda = \text{constant}. \tag{ii}$$

Moreover, we showed that the principal stretches were

$$\lambda_1 = r'(x_1), \qquad \lambda_2 = r(x_1)\theta'(x_2), \qquad \lambda_3 = \Lambda, \tag{iii}$$

and the deformation gradient tensor was

$$\mathbf{F} = \lambda_1(\mathbf{e}_r \otimes \mathbf{e}_1) + \lambda_2(\mathbf{e}_\theta \otimes \mathbf{e}_2) + \lambda_3(\mathbf{e}_z \otimes \mathbf{e}_3). \tag{iv}$$

Here $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are the basis vectors associated with cylindrical polar coordinates (r, θ, z) in the deformed configuration, i.e.

$$\mathbf{e}_r = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2, \qquad \mathbf{e}_\theta = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2, \qquad \mathbf{e}_z = \mathbf{e}_3. \tag{v}$$

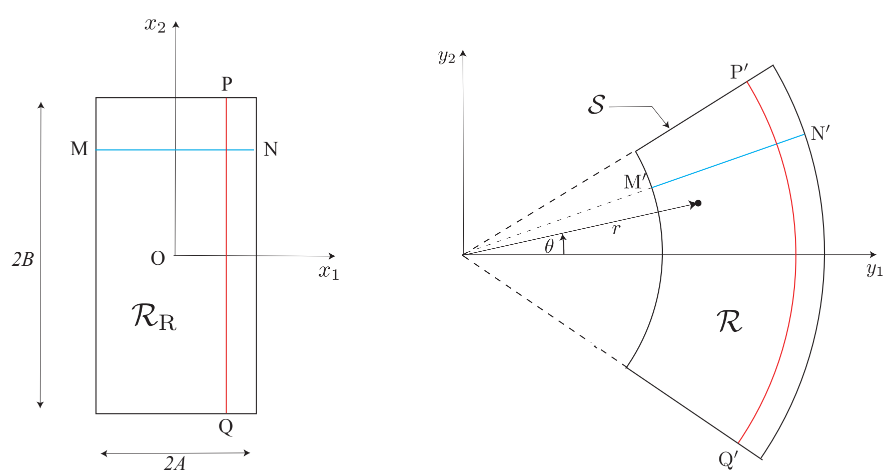


Figure 3.17: In a reference configuration the body occupies the rectangular parallelepiped region $\mathcal{R}_{\rm R} = \{(x_1, x_2, x_3) : -A \le x_1 \le A, -B \le x_2 \le B, -C \le x_3 \le C\}$ (left). The body undergoes a bending deformation in the x_1, x_2-plane . In the deformed configuration it occupies the region $\mathcal{R}$ (right).

Assume that the Piola stress tensor field is given by (recall (3.59), (3.60) and cf. (iv))

$$\mathbf{S} = \sigma_1(\mathbf{e}_r \otimes \mathbf{e}_1) + \sigma_2(\mathbf{e}_\theta \otimes \mathbf{e}_2) + \sigma_3(\mathbf{e}_z \otimes \mathbf{e}_3); \tag{vi}$$

moreover, since the principal stretches are independent of x_3, assume that the same is true of the σ_i's:

$$\sigma_i = \sigma_i(x_1, x_2), \qquad i = 1, 2, 3. \tag{vii}$$

Keep in mind that since we do not have a constitutive relation, equation (vi) does not follow from (iv). Exercise: Calculate the corresponding form of the Cauchy stress tensor.

Determine the restrictions on the functions $\sigma_i(x_1, x_2)$ arising from the following requirements: (a) the two curved boundaries of $\mathcal{R}$ are traction free, (b) the resultant force on the top inclined flat face $\mathcal{S}$ of $\mathcal{R}$ vanishes, (c) the resultant moment about the origin of the traction distribution on $\mathcal{S}$ is $\mathbf{m} = m\mathbf{e}_3$, and (d) the body is in equilibrium with no body forces.

Remark: At the end of the solution, we will make some further simplifications by using a constitutive relation (though we have not yet talked about constitutive relations!)

Solution:

(a) The outer curved boundary of $\mathcal{R}$ is the image of the flat boundary $x_1 = A$ of $\mathcal{R}_{\mathrm{R}}$. The Piola traction on this surface is given by $\mathbf{s} = \mathbf{S}\mathbf{n}_{\mathrm{R}}$ together with $\mathbf{n}_R = \mathbf{e}_1, x_1 = A$ and (vi):

$$\mathbf{s} = \mathbf{S}\mathbf{n}_R = \sigma_1 \mathbf{e}_r = \sigma_1(A, x_2)\mathbf{e}_r. \tag{viii}$$

Similarly, the inner curved boundary of $\mathcal{R}$ corresponds to the flat surface $x_1 = -A$ of $\mathcal{R}_{\mathrm{R}}$. The Piola traction on this surface is given by $\mathbf{s} = \mathbf{S}\mathbf{n}_{\mathrm{R}}, \mathbf{n}_R = -\mathbf{e}_1, x_1 = -A$ and (vi):

$$\mathbf{s} = \mathbf{S}\mathbf{n}_R = -\sigma_1(-A, x_2)\mathbf{e}_r. \tag{ix}$$

Observe that the traction $\mathbf{s}$ on these surfaces acts in the radial direction $\mathbf{e}_r$. Since $\mathbf{s}\, dA_x = \mathbf{t}\, dA_y$, so does the traction $\mathbf{t}$. Since these curved surfaces are traction-free, we must have

$$\sigma_1(\pm A, x_2) = 0 \qquad \text{for} \quad -B \le x_2 \le B. \qquad \square \tag{x}$$

(b) The top inclined flat boundary $\mathcal{S}$ of $\mathcal{R}$ is the image of the top horizontal surface $x_2 = B$ of $\mathcal{R}_{\mathrm{R}}$. The Piola traction on this surface can be calculated from $\mathbf{s} = \mathbf{S}\mathbf{n}_{\mathrm{R}}$ with $\mathbf{n}_R = \mathbf{e}_2, x_2 = B$ and (vi):

$$\mathbf{s} = \mathbf{S}\mathbf{n}_R = \sigma_2(x_1, B)\mathbf{e}_\theta. \tag{xi}$$

Observe that the traction $\mathbf{s}$ on $x_2 = B$ acts in the circumferential direction $\mathbf{e}_\theta$, and therefore so does the traction $\mathbf{t}$ on the corresponding surface of $\mathcal{R}$.

Remark: The unit vector $\mathbf{e}_\theta = \mathbf{e}_\theta(\theta)$ depends in general on θ, and since $\theta = \theta(x_2)$ by (i), it depends on x_2: $\mathbf{e}_\theta = \mathbf{e}_\theta(\theta(x_2))$. However, it is constant on the surface $x_2 = B$ since the angle $\theta = \theta(B)$ is constant there.

Keeping in mind that $\mathcal{S}$ denotes the top inclined surface of $\mathcal{R}$ and $\mathcal{S}_{\mathrm{R}}$ is its pre-image in the reference configuration, the resultant force on $\mathcal{S}$ is

$$= \int_{\mathcal{S}} \mathbf{t}\, dA_y \overset{(3.52)}{=} \int_{\mathcal{S}_{\mathrm{R}}} \mathbf{s}\, dA_x = \int_{-A}^{A}\int_{-C}^{C} \mathbf{s}\, dx_3 dx_1 = 2C \int_{-A}^{A} \mathbf{s}\, dx_1 \overset{(xi)}{=} 2C \int_{-A}^{A} \sigma_2(x_1, B)\mathbf{e}_\theta\, dx_1 =$$
$$= \left(2C \int_{-A}^{A} \sigma_2(x_1, B)\, dx_1 \right) \mathbf{e}_\theta.$$

Observe that we were able to take $\mathbf{e}_\theta$ out of the integral since $\mathbf{e}_\theta$ here is $\mathbf{e}_\theta(\theta(B))$ and so did not depend on x_1. Therefore, the resultant force on $\mathcal{S}$ vanishes when

$$\int_{-A}^{A} \sigma_2(x_1, B)\, dx_1 = 0. \qquad \square \tag{xii}$$

We will be able to further simplify this boundary condition after deriving the equilibrium equations in part (d) below and using (x) and a certain constitutive relation.

(c) The position vector of a generic particle in $\mathcal{R}$ is $\mathbf{y} = r\mathbf{e}_r$. The resultant moment about O due to the traction distribution on $\mathcal{S}$ is therefore

$$\begin{aligned}\mathbf{m} \; &= \int_{\mathcal{S}} \mathbf{y}\times\mathbf{t}\, dA_y \overset{(3.52)}{=} \int_{\mathcal{S}_{\rm R}} \mathbf{y}\times\mathbf{s}\, dA_x \overset{(xi)}{=} \int_{\mathcal{S}_{\rm R}} \mathbf{y}\times\sigma_2\mathbf{e}_\theta\, dA_x = \int_{\mathcal{S}_{\rm R}} r\mathbf{e}_r\times\sigma_2\mathbf{e}_\theta\, dA_x = \\ &= \mathbf{e}_z \int_{\mathcal{S}_{\rm R}} r\sigma_2\, dA_x = \mathbf{e}_z \int_{-A}^{A} r\sigma_2\, 2C dx_1 = \left(2C\int_{-A}^{A} r(x_1)\sigma_2(x_1,B)\, dx_1\right)\mathbf{e}_z.\end{aligned}$$

We are told that to moment on this surface is $m\mathbf{e}_z$, whence

$$m = 2C\int_{-A}^{A} r(x_1)\sigma_2(x_1,B)\, dx_1. \qquad \square \tag{xiii}$$

(d) Next we enforce the equilibrium equation $\mathrm{Div}\,\mathbf{S} = \mathbf{o}$. We will do this in rectangular cartesian coordinates as $\partial S_{ij}/\partial x_j = 0$, and for this we must first determine the components S_{ij} of $\mathbf{S}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Substituting (v) into (vi) and simplifying leads to

$$\mathbf{S} = \sigma_1\cos\theta(\mathbf{e}_1\otimes\mathbf{e}_1) + \sigma_1\sin\theta(\mathbf{e}_2\otimes\mathbf{e}_1) - \sigma_2\sin\theta(\mathbf{e}_1\otimes\mathbf{e}_2) + \sigma_2\cos\theta(\mathbf{e}_2\otimes\mathbf{e}_2) + \sigma_3(\mathbf{e}_3\otimes\mathbf{e}_3). \tag{xiv}$$

From this and $\mathbf{S} = S_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ we can read off the cartesian components of Piola stress:

$$\begin{gathered} S_{11} = \sigma_1\cos\theta, \quad S_{12} = -\sigma_2\sin\theta, \quad S_{21} = \sigma_1\sin\theta, \quad S_{22} = \sigma_2\cos\theta, \quad S_{33} = \sigma_3, \\ S_{13} = S_{31} = S_{23} = S_{32} = 0. \end{gathered} \tag{xv}$$

We now substitute the stress components (xv) into the equilibrium equations $\partial S_{ij}/\partial x_j = 0$ keeping in mind that $\theta = \theta(x_2)$, $\sigma_i = \sigma_i(x_1, x_2)$:

$$\begin{aligned} &\frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{12}}{\partial x_2} + \frac{\partial S_{13}}{\partial x_3} = 0 \qquad \Rightarrow \quad \frac{\partial \sigma_1}{\partial x_1}\cos\theta - \frac{\partial\sigma_2}{\partial x_2}\sin\theta - \sigma_2\cos\theta\,\theta' = 0,\\ &\frac{\partial S_{21}}{\partial x_1} + \frac{\partial S_{22}}{\partial x_2} + \frac{\partial S_{23}}{\partial x_3} = 0 \qquad \Rightarrow \quad \frac{\partial \sigma_1}{\partial x_1}\sin\theta + \frac{\partial\sigma_2}{\partial x_2}\cos\theta - \sigma_2\sin\theta\,\theta' = 0,\\ &\frac{\partial S_{31}}{\partial x_1} + \frac{\partial S_{32}}{\partial x_2} + \frac{\partial S_{33}}{\partial x_3} = 0 \qquad \Rightarrow \quad 0 = 0,\end{aligned}$$

These can be combined and simplified as follows: Multiplying the first equation by $\sin\theta$, the second by $\cos\theta$ and adding, and similarly multiplying the first equation by $\cos\theta$, the second by $\sin\theta$ and subtracting leads to the following pair of partial differential equations to be obeyed by $\sigma_1(x_1,x_2), \sigma_2(x_1,x_2), \theta(x_2)$:

$$\left.\begin{aligned} &\frac{\partial\sigma_1}{\partial x_1} - \sigma_2\theta' = 0,\\ &\frac{\partial\sigma_2}{\partial x_2} = 0, \end{aligned}\right\} \qquad \text{for} -A \le x_1 \le A, \quad -B \le x_2 \le B. \qquad \square \tag{xvi}$$

Remark: We now simplify the preceding results further by assuming a specific form of the constitutive relation. Suppose that the constitutive relation tells us that each stress component σ_i is a function of the three principal stretches:

$$\sigma_i = \widehat{\sigma}_i(\lambda_1, \lambda_2, \lambda_3), \qquad i = 1, 2, 3.$$

The specific functions $\widehat{\sigma}_i$ here will depend on the material. For our present purposes we shall assume (only) that $\widehat{\sigma}_2$ *does* depend on λ_2, or said differently, σ_2 is not independent of λ_2, i.e.

$$\frac{\partial \widehat{\sigma}_2}{\partial \lambda_2} \neq 0.$$

Equation $(xvi)_2$ can now be simplified as follows using the constitutive relation $\sigma_2 = \widehat{\sigma}_2(\lambda_1, \lambda_2, \lambda_3)$:

$$\frac{\partial \sigma_2}{\partial x_2} = \frac{\partial \widehat{\sigma}_2}{\partial \lambda_1}\cancel{\frac{\partial \lambda_1}{\partial x_2}} + \frac{\partial \widehat{\sigma}_2}{\partial \lambda_2}\frac{\partial \lambda_2}{\partial x_2} + \frac{\partial \widehat{\sigma}_2}{\partial \lambda_3}\cancel{\frac{\partial \lambda_3}{\partial x_2}} = \frac{\partial \widehat{\sigma}_2}{\partial \lambda_2}\frac{\partial \lambda_2}{\partial x_2} = 0$$

where we have used the fact that according to (iii), λ_1 and λ_3 do not depend on x_2. Since we assumed $\partial \widehat{\sigma}_2 / \partial \lambda_2 \neq 0$, this gives

$$\frac{\partial \lambda_2}{\partial x_2} = 0 \qquad \overset{(iii)}{\Rightarrow} \qquad r(x_1)\theta''(x_2) = 0 \quad \overset{(ii)_1}{\Rightarrow} \quad \theta''(x_2) = 0 \quad \Rightarrow \quad \theta(x_2) \overset{(ii)_4}{=} \frac{\beta x_2}{B}, \qquad (xvii)$$

where β is a to-be-determined constant of integration with the geometric meaning: $\theta(\pm B) = \pm\beta$.

From $(xvii)$ and (iii) we see that all three stretches are independent of x_2, and so by the constitutive relation $\sigma_i = \widehat{\sigma}_i(\lambda_1, \lambda_2, \lambda_3)$, so are the stress components σ_i:

$$\sigma_i = \sigma_i(x_1), \qquad i = 1, 2, 3. \qquad (xviii)$$

The remaining equilibrium equation $(xvi)_1$ can now be written as

$$\sigma_1'(x_1) - \frac{\beta}{B}\sigma_2(x_1) = 0. \qquad \Box \qquad (xvi)$$

(b) (continued) The boundary condition (xii) on the top inclined surface $\mathcal{S}$ can be now be shown to be automatic since

$$\int_{-A}^{A} \sigma_2(x_1)\,dx_1 \overset{(xvi)}{=} \frac{B}{\beta}\int_{-A}^{A} \sigma_1'\,dx_1 = \frac{B}{\beta}\Big(\sigma_1(A) - \sigma_1(-A)\Big) \overset{(x)}{=} 0.$$

Please revisit the last part of this solution once we have discussed constitutive relations.

3.8 Rate of working. Stress power.

We now derive a relation between the rate of external working on a part of the body and the rate of internal working within that part. This analysis, like everything else so far, is independent of the constitutive relation and is valid for *all* materials. It is worth emphasizing that the relation to be derived is *not* the first law of thermodynamics – it is a relation that is entirely mechanical in character and relates internal and external working.

Since we want to calculate the *rate* of working, we have to consider particle velocity, and for this we must admit time t into our analysis. Accordingly we now consider a time-dependent quasi-static motion: a family of deformations[11] $\mathbf{y}(\mathbf{x}, t)$ with time t being a parameter. By saying the motion is quasi-static we mean that the *equilibrium* equations hold at each instant t, inertial effects being omitted.

The velocity of a particle $\mathbf{x}$ is the rate of change of the position of that particle with respect to time:

$$\mathbf{v}(\mathbf{x}, t) = \frac{\partial \mathbf{y}}{\partial t}(\mathbf{x}, t). \tag{3.68}$$

Since $F_{ij} = \partial y_i / \partial x_j$ we can write

$$\dot{F}_{ij} := \frac{\partial}{\partial t} F_{ij}(\mathbf{x}, t) = \frac{\partial}{\partial t}\frac{\partial y_i}{\partial x_j}(\mathbf{x}, t) = \frac{\partial}{\partial x_j}\frac{\partial y_i}{\partial t}(\mathbf{x}, t) = \frac{\partial v_i}{\partial x_j}(\mathbf{x}, t) \quad \Leftrightarrow \quad \dot{\mathbf{F}} = \operatorname{Grad} \mathbf{v}, \tag{3.69}$$

where $\dot{\mathbf{F}}$ is the time rate of change of $\mathbf{F}(\mathbf{x}, t)$ at a fixed particle $\mathbf{x}$ and $\operatorname{Grad}\mathbf{v}$ is the 2-tensor with cartesian components $\partial v_i / \partial x_j$.

Consider a part of the body that occupies a region $\mathcal{D}_t$ at time t. Let $p(\mathcal{D}_t)$ denote the *rate at which the external forces on* $\mathcal{D}_t$ *do work*:

$$p(\mathcal{D}_t) = \int_{\partial \mathcal{D}_t} \mathbf{t} \cdot \mathbf{v} \, \mathrm{d}A_y + \int_{\mathcal{D}_t} \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}V_y \ ; \tag{3.70}$$

see (3.3). By using (3.46) and (3.52), we can express $p(\mathcal{D}_t)$ in referential form as

$$p(\mathcal{D}_t) = \int_{\partial \mathcal{D}_{\mathrm{R}}} \mathbf{s} \cdot \mathbf{v} \, \mathrm{d}A_x + \int_{\mathcal{D}_{\mathrm{R}}} \mathbf{b}_{\mathrm{R}} \cdot \mathbf{v} \, \mathrm{d}V_x \ , \tag{3.71}$$

where $\mathcal{D}_{\mathrm{R}}$ is the region occupied by the part being considered in the reference configuration. Note that $\mathcal{D}_t$ evolves with time but $\mathcal{D}_{\mathrm{R}}$ does not (because we are concerned with a fixed set of particles). It is now convenient to work in terms of components (in some fixed orthonormal

[11] a one-parameter family of deformations,

basis). Then we have

$$
\begin{aligned}
p(\mathcal{D}_t) \;&=\; \int_{\partial\mathcal{D}_\mathrm{R}} s_i v_i \,\mathrm{d}A_x + \int_{\mathcal{D}_\mathrm{R}} b_i^R v_i \,\mathrm{d}V_x \overset{(3.51)}{=} \int_{\partial\mathcal{D}_\mathrm{R}} S_{ij} n_j^R v_i \,\mathrm{d}A_x + \int_{\mathcal{D}_\mathrm{R}} b_i^R v_i \,\mathrm{d}V_x = \\
&\overset{(1.177)}{=} \int_{\mathcal{D}_\mathrm{R}} \frac{\partial}{\partial x_j}\Big(S_{ij} v_i\Big) \,\mathrm{d}V_x + \int_{\mathcal{D}_\mathrm{R}} b_i^R v_i \,\mathrm{d}V_x = \\
&= \int_{\mathcal{D}_\mathrm{R}} \left[\frac{\partial S_{ij}}{\partial x_j} v_i + S_{ij}\frac{\partial v_i}{\partial x_j} \;+\; b_i^R v_i\right] \mathrm{d}V_x \;= \int_{\mathcal{D}_\mathrm{R}} \left[\left(\frac{\partial S_{ij}}{\partial x_j} + b_i^R\right) v_i + S_{ij}\frac{\partial v_i}{\partial x_j}\right] \mathrm{d}V_x \\
&\overset{(3.65)_1}{=} \int_{\mathcal{D}_\mathrm{R}} S_{ij}\frac{\partial v_i}{\partial x_j} \mathrm{d}V_x \overset{(3.69)}{=} \int_{\mathcal{D}_\mathrm{R}} S_{ij}\dot{F}_{ij} \mathrm{d}V_x = \int_{\mathcal{D}_\mathrm{R}} \mathbf{S}\cdot\dot{\mathbf{F}} \,\mathrm{d}V_x.
\end{aligned}
\tag{3.72}
$$

Thus from (3.71) and (3.72) we have the following rate of working identity:

$$
\int_{\partial\mathcal{D}_\mathrm{R}} \mathbf{s}\cdot\mathbf{v}\,\mathrm{d}A_x + \int_{\mathcal{D}_\mathrm{R}} \mathbf{b}_\mathrm{R}\cdot\mathbf{v}\,\mathrm{d}V_x = \int_{\mathcal{D}_\mathrm{R}} \mathbf{S}\cdot\dot{\mathbf{F}}\,\mathrm{d}V_x. \tag{3.73}
$$

Equation (3.73) states that the rate of external work on a part of the body (the left-hand side) equals the rate of internal work within that part (the right-hand side). The rate of working by the internal stresses per unit reference volume, i.e. $\mathbf{S}\cdot\dot{\mathbf{F}}$, is called the **stress power**:

$$
\boxed{\text{Stress power } \;= \mathbf{S}\cdot\dot{\mathbf{F}}.} \tag{3.74}
$$

The stress power accounts for both stored and dissipated energy. The integral involving the stress power on the right-hand side of (3.73) *cannot* in general be written as the time derivative of the volume integral of some scalar field.

Problem 3.8.1. Evaluate the stress power $\mathbf{S}\cdot\dot{\mathbf{F}}$ for (a) a simple shear deformation $y_1 = x_1 + kx_2, y_2 = x_2, y_3 = x_3$; and (b) a uniaxial stress state $\mathbf{S} = S_{11}\,\mathbf{e}_1\otimes\mathbf{e}_1$.

Solution:

$$
(a)\quad \mathbf{S}\cdot\dot{\mathbf{F}} = \big(S_{ij}\,\mathbf{e}_i\otimes\mathbf{e}_j\big)\cdot\big(\dot{k}\,\mathbf{e}_1\otimes\mathbf{e}_2\big) = S_{ij}\dot{k}(\mathbf{e}_i\cdot\mathbf{e}_1)(\mathbf{e}_j\cdot\mathbf{e}_2) = S_{ij}\dot{k}\,\delta_{i1}\delta_{2j} = S_{12}\,\dot{k},
$$

$$
(b)\quad \mathbf{S}\cdot\dot{\mathbf{F}} = \big(S_{11}\,\mathbf{e}_1\otimes\mathbf{e}_1\big)\cdot\big(\dot{F}_{ij}\,\mathbf{e}_i\otimes\mathbf{e}_j\big) = S_{11}\dot{F}_{ij}(\mathbf{e}_1\cdot\mathbf{e}_i)(\mathbf{e}_1\cdot\mathbf{e}_j) = S_{11}\dot{F}_{ij}\,\delta_{i1}\delta_{2j} = S_{11}\dot{F}_{11},
$$

having used $\mathbf{F} = \mathbf{I} + k\,\mathbf{e}_1\otimes\mathbf{e}_2$ in (a).

Problem 3.8.2. Establish the following spatial form of the rate of working identity:

$$\int_{\partial\mathcal{D}_t} \mathbf{t}\cdot\mathbf{v}\,\mathrm{d}A_y + \int_{\mathcal{D}_t} \mathbf{b}\cdot\mathbf{v}\,\mathrm{d}V_y = \int_{\mathcal{D}_t} \mathbf{T}\cdot\mathbf{D}\,\mathrm{d}V_y\ , \tag{3.75}$$

where $\mathbf{D}$ is defined by

$$\mathbf{D} := \frac{1}{2}(\mathbf{L}+\mathbf{L}^T) \qquad \text{where} \quad \mathbf{L} := \operatorname{grad}\mathbf{v}. \tag{3.76}$$

In cartesian components

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right), \qquad L_{ij} = \frac{\partial v_i}{\partial y_j}. \tag{3.77}$$

The (kinematic) tensors $\mathbf{D}$ and $\mathbf{L}$ are known as the *stretching tensor* and the *velocity gradient tensor* respectively. Note that the gradient here is with respect to the spatial position $\mathbf{y}$ and it is understood that the velocity field has been expressed in spatial form as $\mathbf{v}(\mathbf{y},t)$; see Section 2.8. It follows from the right-hand side of (3.75) that the stress power, the rate of internal working per unit *reference* volume, can be written as

$$\boxed{\text{Stress power } = J\,\mathbf{T}\cdot\mathbf{D}.} \tag{3.78}$$

Solution: Since the relation between $\mathbf{x}$ and $\mathbf{y}$ is one-to-one, the relation $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x},t)$ can be inverted at each instant t to give $\mathbf{x} = \overline{\mathbf{x}}(\mathbf{y},t)$. Thus the referential and spatial descriptions of the velocity field, $\widehat{\mathbf{v}}(\mathbf{x},t)$ and $\overline{\mathbf{v}}(\mathbf{y},t)$, are related by

$$\overline{\mathbf{v}}(\mathbf{y},t) = \widehat{\mathbf{v}}(\overline{\mathbf{x}}(\mathbf{y},t),t), \qquad \widehat{\mathbf{v}}(\mathbf{x},t) = \overline{\mathbf{v}}(\widehat{\mathbf{y}}(\mathbf{x},t),t).$$

Recall from (3.69) the relation $\dot{F}_{ij} = \frac{\partial \widehat{v}_i}{\partial x_j}$. and so by using the chain rule

$$\dot{F}_{ij} = \frac{\partial \widehat{v}_i}{\partial x_j} = \frac{\partial \overline{v}_i}{\partial y_k}\frac{\partial y_k}{\partial x_j} \overset{(3.77)_1}{=} L_{ik}F_{kj} \qquad \Leftrightarrow \qquad \dot{\mathbf{F}} = \mathbf{L}\mathbf{F}. \tag{3.79}$$

This is a relation between the time rate of change of the deformation gradient tensor (at a fixed particle $\mathbf{x}$) and the velocity gradient tensor.

It now follows that

$$\mathbf{S}\cdot\dot{\mathbf{F}} \overset{(3.79)}{=} \mathbf{S}\cdot\mathbf{L}\mathbf{F} \overset{(3.50)}{=} J\,\mathbf{T}\,\mathbf{F}^{-T}\cdot\mathbf{L}\mathbf{F} \overset{(1.123)}{=} J\mathbf{T}\cdot\mathbf{L}.$$

However,

$$\mathbf{T}\cdot\mathbf{L} = \mathbf{T}\cdot\left[\frac{1}{2}(\mathbf{L}+\mathbf{L}^T) + \frac{1}{2}(\mathbf{L}-\mathbf{L}^T)\right] = \mathbf{T}\cdot\mathbf{D} + \frac{1}{2}\mathbf{T}\cdot(\mathbf{L}-\mathbf{L}^T) = \mathbf{T}\cdot\mathbf{D},$$

where in getting to the last equality we used the result from (1.141) since $\mathbf{T}$ is symmetric and $\frac{1}{2}(\mathbf{L}-\mathbf{L}^T)$ is skew symmetric. Thus from the two preceding equations we have

$$\mathbf{S}\cdot\dot{\mathbf{F}} = J\mathbf{T}\cdot\mathbf{D}.$$

Using this in (3.73), together with $J\,dV_x = dV_y$, (3.70) and (3.71) yields (3.75)

3.8.1 Work Conjugate Stress-Strain Pairs.

Consider a body undergoing an arbitrary quasi-static motion. Suppose that the stress power $\mathbf{S} \cdot \dot{\mathbf{F}}$ can be expressed in the form $\boldsymbol{\Sigma} \cdot \dot{\mathbf{E}}$ where $\mathbf{E}$ is some strain measure in the sense of Section 2.6:

$$\mathbf{S} \cdot \dot{\mathbf{F}} = \boldsymbol{\Sigma} \cdot \dot{\mathbf{E}}. \tag{3.80}$$

The components of $\boldsymbol{\Sigma}$ will necessarily have the dimension of stress. We say that the stress $\boldsymbol{\Sigma}$ and the strain $\mathbf{E}$ are work-conjugate[12]. This conjugacy reflects a special relationship between the stress $\boldsymbol{\Sigma}$ and strain $\mathbf{E}$. As we shall see when studying the constitutive behavior of an elastic material, the constitutive relation for the stress $\boldsymbol{\Sigma}$ is most naturally written in terms of the strain $\mathbf{E}$.

For example, consider the family of Lagrangian strain tensors

$$\mathbf{E}^{(0)} = \ln \mathbf{U}, \qquad\qquad \mathbf{E}^{(n)} = \frac{1}{n}\Big(\mathbf{U}^n - \mathbf{I}\Big), \quad n \neq 0.$$

Can one find a family of corresponding stress tensors $\mathbf{S}^{(n)}$ such that the stress power $=$ $\mathbf{S}^{(n)} \cdot \dot{\mathbf{E}}^{(n)}$?

Consider the case $n = 2$, i.e. the Green Saint-Venant strain tensor $\mathbf{E}^{(2)}$. We want to find a tensor $\mathbf{S}^{(2)}$ such that the

$$\text{Stress power} = \mathbf{S} \cdot \dot{\mathbf{F}} = \mathbf{S}^{(2)} \cdot \dot{\mathbf{E}}^{(2)}. \tag{i}$$

Since $\mathbf{E}^{(2)}$ is symmetric, there is no loss of generality in assuming $\mathbf{S}^{(2)}$ to be symmetric. (Why?) Differentiating $\mathbf{E}^{(2)} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$ with respect to t gives

$$\dot{\mathbf{E}}^{(2)} = \frac{1}{2}\dot{\mathbf{F}}^T\mathbf{F} + \frac{1}{2}\mathbf{F}^T\dot{\mathbf{F}}. \tag{ii}$$

Now substitute (ii) into (i) and simplify:

$$\begin{aligned}\mathbf{S} \cdot \dot{\mathbf{F}} &= \mathbf{S}^{(2)} \cdot \dot{\mathbf{E}}^{(2)} = \tfrac{1}{2}\mathbf{S}^{(2)} \cdot \dot{\mathbf{F}}^T\mathbf{F} + \tfrac{1}{2}\mathbf{S}^{(2)} \cdot \mathbf{F}^T\dot{\mathbf{F}} \overset{(1.123)}{=} \tfrac{1}{2}\mathbf{S}^{(2)}\mathbf{F}^T \cdot \dot{\mathbf{F}}^T + \tfrac{1}{2}\mathbf{F}\mathbf{S}^{(2)} \cdot \dot{\mathbf{F}} = \\ &= \tfrac{1}{2}\mathbf{F}(\mathbf{S}^{(2)})^T \cdot \dot{\mathbf{F}} + \tfrac{1}{2}\mathbf{F}\mathbf{S}^{(2)} \cdot \dot{\mathbf{F}} = \mathbf{F}\mathbf{S}^{(2)} \cdot \dot{\mathbf{F}},\end{aligned}$$

where in getting to the second line we used $\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^T \cdot \mathbf{B}^T$ and in the last step we used the symmetry of $\mathbf{S}^{(2)}$. Thus $\mathbf{S} \cdot \dot{\mathbf{F}} = \mathbf{F}\mathbf{S}^{(2)} \cdot \dot{\mathbf{F}}$ and so

$$\mathbf{S}^{(2)} = \mathbf{F}^{-1}\mathbf{S}. \tag{3.81}$$

[12] Note that $\mathbf{F}$ is not a strain. Thus one usually does not refer to the pair $\mathbf{S}, \dot{\mathbf{F}}$ as being work conjugate.

The symmetric tensor $\mathbf{S}^{(2)}$ is known as the second Piola-Kirchhoff stress tensor. It is conjugate to the Green Saint-Venant strain tensor.

The case of general n is discussed in Chapter 3.5 of Ogden [5].

Exercises: Problems 3.32, 3.33, 3.34 and 3.35.

3.8.2 Some other stress tensors.

In addition to the Cauchy and Piola stress tensors, various other stress measures are used in the literature. Some examples are

$$\begin{array}{ll}
\mathbf{T} & \text{Cauchy stress tensor,} \\
\mathrm{J}\mathbf{T} & \text{Kirchhoff stress tensor,} \\
\mathbf{S} = \mathrm{J}\mathbf{T}\mathbf{F}^{-T} & \text{Piola stress tensor,} \\
\mathbf{S}^{(2)} = \mathrm{J}\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} = \mathbf{F}^{-1}\mathbf{S} & 2^{\text{nd}}\ \text{Piola} - \text{Kirchhoff stress tensor,} \\
\mathbf{S}^{T} = \mathrm{J}\mathbf{F}^{-1}\mathbf{T} & \text{Nominal stress tensor,} \\
\mathbf{S}^{(1)} = \frac{1}{2}\left(\mathbf{S}^{T}\mathbf{R} + \mathbf{R}^{T}\mathbf{S}\right) & \text{Biot stress tensor.}
\end{array} \tag{3.82}$$

Even though many of these stress tensors have no simple physical significance, they are sometimes useful in, say, carrying out computations.

Exercise: Show that $\mathbf{S}^{(1)} \cdot \mathbf{U} = \mathbf{S} \cdot \mathbf{F}$. Note that this equation involves $\mathbf{U}$ and $\mathbf{F}$ *not* $\dot{\mathbf{U}}$ and $\dot{\mathbf{F}}$.

3.9 Linearization.

We now specialize the preceding analyses to the case where the deformed configuration is close to the reference configuration in the sense that the displacement gradient tensor $\mathbf{H} = \mathbf{F} - \mathbf{I} = \nabla\mathbf{u}$ is small: $|\mathbf{H}| = |\nabla\,\mathbf{u}| \ll 1$. It is natural therefore to work with the formulation with respect to the (fixed) reference configuration:

$$\mathrm{Div}\,\mathbf{S} + \mathbf{b}_{\mathrm{R}} = \mathbf{0}, \qquad \mathbf{S}\mathbf{F}^{T} = \mathbf{F}\mathbf{S}^{T} \qquad \text{for all } \mathbf{x} \in \mathcal{R}_{\mathrm{R}}. \tag{3.83}$$

Since $\mathbf{F} = \mathbf{I} + \mathbf{H}$ we see immediately that to leading order the moment equilibrium equation $(3.83)_2$ reduces to $\mathbf{S} \doteq \mathbf{S}^{T}$ and so the Piola stress tensor is symmetric to leading order at an infinitesimal deformation. In fact, by using $\mathbf{F} = \mathbf{I} + \mathbf{H}$ and $J = \det(\mathbf{I} + \mathbf{H}) = 1 + \operatorname{tr}\mathbf{H} + \mathrm{O}(|\mathbf{H}|^2)$, it follows from $\mathbf{T} = J^{-1}\mathbf{S}\mathbf{F}^T$ that

$$\mathbf{S} = \mathbf{T} \ + \mathrm{O}(|\mathbf{H}|). \tag{3.84}$$

Thus to leading order, the Piola stress tensor and the Cauchy stress tensor do not differ in infinitesimal deformations. Similarly the body force density $\mathbf{b} = \mathbf{b}_{\mathrm{R}} + \mathrm{O}(|\mathbf{H}|)$.

For clarity we shall use the symbol $\boldsymbol{\sigma}$ for the stress tensor in the linearized theory. The stress component σ_{ij} is the i^{th} component of force per unit area on the surface normal to $\mathbf{e}_j$ where we do not need to distinguish between the deformed and reference configurations.

The (force) equilibrium equation now reads

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{b}_{\mathrm{R}} = \mathbf{0} \qquad \text{for } \mathbf{x} \in \mathcal{R}_{\mathrm{R}}, \tag{3.85}$$

and the moment equilibrium equation tells us that

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \qquad \text{for } \mathbf{x} \in \mathcal{R}_{\mathrm{R}}. \tag{3.86}$$

Note that these field equations hold on the region $\mathcal{R}_{\mathrm{R}}$ occupied by the body in the reference configuration. Similarly the traction-stress relation is

$$\mathbf{t} = \boldsymbol{\sigma}\,\mathbf{n}_{\mathrm{R}}. \tag{3.87}$$

Thus in conclusion, for infinitesimal deformations we will work with the stress tensor field $\boldsymbol{\sigma}(\mathbf{x})$ and do not need to consider the deformed configuration in formulating any of the fundamental principles for stress. Reviewing the preceding material in this chapter we see, for example, that we can interpret the stress components σ_{ij} as in Figure 3.8 with T_{ij} replaced by σ_{ij} and we do not need to address whether the planes shown in the figure are in the reference or deformed configurations. Similarly in Problem 3.3.2 we can take the prismatic region there to be the region the body occupies in the reference configuration.

3.10 Some other coordinate systems.

3.10.1 Cylindrical polar coordinates.

In this section we express the equilibrium equation $\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{o}$ in terms of cylindrical polar coordinates:

– Let (y_1, y_2, y_3) denote the rectangular cartesian coordinates of a particle in the deformed configuration and let (r, θ, z) be its corresponding cylindrical polar coordinates. Then

$$y_1 = r\cos\theta, \qquad y_2 = r\sin\theta, \qquad y_3 = z, \tag{3.88}$$

and the associated basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are related by

$$\mathbf{e}_r = \mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta\ , \qquad \mathbf{e}_\theta = -\mathbf{e}_1\sin\theta + \mathbf{e}_2\cos\theta\ , \qquad \mathbf{e}_z = \mathbf{e}_3. \tag{3.89}$$

– The stress tensor $\mathbf{T}(\mathbf{y})$ can be written in terms of its cylindrical polar components as

$$\begin{aligned}\mathbf{T} \ &= T_{rr}\mathbf{e}_r\otimes\mathbf{e}_r + T_{r\theta}\mathbf{e}_r\otimes\mathbf{e}_\theta + T_{rz}\mathbf{e}_r\otimes\mathbf{e}_z +\\ &+T_{\theta r}\mathbf{e}_\theta\otimes\mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta\otimes\mathbf{e}_\theta + T_{\theta z}\mathbf{e}_\theta\otimes\mathbf{e}_z +\\ &+T_{zr}\mathbf{e}_z\otimes\mathbf{e}_r + T_{z\theta}\mathbf{e}_z\otimes\mathbf{e}_\theta + T_{zz}\mathbf{e}_z\otimes\mathbf{e}_z.\end{aligned} \tag{3.90}$$

For reasons that we will explain shortly, we have not enforced the symmetry of $\mathbf{T}$ in writing (3.90).

–The three scalar equilibrium equations we want to derive are the $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_z$ components of the vector equilibrium equation $\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{o}$. The three components of the vector $\operatorname{div}\mathbf{T}$ are

$$\mathbf{e}_r\cdot\operatorname{div}\mathbf{T}, \qquad \mathbf{e}_\theta\cdot\operatorname{div}\mathbf{T}, \qquad \mathbf{e}_z\cdot\operatorname{div}\mathbf{T}, \tag{3.91}$$

and so $\operatorname{div}\mathbf{T}$ itself can be expressed in terms of these components and the basis vectors $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_z$ as

$$\operatorname{div}\mathbf{T} = \big(\mathbf{e}_r\cdot\operatorname{div}\mathbf{T}\big)\mathbf{e}_r + \big(\mathbf{e}_\theta\cdot\operatorname{div}\mathbf{T}\big)\mathbf{e}_\theta + \big(\mathbf{e}_z\cdot\operatorname{div}\mathbf{T}\big)\mathbf{e}_z. \tag{3.92}$$

Our goal therefore is to calculate the three terms in (3.91). We shall do this using the vector identity established in Problem 1.8.1, viz.

$$\mathbf{v}\cdot\operatorname{div}\mathbf{T} = \operatorname{div}(\mathbf{T}^T\mathbf{v}) - \mathbf{T}\cdot\operatorname{grad}\mathbf{v}, \tag{3.93}$$

that holds for any vector field $\mathbf{v}(\mathbf{y})$ and tensor field $\mathbf{T}(\mathbf{y})$. Observe that the right-hand side of (3.93) involves the divergence and gradient of two *vector* fields, and we previously calculated expressions for these (in cylindrical polar coordinates) in Section 1.8.6; see equations (1.190) and (1.189).

First take $\mathbf{v} = \mathbf{e}_r$ in the identity (3.93) and calculate the two terms on its right-hand side. By taking $\mathbf{u} = \mathbf{e}_r$ in (1.189) (together with the obvious change of notation from (R, Θ, Z) to (r, θ, z)) we get

$$\operatorname{grad}\mathbf{e}_r = \frac{1}{r}\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta \qquad \overset{(3.90)}{\Rightarrow} \qquad \mathbf{T}\cdot\operatorname{grad}\mathbf{e}_r = \frac{T_{\theta\theta}}{r}, \tag{i}$$

which is the second term on the right-hand side of (3.93). The first term can be evaluated as follows:

$$\mathrm{div}\,(\mathbf{T}^T\mathbf{e}_r) \overset{(3.90)}{=} \mathrm{div}\,(T_{rr}\mathbf{e}_r + T_{r\theta}\mathbf{e}_\theta + T_{rz}\mathbf{e}_z) \overset{(1.190)}{=} \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}T_{rr} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z}. \qquad (ii)$$

Substituting (i) and (ii) into (3.93) yields

$$\mathbf{e}_r \cdot \mathrm{div}\,\mathbf{T} = \mathrm{div}\,\mathbf{T}^T\mathbf{e}_r - \mathbf{T}\cdot\mathrm{grad}\,\mathbf{e}_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr}-T_{\theta\theta}}{r}. \qquad (iii)$$

A parallel calculation with $\mathbf{v} = \mathbf{e}_\theta$ yields

$$\mathrm{grad}\,\mathbf{e}_\theta = -\frac{1}{r}\,\mathbf{e}_r\otimes\mathbf{e}_\theta \qquad \overset{(3.90)}{\Rightarrow} \qquad \mathbf{T}\cdot\mathrm{grad}\,\mathbf{e}_\theta = -\frac{T_{r\theta}}{r}, \qquad (iv)$$

$$\mathrm{div}\,(\mathbf{T}^T\mathbf{e}_\theta) \overset{(3.90)}{=} \mathrm{div}\,(T_{\theta r}\mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta + T_{\theta z}\mathbf{e}_z) \overset{(1.190)}{=} \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}T_{\theta r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z}, \qquad (v)$$

$$\mathbf{e}_\theta \cdot \mathrm{div}\,\mathbf{T} = \mathrm{div}\,\mathbf{T}^T\mathbf{e}_\theta - \mathbf{T}\cdot\mathrm{grad}\,\mathbf{e}_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{T_{r\theta}+T_{\theta r}}{r}. \qquad (vi)$$

And $\mathbf{v} = \mathbf{e}_z$ leads to

$$\mathrm{grad}\,\mathbf{e}_z = \mathbf{o} \qquad \overset{(3.90)}{\Rightarrow} \qquad \mathbf{T}\cdot\mathrm{grad}\,\mathbf{e}_\theta = 0, \qquad (vii)$$

$$\mathrm{div}\,(\mathbf{T}^T\mathbf{e}_z) \overset{(3.90)}{=} \mathrm{div}\,(T_{zr}\mathbf{e}_r + T_{z\theta}\mathbf{e}_\theta + T_{zz}\mathbf{e}_z) \overset{(1.190)}{=} \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}T_{zr} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z}, \qquad (viii)$$

$$\mathbf{e}_z \cdot \mathrm{div}\,\mathbf{T} = \mathrm{div}\,\mathbf{T}^T\mathbf{e}_z - \mathbf{T}\cdot\mathrm{grad}\,\mathbf{e}_z = \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r}. \qquad (ix)$$

Finally we substitute (iii), (vi) and (ix) into (3.92) to get

$$\begin{aligned}\mathrm{div}\,\mathbf{T} \;&= \left(\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr}-T_{\theta\theta}}{r}\right)\mathbf{e}_r + \\ &+ \left(\frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{T_{r\theta}+T_{\theta r}}{r}\right)\mathbf{e}_\theta + \\ &+ \left(\frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r}\right)\mathbf{e}_z, \qquad \square\end{aligned} \qquad (3.94)$$

which gives us the divergence of a tensor field $\mathbf{T}(\mathbf{y})$ in cylindrical polar coordinates.

– **Remark:** We have (deliberately) not used the symmetry of $\mathbf{T}$ in the preceding calculations and formulae. As a result we can appropriate (3.94) (with the appropriate change of notation) to evaluate $\mathrm{Div}\,\mathbf{S}$.

– The equilibrium equation div $\mathbf{T} + \mathbf{b} = \mathbf{o}$ obeyed by the Cauchy stress tensor field $\mathbf{T}(\mathbf{y})$ can now be written in cylindrical polar coordinates as

$$\begin{aligned}
\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r} + b_r &= 0, \\
\frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{T_{r\theta} + T_{\theta r}}{r} + b_\theta &= 0, \\
\frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r} + b_z &= 0\ ,
\end{aligned} \tag{3.95}$$

where $\mathbf{b} = b_r\mathbf{e}_r + b_\theta\mathbf{e}_\theta + b_z\mathbf{e}_z$.

Exercise: Problem 3.26, Problem 3.27.

3.10.2 Spherical polar coordinates.

– Let (y_1, y_2, y_3) denote the rectangular cartesian coordinates of a particle in the deformed configuration and let (r, θ, ϕ) be its spherical polar coordinates. Then

$$y_1 = r\sin\theta\cos\phi, \qquad y_2 = r\sin\theta\sin\phi, \qquad y_3 = r\cos\theta. \tag{3.96}$$

$$0 \le r < \infty, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi < \pi.$$

The associated basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ are related by

$$\left.\begin{aligned}
\mathbf{e}_r &= (\sin\theta\cos\phi)\ \mathbf{e}_1 + (\sin\theta\sin\phi)\ \mathbf{e}_2 + \cos\theta\ \mathbf{e}_3, \\
\mathbf{e}_\theta &= (\cos\theta\cos\phi)\ \mathbf{e}_1 + (\cos\theta\sin\phi)\ \mathbf{e}_2 - \sin\theta\ \mathbf{e}_3, \\
\mathbf{e}_\phi &= -\sin\phi\ \mathbf{e}_1 + \cos\phi\ \mathbf{e}_2,
\end{aligned}\right\} \tag{3.97}$$

– Let $T_{rr}, T_{r\theta}, T_{r\phi}, \ldots$ be the components of the Cauchy stress tensor $\mathbf{T}$ in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$:

$$\mathbf{T} = T_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + T_{r\theta}\mathbf{e}_r \otimes \mathbf{e}_\theta + T_{r\phi}\mathbf{e}_r \otimes \mathbf{e}_\phi + T_{\theta r}\mathbf{e}_\theta \otimes \mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta \ldots\ .$$

– The equilibrium equation div $\mathbf{T} + \mathbf{b} = \mathbf{o}$ obeyed by the Cauchy stress tensor field $\mathbf{T}(\mathbf{y})$ can be shown to be

$$\begin{aligned}
\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\phi}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{2T_{rr} - T_{\phi\phi} - T_{\theta\theta} + T_{r\phi}\cot\phi}{r} + b_r &= 0, \\
\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\theta}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{3T_{r\theta} + 2T_{\theta\phi}\cot\phi}{r} + b_\theta &= 0, \\
\frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{\phi\theta}}{\partial \theta} + \frac{3T_{r\phi} + (T_{\phi\phi} - T_{\theta\theta})\cot\phi}{r} + b_\phi &= 0.
\end{aligned} \tag{3.98}$$

3.10.3 Worked examples

Problem 3.10.1. *(Combined axial and azimuthal shear of a tube)* An elastic body in a reference configuration occupies a hollow circular cylindrical region of unit length and inner and outer radii A and B respectively. Its outer surface $R = B$ is held fixed. A rigid solid cylinder of radius A is inserted into the cavity, and firmly bonded to the hollow elastic cylinder on their common interface $R = A$. A force $F\mathbf{e}_z$ in the axial direction and a torque $T\mathbf{e}_z$ about the axis are applied on the rigid cylinder. Assume that the resulting traction between the cylinders is uniformly distributed on their common interface. The resulting deformation involves axial and azimuthal shear, the kinematics of which were analyzed in Problem 2.17. The deformation was $r = R, \theta = \Theta + \phi(R), z = Z + w(R)$. Here we analyze the stress field.

In view of symmetry, assume that the Cauchy stress components in cylindrical coordinates (as given by a suitable isotropic constitutive relation) depend only on the r coordinate. (a) Simplify and solve the equilibrium equations to the extent possible. (b) Determine the boundary conditions on stress at $r = A$. (c) Using your answers from (a) and (b), determine the stress fields $T_{rz}(r)$ and $T_{r\theta}(r)$ in the elastic body.

Solution:

(a) We are told that the Cauchy stress components in cylindrical polar coordinates depend only of r (and not θ and z). The equilibrium equations (3.95) (in the absence of body force) then specialize to

$$\frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0. \qquad \frac{dT_{r\theta}}{dr} + 2\frac{T_{r\theta}}{r} = 0, \qquad \frac{dT_{rz}}{dr} + \frac{T_{rz}}{r} = 0. \tag{i}$$

The second and third of these equations can be written as

$$\frac{d}{dr}(r^2 T_{r\theta}) = 0, \qquad \frac{d}{dr}(rT_{rz}) = 0, \tag{ii}$$

which can be integrated to obtain

$$T_{r\theta}(r) = \frac{c_1}{r^2}, \qquad T_{rz}(r) = \frac{c_2}{r}, \qquad A \le r \le B, \qquad \Box \tag{iii}$$

where c_1 and c_2 are constants of integration (to be found using the boundary conditions).

(b) We now consider the equilibrium of the rigid cylinder. *We shall proceed vectorially but strongly encourage the reader to derive the results* (x) *and* (xi) *below using physical arguments.* Force balance requires

$$F\mathbf{e}_z + \int_{\mathcal{S}} \mathbf{t}\, dA_y = \mathbf{o}, \tag{iv}$$

where $\mathcal{S}$ is the interface $r = A$ between the cylinders. The traction (on the rigid cylinder) at this surface can be calculated using $\mathbf{t} = \mathbf{T}\mathbf{n}$, $\mathbf{n} = \mathbf{e}_r$ and $r = A$:

$$\mathbf{t} = T_{rr}(A)\mathbf{e}_r + T_{r\theta}(A)\mathbf{e}_\theta + T_{rz}(A)\mathbf{e}_z. \tag{v}$$

Substituting (v) into (iv) and using $dA_y = Ad\theta$ gives

$$F\mathbf{e}_z + \int_0^{2\pi} \Big[T_{rr}(A)\mathbf{e}_r + T_{r\theta}(A)\mathbf{e}_\theta + T_{rz}(A)\mathbf{e}_z\Big] Ad\theta = \mathbf{o}.$$

Keeping in mind that the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ depend on θ but $\mathbf{e}_z$ does not, we rewrite this as

$$F\mathbf{e}_z + AT_{rr}(A)\int_0^{2\pi} \mathbf{e}_r(\theta)d\theta + AT_{r\theta}(A)\int_0^{2\pi} \mathbf{e}_\theta(\theta)d\theta + AT_{rz}(A)\mathbf{e}_z \int_0^{2\pi} d\theta = \mathbf{o}.$$

Since

$$\mathbf{e}_r(\theta) = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2, \qquad \mathbf{e}_\theta(\theta) = -\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2,$$

the first and second integrals vanish and we are left with

$$F\mathbf{e}_z + AT_{rz}(A)\mathbf{e}_z \int_0^{2\pi} d\theta = \mathbf{o} \qquad \Rightarrow \qquad F\mathbf{e}_z + 2\pi AT_{rz}(A)\mathbf{e}_z = \mathbf{o}.$$

This leads to the boundary condition

$$T_{rz}(A) = -\frac{F}{2\pi A}. \qquad \square \tag{vi}$$

We next consider moment balance of the rigid cylinder which requires

$$T\mathbf{e}_z + \int_{\mathcal{S}} \mathbf{y} \times \mathbf{t}\, dA_y = \mathbf{o}. \tag{vii}$$

Since $\mathbf{y} = r\mathbf{e}_r = A\mathbf{e}_r$ at a point on $\mathcal{S}$, we have

$$\mathbf{y} \times \mathbf{t} = A\mathbf{e}_r \times \Big[T_{rr}(A)\mathbf{e}_r + T_{r\theta}(A)\mathbf{e}_\theta + T_{rz}(A)\mathbf{e}_z\Big] = AT_{r\theta}(A)\mathbf{e}_z - AT_{rz}(A)\mathbf{e}_\theta \quad \text{on } \mathcal{S}. \tag{viii}$$

Substituting $(viii)$ into (vii) and simplifying the integrals as above leads to

$$T\mathbf{e}_z + A^2T_{r\theta}(A)\mathbf{e}_z \int_0^{2\pi} d\theta = \mathbf{o}$$

from which we obtain the boundary condition

$$T_{r\theta}(A) = -\frac{T}{2\pi A^2}. \qquad \square \tag{ix}$$

(c) On using the boundary condition (vi) in the stress field $(iii)_2$ we get $c_2 = -F/(2\pi)$ and so the shear stress field $T_{rz}(r)$ in the elastic body is

$$T_{rz}(r) = -\frac{F}{2\pi r}, \qquad A \le r \le B. \qquad \square \tag{x}$$

Similarly from (ix) and $(iii)_1$ we find

$$T_{r\theta}(r) = -\frac{T}{2\pi r^2}, \qquad A \le r \le B. \qquad \square \tag{xi}$$

3.11 Exercises.

Unless explicitly told otherwise neglect body forces and inertial effects, and assume all components of vectors and tensors to be with respect to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ that has been implicitly chosen.

Problem 3.1. A uniform, heavy, elastic rope has length $2L_{\mathrm{R}}$ when it is in a stress-free reference configuration (e.g. lying on a rigid horizontal table). Its weight is W. In the reference configuration we can identify the rope with the straight line segment

$$\mathcal{R}_{\mathrm{R}} = \{x \,:\, -L_{\mathrm{R}} \le x \le L_{\mathrm{R}}\}.$$

The rope is placed over a rigid frictionless semi-circular cylinder of radius R as depicted in Figure 3.18 and lies in a vertical plane with gravity acting downwards. In the deformed configuration the rope is identified with the circular arc

$$\mathcal{R} = \{(y_1, y_2) \,:\, y_1 = R\sin\theta,\, y_2 = R\cos\theta,\, -\theta_* \le \theta \le \theta_*\}.$$

where $\theta_* = L_*/R \in (0, \pi/2)$ and the deformed length $2L_*$ are unknown.

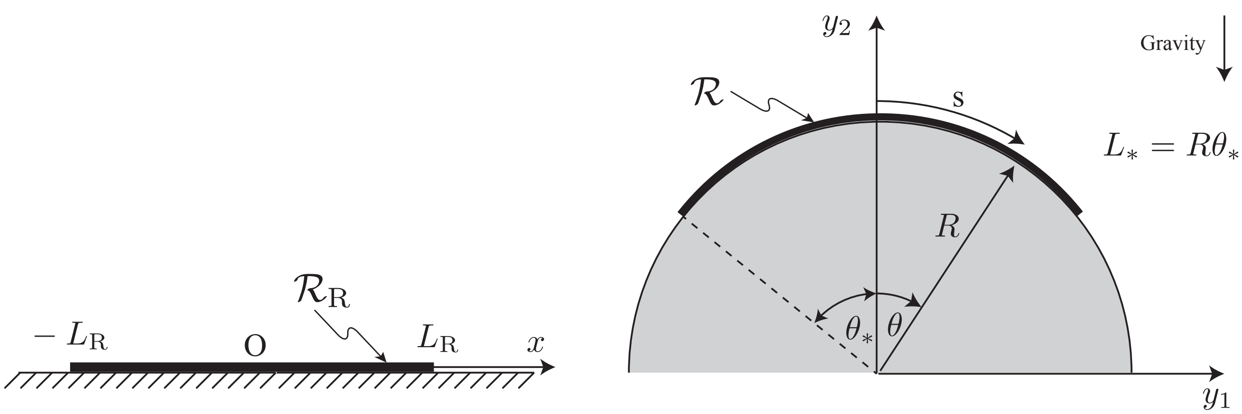


Figure 3.18: A heavy elastic rope in reference and deformed configurations.

The rope is very flexible and so the only internal force within it is a force τ in the direction tangent to the rope, and there is no internal moment. The constitutive relation of the rope, relating the internal normal force τ to the stretch λ, is

$$\tau = \mu \ln \lambda \tag{i}$$

where the constant parameter μ is a material property representing the elastic modulus and λ is the stretch with respect to the unstressed reference configuration. (a) Write down the equation that enforces the equilibrium of a finite (not infinitesimal) segment of the rope and from it, by localization, derive the equilibrium field equation that the force field $\tau(s)$ must obey. (b) Calculate the internal force $\tau(s)$ in terms of L_* and the given parameters. Sketch a graph of $\tau(s)$ versus the arc length s. *Keep in mind that the weight per unit deformed length of the rope is not* $w_{\rm R}$. (c) You want to determine the deformed length $2L_*$ of the rope: to this end, derive an equation in which L_* is the only unknown.

Solution: Suppose that the deformation takes the particle located at x in the reference configuration into the point with arc length s in the deformed configuration. This deformation is described by

$$s = s(x), \qquad 0 \leq x \leq L_{\rm R}. \tag{ii}$$

If dx and ds denote the respective lengths of an infinitesimal material fiber in the reference and deformed configurations, the stretch λ along the rope is their ratio:

$$\lambda(x) = \frac{ds}{dx}(x). \tag{iii}$$

Any function of the (Lagrangian coordinate) x can be written in terms of the (Eulerian coordinate) s by using (ii) and its inverse. The (unknown) deformed length of the rope is twice

$$L_* = s(L_{\rm R}). \tag{iv}$$

The angle θ shown in the figure is related to the arc length s by $s = R\theta$ so that

$$\theta(s) = s/R; \tag{v}$$

one can use either s or θ equivalently to characterize the position of a particle in the deformed configuration.

– The weight density of the rope in the reference configuration (i.e. weight per unit *undeformed* length) is

$$w_R = W/(2L_R).$$

If w denotes the weight density in the deformed configuration (i.e. weight per unit deformed length), mass conservation requires $w_{\rm R}dx = wds$ which by (iii) yields

$$w(s)\lambda(s) = w_{\rm R}. \tag{vi}$$

(a) Let $\tau(s)$ be the internal force (expected to be tensile) in the rope at the location s. In this elementary problem, the easiest way in which to derive the equilibrium equation ($viii$) is by drawing the free-body-diagram of an infinitesimal segment $[s, s + ds]$ of the rope *in the deformed configuration* and enforcing its equilibrium[13]. We leave that as an exercise. Instead, here we derive the equilibrium equation following the general approach that can be used for more complicated problems.

The unit tangent vector to the rope in the direction of increasing arc length is $\mathbf{e}_\theta$ and the unit normal vector is $\mathbf{e}_r$ where

$$\mathbf{e}_r = \sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2, \qquad \mathbf{e}_\theta = \cos\theta\mathbf{e}_1 - \sin\theta\mathbf{e}_2; \tag{a}$$

[13]Let θ and $\theta + d\theta$ be the angles between the radii at each end of this segment and the vertical. The normal reaction force on this segment makes an angle $\theta + d\theta/2$ with the vertical. Force balance in the direction perpendicular to the normal reaction gives $\tau(s+ds)\sin(\pi/2 - d\theta/2) - \tau(s)\sin(\pi/2 - d\theta/2) + w(s)ds\sin(\theta + d\theta/2) = 0$ which simplifies to the equilibrium equation (vii).

see Figure 3.18. Note that

$$\frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r. \qquad (b)$$

Let $\mathbf{t}(s)$ be the internal normal force (vector) at s applied by the part of the rope ahead of s on the part behind s. Let $\eta(s)\mathbf{e}_r(s)$ be the normal reaction force per unit length applied by the cylinder on the rope. The weight per unit length of the rope is $-w(s)\mathbf{e}_2$. Thus the "body force", i.e. the force distributed along the length of the rope, is $\mathbf{b} = \eta\mathbf{e}_r - w\mathbf{e}_2$. Then the equilibrium of the arbitrary segment $[s_1, s_2]$ of the rope requires

$$\mathbf{t}(s_2) - \mathbf{t}(s_1) + \int_{s_1}^{s_2} \Big[\eta\mathbf{e}_r - w\mathbf{e}_2\Big] ds = \mathbf{o}.$$

In preparation for localizing this, we write the boundary term $\mathbf{t}(s_2) - \mathbf{t}(s_1)$ as an integral over $[s_1, s_2]$. This yields

$$\int_{s_1}^{s_2} \frac{d\mathbf{t}}{ds} ds + \int_{s_1}^{s_2} \Big[\eta\mathbf{e}_r - w\mathbf{e}_2\Big] ds = \mathbf{o}.$$

Since this holds for all s_1 and s_2 (within the rope) we can localize it to obtain the following equilibrium equation at any location s:

$$\frac{d\mathbf{t}}{ds} + \eta\mathbf{e}_r - w\mathbf{e}_2 = \mathbf{o}. \qquad (c)$$

The first term can be simplified using $\mathbf{t}(s) = \tau(s)\mathbf{e}_\theta(s)$ (note that θ is measured CW and this determines the direction of $\mathbf{e}_\theta$):

$$\frac{d\mathbf{t}}{ds} = \frac{d}{ds}(\tau(s)\mathbf{e}_\theta(s)) = \frac{d\tau}{ds}\mathbf{e}_\theta + \tau\frac{d\mathbf{e}_\theta}{ds} = \frac{d\tau}{ds}\mathbf{e}_\theta + \tau\frac{d\mathbf{e}_\theta}{d\theta}\frac{d\theta}{ds} \overset{(b)}{=} \frac{d\tau}{ds}\mathbf{e}_\theta + \tau(-\mathbf{e}_r)\left(\frac{1}{R}\right),$$

where in the last step we also used $s = R\theta$. Thus (c) can be written as

$$\frac{d\tau}{ds}\mathbf{e}_\theta - \frac{\tau}{R}\mathbf{e}_r + \eta\mathbf{e}_r - w\mathbf{e}_2 = \mathbf{o}. \qquad (d)$$

This is the (vector) equilibrium equation. Since

$$\mathbf{e}_2 = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta$$

we can write (d) as

$$\left(\frac{d\tau}{ds} + w\sin\theta\right)\mathbf{e}_\theta + \left(\eta - w\cos\theta - \frac{\tau}{R}\right)\mathbf{e}_r = 0$$

which gives the two scalar equilibrium equations in the $\mathbf{e}_\theta$ and $\mathbf{e}_r$ directions:

$$\frac{d\tau}{ds}(s) + w(s)\sin\theta(s) = 0, \qquad \eta(s) - w(s)\cos\theta(s) - \frac{\tau(s)}{R} = 0. \qquad (vii)$$

The first of these is the equilibrium equation we will find useful. In this problem we are not interested in the normal reaction force η which can be calculated from the second.

(b) Substituting $(i), (v)$ and (vi) into $(vii)_1$ gives the following ordinary differential equation for $\lambda(s)$:

$$\frac{\mu}{\lambda}\frac{d\lambda}{ds} = -\frac{w_{\rm R}}{\lambda}\sin(s/R) \qquad \Rightarrow \qquad \frac{d\lambda}{ds} = -\frac{w_{\rm R}}{\mu}\sin(s/R). \qquad (viii)$$

We shall integrate this from an arbitrary location in the rope to the right-hand free end. Since the internal force in the rope vanishes at the free ends, $\tau = 0$ at $s = L_*$, it follows from the constitutive relation (i) that the rope is unstretched there:

$$\lambda = 1 \quad \text{at} \quad s = L_*. \qquad (ix)$$

Integrating $(viii)$ from s to L_* and using the boundary condition (ix) leads to

$$\lambda(s) = 1 - \alpha^2\Big(\cos(L_*/R) - \cos(s/R)\Big) \tag{x}$$

where we have introduced the known nondimensional parameter

$$\alpha := \sqrt{w_{\rm R} R/\mu}.$$

Thus the internal force in the rope is $\tau(s) = \mu \ln \lambda(s)$ with $\lambda(s)$ given by (x). $\square$

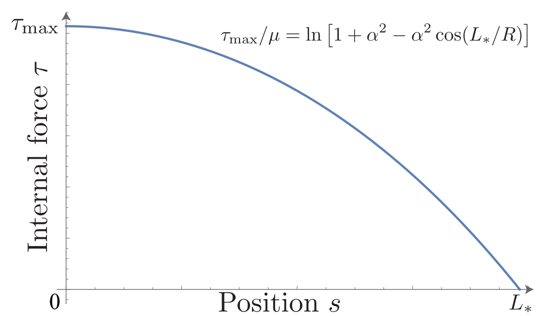


Figure 3.19: The internal force $\tau(s) = \mu\lambda(s)$ versus s with $\lambda(s)$ given by (x).

(c) It remains to find L_* for which we integrate $\lambda = ds/dx$ from the midpoint $(x = 0, s = 0)$ to the right-hand end $(x = L_{\rm R}, s = L_*)$:

$$\int_0^{L_{\rm R}} dx = \int_0^{L_*} \frac{1}{\lambda(s)}\, ds. \tag{xi}$$

Substituting (x) into (xi) gives

$$L_{\rm R} = \int_0^{L_*} \frac{1}{1-\alpha^2\Big(\cos(L_*/R) - \cos(s/R)\Big)}\, ds. \qquad \square$$

This is an equation for L_*. If so desired, we can change variables and write this as

$$\frac{L_{\rm R}}{R} = \int_0^{\theta_*} \frac{1}{1-\alpha^2\big(\cos\theta_* - \cos\theta\big)}\, d\theta, \qquad \square$$

which is an equation for $\theta_* = L_*/R$.

Problem 3.2. Suppose that the region $\mathcal{R}$ occupied by a certain body in its deformed configuration is a prismatic cylinder of length L and equilateral triangular cross section of height $3a$ as shown in Figure 3.20. The coordinate axes $\{y_1, y_2, y_3\}$ are as shown in the figure.

The Cauchy stress field in the cylinder is known to be

$$[T(y_1,y_2,y_3)] = \begin{pmatrix} 0 & 0 & T_{13} \\ 0 & 0 & T_{23} \\ T_{31} & T_{32} & 0 \end{pmatrix} \qquad \text{where} \qquad \left.\begin{aligned} T_{13} &= T_{31} &&= K y_2(y_1 - a), \\ T_{23} &= T_{32} &&= \tfrac{K}{2}\left(y_1^2 + 2a y_1 - y_2^2\right), \end{aligned}\right\} \tag{i}$$

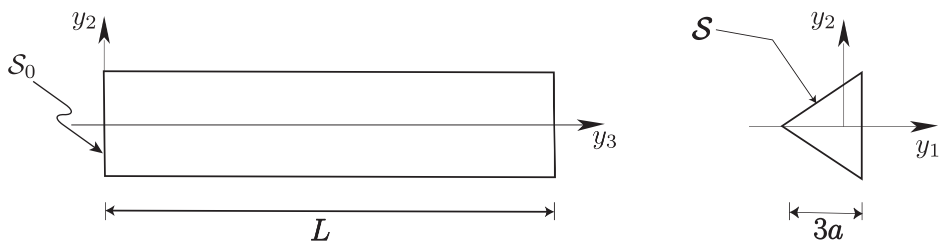


Figure 3.20: The region $\mathcal{R}$ occupied by the deformed body is a prismatic cylinder of length L and equilateral triangular cross section of height $3a$. $\mathcal{S}$ denotes one of its lateral surfaces.

where the constant K is a given loading parameter.

Calculate

(a) the applied load distribution (traction) on the lateral surfaces, and

(b) the *resultant* force and moment on the end $y_3 = 0$?

Problem 3.3. (Spencer) The region $\mathcal{R}$ occupied by a certain body in its deformed configuration is a right circular cylinder of length l and radius a:

$$\mathcal{R} = \{(y_1, y_2, y_3) \mid y_1^2 + y_2^2 \le a^2, \ \ -l \le y_3 \le 0\}.$$

Suppose the matrix of components of the Cauchy stress tensor field in the cylinder is

$$[T(y_1, y_2, y_3)] = \begin{pmatrix} 0 & 0 & -\alpha y_2 \\ 0 & 0 & \alpha y_1 \\ -\alpha y_2 & \alpha y_1 & \beta + \gamma y_1 + \eta y_2 \end{pmatrix}, \qquad (i)$$

where α, β, γ and η are constants. The components here have been taken with respect to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_3$ is aligned with the axis of the cylinder.

(a) Verify that this stress field satisfies all requirements for equilibrium in the interior of the body.

(b) Verify that the curved surface of the cylinder is traction-free.

(c) Calculate the traction on the end $y_3 = 0$. Hence calculate the resultant force and couple acting on the cylinder at the end $y_3 = 0$. Show that the parameters α, β, γ and η describe, respectively, a couple twisting the cylinder about the y_3-axis, a force pulling on the cylinder in the y_3-direction, a couple bending the cylinder about the y_2-axis, and a couple bending the cylinder about the y_1-axis.

(d) Consider the special case $\gamma = \eta = 0$ where there is no bending. Calculate the principal components of stress at an arbitrary point in the body. Calculate the value of the largest normal stress in the cylinder.

(e) Given a circular cylinder that is subjected to some prescribed traction on its boundary leading to axial loading, twisting and bending, does it necessarily follow that the stress field in the body has to be the stress field in (i)?

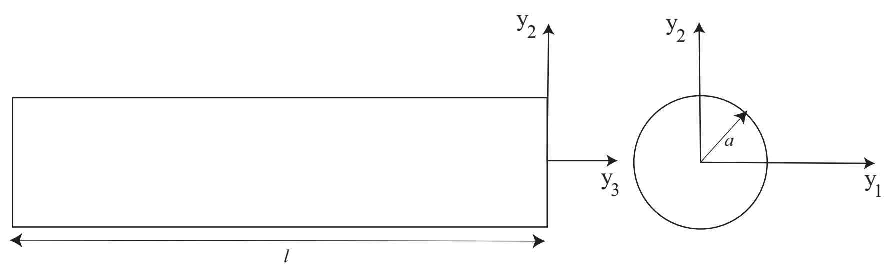


Figure 3.21: Figure for Problem 3.3: A right circular cylinder of length ℓ and radius a.

Solution:

(a) For force balance we must have $\partial T_{ij}/\partial y_j = 0$ and for moment balance we need $T_{ij} = T_{ji}$. Clearly the given matrix of stress components is symmetric. Therefore to check whether this stress field is in equilibrium we only need to substitute (i) into the equilibrium equations $\partial T_{ij}/\partial y_j = 0$:

$$\partial T_{11}/\partial y_1 + \partial T_{12}/\partial y_2 + \partial T_{13}/\partial y_3 = 0,$$
$$\partial T_{21}/\partial y_1 + \partial T_{22}/\partial y_2 + \partial T_{23}/\partial y_3 = 0,$$
$$\partial T_{31}/\partial y_1 + \partial T_{32}/\partial y_2 + \partial T_{33}/\partial y_3 = 0.$$

It is readily seen that these equations hold trivially since each term in each equation is identically zero for the stress field (i).

(b) The components of the unit outward normal vector $\mathbf{n}$ on the curved surface $y_1^2 + y_2^2 = a^2$ can be written as

$$n_1 = \cos\theta, \quad n_2 = \sin\theta, \quad n_3 = 0, \qquad 0 \le \theta < 2\pi, \tag{ii}$$

and so the traction on this surface can be calculated using $t_i = T_{ij}n_j$. The first two of these equations (i.e. $i = 1$ and $i = 2$) vanish trivially for the stress field (i):

$$t_1 = T_{11}n_1 + T_{12}n_2 + T_{13}\,\cancel{n_3} = 0; \quad t_2 = T_{21}n_1 + T_{22}n_2 + T_{23}\,\cancel{n_3} = 0.$$

The third equation simplifies as follows

$$t_3 = T_{31}n_1 + T_{32}n_2 + T_{33}\cancel{n_3} \overset{(i),(ii)}{=} (-\alpha y_2)\cos\theta + (\alpha y_1)\sin\theta = (-\alpha a\sin\theta)\cos\theta + (\alpha a\cos\theta)\sin\theta = 0$$

where we have used the fact that $y_1 = a\cos\theta, y_2 = a\sin\theta$ on the curved boundary.

Thus we conclude that $t_1 = t_2 = t_3 = 0$ on the curved boundary whence it is traction-free.

(c) Let $\mathcal{S}_0$ denote the cross section at $y_3 = 0$. We know (how?) that

$$\int_{\mathcal{S}_0} dA_y = \pi a^2, \qquad \int_{\mathcal{S}_0} y_1\, dA_y = \int_{\mathcal{S}_0} y_2\, dA_y = \int_{\mathcal{S}_0} y_1 y_2\, dA_y = 0, \qquad \int_{\mathcal{S}_0} y_1^2\, dA_y = \int_{\mathcal{S}_0} y_2^2\, dA_y = \frac{1}{4}\pi a^4. \tag{iii}$$

The components of the unit outward normal vector $\mathbf{n}$ on $\mathcal{S}_0$ are $n_1 = n_2 = 0, n_3 = 1$. Therefore the traction components $t_i = T_{ij}n_j$ on this surface specialize to

$$t_1 = T_{13}n_3 = -\alpha y_2, \quad t_2 = T_{23}n_3 = \alpha y_1, \quad t_3 = T_{33}n_3 = \beta + \gamma y_1 + \eta y_2. \tag{iv}$$

The resultant force $\mathbf{f}$ on $\mathcal{S}_0$ is given by the integral of $\mathbf{t}$ over $\mathcal{S}_0$ and so using (iii)

$$\begin{aligned}
f_1 &= \int_{\mathcal{S}_0} t_1\, dA_y = \int_{\mathcal{S}_0} T_{1j}n_j\, dA_y = \int_{\mathcal{S}_0} T_{13}\, dA_y = \int_{\mathcal{S}_0} (-\alpha y_2)\, dA_y &&= 0,\\
f_2 &= \int_{\mathcal{S}_0} t_2\, dA_y = \int_{\mathcal{S}_0} T_{2j}n_j\, dA_y = \int_{\mathcal{S}_0} T_{23}\, dA_y = \int_{\mathcal{S}_0} (\alpha y_1)\, dA_y &&= 0\\
f_3 &= \int_{\mathcal{S}_0} t_3\, dA_y = \int_{\mathcal{S}_0} T_{3j}n_j\, dA_y = \int_{\mathcal{S}_0} T_{33}\, dA_y = \int_{\mathcal{S}_0} (\beta + \gamma y_1 + \eta y_2)\, dA_y &&= \beta\pi a^2.
\end{aligned}$$

Therefore the resultant force on $\mathcal{S}_0$ is a pure axial force in the y_3-direction:

$$\mathbf{f} = \beta\pi a^2\ \mathbf{e}_3. \tag{v}$$

Turning next to the resultant moment on this face we recall that in general the resultant moment $\mathbf{m}$ is given by the integral of $\mathbf{y} \times \mathbf{t}$ over the surface $\mathcal{S}_0$. Therefore

$$\begin{aligned}
m_1 &= \int_{\mathcal{S}_0} e_{1jk} y_j t_k\, dA_y = \int_{\mathcal{S}_0} (e_{123} y_2 t_3 + e_{132}\, \cancel{y_3}\, t_2)\, dA_y = \int_{\mathcal{S}_0} y_2 t_3\, dA_y,\\
m_2 &= \int_{\mathcal{S}_0} e_{2jk} y_j t_k\, dA_y = \int_{\mathcal{S}_0} (e_{213} y_1 t_3 + e_{231}\, \cancel{y_3}\, t_1)\, dA_y = \int_{\mathcal{S}_0} -y_1 t_3\, dA_y,\\
m_3 &= \int_{\mathcal{S}_0} e_{3jk} y_j t_k\, dA_y = \int_{\mathcal{S}_0} (e_{312} y_1 t_2 + e_{321}\, y_2\, t_1)\, dA_y = \int_{\mathcal{S}_0} (-y_2 t_1 + y_1 t_2)\, dA_y.
\end{aligned}$$

Substituting (iv) into this and using (iii) to evaluate the integrals leads to

$$\begin{aligned}
m_1 &= \int_{\mathcal{S}_0} y_2 t_3\, dA_y = \int_{\mathcal{S}_0} y_2(\beta + \gamma y_1 + \eta y_2)\, dA_y = \eta\frac{\pi a^4}{4},\\
m_2 &= \int_{\mathcal{S}_0} -y_1 t_3\, dA_y = \int_{\mathcal{S}_0} -y_1(\beta + \gamma y_1 + \eta y_2)\, dA_y = -\gamma\frac{\pi a^4}{4},\\
m_3 &= \int_{\mathcal{S}_0} (-y_2 t_1 + y_1 t_2)\, dA_y = \int_{\mathcal{S}_0} (\alpha y_2^2 + \alpha y_1^2)\, dA_y = \alpha\frac{\pi a^4}{2}.
\end{aligned}$$

Therefore the resultant moment on $\mathcal{S}_0$ is

$$\mathbf{m} = \frac{\pi a^4}{4}\big(\eta\mathbf{e}_1 - \gamma\mathbf{e}_2 + 2\alpha\mathbf{e}_3\big). \tag{vi}$$

Therefore from equations (v) and (vi) we conclude that the parameters α, β, γ and η describe, respectively, a couple twisting the cylinder about the y_3-axis, a force pulling the cylinder in the y_1-direction, a couple bending the cylinder about the y_2-axis, and a couple bending the cylinder about the y_1-axis.

(d) The principal stresses at an arbitrary point in the body are found by calculating the eigenvalues T of the given stress tensor:

$$\det\left[\mathbf{T} - \tau I\right] = \det\begin{pmatrix} -\tau & 0 & -\alpha y_2 \\ 0 & -\tau & \alpha y_1 \\ -\alpha y_2 & \alpha y_1 & \beta - \tau \end{pmatrix} = -\tau^3 + \beta\tau^2 + \alpha^2 y_2(y_1 + y_2)\tau = 0.$$

The principal stresses, i.e. the roots of this cubic equation are

$$\tau = 0, \qquad \tau = \frac{1}{2}\left\{\beta \pm \sqrt{\beta^2 + 4\alpha^2(y_1^2 + y_2^2)}\right\}$$

Therefore the largest principal stress is

$$\tau(y_1, y_2, y_3) = \frac{1}{2}\left\{\beta + \sqrt{\beta^2 + 4\alpha^2(y_1^2 + y_2^2)}\right\} > 0. \qquad \square$$

This is the largest principal stress at any point (y_1, y_2, y_3). In order to find the maximum principal stress from among all points in the body, we need to maximize T as a function of (y_1, y_2, y_3). Clearly this occurs at the outer surface where $y_1^2 + y_2^2 = a^2$ and has the value

$$\tau_{max} = \frac{1}{2}\left\{\beta + \sqrt{\beta^2 + 4\alpha^2 a^2}\right\}. \qquad \square$$

(e) No. Prescribing only the resultant force and moment at the ends, and not the detailed traction distribution, does not adequately describe the boundary conditions of elasticity theory. There are other stress fields that also have these same resultants (and are in equilibrium and maintain a traction free curved surface.)

Problem 3.4. The normal stress at some point on a surface is

$$T_{\text{normal}}(\mathbf{n}) = \mathbf{t}(\mathbf{n}) \cdot \mathbf{n},$$

where the unit vector $\mathbf{n}$ is perpendicular to the surface at that point. From among all planes through that point, on which is $T_{\text{normal}}(\mathbf{n})$ a maximum and what is its value on that plane?

Solution: The normal stress on the plane perpendicular to $\mathbf{n}$ is

$$T_{\text{normal}}(\mathbf{n}) = \mathbf{t}(\mathbf{n}) \cdot \mathbf{n} = \mathbf{T}\mathbf{n} \cdot \mathbf{n} = T_{ij} n_i n_j. \qquad (i)$$

It is convenient to work in a principal basis for $\mathbf{T}$. When the components of $\mathbf{T}$ and $\mathbf{n}$ are taken with respect to such a basis, (i) reads

$$T_{\text{normal}}(\mathbf{n}) = T_{ij} n_i n_j = \tau_1\, n_1^2 + \tau_2\, n_2^2 + \tau_3\, n_3^2. \qquad (ii)$$

There is no loss of generality in supposing that $\tau_1 \geq \tau_2 \geq \tau_3$. For simplicity we shall exclude the case where two principal stresses are equal and so restrict attention to

$$\tau_1 > \tau_2 > \tau_3. \qquad (iii)$$

Exercise: consider the cases where either two, or all three, principal stresses coincide.

Solution 1: We want to maximize $T_{\text{normal}}(\mathbf{n})$ over all unit vectors $\mathbf{n}$. We account for the fact that there is a constraint $|\mathbf{n}|^2 = n_1 n_i = 1$ on the set of vectors over which we carry out the extremization by including (in the standard way) a Lagrange multiplier τ and considering the modified function

$$\mathcal{T}(\mathbf{n}) = \tau_1\, n_1^2 + \tau_2\, n_2^2 + \tau_3\, n_3^2 - \tau(n_1^2 + n_2^2 + n_3^2 - 1). \qquad (iv)$$

Setting the derivative of (v) with respect to each n_i equal to zero gives the three equations

$$n_1(\tau_1 - \tau) = 0, \qquad n_2(\tau_2 - \tau) = 0 \qquad n_3(\tau_3 - \tau) = 0. \qquad (v)$$

One extremum corresponds to $\tau = \tau_1, n_2 = n_3 = 0$ (note that $n_2 = n_3 = 0$ is in fact the eigenvector associated with the eigenvalue $\tau = \tau_1$). Evaluating (ii) at this solution gives $T_{\text{normal}}(\mathbf{n}) = \tau_1$. Likewise the other extrema correspond to $T_{\text{normal}}(\mathbf{n}) = \tau_2$ with $n_2 = 1, n_1 = n_3 = 0$; and $T_{\text{normal}}(\mathbf{n}) = \tau_3$ with $n_3 = 1, n_1 = n_2 = 0$. Thus, in view of (iii), we conclude that the maximum value of $T_{\text{normal}}(\mathbf{n})$ is the largest eigenvalue τ_1 and that it acts on a plane normal to the corresponding eigenvector. Likewise the minimum value of $T_{\text{normal}}(\mathbf{n})$ is the smallest eigenvalue τ_3 and it acts on a plane normal to the associated eigenvector.

Solution 2: In view of (iii), if we replace τ_2 and τ_3 by τ_1 in (ii) we conclude that

$$T_{\text{normal}}(\mathbf{n}) = \tau_1\, n_1^2 + \tau_2\, n_2^2 + \tau_3\, n_3^2 \le \tau_1\, n_1^2 + \tau_1\, n_2^2 + \tau_1\, n_3^2 = \tau_1\,(n_1^2 + n_2^2 + n_3^2) = \tau_1.$$

Thus τ_1 is an upper bound on the value of $T_{\text{normal}}(\mathbf{n})$. However $T_{\text{normal}}(\mathbf{n})$ takes the value τ_1 (when $n_1 = 1, n_2 = n_3 = 0$) and so achieves this upper bound. Thus the maximum value of $T_{\text{normal}}(\mathbf{n})$ is τ_1.

Likewise on replacing τ_1 and τ_2 by τ_3 in (ii) we conclude that

$$T_{\text{normal}}(\mathbf{n}) = \tau_1\, n_1^2 + \tau_2\, n_2^2 + \tau_3\, n_3^2 \ge \tau_3\, n_1^2 + \tau_3\, n_2^2 + \tau_3\, n_3^2 = \tau_3\,(n_1^2 + n_2^2 + n_3^2) = \tau_3.$$

Thus τ_3 is a lower bound on the value of $T_{\text{normal}}(\mathbf{n})$. However $T_{\text{normal}}(\mathbf{n})$ takes the value τ_3 (when $n_1 = n_2 = 0, n_3 = 1$) and so achieves this lower bound. Thus the minimum value of $T_{\text{normal}}(\mathbf{n})$ is τ_3.

Problem 3.5. The magnitude of the resultant shear stress at a point on a surface perpendicular to the unit vector $\mathbf{n}$ is

$$T_{\text{shear}}(\mathbf{n}) = \sqrt{\mathbf{t}(\mathbf{n}) \cdot \mathbf{t}(\mathbf{n}) - (\mathbf{t}(\mathbf{n}) \cdot \mathbf{n})^2}.$$

From among all planes through that point, on which is $T^2_{\text{shear}}(\mathbf{n})$ a maximum and what is its value on that plane?

Solution: Denote the principal Cauchy stresses by τ_1, τ_2, τ_3 and the corresponding principal directions by $\mathfrak{t}_1$, $\mathfrak{t}_2$ and $\mathfrak{t}_3$. Then the Cauchy stress tensor can be written as

$$\mathbf{T} = \tau_1\, \mathfrak{t}_1 \otimes \mathfrak{t}_1 \; + \; \tau_2\, \mathfrak{t}_2 \otimes \mathfrak{t}_2 \; + \; \tau_3\, \mathfrak{t}_3 \otimes \mathfrak{t}_3. \qquad (i)$$

Let n_1, n_2 and n_3 be the components of an arbitrary unit vector $\mathbf{n}$ in the principal basis $\{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3\}$:

$$\mathbf{n} = n_1\, \mathfrak{t}_1 \; + \; n_2\, \mathfrak{t}_2 \; + \; n_3\, \mathfrak{t}_3. \qquad (ii)$$

In carrying out the calculations below we shall use

$$\mathbf{t}_i \cdot \mathbf{t}_j = \delta_{ij} \tag{iii}$$

since $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3$ are mutually orthogonal unit vectors.

The traction on the plane normal to the unit vector $\mathbf{n}$ is

$$\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n} \overset{(i),(ii)}{=} (\tau_1\, \mathbf{t}_1\otimes\mathbf{t}_1 + \tau_2\, \mathbf{t}_2\otimes\mathbf{t}_2 + \tau_3\, \mathbf{t}_3\otimes\mathbf{t}_3)(n_1\, \mathbf{t}_1 + n_2\, \mathbf{t}_2 + n_3\, \mathbf{t}_3) \overset{(iii)}{=} \tau_1 n_1\, \mathbf{t}_1 + \tau_2 n_2\, \mathbf{t}_2 + \tau_3 n_3\, \mathbf{t}_3. \tag{v}$$

Therefore the normal stress on this plane is

$$T_{\text{normal}} = \mathbf{t}(\mathbf{n})\cdot\mathbf{n} \overset{(v),(ii)}{=} (\tau_1 n_1\, \mathbf{t}_1 + \tau_2 n_2\, \mathbf{t}_2 + \tau_3 n_3\, \mathbf{t}_3)(n_1\, \mathbf{t}_1 + n_2\, \mathbf{t}_2 + n_3\, \mathbf{t}_3) \overset{(iii)}{=} \tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2. \tag{vi}$$

Likewise

$$T_{\text{shear}}^2 + T_{\text{normal}}^2 = \mathbf{t}\cdot\mathbf{t} = \mathbf{T}\mathbf{n}\cdot\mathbf{T}\mathbf{n} = T_{ij}n_jT_{ik}n_k = \tau_1^2 n_1^2 + \tau_2^2 n_2^2 + \tau_3^2 n_3^2. \tag{vii}$$

Solution 1: Reference: D.H. Warner, An elementary derivation of the maximum shear stress in a three dimensional state of stress, *Journal of Elasticity*, 152(2022), pp. 179-182.

Without loss of generality we can order the principal Cauchy stresses as

$$\tau_1 \ge \tau_2 \ge \tau_3.$$

On using $n_1^2 + n_2^2 + n_3^2 = 1$ to eliminate n_3^2 from (vi) and (vii) we get

$$T_{\text{normal}} = (\tau_1 - \tau_3)n_1^2 + (\tau_2 - \tau_3)n_2^2 + \tau_3, \qquad T_{\text{shear}}^2 + T_{\text{normal}}^2 = (\tau_1^2 - \tau_3^2)n_1^2 + (\tau_2^2 - \tau_3^2)n_2^2 + \tau_3^2.$$

Eliminating n_1^2 from these two equations and rearranging terms leads to

$$T_{\text{shear}}^2 = \left(\frac{\tau_1 - \tau_3}{2}\right)^2 - \left(T_{\text{normal}} - \frac{\tau_1 + \tau_3}{2}\right)^2 - (\tau_1 - \tau_2)(\tau_2 - \tau_3)n_2^2.$$

In view of the ordering of the principal stresses this implies that

$$T_{\text{shear}}^2 \le \left(\frac{\tau_1 - \tau_3}{2}\right)^2 \qquad \Rightarrow \qquad |T_{\text{shear}}| \le \frac{\tau_1 - \tau_3}{2}.$$

Thus $\frac{1}{2}(\tau_1 - \tau_3)$ is an upper bound on $|T_{\text{shear}}|$. That $|T_{\text{shear}}|$ achieves this upper bound can be readily verified by taking $n_1 = n_3 = 1/\sqrt{3}, n_2 = 0$ in (vi) and (vii) which gives $T_{\text{normal}} = (\tau_1 + \tau_3)/2$ and $T_{\text{shear}}^2 = \big[(\tau_1 - \tau_3)/2\big]^2$. Thus

$$|T_{\text{shear}}|_{\text{max over unit vectors } \mathbf{n}} = \frac{\tau_1 - \tau_3}{2}.$$

Solution 2: Using calculus: our task is to maximizes T_{shear}^2 over all unit vectors $\mathbf{n}$ where

$$T_{\text{shear}}^2 = |\mathbf{t}(\mathbf{n})|^2 - T_{\text{normal}}^2 = \tau_1^2 n_1^2 + \tau_2^2 n_2^2 + \tau_3^2 n_3^2 - (\tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2)^2. \tag{viii}$$

We incorporate the constraint $|\mathbf{n}| = 1$ in the usual way by introducing a Lagrange multiplier τ and constructing the related function

$$\mathcal{L}(n_1, n_2, n_3) = T_{\text{shear}}^2 - \tau(n_1^2 + n_2^2 + n_3^2 - 1). \tag{ix}$$

On setting $\partial\mathcal{L}/\partial n_i = 0$ for $i = 1, 2, 3$ we get

$$\left.\begin{aligned} n_1\left[\tau_1^2 - 2\tau_1(\tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2) + \tau\right] &= 0, \\ n_2\left[\tau_2^2 - 2\tau_2(\tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2) + \tau\right] &= 0, \\ n_3\left[\tau_3^2 - 2\tau_3(\tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2) + \tau\right] &= 0. \end{aligned}\right\} \qquad (x)$$

We must solve (x) for $\mathbf{n}$ and then evaluate $(viii)$ at this $\mathbf{n}$ (or these $\mathbf{n}$'s).

We shall only consider the case where the principal stresses are distinct.

Since $\mathbf{n}$ is a unit vector, all three n's cannot vanish. First consider solutions of (x) (if any) where two of the n's are zero, e.g.

$$n_2 = n_3 = 0 \qquad \Rightarrow \qquad n_1 = \pm 1. \qquad (xi)$$

Substituting (xi) into $(viii)$ gives

$$T^2_{\text{shear}} = \tau_1^2 n_1^2 - (\tau_1 n_1^2)^2 = 0. \qquad (xii)$$

Of course the cases $n_1 = n_2 = 0$ and $n_3 = n_1 = 0$ lead to the same result. So this extremum does not provide the maximum value of T^2_{shear}.

Next consider solutions where only one of the n's is zero. e.g.

$$n_3 = 0, \quad n_1 \neq 0, \quad n_2 \neq 0 \qquad (xiii)$$

Then (x) reduces to

$$\left.\begin{aligned} \tau_1^2 - 2\tau_1(\tau_1 n_1^2 + \tau_2 n_2^2) + \tau = 0 \qquad &\Rightarrow \qquad \tau_1^2 - 2\tau_1^2 n_1^2 - 2\tau_1\tau_2 n_2^2 + \tau = 0, \\ \tau_2^2 - 2\tau_2(\tau_1 n_1^2 + \tau_2 n_2^2) + \tau = 0 \qquad &\Rightarrow \qquad \tau_2^2 - 2\tau_1\tau_2 n_1^2 - 2\tau_2^2 n_2^2 + \tau = 0 \end{aligned}\right\} \qquad (xiv)$$

On subtracting the first equation from the second (to eliminate τ) and using $n_1^2 + n_2^2 = 1$ one ends up with

$$(\tau_1 - \tau_2)^2(n_1^2 - n_2^2) = 0. \qquad (xv)$$

Since $\tau_1 \neq \tau_2$ (we are considering the case where the principal stresses are distinct) this yields

$$n_1 = \pm n_2. \qquad (xvi)$$

So we have $n_1 = \pm n_2, n_3 = 0$ and since $\mathbf{n}$ is a unit vector this gives

$$n_1 = \pm\frac{1}{\sqrt{2}}, \qquad n_2 = \pm\frac{1}{\sqrt{2}}, \qquad n_3 = 0. \qquad (xvii)$$

Note that this direction bisects the angle between $\mathfrak{t}_1$ and $\mathfrak{t}_2$ and lies in the plane spanned by $\mathfrak{t}_1$ and $\mathfrak{t}_2$. This is a direction that extremizes T^2_{shear}. To find the value of T^2_{shear} at this extremum we substitute $(xvii)$ into $(viii)$ to get

$$T^2_{\text{shear}} = \tau_1^2/2 + \tau_2^2/2 - (\tau_1/2 + \tau_2/2)^2 = \frac{1}{4}(\tau_1 - \tau_2)^2,$$

from which we have

$$T_{\text{shear}} = \frac{1}{2}|\tau_1 - \tau_2|.$$

The cases $n_1 = 0, n_2 \neq 0, n_3 \neq 0$ and $n_2 = 0, n_3 \neq 0, n_1 \neq 0$ can be handled similarly and one is eventually led to the conclusion that the maximum resultant shear stress is given by the maximum of

$$\frac{1}{2}|\tau_1 - \tau_2|, \qquad \frac{1}{2}|\tau_2 - \tau_3|, \qquad \frac{1}{2}|\tau_3 - \tau_1|,$$

and the normal to the plane on which it acts bisects the two associated principal directions.

It remains to consider the case where none of the n's vanish. Cancelling the n's outside the square bracket in (x) and subtracting the first equation from the second, and the second from the third, leads to two equations (that do not involve τ). The analysis of these equations (together with the requirement $|\mathbf{n}| = 1$ is left as an exercise.

Problem 3.6.

(a) For every stress tensor $\mathbf{T}$, is there a plane on which the magnitude of the resultant shear stress $T_{\text{shear}}(\mathbf{n})$ vanishes?

(b) For every stress tensor $\mathbf{T}$, is there a plane on which the normal stress $T_{\text{normal}}(\mathbf{n})$ vanishes?

(c) Suppose that the principal stresses τ_1, τ_2, τ_3 at some point in a body are all non-zero and $\tau_2 = \tau_3$. Find necessary and sufficient conditions on τ_1 and τ_2 under which there is a plane on which the normal stress vanishes.

Solution:

(a) Since $\mathbf{T}$ is symmetric it necessarily has a principal basis, and the matrix of components of $[T]$ in that basis is diagonal. Thus the shear stress vanishes on any plane perpendicular to a principal direction.

(b) From (3.40) (in a principal basis for $\mathbf{T}$ we have)

$$T_{\text{normal}}(\mathbf{n}) = \tau_1\, n_1^2 + \tau_2\, n_2^2 + \tau_3\, n_3^2. \qquad (i)$$

Suppose that all three principal stresses are positive: $\tau_1 > 0, \tau_2 > 0, \tau_3 > 0$. Then T_{normal} is the sum of three non-negative terms at least one of which is strictly positive. (Since $\mathbf{n}$ is a unit vector, all three n_i's cannot vanish.) Thus in this case there is no direction $\mathbf{n}$ for which T_{normal} vanishes and so in general there is not a direction $\mathbf{n}$ for which T_{normal} vanishes.

An explicit example of this is provided by a pure hydrostatic stress $\mathbf{T} = \tau\mathbf{I}, \tau \neq 0$. The normal stress on the plane perpendicular to an arbitrary direction $\mathbf{n}$ is $T_{\text{normal}}(\mathbf{n}) = \mathbf{T}\mathbf{n} \cdot \mathbf{n} = \tau\mathbf{n} \cdot \mathbf{n} = \tau$ which is nonzero for all $\mathbf{n}$.

(c) In this case, set $\tau_2 = \tau_3$ in (ii) and use $n_1^2 + n_2^2 + n_3^2 = 1$ to get

$$T_{\text{normal}}(\mathbf{n}) = \tau_1\, n_1^2 + \tau_2\,(n_2^2 + n_3^2) = \tau_1\, n_1^2 + \tau_2\,(1 - n_1^2) = (\tau_1 - \tau_2)n_1^2 + \tau_2. \qquad (ii)$$

Suppose that T_{normal} vanishes:

$$(\tau_1 - \tau_2)n_1^2 + \tau_2 = 0. \qquad (iii)$$

Since $\tau_2 \neq 0$, it follows that $(\tau_1 - \tau_2)n_1^2 \neq 0$ and therefore that

$$n_1 \neq 0, \qquad \text{and} \qquad \tau_1 \neq \tau_2. \tag{iv}$$

From (iii),

$$n_1^2 = \frac{\tau_2}{\tau_2 - \tau_1}, \tag{v}$$

where we were able to divide by $\tau_2 - \tau_1$ because of $(iv)_2$. Since $n_1 (\neq 0)$ is one component of a unit vector, we know that $0 < n_1^2 \leq 1$. Therefore (v) will define an acceptable value for n_1 provided

$$0 < \frac{\tau_2}{\tau_2 - \tau_1} \leq 1. \tag{vi}$$

This yields

$$\mathsf{Case}\ \tau_2 > \tau_1: \qquad (vi) \quad \Rightarrow \ 0 < \tau_2 \leq \tau_2 - \tau_1 \quad \Rightarrow \quad \tau_2 > 0 \geq \tau_1 \quad \Rightarrow \quad \tau_2 > 0 > \tau_1. \tag{vii}$$

where in the last step we used the fact that $\tau_1 \neq 0$ (given), and

$$\mathsf{Case}\ \tau_2 < \tau_1: \qquad (vi) \quad \Rightarrow \ 0 > \tau_2 \geq \tau_2 - \tau_1 \quad \Rightarrow \quad \tau_1 \geq 0 > \tau_2 \quad \Rightarrow \quad \tau_1 > 0 > \tau_2. \tag{viii}$$

Therefore it is necessary that τ_1 and τ_2 have opposite signs.

Conversely suppose τ_1 and τ_2 have opposite signs. Then one can reverse the preceding steps and go from the last inequality in (vii) or $(viii)$ to (vi). When (vi) holds (v) defines a real direction

$$n_1 = \sqrt{\frac{\tau_2}{\tau_2 - \tau_1}}$$

for which T_{normal} vanishes (n_2 and n_3 being arbitrary except for making a unit vector).

Thus necessary and sufficient for there being a plane on which T_{normal} vanishes is that τ_1 and τ_2 have opposite signs. □

Problem 3.7. Suppose the traction $\mathbf{t}(\mathbf{n})$ on every plane through a given point has the same direction $\mathbf{a}$, i.e. suppose that $\mathbf{t}(\mathbf{n}) = \tau(\mathbf{n})\,\mathbf{a}$ for all unit vectors $\mathbf{n}$ where $\mathbf{a}$ is a constant unit vector. What is the form of the most general stress tensor $\mathbf{T}$ that is consistent with this?

Solution In this solution we shall suspend the usual rules of indicial notation. Pick a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_1 = \mathbf{a}$. Since $\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n} = \tau(\mathbf{n})\,\mathbf{a}$ for all unit vectors $\mathbf{n}$, we pick $\mathbf{n} = \mathbf{e}_j$ to get $\mathbf{T}\mathbf{e}_j = \tau(\mathbf{e}_j)\,\mathbf{e}_1$. Then from (3.14) it follows that

$$T_{ij} = \mathbf{T}\mathbf{e}_j \cdot \mathbf{e}_i = \tau(\mathbf{e}_j)\,\mathbf{e}_1 \cdot \mathbf{e}_i = \tau(\mathbf{e}_j)\delta_{1j} \qquad \text{for all } i, j, = 1, 2, 3.$$

Therefore all components T_{ij} vanish except $T_{11} = \tau(\mathbf{e}_1)$. Therefore

$$\mathbf{T} = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = T_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 = \tau(\mathbf{e}_1)\mathbf{e}_1 \otimes \mathbf{e}_1,$$

and so returning to the vector $\mathbf{a} = \mathbf{e}_1$ we obtain $\mathbf{T} = \tau(\mathbf{a})\mathbf{a} \otimes \mathbf{a}$. □

Problem 3.8. In this problem you are to show that the Cauchy stress tensor at a point in a body is fully determined by the traction on any three linearly independent planes. Specifically: consider the three planes (through some point in the body) normal to the respective unit vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$. Suppose the traction on each of these planes is $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ respectively.

(a) Write down (in terms of $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$) the Cauchy stress tensor in the case where the unit vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are mutually orthogonal.

(b) Write down the Cauchy stress tensor in the case where the unit vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are linearly independent (but not necessarily mutually orthogonal).

(c) Can you write down the Cauchy stress tensor in the case where the unit vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are linearly dependent (with $\mathbf{n}_1, \mathbf{n}_2$ being linearly independent)? Explain.

Solution: We must construct a tensor $\mathbf{T}$ for which

$$\mathbf{T}\mathbf{n}_i = \mathbf{t}_i, \tag{i}$$

and, moreover, such that $\mathbf{T}\mathbf{n}$ is completely determined for all unit vectors $\mathbf{n}$.

(a) In this case $\mathbf{n}_i \cdot \mathbf{n}_j = \delta_{ij}$ Thus clearly

$$\mathbf{T} = \mathbf{t}_1 \otimes \mathbf{n}_1 + \mathbf{t}_2 \otimes \mathbf{n}_2 + \mathbf{t}_3 \otimes \mathbf{n}_3 = \mathbf{t}_j \otimes \mathbf{n}_j,$$

since $\mathbf{T}\mathbf{n}_i = (\mathbf{t}_j \otimes \mathbf{n}_j)\mathbf{n}_i = (\mathbf{n}_j \cdot \mathbf{n}_i)\mathbf{t}_j = \delta_{ij}\mathbf{t}_j = \mathbf{t}_i$. For an arbitrary vector $\mathbf{n} = n_i\mathbf{n}_i$ we have $\mathbf{T}\mathbf{n} = n_i(\mathbf{t}_j \otimes \mathbf{n}_j)\mathbf{n}_i = n_i\delta_{ij}\mathbf{t}_j = n_i\mathbf{t}_i$.

(b) We (continue to) want to find a tensor $\mathbf{T}$ such that (i) holds. If we can find 3 vectors $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ such that $\mathbf{m}_i \cdot \mathbf{n}_j = \delta_{ij}$ (keep in mind that in this part of the problem $\mathbf{n}_i \cdot \mathbf{n}_j \neq \delta_{ij}$) then, since $(\mathbf{t}_j \otimes \mathbf{m}_j)\mathbf{n}_i = (\mathbf{m}_j \cdot \mathbf{n}_i)\mathbf{t}_j = \delta_{ij}\mathbf{t}_j = \mathbf{t}_i$ it would follow that $\mathbf{T} = \mathbf{t}_i \otimes \mathbf{m}_i$. Thus our task is to determine three such vectors $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$.

Since $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are linearly independent, it follows from Problem 1.7 that $\mathbf{n}_i \times \mathbf{n}_j \neq \mathbf{o}$ and $(\mathbf{n}_i \times \mathbf{n}_j) \cdot \mathbf{n}_k \neq 0$ for distinct i, j, k. Therefore we can define three non-zero vectors $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ related to $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ by[14]

$$\mathbf{m}_1 = \frac{\mathbf{n}_2 \times \mathbf{n}_3}{(\mathbf{n}_2 \times \mathbf{n}_3) \cdot \mathbf{n}_1}, \qquad \mathbf{m}_2 = \frac{\mathbf{n}_3 \times \mathbf{n}_1}{(\mathbf{n}_3 \times \mathbf{n}_1) \cdot \mathbf{n}_2}, \qquad \mathbf{m}_3 = \frac{\mathbf{n}_1 \times \mathbf{n}_2}{(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{n}_3}. \tag{ii}$$

Observe that the vector $\mathbf{m}_1$ is perpendicular to the vectors $\mathbf{n}_2$ and $\mathbf{n}_3$ and its length is such that $\mathbf{m}_1 \cdot \mathbf{n}_1 = 1$, i.e.

$$\mathbf{m}_1 \cdot \mathbf{n}_1 = 1, \qquad \mathbf{m}_1 \cdot \mathbf{n}_2 = \mathbf{m}_1 \cdot \mathbf{n}_3 = 0. \tag{iii}$$

The vectors $\mathbf{m}_2$ and $\mathbf{m}_3$ are related analogously to $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$. Therefore

$$\mathbf{m}_i \cdot \mathbf{n}_j = \delta_{ij}. \tag{iv}$$

It now follows from the remarks in the first paragraph that

$$\mathbf{T} = \mathbf{t}_j \otimes \mathbf{m}_j. \qquad \square$$

[14] The vectors $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ are said to be **reciprocal** to the vectors $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$.

For an arbitrary vector $\mathbf{n} = n_i \mathbf{n}_i$ we have $\mathbf{T}\mathbf{n} = n_i(\mathbf{t}_j \otimes \mathbf{m}_j)\mathbf{n}_i \overset{(iv)}{=} n_i \mathbf{t}_i$.

Remark: In the special case where each vector of the set $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ is perpendicular to the other two, one sees from (ii) that $\mathbf{m}_i = \mathbf{n}_i/|\mathbf{n}_i|^2 = \mathbf{n}_i$ since $\mathbf{n}_i$ is a unit vector.

(c) No. The vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ lie in a plane and we have no information on the traction acting *on* that plane, i.e. we do not have enough information to calculate $\mathbf{T}\mathbf{n}$ when $\mathbf{n}$ is perpendicular to the plane containing $\mathbf{n}_1, \mathbf{n}_2$ and $\mathbf{n}_3$.

To demonstrate this formally it suffices to provide a single counter example. Consider the case where $\mathbf{n}_1$ is perpendicular to $\mathbf{n}_2$, and $\mathbf{n}_3$ lies in the plane defined by $\mathbf{n}_1$ and $\mathbf{n}_2$. Suppose further that

$$\mathbf{t}_i = \tau \mathbf{n}_i, \qquad i = 1, 2, 3, \tag{v}$$

for some (given) scalar τ. The question is whether (v) fully determines the stress tensor $\mathbf{T}$. Consider the tensor

$$\mathbf{T} = \tau \mathbf{n}_1 \otimes \mathbf{n}_1 + \tau \mathbf{n}_2 \otimes \mathbf{n}_2 + T_{33} \mathbf{n} \otimes \mathbf{n},$$

where $\mathbf{n}$ is a unit vector perpendicular to each $\mathbf{n}_i, i = 1, 2, 3$, and T_{33} is arbitrary. One can readily verify that $\mathbf{T}\mathbf{n}_i = \tau \mathbf{n}_i$ for $i = 1, 2, 3$ and therefore that $\mathbf{T}$ is consistent with all the given information in (v). However due to the presence of the arbitrary term T_{33}, the stress tensor $\mathbf{T}$ is not fully determined by the given information in (v), e.g. the traction on the plane normal to $\mathbf{n}$ is $T_{33}\mathbf{n}$ and is undetermined.

Problem 3.9. (Based on Chadwick) The Cauchy stress tensor (at a certain point in a body) is

$$\mathbf{T} = \alpha \mathbf{I} + \beta(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3) \tag{i}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants.

(a) Calculate the principal stresses and corresponding principal directions.

(b) Calculate the maximum (over all $\mathbf{n}$) of the resultant shear stress magnitude $T_{\text{shear}}(\mathbf{n})$.

(c) Find necessary and sufficient conditions under which there is a plane on which the normal stress vanishes.

(d) Find necessary and sufficient conditions under which there is a plane on which the traction vanishes; assume $\alpha \neq \beta$.

Solution:

(a) The principal stresses are the eigenvalues of $\mathbf{T}$ and so we must find τ such that

$$\det(\mathbf{T} - \tau\, \mathbf{I}) = \det \begin{pmatrix} \alpha - \tau & \beta & \beta \\ \beta & \alpha - \tau & \beta \\ \beta & \beta & \alpha - \tau \end{pmatrix} = 0 \tag{ii}$$

Expanding the determinant and simplifying leads to

$$(\alpha - \tau + 2\beta)(\alpha - \tau - \beta)(\alpha - \tau - \beta) = 0,$$

from which we conclude that the principal stresses are

$$\tau_1 = \alpha + 2\beta, \quad \tau_2 = \tau_3 = \alpha - \beta. \tag{iii}$$

To find the principal direction associated with, say τ_1, we must find a unit vector $\mathbf{t}$ such that $\mathbf{T}\mathbf{t} = \tau_1\mathbf{t}$:

$$\begin{pmatrix} \alpha - \tau_1 & \beta & \beta \\ \beta & \alpha - \tau_1 & \beta \\ \beta & \beta & \alpha - \tau_1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives three scalar algebraic equations (two of which are independent) from which to find s_1, s_2, s_3. Solving them (together with $s_1^2 + s_2^2 + s_3^2 = 1$) leads to

$$s_1 = s_2 = s_3 = \frac{1}{\sqrt{3}},$$

and so the principal direction associated with τ_1 is

$$\mathbf{t}_1 = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3).$$

The principal directions associated with τ_2 and τ_3 are found similarly (with some attention to the fact that the eigenvalue is repeated: $\tau_2 = \tau_3$. One finds

$$\mathbf{t}_2 = \frac{1}{\sqrt{6}}(\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3), \qquad \mathbf{t}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2).$$

(b) The maximum value of the magnitude of the resultant shear stress, $(T_{\text{shear}})_{\max}$, is given by the largest of

$$\frac{1}{2}|\tau_1 - \tau_2|, \qquad \frac{1}{2}|\tau_2 - \tau_3|, \qquad \frac{1}{2}|\tau_3 - \tau_1|.$$

By substituting (iii) into this one finds

$$(T_{\text{shear}})_{\max} = \frac{3}{2}|\beta|.$$

(c) The normal stress on the plane perpendicular to $\mathbf{n}$ is

$$T_{\text{normal}} = \mathbf{t}(\mathbf{n}) \cdot \mathbf{n} = \mathbf{T}\mathbf{n} \cdot \mathbf{n} = T_{ij} n_i n_j. \tag{iv}$$

It is convenient to work in the principal basis $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ and take all components with respect to that basis. Equation (iv) then reads

$$T_{\text{normal}} = \tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2. \tag{v}$$

Substituting (iii) into (v) and simplifying leads to

$$\begin{aligned} T_{\text{normal}} &= (\alpha + 2\beta)n_1^2 + (\alpha - \beta)(n_2^2 + n_3^2) = (\alpha + 2\beta)n_1^2 + (\alpha - \beta)(1 - n_1^2) = \\ &= 3\beta\, n_1^2 + \alpha - \beta, \end{aligned}$$

where we have used the fact that $n_1^2 + n_2^2 + n_3^2 = 1$. We are told that this vanishes on some plane and so

$$n_1^2 = \frac{\beta - \alpha}{3\beta}.$$

In order that this give an acceptable value of n_1 it is necessary and sufficient that $0 \leq n_1^2 < 1$:

$$0 \leq \frac{\beta - \alpha}{3\beta} < 1 \qquad \Leftrightarrow \qquad -2 < \frac{\alpha}{\beta} \leq 1$$

Question: why is $n_1^2 = 1$ disallowed?

(d) In this case $\mathbf{Tn} = \mathbf{o}$ and $\mathbf{n} \neq \mathbf{o}$ and therefore it is necessary that $\mathbf{T}$ be singular. Thus

$$\det \mathbf{T} = (\alpha - \beta)^2(\alpha + 2\beta) = 0 \qquad \Rightarrow \quad \alpha = -2\beta, \qquad \Box \tag{viii}$$

since we are told that $\alpha \neq \beta$. Conversely when $(viii)$ holds one can readily verify that $\mathbf{Tn} = \mathbf{o}$ for

$$\mathbf{n} = \frac{1}{\sqrt{3}}\,\mathbf{e}_1 + \frac{1}{\sqrt{3}}\,\mathbf{e}_2 + \frac{1}{\sqrt{3}}\,\mathbf{e}_3.$$

Problem 3.10. (Ogden) The resultant shear stress on a plane perpendicular to an arbitrary direction $\mathbf{n}$ has magnitude $T_{\text{shear}}(\mathbf{n})$. It was defined in equation (3.7) (and depicted in Figure 3.5) to be

$$T_{\text{shear}}(\mathbf{n}) = \sqrt{\mathbf{t}(\mathbf{n}) \cdot \mathbf{t}(\mathbf{n}) - [\mathbf{t}(\mathbf{n}) \cdot \mathbf{n}]^2}. \tag{i}$$

(a) Show that the expression (i) can be written as

$$T^2_{\text{shear}} = (\tau_1 - \tau_2)^2 n_1^2 n_2^2 + (\tau_2 - \tau_3)^2 n_2^2 n_3^2 + (\tau_3 - \tau_1)^2 n_3^2 n_1^2, \tag{3.99}$$

where τ_1, τ_2, τ_3 are the principal Cauchy stresses and n_1, n_2, n_3 are the components of the arbitrary direction $\mathbf{n}$ in the principal basis for $\mathbf{T}$.

(b) Show that the average value of (3.99) over all possible directions $\mathbf{n}$ is

$$\frac{1}{15}\Big[(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2\Big]. \tag{iii}$$

(c) The particular plane whose direction is equally inclined to the principal axes of $\mathbf{T}$ is known as the *octahedral plane*. Calculate the magnitude of the resultant shear stress on the octahedral plane. How is it related to (iii)?

Solution: The summation convention is suspended in this solution. All components are taken with respect to a principal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbf{T}$.

(a) We need to calculate $\mathbf{t} \cdot \mathbf{n}$ and $\mathbf{t} \cdot \mathbf{t}$:

$$\mathbf{t} = \mathbf{Tn} = \left(\sum_{i=1}^{3} \tau_i (\mathbf{e}_i \otimes \mathbf{e}_i)\right)\mathbf{n} = \sum_{i=1}^{3} \tau_i n_i \mathbf{e}_i, \tag{iv}$$

$$\mathbf{t}\cdot\mathbf{n} = \mathbf{T}\mathbf{n}\cdot\mathbf{n} = \left(\sum_{i=1}^{3}\tau_i n_i \mathbf{e}_i\right)\cdot\mathbf{n} = \sum_{i=1}^{3}\tau_i n_i n_i = \tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2, \qquad (v)$$

$$\mathbf{t}\cdot\mathbf{t} = \mathbf{T}\mathbf{n}\cdot\mathbf{T}\mathbf{n} = \mathbf{T}^T\mathbf{T}\mathbf{n}\cdot\mathbf{n} = \mathbf{T}^2\mathbf{n}\cdot\mathbf{n} = \tau_1^2 n_1^2 + \tau_2^2 n_2^2 + \tau_3^2 n_3^2, \qquad (vi)$$

where in writing the last expression in (vi) we simply recognized the similarity with $\mathbf{Tn}\cdot\mathbf{n}$ that we calculated in (v). Substituting (v) and (vi) into (i)

$$\begin{aligned}
T_{\text{shear}}^2 &= \tau_1^2 n_1^2 + \tau_2^2 n_2^2 + \tau_3^2 n_3^2 - (\tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2)^2 = \\
&= \tau_1^2 n_1^2(1-n_1^2) + \tau_2^2 n_2^2(1-n_2^2) + \tau_3^2 n_3^2(1-n_3^2) - 2\tau_1\tau_2 n_1^2 n_2^2 - 2\tau_2\tau_3 n_2^2 n_3^2 - 2\tau_3\tau_1 n_3^2 n_1^2 = \\
&= \tau_1^2 n_1^2(n_2^2+n_3^2) + \tau_2^2 n_2^2(n_3^2+n_1^2) + \tau_3^2 n_3^2(n_1^2+n_2^2) - 2\tau_1\tau_2 n_1^2 n_2^2 - 2\tau_2\tau_3 n_2^2 n_3^2 - 2\tau_3\tau_1 n_3^2 n_1^2 = \\
&= (\tau_1-\tau_2)^2 n_1^2 n_2^2 + (\tau_2-\tau_3)^2 n_2^2 n_3^2 + (\tau_3-\tau_1)^2 n_3^2 n_1^2.
\end{aligned}$$

(b) Using spherical polar coordinates,

$$n_1 = \sin\phi\cos\theta, \qquad n_2 = \sin\phi\sin\theta, \qquad n_3 = \cos\phi. \qquad (vii)$$

With $\mathcal{S}$ denoting the surface of a unit sphere, the area dA of an infinitesimal patch on $\mathcal{S}$ is

$$dA = (r\,d\phi)(r\sin\phi\,d\theta) = r^2\sin\phi\,d\phi d\theta = \sin\phi\,d\phi d\theta \qquad \text{since the radius } r = 1. \qquad (viii)$$

The average value that we seek is

$$\frac{1}{4\pi}\int_{\mathcal{S}} T_{\text{shear}}^2\,dA = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} T_{\text{shear}}^2\ \sin\phi\,d\phi d\theta, \qquad (ix)$$

where 4π is the total surface area of the unit sphere. Substituting (vii) into (ii), the result into (ix) and evaluating the integrals leads to (iii).

(c) By setting $n_1 = n_2 = n_3 = 1/\sqrt{3}$ in (ii) we get

$$T_{\text{shear}}^2\big|_{\text{octahedral}} = \frac{1}{9}\big[(\tau_1-\tau_2)^2 + (\tau_2-\tau_3)^2 + (\tau_3-\tau_1)^2\big].$$

which differs from (i) by (just) a factor $5/3$.

Problem 3.11. (Atkin and Fox, Ogden) The stress field in a body is known to be uniaxial in the (fixed) direction $\mathbf{m}$ but not necessarily uniform, i.e. it is known to have the form

$$\mathbf{T}(\mathbf{y}) = \tau(\mathbf{y})\,\mathbf{m}\otimes\mathbf{m}, \qquad (i)$$

where $\mathbf{m}$ is a constant unit vector and $\tau(\mathbf{y})$ is a scalar-valued function. The body is in equilibrium and there are no body forces.

(a) Show that the vector $\operatorname{grad}\tau$ must be perpendicular to $\mathbf{m}$.

(b) Show that $\tau(\mathbf{y})$ must be independent of $\mathbf{y}\cdot\mathbf{m}$, i.e. if you pick a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_3 = \mathbf{m}$, then $\tau(\mathbf{y}) = \tau(y_1, y_2)$.

(c) Show that the traction $\mathbf{t}(\mathbf{n})$ on any plane is parallel to $\mathbf{m}$.

(d) Specialize the traction from part (c) to the cases where $\mathbf{n}$ is parallel to $\mathbf{m}$ and $\mathbf{n}$ is perpendicular to $\mathbf{m}$.

(e) Show that

$$T^2_{\text{shear}}(\mathbf{n}) = \tau^2 \left[1 - (\mathbf{m}\cdot\mathbf{n})^2\right](\mathbf{m}\cdot\mathbf{n})^2.$$

Calculate the maximum and average values of this over all directions $\mathbf{n}$.

Problem 3.12. Consider two bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ related in the usual way by $\mathbf{e}'_i = Q_{ij}\mathbf{e}_j$ where $[Q]$ is an orthogonal matrix. Recall from (3.14) that the components of $\mathbf{T}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are defined by

$$T_{ij} = \mathbf{t}(\mathbf{e}_j)\cdot\mathbf{e}_i = \mathbf{T}\mathbf{e}_j\cdot\mathbf{e}_i,$$

and therefore its components in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are, by definition,

$$T'_{ij} = \mathbf{t}(\mathbf{e}'_j)\cdot\mathbf{e}'_i = \mathbf{T}\mathbf{e}'_j\cdot\mathbf{e}'_i.$$

Verify that the matrices $[T]$ and $[T']$ are related by the basis transformation rule for a 2-tensor.

Solution: Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ be two bases related in the usual way by an orthogonal matrix $[Q]$:

$$\mathbf{e}'_i = Q_{ij}\mathbf{e}_j, \qquad \mathbf{e}_i = Q_{ji}\mathbf{e}'_j. \tag{i}$$

First, let t_i, n_i and T_{ij} be the components[15] of $\mathbf{t}, \mathbf{n}$ and $\mathbf{T}$ in the first basis and t'_i, n'_i and T'_{ij} the corresponding components in the second. Then by (3.19)

$$t_i = T_{ji}n_j, \qquad t'_i = T'_{ji}n'_j. \tag{ii}$$

Second, since $\mathbf{t}$ and $\mathbf{n}$ are vectors, their components in the two bases are related by the usual 1-tensor transformation rule

$$t'_i = Q_{ij}t_j, \qquad n'_i = Q_{ij}n_j. \tag{iii}$$

Substituting (iii) into $(ii)_2$ gives

$$Q_{ij}t_j = T'_{ji}Q_{jk}n_k$$

which by using $(ii)_1$ yields

$$Q_{ij}T_{kj}n_k = T'_{ji}Q_{jk}n_k.$$

Since this holds for all unit vectors $\{n\}$, and $[Q]$, $[T]$ and $[T']$ are independent of $\{n\}$, it follows that

$$Q_{ij}T_{kj} = T'_{ji}Q_{jk}.$$

[15] By (3.14), the components of stress in the two bases are

$$T_{ij} = t_j(\mathbf{e}_i), \qquad T'_{ij} = t'_j(\mathbf{e}'_i).$$

In matrix form this reads $[T][Q]^T = [Q]^T[T']$ whence

$$[T'] = [Q][T][Q]^T$$

where we have used $[Q]^T = [Q]^{-1}$ since $[Q]$ is orthogonal. This is the basis transformation rule for the components of a 2-tensor.

Problem 3.13. (Ogden) The *mean Cauchy stress* in a body is defined as

$$\overline{\mathbf{T}} := \frac{1}{\mathsf{vol}} \int_{\mathcal{R}} \mathbf{T}(\mathbf{y})\ dV_y$$

where vol is the volume of the region $\mathcal{R}$ occupied by the body.

(a) Given that the body is in equilibrium, show that one can express $\overline{\mathbf{T}}$ in the alternative form

$$\overline{\mathbf{T}} = \frac{1}{2\,\mathsf{vol}} \int_{\mathcal{R}} (\mathbf{b}\otimes\mathbf{y} + \mathbf{y}\otimes\mathbf{b})\ dV_y + \frac{1}{2\,\mathsf{vol}} \int_{\partial\mathcal{R}} (\mathbf{t}\otimes\mathbf{y} + \mathbf{y}\otimes\mathbf{t})\ dA_y. \tag{3.100}$$

This shows the following important property of the mean stress: it is *fully* determined by the traction on the boundary of the body (and the prescribed body force field).

(b) Suppose the body force vanishes and the traction on the boundary $\partial\mathcal{R}$ is $\mathbf{t}(\mathbf{y}, \mathbf{n}) = \tau\mathbf{n}$ where τ is a constant; note that the traction is perpendicular to the boundary at each point on $\partial\mathcal{R}$. Show that $\overline{\mathbf{T}} = \tau\,\mathbf{I}$.

(c) Suppose the body force vanishes and the traction on the boundary $\partial\mathcal{R}$ is $\mathbf{t}(\mathbf{y}, \mathbf{n}) = \tau\big[(\mathbf{a}\cdot\mathbf{n})\mathbf{a}\big]$ where the unit vector $\mathbf{a}$ and scalar τ are constants; note that the traction at every point on the boundary is in the same direction $\mathbf{a}$. Show that $\overline{\mathbf{T}} = \tau\mathbf{a}\otimes\mathbf{a}$.

(d) The body depicted in Figure 3.22 contains a cavity, and it is subjected to a uniform internal pressure p_1 on the cavity surface $\mathcal{S}_1$ and a uniform external pressure p_2 on the outer surface $\mathcal{S}_2$. The cavity volume is V_1 and the body has volume V_b. Neglect body forces. Show that

$$\overline{\mathbf{T}} = -\frac{p_2 V_2 - p_1 V_1}{V_2 - V_1}\,\mathbf{I} \qquad \text{where} \quad V_2 = V_1 + V_b.$$

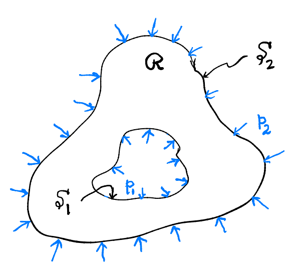


Figure 3.22: Body with a cavity subject to a uniform internal pressure p_1 and a uniform external pressure p_2.

Solution:

(a) In cartesian components, we want to show that

$$\int_{\mathcal{R}} (b_i y_j + y_i b_j)\ dV_y + \int_{\partial\mathcal{R}} (t_i y_j + y_i t_j)\ dA_y = 2 \int_{\mathcal{R}} T_{ij}\ dV_y. \qquad (i)$$

We simplify the left-hand side by substituting the traction-stress relation $t_i = T_{ij} n_j$, using the divergence theorem and then collecting terms:

$$\begin{aligned}
\text{Left-hand side} &= \textstyle\int_{\mathcal{R}} (b_i y_j + y_i b_j)\ dV_y + \int_{\partial\mathcal{R}} (T_{ik} n_k y_j + y_i T_{jk} n_k)\ dA_y = \\
&= \textstyle\int_{\mathcal{R}} (b_i y_j + y_i b_j)\ dV_y + \int_{\mathcal{R}} \frac{\partial}{\partial y_k} (T_{ik} y_j + y_i T_{jk})\ dV_y = \\
&= \textstyle\int_{\mathcal{R}} \left(\left[b_i + \frac{\partial T_{ik}}{\partial y_k} \right] y_j + \left[b_j + \frac{\partial T_{jk}}{\partial y_k} \right] y_i \right)\ dV_y + \int_{\mathcal{R}} \left(T_{ik} \frac{\partial y_j}{\partial y_k} + T_{jk} \frac{\partial y_i}{\partial y_k} \right)\ dV_y = \\
&= \textstyle\int_{\mathcal{R}} \left(T_{ik} \delta_{jk} + T_{jk} \delta_{ik} \right)\ dV_y = 2 \int_{\mathcal{R}} T_{ij}\ dV_y, \qquad \square
\end{aligned}$$

where in going from the third line to the fourth we dropped the first integral by using the equilibrium equations $\partial T_{ij}/\partial y_j + b_i = 0$.

(b) We substitute $t_i = \tau n_i$ into (i) (and drop the body force terms)

$$\int_{\mathcal{R}} T_{ij}\ dV_y = \frac{1}{2} \int_{\partial\mathcal{R}} (t_i y_j + t_j y_i)\ dA_y = \frac{\tau}{2} \int_{\partial\mathcal{R}} (n_i y_j + n_j y_i)\ dA_y = \frac{\tau}{2} \int_{\mathcal{R}} \left(\frac{\partial y_j}{\partial y_i} + \frac{\partial y_i}{\partial y_j} \right)\ dV_y = \tau \delta_{ij} \int_{\mathcal{R}}\ dV_y \qquad (ii)$$

having used the divergence theorem in the penultimate step. This gives the result $\overline{\mathbf{T}} = \tau\,\mathbf{I}$.

(c) We substitute $t_i = \tau a_k n_k a_i$ into (i) (and drop the body force terms)

$$\begin{aligned}
\textstyle\int_{\mathcal{R}} T_{ij}\ dV_y &= \textstyle\frac{1}{2} \int_{\partial\mathcal{R}} (t_i y_j + t_j y_i)\ dA_y = \frac{\tau}{2} \int_{\partial\mathcal{R}} (a_k n_k a_i y_j + a_k n_k a_j y_i)\ dA_y = \\
&= \textstyle\frac{\tau}{2} \int_{\mathcal{R}} \left(a_k a_i \frac{\partial y_j}{\partial y_k} + a_k a_j \frac{\partial y_i}{\partial y_k} \right)\ dV_y = \frac{\tau}{2} \int_{\mathcal{R}} \left(a_k a_i \delta_{jk} + a_k a_j \delta_{ik} \right)\ dV_y = \tau a_i a_j \int_{\mathcal{R}}\ dV_y
\end{aligned} \qquad (iii)$$

having used the divergence theorem in getting to the second line. This gives the result $\overline{\mathbf{T}} = \tau\,\mathbf{a} \otimes \mathbf{a}$.

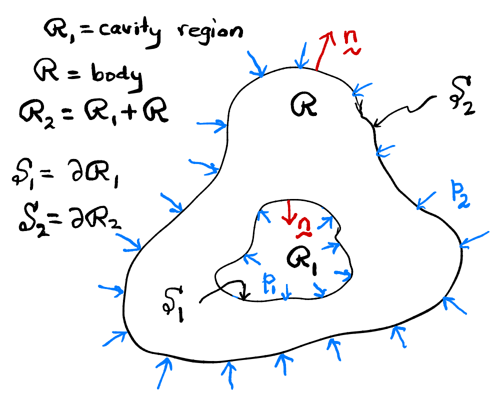


Figure 3.23: Regions occupied and boundaries thereof of a body with a cavity.

(d) Setting $\mathbf{b} = \mathbf{o}$, $\mathsf{vol} = V_b$ and $\partial\mathcal{R} = \mathcal{S}_1 + \mathcal{S}_2$ in (3.100) gives

$$\overline{T}_{ij} = \left[\frac{1}{2V_b}\int_{\mathcal{S}_1}(t_i y_j + y_i t_j)\ dA_y\right] + \left[\frac{1}{2V_b}\int_{\mathcal{S}_2}(t_i y_j + y_i t_j)\ dA_y\right]. \qquad (iv)$$

When a pressure p acts on the surface of a body, the traction is $\mathbf{t} = -p\mathbf{n}$ where the unit vector $\mathbf{n}$ is normal to the surface and points out of the body. Thus in this problem

$$t_i = -p_1 n_i \quad \text{on } \mathcal{S}_1, \qquad t_i = -p_2 n_i \quad \text{on } \mathcal{S}_2, \qquad (v)$$

where the outward pointing unit normal vectors $\mathbf{n}$ associated with the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are as shown in Figure 3.23.

We now evaluate the terms on the right-hand side of (iv). First,

$$\int_{\mathcal{S}_2}(t_i y_j + y_i t_j)\ dA_y \overset{(v)_2}{=} -p_2\int_{\mathcal{S}_2}(n_i y_j + y_i n_j)\ dA_y \overset{*}{=} -p_2\int_{\mathcal{R}_2}(\delta_{ji} + \delta_{ij})\ dV_y = -2p_1\delta_{ij}\int_{\mathcal{R}_2} dV_y = -2p_2V_2\delta_{ij},$$

where in step $*$ we used the divergence theorem and $\partial y_i/\partial y_j = \delta_{ij}$, and $\mathcal{R}_2$ is the region occupied by the body *plus the cavity* and V_2 is the volume of the body *plus the cavity.* Next,

$$\begin{aligned}\int_{\mathcal{S}_1}(t_i y_j + y_i t_j)\ dA_y \quad &\overset{(v)_1}{=} \quad -p_1\int_{\mathcal{S}_1}(n_i y_j + y_i n_j)\ dA_y \overset{**}{=} p_1\int_{\mathcal{S}_1}\Big((-n_i)y_j + y_i(-n_j)\Big)\ dA_y = \\ &\overset{\star}{=} \quad p_1\int_{\mathcal{R}_1}(\delta_{ji} + \delta_{ij})\ dV_y = 2p_1\delta_{ij}\int_{\mathcal{R}_1} dV_y = 2p_1V_1\delta_{ij};\end{aligned}$$

in step $**$, because we intend to apply the divergence theorem to the region $\mathcal{R}_1$ occupied by the cavity, and since the normal vector on $\partial\mathcal{R}_1$ that points out of $\mathcal{R}_1$ is $-\mathbf{n}$, we converted $\mathbf{n}$ to $-\ \mathbf{n}$; in step $\star$ we used the divergence theorem and $\partial y_i/\partial y_j = \delta_{ij}$, and V_1 is the volume of the cavity region $\mathcal{R}_1$.

Substituting the preceding two expressions into (iv) gives

$$\overline{T}_{ij} = \left[\frac{1}{2V_b}\ 2p_1V_1\delta_{ij}\right] + \left[-\frac{1}{2V_b}\ 2p_2V_2\delta_{ij}\right] = -\frac{p_2V_2 - p_1V_1}{V_b}\delta_{ij}. \qquad \square$$

Problem 3.14. Define the *mean Piola stress* in a body by

$$\overline{\mathbf{S}} = \frac{1}{\mathsf{vol}}\int_{\mathcal{R}_\mathrm{R}} \mathbf{S}(\mathbf{x})\, dV_x. \qquad (i)$$

Show that

$$\overline{\mathbf{S}}^T = \frac{1}{\mathsf{vol}}\left[\int_{\partial\mathcal{R}_\mathrm{R}} \mathbf{x}\otimes\mathbf{S}\mathbf{n}_\mathrm{R}\, dA_x + \int_{\mathcal{R}_\mathrm{R}} \mathbf{x}\otimes\mathbf{b}_\mathrm{R}\, dV_x\right] \qquad (ii)$$

and therefore that the average Piola stress tensor field in a body depends only on the traction $\mathbf{S}(\mathbf{x})\mathbf{n}_\mathrm{R}(\mathbf{x})$ on the boundary $\partial\mathcal{R}_\mathrm{R}$ and the body force field $\mathbf{b}_\mathrm{R}(\mathbf{x})$ in $\mathcal{R}_\mathrm{R}$. Show also that

$$\int_{\mathcal{R}_\mathrm{R}} \mathbf{F}\mathbf{S}^T\, dV_x = \int_{\partial\mathcal{R}_\mathrm{R}} \mathbf{y}(\mathbf{x})\otimes\mathbf{S}\mathbf{n}_\mathrm{R}\, dA_x + \int_{\mathcal{R}_\mathrm{R}} \mathbf{y}(\mathbf{x})\otimes\mathbf{b}_\mathrm{R}\, dV_x. \qquad (iii)$$

Problem 3.15. In a reference configuration a body occupies the unit cube

$$\mathcal{R}_{\rm R} = \{(x_1, x_2, x_3) \,:\, 0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ 0 \le x_3 \le 1\},$$

and undergoes the deformation

$$y_1 = \lambda x_1 + k x_2, \qquad y_2 = \lambda^{-1} x_2, \qquad y_3 = x_3. \tag{i}$$

where λ and k are known constants.

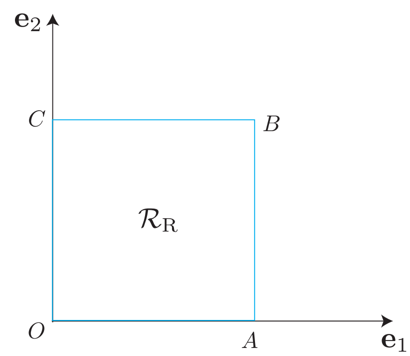


Figure 3.24: Side view of the region $\mathcal{R}_{\rm R}$. (Problem 3.15.)

(a) Sketch the region occupied by the body in the deformed configuration noting the lengths of the edges.

(b) Suppose that the matrix of components of the Cauchy stress is

$$[T] = \begin{pmatrix} -p + \mu(\lambda^2 + k^2) & \mu k \lambda^{-1} & 0 \\ \mu k \lambda^{-1} & -p + \mu \lambda^{-2} & 0 \\ 0 & 0 & -p + \mu \end{pmatrix}, \tag{ii}$$

where μ is a known constant and p is an unknown constant.

The deformed images of the faces $x_3 = 0$ and $x_3 = 1$ are known to be traction-free. Simplify the expression (ii) for $[T]$ accordingly.

(c) Calculate the Piola stress tensor.

(d) Calculate the force (vector) that must be applied on the deformed image of the face $x_2 = 1$. Do this using both the Piola and Cauchy tractions.

(e) Determine the (true) Cauchy traction that must be applied on the deformed image of the face $x_1 = 1$.

Solution:

(a) We see from (i) that particles do not displace in the x_3-direction. Moreover, the u_1 and u_2 displacement components do not depend on x_3. Thus this deformation is planar (in the x_1, x_2-plane) meaning every

section $x_3 = \mathsf{constant}$ deforms identically and in-plane. Thus, in sketching the body we can simply look at the x_1, x_2-plane. Consider the four points O, A, B and C. In the reference configuration they have coordinates $(x_1, x_2, x_3) = (0,0,0), (1,0,0), (1,1,0)$ and $(0,1,0)$. Substituting these into (i) gives the coordinates of the points O', A', B' and C' in the deformed configuration $(y_1, y_2, y_3) = (0,0,0), (\lambda, 0, 0), (\lambda + k, \lambda^{-1}, 0)$ and $(k, \lambda^{-1}, 0)$. Figure 4.10 shows a view of $\mathcal{R}_{\mathrm{R}}$ and $\mathcal{R}$ looking down the x_3-axis.

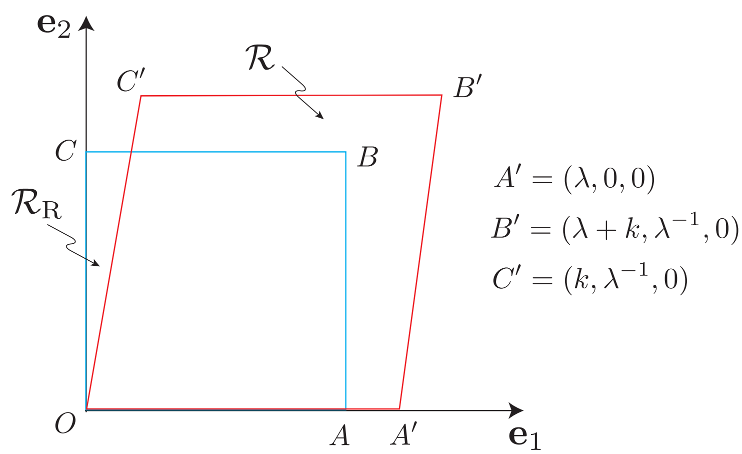


Figure 3.25: Side view of cube of the regions $\mathcal{R}_{\mathrm{R}}$ and $\mathcal{R}$. The body has been biaxially stretched and sheared.

The lengths of the edges are

$$|OA'| = |B'C'| = \lambda, \qquad |A'B'| = |C'O| = \sqrt{k^2 + \lambda^{-2}} \qquad \square \tag{iii}$$

(b) The outward pointing unit normal vector on the plane $x_3 = 1$ is $\mathbf{e}_3$. Therefore the Cauchy traction components acting on it are T_{13}, T_{23} and T_{33} and we are told that they vanish. The first two vanish automatically. In order that (ii) obeys the requirement $T_{33} = 0$ one must have

$$p = \mu. \tag{iv}$$

Substituting (iv) in (ii) leads to

$$[T] = \begin{pmatrix} \mu(\lambda^2 + k^2 - 1) & \mu k \lambda^{-1} & 0 \\ \mu k \lambda^{-1} & \mu(\lambda^{-2} - 1) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \square$$

or equivalently

$$\mathbf{T} = \mu(\lambda^2 + k^2 - 1)\mathbf{e}_1 \otimes \mathbf{e}_1 + \mu k \lambda^{-1}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \mu(\lambda^{-2} - 1)\mathbf{e}_2 \otimes \mathbf{e}_2. \qquad \square \tag{v}$$

(c) The components of the Piola stress tensor can be found from $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$ after first finding J and $\mathbf{F}^{-1}$. Differentiating (i) gives the components $F_{ij} = \partial y_i / \partial x_j$ of the deformation gradient tensor leading to

$$\mathbf{F} = \lambda \mathbf{e}_1 \otimes \mathbf{e}_1 + k \mathbf{e}_1 \otimes \mathbf{e}_2 + \lambda^{-1} \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{vi}$$

from which we find

$$\mathbf{F}^{-1} = \lambda^{-1}\mathbf{e}_1 \otimes \mathbf{e}_1 - k\mathbf{e}_1 \otimes \mathbf{e}_2 + \lambda\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \qquad J = \det \mathbf{F} = 1. \tag{vii}$$

Therefore by substituting (v) and (vii) into $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$ and simplifying leads to

$$\begin{aligned}\mathbf{S} &= J\mathbf{T}\mathbf{F}^{-T} = \Big[\mu(\lambda^2 + k^2 - 1)\mathbf{e}_1 \otimes \mathbf{e}_1 + \mu k\lambda^{-1}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \mu(\lambda^{-2} - 1)\mathbf{e}_2 \otimes \mathbf{e}_2\Big] \\ &\qquad \Big[\lambda^{-1}\mathbf{e}_1 \otimes \mathbf{e}_1 - k\mathbf{e}_2 \otimes \mathbf{e}_1 + \lambda\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3\Big] = \qquad (viii) \\ &= \mu(\lambda - \lambda^{-1})\mathbf{e}_1 \otimes \mathbf{e}_1 + \mu k(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \mu(\lambda^{-1} - \lambda)\mathbf{e}_2 \otimes \mathbf{e}_2. \qquad \square\end{aligned}$$

(d) On the surface $x_2 = 1$ (whose edge is BC in Figure 4.10) the unit outward normal vector is

$$\mathbf{n}_R = \mathbf{e}_2, \tag{ix}$$

and so from $\mathbf{s} = \mathbf{S}\mathbf{n}_{\mathrm{R}}$, the components of the Piola traction are

$$s_1 = S_{12} = \mu k, \qquad s_2 = S_{22} = \mu(\lambda^{-1} - \lambda), \qquad s_3 = S_{32} = 0. \tag{x}$$

Thus the force acting on this surface can be calculated from $\mathsf{force} = \mathbf{s}\,A_x$ where for this surface $A_x = 1$. Thus

$$\text{force} = \mu k\,\mathbf{e}_1 + \mu(\lambda^{-1} - \lambda)\,\mathbf{e}_2. \qquad \square \tag{xi}$$

Alternatively consider the deformed configuration. On the surface $y_2 = \lambda^{-1}$ (whose edge is $B'C'$ in Figure 4.10) the unit outward normal vector is

$$\mathbf{n} = \mathbf{e}_2, \tag{xii}$$

and so from $\mathbf{t} = \mathbf{T}\mathbf{n}$ the components of the Cauchy traction are

$$t_1 = T_{12} = \mu k\lambda^{-1}, \qquad t_2 = T_{22} = \mu(\lambda^{-2} - 1), \qquad t_3 = T_{32} = 0. \tag{xiii}$$

Thus the force can be calculated from $\mathsf{force} = \mathbf{t}\,A_y$ where for this surface $A_y = \lambda$. This leads to (of course) the same result

$$\text{force} = \mu k\mathbf{e}_1 + \mu(\lambda^{-1} - \lambda)\mathbf{e}_2. \qquad \square \tag{xiv}$$

(e) Finally consider the surface $x_1 = 1$ (whose edge is AB in Figure 4.10). The unit outward normal vector on this is

$$\mathbf{n}_R = \mathbf{e}_1. \tag{xv}$$

From $\mathbf{s} = \mathbf{S}\mathbf{n}_{\mathrm{R}}$ we get the components of the Piola traction:

$$s_1 = S_{11} = \mu(\lambda - \lambda^{-1}), \qquad s_2 = S_{21} = \mu k, \qquad s_3 = S_{31} = 0. \tag{xvi}$$

The (true) Cauchy traction $\mathbf{t}$ and the Piola traction $\mathbf{s}$ are related by $\mathbf{t}\,A_y = \mathbf{s}\,A_x$, where for the surface under consideration here, $A_x = 1$ and $A_y = \sqrt{k^2 + \lambda^{-2}}$. Thus

$$\mathbf{t} = \frac{A_x}{A_y}\,\mathbf{s} = \frac{1}{\sqrt{k^2 + \lambda^{-2}}}\,\Big[\mu(\lambda - \lambda^{-1})\mathbf{e}_1 + \mu k\mathbf{e}_2\Big]. \qquad \square \tag{xvii}$$

Alternatively one can calculate $\mathbf{t}$ using $\mathbf{t} = \mathbf{T}\mathbf{n}$ where $\mathbf{n}$ is the outward unit normal to the deformed image of $x_1 = 1$ (whose edge is $A'B'$ in Figure 4.10).

Problem 3.16. Consider a body that occupies a unit cube in a reference configuration:

$$\mathcal{R}_{\rm R} = \{(x_1, x_2, x_3) \mid 0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ 0 \le x_3 \le 1\}.$$

It is subjected to the following homogeneous deformation:

$$y_1 = x_1 + kx_2, \quad y_2 = \lambda x_2, \quad y_3 = x_3, \tag{i}$$

where k and λ are constants. The Piola stress field in the body is uniform and its matrix of components is

$$[S] = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}. \tag{ii}$$

Consider a surface $\mathcal{S}_{\rm R}$ in the reference configuration characterized by $x_1 + x_2 = 1$. The deformation carries $\mathcal{S}_{\rm R} \to \mathcal{S}$.

Without calculating the Cauchy stress tensor, determine

(a) the force (vector) that acts on $\mathcal{S}$,

(b) the true (Cauchy) traction on $\mathcal{S}$, and

(c) the normal component of true (Cauchy) traction on $\mathcal{S}$.

Next, calculate the matrix of components $[T]$ of the Cauchy stress tensor and recalculate your answers to parts $(a) - (c)$.

Solution: The matrix of components of the deformation gradient tensor can be calculated from (i). It, its inverse and determinant are

$$[F] = \begin{pmatrix} 1 & k & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad [F]^{-1} = \begin{pmatrix} 1 & -k/\lambda & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad J = \det[F] = \lambda. \tag{iii}$$

A unit vector normal to the surface $\mathcal{S}_{\rm R}$ is

$$\mathbf{n}_{\rm R} = (1/\sqrt{2})\mathbf{e}_1 + (1/\sqrt{2})\mathbf{e}_2. \tag{iv}$$

Therefore a unit vector normal to its deformed image $\mathcal{S}$ is found from $(iii), (iv)$ and Nanson's formula to be

$$\mathbf{n} = \frac{\mathbf{F}^{-T}\mathbf{n}_{\rm R}}{|\mathbf{F}^{-T}\mathbf{n}_{\rm R}|} = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2, \tag{v}$$

where

$$\cos\phi = \frac{\lambda}{\sqrt{\lambda^2 + (1-k)^2}}, \qquad \sin\phi = \frac{1-k}{\sqrt{\lambda^2 + (1-k)^2}}. \tag{vi}$$

The areas A_x and A_y of the surfaces $\mathcal{S}_{\rm R}$ and $\mathcal{S}$ are

$$A_x = \sqrt{2}, \qquad\qquad A_y = A_x J|\mathbf{F}^{-T}\mathbf{n}_{\rm R}| = \lambda/\cos\phi. \tag{vii}$$

(a) The resultant force on the deformed surface $\mathcal{S}$ is given by

$$\text{force } = \mathbf{s}A_x = \mathbf{S}\mathbf{n}_{\rm R}A_x \overset{(ii),(iv),(vii)}{=} (S_{11}+S_{12})\mathbf{e}_1 + (S_{21}+S_{22})\mathbf{e}_2 + (S_{31}+S_{32})\mathbf{e}_3. \qquad \square$$

(b) Since the resultant force $= \mathbf{s}A_x = \mathbf{t}A_y$ we find the Cauchy (true) traction to be

$$\mathbf{t} = \mathbf{s}A_x/A_y = \frac{\cos\phi}{\lambda}\Big[(S_{11}+S_{12})\mathbf{e}_1 + (S_{21}+S_{22})\mathbf{e}_2 + (S_{31}+S_{32})\mathbf{e}_3\Big]. \qquad \square \tag{viii}$$

(c) The (true) normal stress on the plane $\mathcal{S}$ is given by (v) and $(viii)$ as

$$T_{\rm normal} = \mathbf{t}\cdot\mathbf{n} = \frac{\cos\phi}{\lambda}\Big[(S_{11}+S_{12})\cos\phi + (S_{21}+S_{22})\sin\phi\Big]. \tag{$\square$}$$

From $(ii), (iii)$ and $[T] = J^{-1}[S][F]^T$,

$$[T] = \frac{1}{\lambda}\begin{pmatrix} S_{11}+kS_{12} & \lambda S_{12} & S_{13} \\ S_{21}+kS_{22} & \lambda S_{22} & S_{23} \\ S_{31}+kS_{32} & \lambda S_{32} & S_{33} \end{pmatrix}. \tag{ix}$$

The Cauchy traction can now be calculated from $(v), (ix)$ and $\{t\} = [T]\{n\}$ leading to $(viii)$. (In carrying out this calculation it will be helpful to note from (vi) that $(1-k)\cos\phi = \lambda\sin\phi$.) The force is then given by $\{t\}A_y$

Problem 3.17. (*Symmetry of the Cauchy stress tensor.*) Take the vector product of the moment balance law (3.9) with an arbitrary constant vector, and use the vector identity (1.191) (page 83) to show that

$$\int_{\partial\mathcal{D}} (\mathbf{y}\otimes\mathbf{t} - \mathbf{t}\otimes\mathbf{y})\,dA_y + \int_{\mathcal{D}} (\mathbf{y}\otimes\mathbf{b} - \mathbf{b}\otimes\mathbf{y})\,dV_y = \mathbf{0}. \tag{i}$$

From (i), $\mathbf{t} = \mathbf{T}\mathbf{n}$ and the divergence theorem show that

$$\int_{\partial\mathcal{D}} \Big[(\mathbf{T}^T - \mathbf{T} + \mathbf{y}\otimes(\operatorname{div}\mathbf{T} + \mathbf{b}) - (\operatorname{div}\mathbf{T} + \mathbf{b})\otimes\mathbf{y})\Big]\,dV_y = \mathbf{0}. \tag{ii}$$

Using the equilibrium equation (3.27) in (ii), followed by localization, tells us that $\mathbf{T} = \mathbf{T}^T$.

Solution: On taking the vector product of the moment balance law (3.9) with an arbitrary constant vector $\mathbf{a}$ we get

$$\int_{\partial\mathcal{D}} (\mathbf{y}\times\mathbf{t})\times\mathbf{a}\,dA_y + \int_{\mathcal{D}} (\mathbf{y}\times\mathbf{b})\times\mathbf{a}\,dV_y = \mathbf{o},$$

which by the vector identity (1.191) (page 83) leads to

$$\int_{\partial\mathcal{D}} (\mathbf{y}\otimes\mathbf{t} - \mathbf{t}\otimes\mathbf{y})\mathbf{a}\,dA_y + \int_{\mathcal{D}} (\mathbf{y}\otimes\mathbf{b} - \mathbf{b}\otimes\mathbf{y})\mathbf{a}\,dV_y = \mathbf{0}.$$

Since this must hold for all vectors $\mathbf{a}$ it follows that (i) must hold. Next, by using $\mathbf{t} = \mathbf{T}\mathbf{n}$ and the divergence theorem one can readily show that (e.g. using components)

$$\int_{\partial\mathcal{D}} \mathbf{y}\otimes\mathbf{t}\,dA_y = \int_{\partial\mathcal{D}} \mathbf{y}\otimes\mathbf{T}\mathbf{n}\,dA_y = \int_{\mathcal{D}} (\mathbf{T}^T + \mathbf{y}\otimes \operatorname{div}\mathbf{T})\,dV_y, \tag{iii}$$

$$\int_{\partial\mathcal{D}} \mathbf{t}\otimes\mathbf{y}\,dA_y = \int_{\partial\mathcal{D}} \mathbf{T}\mathbf{n}\otimes\mathbf{y}\,dA_y = \int_{\mathcal{D}} (\mathbf{T} + \operatorname{div}\mathbf{T}\otimes\mathbf{y})\,dV_y. \tag{iv}$$

Substituting (iii) and (iv) into (i) yields (ii). Finally, on using the equilibrium equation $\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{o}$, (ii) reduces it to

$$\int_{\partial\mathcal{D}} (\mathbf{T}^T - \mathbf{T})\,dV_y = \mathbf{0},$$

that by localization leads to $\mathbf{T} = \mathbf{T}^T$.

Problem 3.18. Suppose one does not postulate the force balance law (3.8). Moment balance *about an arbitrary pivot point* $\mathbf{z}$ requires

$$\int_{\partial D} (\mathbf{y}-\mathbf{z})\times\mathbf{t}(\mathbf{n})\,dA_y + \int_{D} (\mathbf{y}-\mathbf{z})\times\mathbf{b}\,dV_y = \mathbf{o}, \tag{i}$$

for all parts $\mathcal{D}$ of the body. Show by requiring (i) to hold for *all pivot points* $\mathbf{z}$, that there exists a tensor $\mathbf{T}(\mathbf{y})$ such that $\mathbf{t}(\mathbf{n}) = \mathbf{T}(\mathbf{y})\mathbf{n}$; $\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{o}$; and $\mathbf{T} = \mathbf{T}^T$.

Solution: One can write (i) as

$$\left[\int_{\partial D} \mathbf{y}\times\mathbf{t}(\mathbf{n})\,dA_y + \int_{D} \mathbf{y}\times\mathbf{b}\,dV_y\right] - \mathbf{z}\times\left[\int_{\partial D} \mathbf{t}(\mathbf{n})\,dA_y + \int_{D} \mathbf{b}\,dV_y\right] = \mathbf{o}. \tag{ii}$$

Clearly, (ii) holds for all $\mathbf{z}$ if and only if the force and moment balance laws (3.8) and (3.9) hold. The desired results then follow as in Sections 3.3 and 3.4.

Problem 3.19. Establish the **Principle of Virtual Work**, i.e. show that the equilibrium equation

$$\operatorname{Div}\mathbf{S} + \mathbf{b}_{\mathrm{R}} = \mathbf{0} \tag{i}$$

holds at each $\mathbf{x}\in\mathcal{R}_{\mathrm{R}}$ *if and only if*

$$\int_{\partial\mathcal{R}_{\mathrm{R}}} \mathbf{S}\mathbf{n}_{\mathrm{R}}\cdot\mathbf{w}\,dA_x + \int_{\mathcal{R}_{\mathrm{R}}} \mathbf{b}_{\mathrm{R}}\cdot\mathbf{w}\,dV_x = \int_{\mathcal{R}_{\mathrm{R}}} \mathbf{S}\cdot\nabla\mathbf{w}\,dV_x \tag{ii}$$

for *all arbitrary* smooth enough vector fields $\mathbf{w}(\mathbf{x})$.

Remark: Note that $\mathbf{w}(\mathbf{x})$ is **not** required to be the actual displacement field in the body. It is called a "virtual displacement".

Solution: We first show that (*ii*) implies (*i*). Suppose (*ii*) holds. Then in terms of cartesian components

$$\int_{\partial\mathcal{R}_{\mathrm{R}}} S_{ij} n_j^{\mathrm{R}}\, w_i\, dA_x + \int_{\mathcal{R}_{\mathrm{R}}} b_i^{\mathrm{R}}\, w_i\, dV_x = \int_{\mathcal{R}_{\mathrm{R}}} S_{ij}\, \frac{\partial w_i}{\partial x_j}\, dV_x \qquad (iii)$$

The first term can be written using the divergence theorem as

$$\int_{\partial\mathcal{R}_{\mathrm{R}}} S_{ij} n_j^{\mathrm{R}}\, w_i\, dA_x = \int_{\mathcal{R}_{\mathrm{R}}} \frac{\partial}{\partial x_j}(S_{ij} w_i)\, dV_x = \int_{\mathcal{R}_{\mathrm{R}}} \left[\frac{\partial S_{ij}}{\partial x_j}\, w_i + S_{ij}\, \frac{\partial w_i}{\partial x_j} \right] dV_x.$$

Substituting this into (*iii*) and simplifying leads to

$$\int_{\mathcal{R}_{\mathrm{R}}} \left[\frac{\partial S_{ij}}{\partial x_j} + b_i^{\mathrm{R}} \right] w_i\, dV_x = 0.$$

Since this must hold for all vector fields $\mathbf{w}(\mathbf{x})$ it follows from (the alternative method of localization described in Problem 1.42) that the factor in square brackets in the integrand must vanish. This yields (*i*).

Next we show that (*i*) implies (*ii*). Suppose (*i*) holds. Multiplying it by an arbitrary function $\mathbf{w}(\mathbf{x})$, and integrating the result over $\mathcal{R}_{\mathrm{R}}$ and reversing the steps in the preceding calculation leads to (*ii*). □

Problem 3.20. Consider a material such as a polarized dielectric solid under the action of an electric field, where (in addition to a body force $\mathbf{b}(\mathbf{y})$) there is also a *body couple* $\mathbf{c}(\mathbf{y})$ per unit deformed volume. Also, at any point $\mathbf{y}$ on a surface $\mathcal{S}$ suppose that there is (in addition to the contact force $\mathbf{t}(\mathbf{y},\mathbf{n})$) a *contact couple* $\mathbf{m}(\mathbf{y},\mathbf{n})$ per unit deformed area. Here $\mathbf{n}$ is the unit normal vector at a point on a surface in the deformed body and $\mathbf{m}$ is the couple applied by the material on the positive side of $\mathcal{S}$ on the material on the negative side. (The "positive side" of $\mathcal{S}$ is the side into which $\mathbf{n}$ points.)

Write down the global force and moment balance laws for this case. Show that in addition to the stress tensor $\mathbf{T}$ there is also a *couple stress tensor* $\mathbf{Z}(\mathbf{y})$ such that

$$\mathbf{m} = \mathbf{Z}\mathbf{n}.$$

Derive the local consequences of the force and moment equilibrium principles. Is the stress tensor $\mathbf{T}$ symmetric?

Solution: The couples have no effect on force balance and so we continue to have

$$\int_{\partial\mathcal{D}} \mathbf{t}\, \mathrm{d}A_y + \int_{\mathcal{D}} \mathbf{b}\, \mathrm{d}V_y = \mathbf{0}. \qquad (i)$$

The usual argument thus implies the existence of the Cauchy stress tensor $\mathbf{T}$ such that $\mathbf{t} = \mathbf{T}\mathbf{n}$ and the equilibrium equation $\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{o}$. The couples *do* contribute to the resultant moment and so the balance of moments requires

$$\int_{\partial\mathcal{D}} \mathbf{y}\times\mathbf{t}\, \mathrm{d}A_y + \int_{\mathcal{D}} \mathbf{y}\times\mathbf{b}\, \mathrm{d}V_y + \int_{\partial\mathcal{D}} \mathbf{m}(\mathbf{y},\mathbf{n})\, \mathrm{d}A_y + \int_{\mathcal{D}} \mathbf{c}\, \mathrm{d}V_y = \mathbf{0}. \qquad (ii)$$

Existence of couple stress tensor. Using the fact that $\mathbf{t} = \mathbf{T}\mathbf{n}$ and the divergence theorem allows us to convert the first surface integral in (*ii*) into a volume integral. On applying this balance law to a tetrahedral region and shrinking the region to a point, all the volume integrals vanish and only the contribution from the surface integral of $\mathbf{m}$ remains. Then, mimicking the steps we used to show the existence of the stress tensor $\mathbf{T}$ allows us to conclude that there exists a tensor $\mathbf{Z}(\mathbf{y})$ that is independent of $\mathbf{n}$ such that

$$\mathbf{m}(\mathbf{y}, \mathbf{n}) = \mathbf{Z}(\mathbf{y})\mathbf{n}.$$

$\mathbf{Z}$ is called the couple stress tensor.

Field equations. Substituting $\mathbf{t} = \mathbf{Tn}$ and $\mathbf{m} = \mathbf{Zn}$ into the balance of moments equation (ii) leads to

$$\int_{\partial\mathcal{D}} \mathbf{y} \times \mathbf{Tn}\, \mathrm{d}A_y + \int_{\mathcal{D}} \mathbf{y} \times \mathbf{b}\, \mathrm{d}V_y + \int_{\partial\mathcal{D}} \mathbf{Zn}\, \mathrm{d}A_y + \int_{\mathcal{D}} \mathbf{c}\, \mathrm{d}V_y = \mathbf{0},$$

or in terms of components

$$\int_{\partial\mathcal{D}} e_{ijk} y_j T_{kp} n_p \, \mathrm{d}A_y + \int_{\mathcal{D}} e_{ijk} y_j b_k \, \mathrm{d}A_y + \int_{\partial\mathcal{D}} Z_{ip} n_p \, \mathrm{d}A_y + \int_{\mathcal{D}} c_i \, \mathrm{d}V_y = 0.$$

Using the divergence theorem to convert the surface integrals to volume integrals, using the equilibrium equation $\partial T_{ij}/\partial y_j + b_i = 0$ and then localizing the result in the familiar way leads

$$e_{ijk}\delta_{jp}T_{kp} + Z_{ip,p} + c_i = 0 \qquad \Rightarrow \qquad e_{ijk}T_{kj} + Z_{ip,p} + c_i = 0. \qquad \square$$

This can we written in an alternative, more illuminating form, by first multiplying by e_{ipq} and then using the identity $e_{ipq}e_{ijk} = \delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}$. This leads to

$$(\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj})\, T_{kj} + e_{ipq}\frac{\partial Z_{ij}}{\partial y_j} + e_{ipq}c_i = 0 \qquad \Rightarrow \qquad T_{qp} - T_{pq} + e_{ipq}\frac{\partial Z_{ij}}{\partial y_j} + e_{ipq}c_i = 0. \qquad \square$$

According to this the Cauchy stress $\mathbf{T}$ is not symmetric and that the above equation provides an expression for $\mathbf{T} - \mathbf{T}^T$ in terms of the couple stress and body couple.

Problem 3.21. In Problem 3.20 we encountered couple stresses, and specifically, showed that there is a *couple stress tensor* $\mathbf{Z}$.

(a) Let $\mathbf{Z}_{\mathrm{R}}$ be the referential version of $\mathbf{Z}$, i.e. the tensor analogous to what the Piola stress tensor $\mathbf{S}$ is to the Cauchy stress tensor $\mathbf{T}$. Derive a formula for $\mathbf{Z}_{\mathrm{R}}$.

(b) Similarly derive a formula for the referential body couple $\mathbf{c}_{\mathrm{R}}$.

(c) Derive the field equation obeyed by $\mathbf{Z}_{\mathrm{R}}$ and $\mathbf{c}_{\mathrm{R}}$ corresponding to moment balance in its referential form.

Problem 3.22. (Atkin and Fox) The Cauchy stress field in a certain body is

$$\mathbf{T}(\mathbf{y}) = \tau(\mathbf{y})\ \mathbf{m}(\mathbf{y}) \otimes \mathbf{m}(\mathbf{y}) \qquad \text{where} \quad \mathbf{m}(\mathbf{y}) = \frac{\mathbf{y}}{r}, \quad r = |\mathbf{y}|. \qquad (i)$$

(Assume that the origin lies outside the body so that $r \neq 0$.) Observe that locally, at each point in the body, the stress is in the radial direction $\mathbf{m}$ with magnitude τ. The body is in equilibrium. Show that

$$\mathbf{y} \cdot \operatorname{grad} \tau + 2\tau = 0 \qquad \text{for all} \quad \mathbf{y} \in \mathcal{R}. \tag{ii}$$

Determine $\tau(\mathbf{y})$ (to the extent possible). Hint: Write (ii) in spherical polar coordinates r, θ, φ.

Solution: In preparation for substituting the given stress field into the equilibrium equations, we first differentiate $r^2 = |\mathbf{y}|^2$ and $\mathbf{m} = \mathbf{y}/r$ to get

$$\frac{\partial r}{\partial y_i} = \frac{y_i}{r}, \qquad \frac{\partial m_i}{\partial y_j} = \frac{r^2\delta_{ij} - y_i y_j}{r^3}, \qquad \frac{\partial m_i}{\partial y_i} = \frac{2}{r}. \tag{iii}$$

Substituting (i) into the equilibrium equations $\partial T_{ij}/\partial y_j = 0$ and using (iii) and $m_i = y_i/r$ leads to

$$y_i y_j \frac{\partial \tau}{\partial y_j} + 2\tau y_i = 0,$$

whence (for example by multiplying by y_i and then cancelling the term $r^2 = y_i y_i$) we get

$$y_j \frac{\partial \tau}{\partial y_j} + 2\tau = 0 \qquad \Leftrightarrow \qquad \mathbf{y} \cdot \operatorname{grad} \tau + 2\tau = 0. \qquad \square$$

To solve this differential equation we write it in spherical polar coordinates using $\mathbf{y} = r\mathbf{e}_r$ and the following expression for the gradient of $\tau(r, \theta, \varphi)$:

$$\operatorname{grad} \tau = \frac{\partial \tau}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \tau}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r \sin\theta}\frac{\partial \tau}{\partial \varphi}\mathbf{e}_\varphi.$$

Equation (iv) then leads to

$$r\mathbf{e}_r \cdot \operatorname{grad} \tau + 2\tau = r\frac{\partial \tau}{\partial r} + 2\tau = 0 \qquad \Rightarrow \qquad \frac{\partial}{\partial r}(r^2\tau) = 0 \qquad \Rightarrow \qquad \tau = \frac{c(\theta, \varphi)}{r^2}. \qquad \square$$

where $c(\theta, \varphi)$ is arbitrary.

Solution 2: Since $\mathbf{m} = \mathbf{y}/r = (r\mathbf{e}_r)/r = \mathbf{e}_r$ the given stress field can be written as $\mathbf{T}(r, \theta, \varphi) = T_{rr}(r, \theta, \varphi)\mathbf{e}_r \otimes \mathbf{e}_r$ in spherical polar coordinates. Substituting this into the general equilibrium equations in spherical polar coordinates (3.97) yields

$$\frac{\partial T_{rr}}{\partial r} + 2\frac{T_{rr}}{r} = 0$$

which gives $T_{rr} = c(\theta, \varphi)/r^2$.

Problem 3.23. Consider a very long circular cylindrical tube with inner and outer radii A and B respectively in the reference configuration. The tube is inflated to a pressure p, the outer wall being traction-free. The tube is made of an isotropic material and so the deformation and stress fields are axisymmetric and uniform in the axial direction. The inner and outer radii in the deformed configuration are a and b respectively. Consider a, b and p to be known. Work in cylindrical polar coordinates (r, θ, z) in the deformed configuration.

(a) What are the boundary conditions at $r = a$ and $r = b$?

(b) Specialize the general equilibrium equations (3.95) to the present setting.

(c) Now suppose that the tube is thin-walled, i.e. assume that $t \ll \overline{r}$ where $t = b - a$ is the wall thickness and $\overline{r} = (a+b)/2$ the mean radius. Use the equilibrium equation from part (b) to find an approximate expression for the circumferential Cauchy stress $T_{\theta\theta}$.

Problem 3.24. Reconsider Problem 3.3.4.

(a) Specialize the general equilibrium equations (in cylindrical polar coordinates) (3.95) to the setting of Problem 3.3.4. Assume no body forces.

(b) Show that for *any* smooth function $\phi(r, \theta)$, the stresses given by

$$T_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}, \qquad T_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right), \qquad T_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \tag{viii}$$

satisfy the equilibrium equations from part (a).

(c) Suppose that the shear stress $T_{r\theta}(r, \theta)$ vanishes everywhere in the body. Determine the form of $\phi(r, \theta)$ implied by this and calculate expressions for the two nonzero normal stress components.

(d) Now impose the boundary conditions (v), (vii), (ii) and (iii) (from Problem 3.3.4) and further simplify the form of the stresses.

Solution:

(a) Setting $T_{rz} = T_{\theta z} = T_{zz} = 0$, and $T_{rr} = T_{rr}(r, \theta), T_{r\theta} = T_{r\theta}(r, \theta), T_{\theta\theta} = T_{\theta\theta}(r, \theta)$, and $b_r = b_\theta = b_z = 0$ in (3.95) simplifies the equilibrium equations to

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0, \qquad \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + 2\frac{T_{r\theta}}{r} = 0. \tag{ix}$$

(b) Substituting $(viii)$ into (ix) shows that the equilibrium equations are automatically satisfied.

(c) If the shear stress $T_{r\theta} = 0$ everywhere, $(viii)$ implies

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = 0.$$

Integrating this gives

$$\phi(r, \theta) = r g(\theta) + f(r), \tag{x}$$

where $f(r)$ and $g(\theta)$ are arbitrary functions of r and θ respectively.

(d) Substituting (x) into $(viii)$ gives

$$T_{rr}(r, \theta) = \frac{g''(\theta) + g(\theta)}{r} + \frac{f'(r)}{r}, \qquad T_{r\theta}(r, \theta) = 0, \qquad T_{\theta\theta}(r, \theta) = f''(r). \tag{xi}$$

Turning to the boundary conditions first consider (v) (from Problem 3.3.4), the first of which holds automatically while the second gives

$$\int_a^b T_{\theta\theta}(r,\alpha)\,dr \overset{(xi)}{=} \int_a^b f''(r)\,dr = f'(b) - f'(a) = 0 \qquad \Rightarrow \qquad f'(a) = f'(b). \qquad (xii)$$

Second consider the boundary conditions (vii). This requires

$$\int_a^b rT_{\theta\theta}(r,\alpha)\,dr \overset{(xi)}{=} \int_a^b rf''(r)\,dr = \int_a^b \left[\frac{d}{dr}(rf'(r)) - f'(r)\right]\,dr = m,$$

which upon integration gives

$$bf'(b) - f'(b) - af'(a) + f'(a) = m \qquad \overset{(xii)}{\Rightarrow} \qquad f'(a) = f'(b) = \frac{m}{b-a}. \qquad (xiii)$$

Third consider the boundary conditions (ii), the second of which holds automatically while the first requires

$$T_{rr}(b,\theta) \overset{(xi)}{=} \frac{g''(\theta)+g(\theta)}{b} + \frac{f'(b)}{b} = 0 \qquad \Rightarrow \qquad g''(\theta)+g(\theta) = -f'(b) \qquad (xiv)$$

which implies

$$g(\theta) = c_1\cos\theta + c_2\sin\theta - f'(b) \overset{(xiii)}{=} c_1\cos\theta + c_2\sin\theta - \frac{m}{b-a}. \qquad (xv)$$

Finally the boundary condition (iii) can be examined similarly and it too leads to (xiv).

Substituting (xiv) into (xi) yields the following expression for the stress field:

$$T_{rr}(r,\theta) = \frac{f'(r)-f'(b)}{r}, \qquad T_{r\theta}(r,\theta) = 0, \qquad T_{\theta\theta}(r,\theta) = f''(r). \qquad \square \qquad (xvi)$$

The stress field (xvi) satisfies the equilibrium equations and the given boundary conditions for any function $f(r)$ satisfying $(xiii)$.

Problem 3.25. (Chadwick) (a) A body is in equilibrium with no body forces. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a fixed orthonormal basis with respect to which all components are taken. Show for an *arbitrary* (smooth-enough) vector field $\boldsymbol{\phi}(y_1, y_2)$ that the stress field given by

$$T_{k1} = \frac{\partial \phi_k}{\partial y_2}, \qquad T_{k2} = -\frac{\partial \phi_k}{\partial y_1}, \qquad k = 1,2,3, \qquad (i)$$

obeys the equilibrium equations. (Is the converse necessary? i.e. if a stress field $\mathbf{T}(y_1, y_2)$ is in equilibrium without body forces, must there necessarily exist a vector field $\boldsymbol{\phi}(y_1, y_2)$ such that (i) holds?)

(b) Suppose that $\mathcal{R}$ is a solid prismatic cylinder (whose cross section is not-necessarily circular) with its generators parallel to $\mathbf{e}_3$. Let $\mathcal{C}$ be the closed curve at which the lateral boundary of $\mathcal{R}$ intersects a plane $y_3 = \mathsf{constant}$. Show that the traction on the lateral boundary at a point on $\mathcal{C}$ is

$$\mathbf{t} = \frac{\partial \boldsymbol{\phi}}{\partial s} \qquad (ii)$$

where s is arc length on $\mathcal{C}$.

Solution:

(a) For any smooth function $\phi_1(y_1, y_2)$ it is readily seen by direct substitution that the stress components given by

$$T_{11} = \frac{\partial \phi_1}{\partial y_2}, \qquad T_{12} = -\frac{\partial \phi_1}{\partial y_1}, \tag{iii}$$

satisfy the equilibrium equation

$$\frac{\partial T_{11}}{\partial y_1} + \frac{\partial T_{12}}{\partial y_2} = 0. \tag{iv}$$

In a similar manner one can show that the stress components (i) satisfy the equilibrium equations $\partial T_{i\alpha}/\partial y_\alpha = 0$ for $i = 1, 2, 3$ and $\alpha = 1, 2$; note since $\phi(y_1, y_2)$ is independent of the y_3-coordinate so are the stresses defined by (i).

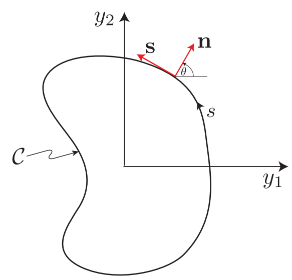


Figure 3.26: Cross section of the prismatic cylinder with arc length s, unit tangent and normal vectors $\mathbf{s}$ and $\mathbf{n}$ on the boundary $\mathcal{C}$ of the cross section.

(b) Observe from Figure 3.26 that the components of the unit tangent vector $\mathbf{s}$ in the direction of increasing arc length s and the unit outward pointing normal vector $\mathbf{n}$ are related by

$$n_1 = s_2, \qquad n_2 = -s_1. \qquad \text{Moreover,} \quad n_3 = 0 \tag{v}$$

since the normal vector on the lateral boundary is orthogonal to $\mathbf{e}_3$. The component t_1 of the traction can be written as

$$t_1 = T_{1j} n_j = T_{11} n_1 + T_{12} n_2 \overset{(iii),(v)}{=} \frac{\partial \phi_1}{\partial y_2} s_2 + \frac{\partial \phi_1}{\partial y_1} s_1 = \nabla \phi_1 \cdot \mathbf{s} = \frac{\partial \phi_1}{\partial s}.$$

Similar calculations can be carried out for the other two traction components. This leads to

$$\mathbf{t} = t_i \mathbf{e}_i = \frac{\partial \phi_i}{\partial s} \mathbf{e}_i = \frac{\partial}{\partial s}(\phi_i \mathbf{e}_i) = \frac{\partial \boldsymbol{\phi}}{\partial s}. \qquad \square$$

Problem 3.26. (See also Problem 3.27.) Let (x_1, x_2, x_3) and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be rectangular cartesian coordinates and the associated basis in the reference configuration; and let (r, θ, z) and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ be cylindrical polar coordinates and the associated basis in the deformed configuration. Let the Piola stress tensor have components in these mixed bases

$$\begin{aligned}\mathbf{S} = \quad & S_{r1}\mathbf{e}_r \otimes \mathbf{e}_1 + S_{r2}\mathbf{e}_r \otimes \mathbf{e}_2 + S_{r3}\mathbf{e}_r \otimes \mathbf{e}_3 + \\ & + S_{\theta 1}\mathbf{e}_\theta \otimes \mathbf{e}_1 + S_{\theta 2}\mathbf{e}_\theta \otimes \mathbf{e}_2 + S_{\theta 3}\mathbf{e}_\theta \otimes \mathbf{e}_3 + \\ & + S_{z1}\mathbf{e}_z \otimes \mathbf{e}_1 + S_{z2}\mathbf{e}_z \otimes \mathbf{e}_2 + S_{z3}\mathbf{e}_z \otimes \mathbf{e}_3.\end{aligned}$$

e.g. see Problem 3.7.1 . Calculate

$$\Big(\mathrm{Div}\,\mathbf{S}\Big)\cdot\mathbf{e}_r, \qquad \Big(\mathrm{Div}\,\mathbf{S}\Big)\cdot\mathbf{e}_\theta, \qquad \Big(\mathrm{Div}\,\mathbf{S}\Big)\cdot\mathbf{e}_z.$$

and hence derive the equilibrium equations in these coordinates.

Problem 3.27. (See also Problem 3.26.) Let (R, Θ, Z) and $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ be cylindrical polar coordinates and the associated basis in the reference configuration; and let (r, θ, z) and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ be cylindrical polar coordinates and the associated basis in the deformed configuration. Let the Piola stress tensor have components in these mixed bases

$$\begin{aligned}\mathbf{S} = \quad & S_{rR}\mathbf{e}_r\otimes\mathbf{e}_R + S_{r\Theta}\mathbf{e}_r\otimes\mathbf{e}_\Theta + S_{rZ}\mathbf{e}_r\otimes\mathbf{e}_Z + \\ & + S_{\theta R}\mathbf{e}_\theta\otimes\mathbf{e}_R + S_{\theta\Theta}\mathbf{e}_\theta\otimes\mathbf{e}_\Theta + S_{\theta Z}\mathbf{e}_\theta\otimes\mathbf{e}_Z + \\ & + S_{zR}\mathbf{e}_z\otimes\mathbf{e}_R + S_{z\Theta}\mathbf{e}_z\otimes\mathbf{e}_\Theta + S_{zZ}\mathbf{e}_z\otimes\mathbf{e}_Z.\end{aligned}$$

Calculate

$$\Big(\mathrm{Div}\,\mathbf{S}\Big)\cdot\mathbf{e}_r, \qquad \Big(\mathrm{Div}\,\mathbf{S}\Big)\cdot\mathbf{e}_\theta, \qquad \Big(\mathrm{Div}\,\mathbf{S}\Big)\cdot\mathbf{e}_z.$$

and hence derive the equilibrium equations in these coordinates.

Problem 3.28. (See Problem 2.32 for an analysis of the kinematics of a piecewise homogeneous deformation.) Consider a planar surface $\mathcal{S}$ that passes through the region $\mathcal{R}$ occupied by a body in the deformed configuration. Let $\mathbf{n}$ be a unit vector normal to $\mathcal{S}$ and let $\mathcal{R}^+$ denote the side into which $\mathbf{n}$ points, $\mathcal{R}^-$ the other side. Thus $\mathcal{S}$ is a planar interface between two parts of the body. Consider the piecewise homogeneous stress field

$$\mathbf{T}(\mathbf{y}) = \begin{cases} \mathbf{T}^+ & \text{for } \mathbf{y}\in\mathcal{R}^+, \\[2ex] \mathbf{T}^- & \text{for } \mathbf{y}\in\mathcal{R}^-, \end{cases}$$

where $\mathbf{T}^\pm$ are constant symmetric tensors. Show that this stress field obeys force and moment balance if and only if

$$\mathbf{T}^+\mathbf{n} - \mathbf{T}^-\mathbf{n} = \mathbf{o}. \tag{3.101}$$

Let $\mathcal{S}_{\mathrm{R}}$ be the image of $\mathcal{S}$ in the reference configuration with $\mathbf{n}_{\mathrm{R}}$ being a unit vector normal to $\mathcal{S}_{\mathrm{R}}$. Assume that the associated deformation is piecewise homogeneous (see Problem 2.32). Show that force and moment balance requires the Piola stress tensor field associated with the aforementioned stress field to obey

$$\mathbf{S}^+\mathbf{n}_{\mathrm{R}} - \mathbf{S}^-\mathbf{n}_{\mathrm{R}} = \mathbf{o}. \tag{3.102}$$

These "jump conditions" plays an important role in studying interfaces between two material phases.

Problem 3.29. Suppose that the Cauchy stress tensor $\mathbf{T}$ and the Eulerian stretch tensor $\mathbf{V}$ are coaxial, i.e. that the principal directions of $\mathbf{T}$ and $\mathbf{V}$ coincide. Show that the Piola stress tensor can be expressed as

$$\mathbf{S} = \sum_{i=1}^{3} \frac{\tau_i J}{\lambda_i} \boldsymbol{\ell}_i \otimes \mathbf{r}_i, \tag{3.103}$$

where, for each $i = 1, 2, 3$, τ_i is a principal Cauchy stress, λ_i a principal stretch, $\boldsymbol{\ell}_i$ a principal direction of $\mathbf{V}$, $\mathbf{r}_i$ a principal direction of $\mathbf{U}$ and $J = \lambda_1, \lambda_2 \lambda_3$.

Solution: From $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$ and $\mathbf{F} = \mathbf{V}\mathbf{R}$ we get

$$\mathbf{S} = J\mathbf{T}(\mathbf{V}\mathbf{R})^{-T} = J\mathbf{T}(\mathbf{R}^{-1}\mathbf{V}^{-1})^T = J\mathbf{T}(\mathbf{R}^T\mathbf{V}^{-1})^T = J\mathbf{T}(\mathbf{V}^{-1})^T(\mathbf{R}^T)^T = J\mathbf{T}\mathbf{V}^{-1}\mathbf{R}. \tag{i}$$

Substituting

$$\mathbf{T} = \sum_{i=1}^{3} \tau_i \boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i, \qquad \mathbf{V}^{-1} = \sum_{i=1}^{3} \lambda_i^{-1} \boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i,$$

into (i) and simplifying (using $\boldsymbol{\ell}_i \cdot \boldsymbol{\ell}_j = \delta_{ij}$) gives

$$\mathbf{S} = J\left(\sum_{i=1}^{3} \tau_i \boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i\right)\left(\sum_{j=1}^{3} \lambda_j^{-1} \boldsymbol{\ell}_j \otimes \boldsymbol{\ell}_j\right)\mathbf{R} = J\left(\sum_{i=1}^{3}\sum_{j=1}^{3} \tau_i \lambda_j^{-1} \delta_{ij} \boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_j\right)\mathbf{R} = J\left(\sum_{i=1}^{3} \tau_i \lambda_i^{-1} \boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i\right)\mathbf{R}.$$

This can be simplified further by using the tensor identity $(\mathbf{a} \otimes \mathbf{b})\mathbf{A} = \mathbf{a} \otimes (\mathbf{A}^T\mathbf{b})$ and the relation $\boldsymbol{\ell}_i = \mathbf{R}\mathbf{r}_i$ between the principal directions of the Eulerian and Lagrangian stretch tensors:

$$\mathbf{S} = J\left(\sum_{i=1}^{3} \tau_i \lambda_i^{-1} \boldsymbol{\ell}_i \otimes \mathbf{R}^T \boldsymbol{\ell}_i\right) = J\left(\sum_{i=1}^{3} \tau_i \lambda_i^{-1} \boldsymbol{\ell}_i \otimes \mathbf{r}_i\right). \qquad \square$$

Problem 3.30. Two symmetric tensors are said to be *coaxial* if their principal axes coincide. Prove that the Cauchy stress tensor $\mathbf{T}$ and the left Cauchy-Green tensor $\mathbf{B}$ are coaxial if and only if the second Piola-Kirchhoff tensor $\mathbf{S}^{(2)}$ is coaxial with the right Cauchy-Green strain tensor $\mathbf{C}$.

Solution: For an alternative proof, use the result of Problem 1.22.

In this solution, the summation convention is suspended. The eigenvectors of $\mathbf{B}$ and $\mathbf{C}$ are $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ and $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ respectively. First suppose that $\mathbf{T}$ is coaxial with $\mathbf{B}$, so that by definition, $\mathbf{T}$ has eigenvectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ and therefore the representation

$$\mathbf{T} = \sum_{i=1}^{3} \tau_i (\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i). \tag{i}$$

We need to show that (i) implies that the second Piola-Kirchhoff stress tensor $\mathbf{S}^{(2)}$ has the representation

$$\mathbf{S}^{(2)} = \sum_{i=1}^{3} s_i (\mathbf{r}_i \otimes \mathbf{r}_i), \tag{ii}$$

so that $\mathbf{S}^{(2)}$ and $\mathbf{C}$ are coaxial.

It follows from (i) that the second Piola-Kirchhoff stress tensor can be expressed as

$$\mathbf{S}^{(2)} \overset{(3.82)}{=} J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} \overset{(i)}{=} \sum_{i=1}^{3} J\tau_i\, \mathbf{F}^{-1}(\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i)\mathbf{F}^{-T} \overset{(1.78)}{=} \sum_{i=1}^{3} J\tau_i\, (\mathbf{F}^{-1}\boldsymbol{\ell}_i) \otimes (\mathbf{F}^{-1}\boldsymbol{\ell}_i). \qquad (iii)$$

By using the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$, the relation $\boldsymbol{\ell}_i = \mathbf{R}\mathbf{r}_i$ and the orthogonality of $\mathbf{R}$,

$$(\mathbf{F}^{-1}\boldsymbol{\ell}_i) \otimes (\mathbf{F}^{-1}\boldsymbol{\ell}_i) = (\mathbf{U}^{-1}\mathbf{R}^T\boldsymbol{\ell}_i) \otimes (\mathbf{U}^{-1}\mathbf{R}^T\boldsymbol{\ell}_i) = (\mathbf{U}^{-1}\mathbf{r}_i) \otimes (\mathbf{U}^{-1}\mathbf{r}_i). \qquad (iv)$$

Since $\mathbf{U}\mathbf{r}_i = \lambda_i\mathbf{r}_i$ we have

$$(\mathbf{U}^{-1}\mathbf{r}_i) \otimes (\mathbf{U}^{-1}\mathbf{r}_i) = (\lambda_i^{-1}\mathbf{r}_i) \otimes (\lambda^{-1}\mathbf{r}_i) = \lambda_i^{-2}(\mathbf{r}_i \otimes \mathbf{r}_i). \qquad (v)$$

Thus on combining $(iii), (iv)$ and (v) we have the following representation for $\mathbf{S}^{(2)}$:

$$\mathbf{S}^{(2)} = \sum_{i=1}^{3} J\tau_i\lambda_i^{-2}\, (\mathbf{r}_i \otimes \mathbf{r}_i), \qquad \square$$

which is of the form (ii). We therefore conclude that $\mathbf{S}^{(2)}$ is coaxial with $\mathbf{C}$. (We also see that the principal values of $\mathbf{S}^{(2)}$ and $\mathbf{T}$ are related by $s_i = J\lambda_i^{-2}\tau_i$.)

The preceding steps can be readily reversed to show that, (ii) implies (i), and therefore that if $\mathbf{S}^{(2)}$ is coaxial with $\mathbf{C}$ then $\mathbf{T}$ is coaxial with $\mathbf{B}$.

Problem 3.31. If the Cauchy stress tensor $\mathbf{T}$ and the left Cauchy-Green tensor $\mathbf{B}$ are coaxial, show that the Biot stress tensor $\mathbf{S}^{(1)}$ is coaxial with the Biot strain tensor $\mathbf{E}^{(1)}$. Is the converse true? (Two symmetric tensors are said to be coaxial if their principal axes coincide.)

Solution: In this solution, the summation convention is suspended.

The eigenvectors of $\mathbf{B}$ are $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ and those of $\mathbf{U}$ (and therefore of $\mathbf{E}^{(1)} = \mathbf{U} - \mathbf{I}$) are $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$. We are told that $\mathbf{T}$ is coaxial with $\mathbf{B}$, so that by the definition of coaxiality, $\mathbf{T}$ also has eigenvectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$, and hence the representation

$$\mathbf{T} = \sum_{i=1}^{3} \tau_i(\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i). \qquad (i)$$

We need to show that (i) implies that the Biot stress tensor $\mathbf{S}^{(1)}$ has the representation

$$\mathbf{S}^{(1)} = \sum_{i=1}^{3} s_i(\mathbf{r}_i \otimes \mathbf{r}_i), \qquad (ii)$$

so that $\mathbf{S}^{(1)}$ and $\mathbf{E}^{(1)}$ are coaxial.

The Biot stress tensor is related to the Cauchy stress tensor by

$$\mathbf{S}^{(1)} \overset{(3.82)}{=} \frac{1}{2}\left(\mathbf{S}^T\mathbf{R} + \mathbf{R}^T\mathbf{S}\right) \overset{(3.50)}{=} \frac{1}{2}\, J\, \left(\mathbf{F}^{-1}\mathbf{T}\mathbf{R} + \mathbf{R}^T\mathbf{T}\mathbf{F}^{-T}\right). \qquad (iii)$$

Based on $(iii)_2$ and (i) we shall simplify the terms $\mathbf{F}^{-1}(\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i)\mathbf{R}$ and $\mathbf{R}^T(\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i)\mathbf{F}^{-T}$:

$$\begin{aligned}\mathbf{F}^{-1}(\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i)\mathbf{R} &\stackrel{(1.78)}{=} (\mathbf{F}^{-1}\boldsymbol{\ell}_i) \otimes (\mathbf{R}^T\boldsymbol{\ell}_i) \stackrel{(*)}{=} (\mathbf{U}^{-1}\mathbf{R}^T\boldsymbol{\ell}_i) \otimes (\mathbf{R}^T\boldsymbol{\ell}_i) = \\ &\stackrel{(**)}{=} (\mathbf{U}^{-1}\mathbf{r}_i) \otimes \mathbf{r}_i \stackrel{(***)}{=} \lambda_i^{-1}\mathbf{r}_i \otimes \mathbf{r}_i, \qquad (iv)\\ \mathbf{R}^T(\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i)\mathbf{F}^{-T} &\stackrel{(1.78)}{=} (\mathbf{R}^T\boldsymbol{\ell}_i) \otimes (\mathbf{F}^{-1}\boldsymbol{\ell}_i) \stackrel{(*)}{=} (\mathbf{R}^T\boldsymbol{\ell}_i) \otimes (\mathbf{U}^{-1}\mathbf{R}^T\boldsymbol{\ell}_i) = \\ &\stackrel{(**)}{=} \mathbf{r}_i \otimes (\mathbf{U}^{-1}\mathbf{r}_i) \stackrel{(***)}{=} \lambda_i^{-1}\mathbf{r}_i \otimes \mathbf{r}_i, \qquad (v)\end{aligned}$$

where in the steps $(*)$ we used $\mathbf{F} = \mathbf{R}\mathbf{U}$ and the orthogonality of $\mathbf{R}$; in the steps $(**)$ we used $\boldsymbol{\ell}_i = \mathbf{R}\mathbf{r}_i$; and in the steps $(***)$ we used $\mathbf{U}\mathbf{r}_i = \lambda_i\mathbf{r}_i$.

Thus on substituting (i) into (iii) and using (iv) and (v) we have the following representation for $\mathbf{S}^{(1)}$:

$$\mathbf{S}^{(1)} = \sum_{i=1}^{3} J\tau_i\lambda_i^{-1}(\mathbf{r}_i \otimes \mathbf{r}_i), \qquad \square$$

which is of the form (ii). We therefore conclude that $\mathbf{S}^{(1)}$ is coaxial with $\mathbf{U}$. (We also see that the principal values of $\mathbf{S}^{(1)}$ and $\mathbf{T}$ are related by $s_i = J\lambda_i^{-1}\tau_i$.)

Problem 3.32. Determine the symmetric stress tensor that is work conjugate to the Biot strain tensor

$$\mathbf{E}^{(1)} = \mathbf{U} - \mathbf{I}.$$

Solution: By the definition of work conjugacy, the stress tensor $\mathbf{S}^{(1)}$ that we seek must be such that

$$\mathbf{S}^{(1)} \cdot \dot{\mathbf{E}}^{(1)} = \mathbf{S} \cdot \dot{\mathbf{F}}. \qquad (i)$$

In the text surrounding (3.81) we found that the second Piola-Kirchhoff stress tensor $\mathbf{S}^{(2)}$ is work conjugate to the Green Saint-Venant strain tensor $\mathbf{E}^{(2)}$, and so our task can be equivalently stated as wanting to find a stress tensor $\mathbf{S}^{(1)}$ such that

$$\mathbf{S}^{(1)} \cdot \dot{\mathbf{E}}^{(1)} = \mathbf{S}^{(2)} \cdot \dot{\mathbf{E}}^{(2)}. \qquad (ii)$$

Differentiating $\mathbf{E}^{(1)} = \mathbf{U} - \mathbf{I}$ and $\mathbf{E}^{(2)} = \frac{1}{2}(\mathbf{U}\mathbf{U} - \mathbf{I})$ with respect to t leads to

$$\dot{\mathbf{E}}^{(1)} = \dot{\mathbf{U}}, \qquad \dot{\mathbf{E}}^{(2)} = \frac{1}{2}(\dot{\mathbf{U}}\mathbf{U} + \mathbf{U}\dot{\mathbf{U}}). \qquad (iii)$$

Substituting (iii) into (ii) yields

$$\mathbf{S}^{(1)} \cdot \dot{\mathbf{U}} = \frac{1}{2}\mathbf{S}^{(2)} \cdot (\dot{\mathbf{U}}\mathbf{U} + \mathbf{U}\dot{\mathbf{U}}).$$

We would like to factor out the term $\dot{\mathbf{U}}$ in the first term on the right-hand side of this equation. Recall that for any tensors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ we have $\mathbf{A} \cdot \mathbf{B}\mathbf{C} = \mathbf{A}\mathbf{C}^T \cdot \mathbf{B}$. Therefore $\mathbf{S}^{(2)} \cdot \dot{\mathbf{U}}\mathbf{U} = \mathbf{S}^{(2)}\mathbf{U} \cdot \dot{\mathbf{U}}$ having used the fact that $\mathbf{U}$ is symmetric. Therefore we can write the preceding equation as

$$\left[\mathbf{S}^{(1)} - \frac{1}{2}\mathbf{S}^{(2)}\mathbf{U} - \frac{1}{2}\mathbf{U}\mathbf{S}^{(2)}\right] \cdot \dot{\mathbf{U}} = 0.$$

Since this must hold for all $\dot{\mathbf{U}}$, and assuming that the terms in the square brackets are independent of $\dot{\mathbf{U}}$, we must have

$$\mathbf{S}^{(1)} = \frac{1}{2}\Big(\mathbf{S}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{S}^{(2)}\Big). \qquad \square \tag{iv}$$

This is the stress tensor work-conjugate to $\mathbf{E}^{(1)}$. We can write this in terms of the Piola stress tensor by using $\mathbf{S}^{(2)} = \mathbf{F}^{-1}\mathbf{S}$ (see (3.81)):

$$\mathbf{S}^{(1)} = \frac{1}{2}\Big(\mathbf{F}^{-1}\mathbf{S}\mathbf{U} + \mathbf{U}\mathbf{F}^{-1}\mathbf{S}\Big). \qquad \square \tag{v}$$

To see that this is in fact the Biot stress tensor introduced at the end of Section 3.7, we must eliminate $\mathbf{F}$ in favor of $\mathbf{R}$ using the polar decomposition. First, the second term can be written as

$$\mathbf{U}\mathbf{F}^{-1}\mathbf{S} = \mathbf{U}(\mathbf{R}\mathbf{U})^{-1}\mathbf{S} = \mathbf{R}^T\mathbf{S}. \tag{vi}$$

Next, on using $\mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T$ (moment balance), we can simplify the first term as

$$\mathbf{F}^{-1}\mathbf{S}\mathbf{U} = \mathbf{F}^{-1}(\mathbf{F}\mathbf{S}^T\mathbf{F}^{-T})\mathbf{U} = \mathbf{S}^T\mathbf{F}^{-T}\mathbf{U} = \mathbf{S}^T\mathbf{R}\mathbf{U}^{-1}\mathbf{U} = \mathbf{S}^T\mathbf{R}. \tag{vii}$$

Substituting $(vi), (vii)$ into (v) yields

$$\mathbf{S}^{(1)} = \frac{1}{2}\Big(\mathbf{S}^T\mathbf{R} + \mathbf{R}^T\mathbf{S}\Big), \qquad \square$$

which is the Biot stress tensor.

Problem 3.33. Determine the stress tensor that is conjugate to the (Lagrangian) logarithmic strain tensor $\ln \mathbf{U}$.

Solution: See the paper by A. Hoger, The stress conjugate to logarithmic strain, *International Journal of Solids and Structures*, **23**(1987), pp. 1645-1656.

Problem 3.34. Pick any Eulerian strain tensor of your choice. Find the stress tensor that is conjugate to it.

Solution: See the paper by Andrew Norris, Eulerian conjugate stress and strain, *J. Mech. Materials Struct.*, **3**(2008), pp. 243-260. In general finding stress tensors conjugate to Eulerian strains is much more difficult than the corresponding problem for Lagrangian strains.

Problem 3.35. Write down expressions for the rate of working of the forces and couples in the settings of Problems 3.20 and 3.21. Rewrite these in the form of a volume integral of the local power.

Problem 3.36. (*Conservation of mass. Rate of change of linear momentum.)* You may find it helpful to review Section 2.11 on the material time derivative and the transport formula.

(a) A body undergoes a motion $\mathbf{y} = \mathbf{y}(\mathbf{x}, t)$. Let $\rho(\mathbf{y}, t)$ be the mass density of the body at the particle that is located at $\mathbf{y}$ at time t. Consider a part of a body, and let $\mathcal{D}_t$ be the region of space it occupies at time t. Note that the region $\mathcal{D}_t$ varies with time. The mass of this part is the integral of $\rho(\mathbf{y}, t)$ over $\mathcal{D}_t$. The conservation of mass requires that the mass of every part of the body be time-independent:

$$\frac{d}{dt}\int_{\mathcal{D}_t} \rho(\mathbf{y}, t)\, dV_y = 0 \qquad \text{for all parts } \mathcal{D}_t. \tag{i}$$

Show that the balance law (i) holds if and only if the following field equation holds,

$$\dot{\rho} + \rho \,\mathrm{div}\, \mathbf{v} = 0 \qquad \text{at all } \mathbf{y} \in \mathcal{R}_t, \tag{3.104}$$

where $\dot{\rho}$ is the material time derivative of ρ and the cartesian components of $\mathrm{div}\, \mathbf{v}$ are $\partial v_i/\partial y_i$.

(b) Let $\rho_R(\mathbf{x})$ be the mass density of the body in a reference configuration. Show that

$$\rho_R = \rho J. \tag{3.105}$$

(c) Let $\mathbf{v}(\mathbf{y}, t)$ be (the spatial description of) the velocity field. It is defined on $\mathcal{R}_t$ at each t. Show that the rate of increase of the linear momentum of the part under consideration is

$$\frac{d}{dt}\int_{\mathcal{D}_t} \rho \mathbf{v}\, dV_y = \int_{\mathcal{D}_t} \rho \dot{\mathbf{v}}\, dV_y \tag{3.106}$$

where $\dot{\mathbf{v}}$ is the material time derivative of $\mathbf{v}$, i.e. the acceleration.

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Chapter 4

Constitutive Relation

In principle, the analyses of deformation in Chapter 2 and stress in Chapter 3 are valid for any continuum, irrespective of the specific material of which the body is composed. However, given the loading applied on a body, the basic equations derived in those chapters are not sufficient for determining the resulting stress and deformation fields. Additional information describing how the stress depends on the deformation is needed, and this comes from considering the behavior of the specific material at hand. This is not surprising since even in the simplest case of a spring, given the force applied on the spring, one cannot determine its elongation without knowing something about the material of which it is made.

Nonlinear elasticity has been, and continues to be, successfully used to study a variety of materials such as biological tissues, "soft materials" more generally including elastomeric materials, and crystalline solids undergoing martensitic phase transformations.

There are two main approaches to constructing continuum-scale constitutive relations. One begins at the atomistic-scale and attempts to deduce the continuum-scale response by some sort of averaging across length and time scales ("coarse graining" using "multi-scale methods"). The other so-called "phenomenological approach" begins directly at the continuum-scale guided by experimental observations and some basic principles. A combination of these two approaches, where micro-mechanical considerations are used to infer the form of the constitutive relation, the details of which are then explored experimentally, is often particularly effective.

In Chapter 8 we shall describe one micro-mechanical model – Cauchy's beautiful derivation of the constitutive relation of a crystalline solid using a simple lattice model in which the atoms interact through a pair potential. A second micro-mechanical model that would have been natural for us to describe is a polymer chain model, and its use in constructing the constitutive relation for a rubber-like material. Unfortunately, this relies crucially on calculating the entropy, a thermodynamic notion that we do not address in these notes. The interested reader can refer to Chapter 9 of Volume II.

In these notes we are concerned with *elastic materials*. We shall assume that the defining characteristic of an elastic material is that it does not dissipate energy (at least when there are no moving singularities in the body such as a propagating crack). Such elastic materials are frequently said to be *hyperelastic* (or Green elastic).

A word on notation: in order to avoid confusion, it will sometimes be helpful to distinguish between functions of different arguments, even when their values represent the same quantity. In particular, we will denote the so-called strain energy density in the various forms $W(\mathbf{x})$, $\widehat{W}(\mathbf{F})$, $\overline{W}(\mathbf{C})$, $\widetilde{W}(I_1, I_2, I_3)$ and $W^*(\lambda_1, \lambda_2, \lambda_3)$. Even though they all represent the elastic energy density, they are different *functions*. When it is not essential that we make the distinction, and there is no chance for confusion, we will simply write $W(\mathbf{x})$, $W(\mathbf{F})$, $W(\mathbf{C})$, $W(I_1, I_2, I_3)$ and $W(\lambda_1, \lambda_2, \lambda_3)$.

Occasionally, we will refer to the time t. When we do so, we will *not* be taking inertial effects into account[1]. Instead, we will simply be considering a one-parameter family of equilibrium deformations, a so-called *quasi-static motion*, with t merely being the parameter. In a quasi-static motion, the stress field obeys the equilibrium equation at each instant t.

A roadmap of this chapter is as follows. In Section 4.2 we characterize an elastic material in terms of its strain energy function $W(\mathbf{F})$. The implications of material frame indifference are explored in Section 4.3. We turn in Section 4.4 to material symmetry with Section 4.4.2 devoted to isotropic materials. (Some anisotropic materials are considered in Chapter 6.) In Section 4.5 materials with internal constraints such as incompressibility and inextensibility are considered. The response in uniaxial tension, simple shear and biaxial plane stress are explored for a general isotropic material in Section 4.7. Restrictions imposed on the strain energy function for reasons of physical reasonableness and mathematical necessity are touched upon in Section 4.6.3. In Section 4.7 we describe a few specific strain energy functions from the literature. Finally, in Section 4.8, we specialize the constitutive relation to infinitesimal deformations and thus derive the stress-strain relation in linear(ized) elasticity.

The discussion of constitutive relations in this chapter, even when limited to elastic

[1]We make an exception briefly when referring to strong ellipticity in Section 4.6.3.

materials, is concise and incomplete. An expanded treatment can be found in the references cited at the end of this chapter and in Chapters 7, 8 (and 9) of Volume II.

4.1 Motivation.

We start by motivating why it is necessary to undertake a careful discussion of the constitutive relation since, based on our experience with linear elasticity theory as undergraduates, it may feel natural to simply write down a relationship between a stress tensor and a strain tensor, say,

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E}) \qquad \text{where} \quad \mathbf{E} = \frac{1}{2}\big(\mathbf{F}^T\mathbf{F} - \mathbf{I}\big). \tag{i}$$

Is this a reasonable constitutive relation?

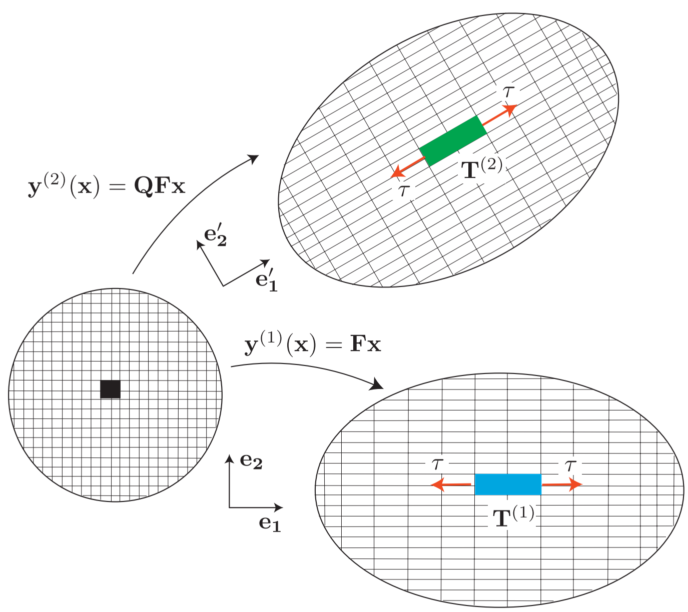

Figure 4.1: A body subjected to two deformations $\mathbf{y}^{(1)}(\mathbf{x}) = \mathbf{Fx}$ and $\mathbf{y}^{(2)}(\mathbf{x}) = \mathbf{QFx}$ that differ by a rigid rotation $\mathbf{Q}$, and the associated stress tensors.

To explore this question, first consider a homogeneous deformation

$$\mathbf{y}^{(1)}(\mathbf{x}) = \mathbf{Fx}. \tag{ii}$$

For illustrative purposes (only), suppose that $\mathbf{F}$ describes a uniaxial stretch in the $\mathbf{e}_1$-direction with equal contraction in the directions normal to it:

$$\mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3).$$

Suppose the associate stress is a uniaxial stress in the $\mathbf{e}_1$-direction:

$$\mathbf{T}^{(1)} = \tau\, \mathbf{e}_1 \otimes \mathbf{e}_1. \tag{iii}$$

Next consider a second deformation, identical to the first followed by an arbitrary rigid rotation $\mathbf{Q}$:

$$\mathbf{y}^{(2)}(\mathbf{x}) = \mathbf{QFx}. \tag{iv}$$

Figure 4.1 shows a cartoon of these two deformations where the small square in the reference configuration has been stretched in the $\mathbf{e}_1$-direction in the first deformation, and has been rotated after stretching in the $\mathbf{e}_1$-direction in the second. On physical grounds, we would expect the stress associated with the second deformation to be a uniaxial stress in the $\mathbf{e}_1'$-direction:

$$\mathbf{T}^{(2)} = \tau\, \mathbf{e}_1' \otimes \mathbf{e}_1'. \tag{v}$$

Here the basis $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ is obtained by rotating the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by $\mathbf{Q}$:

$$\mathbf{e}_i' = \mathbf{Q}\mathbf{e}_i; \tag{vi}$$

see Figure 4.1. While the two stress tensors are distinct, they are related by

$$\mathbf{T}^{(2)} \overset{(v)}{=} \tau \mathbf{e}_1' \otimes \mathbf{e}_1' \overset{(vi)}{=} \tau(\mathbf{Q}\mathbf{e}_1) \otimes (\mathbf{Q}\mathbf{e}_1) = \tau \mathbf{Q}(\mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{Q}^T \overset{(iii)}{=} \mathbf{Q}\mathbf{T}^{(1)}\mathbf{Q}^T. \tag{vii}$$

Using the constitutive relation $(i)_1$ we have

$$\mathbf{T}^{(1)} = \widehat{\mathbf{T}}(\mathbf{E}^{(1)}), \qquad \mathbf{T}^{(2)} = \widehat{\mathbf{T}}(\mathbf{E}^{(2)}), \tag{$viii$}$$

so that from (vii) and $(viii)$,

$$\widehat{\mathbf{T}}(\mathbf{E}^{(2)}) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{E}^{(1)})\mathbf{Q}^T. \tag{ix}$$

However the Green Saint-Venant strain tensors associated with these two deformations are

$$\mathbf{E}^{(1)} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}), \qquad \mathbf{E}^{(2)} = \frac{1}{2}\big((\mathbf{QF})^T(\mathbf{QF}) - \mathbf{I}\big) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \mathbf{E}^{(1)}. \tag{x}$$

Thus from (ix) and (x),

$$\widehat{\mathbf{T}}(\mathbf{E}^{(1)}) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{E}^{(1)})\mathbf{Q}^T,$$

or, since $\mathbf{E}^{(1)}$ is in fact arbitrary,

$$\widehat{\mathbf{T}}(\mathbf{E}) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{E})\mathbf{Q}^T \qquad \text{for all rotations } \mathbf{Q}, \tag{xi}$$

and all strains $\mathbf{E}$. This is a restriction on the form of the function $\widehat{\mathbf{T}}(\mathbf{E})$. In fact, from the result in Problem 1.37, it follows that $\widehat{\mathbf{T}}(\mathbf{E})$ must be a scalar multiple of the identity:

$$\widehat{\mathbf{T}}(\mathbf{E}) = \tau(\mathbf{E})\,\mathbf{I}. \qquad (xii)$$

Thus a constitutive relation of the general form $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E})$ must necessarily be of the particular form $\widehat{\mathbf{T}}(\mathbf{E}) = \tau(\mathbf{E})\,\mathbf{I}$ where the Cauchy stress tensor is hydrostatic (in all deformations) and so the material is a fluid!

Problem 4.1.1. You might say that a shortcoming of the particular constitutive relation (i) above was that the Cauchy stress tensor is associated with the deformed configuration while the Lagrangian strain tensor is associated with the reference configuration. Based on this, replace (i) with the ansatz $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{B})$ where $\mathbf{B}$ is the (Eulerian) left Cauchy-Green tensor and carry out an analysis like the one above. What does this tell you about the form of $\widehat{\mathbf{T}}(\mathbf{B})$?

4.2 An Elastic Material.

– First, we assume that the stress at some particle $\mathbf{x}$ depends only on the deformation of the particles in the immediate neighborhood of that particle. Such a theory is said to be a *local theory*. We know from Chapter 2 that the deformation in the vicinity of a particle is completely characterized by the deformation gradient tensor $\mathbf{F} = \nabla\mathbf{y}$ at that particle. This implies that the stress $\mathbf{S}$ at particle $\mathbf{x}$ depends on the deformation solely through the deformation gradient tensor[2] $\mathbf{F}$ at particle $\mathbf{x}$.

– Second, we assume further that an elastic material *has no memory*[3], and therefore that the stress $\mathbf{S}$ at time t depends only on the value of the deformation gradient tensor $\mathbf{F}$ at that same time t. We are thus led to consider constitutive relations of the form $\mathbf{S}(\mathbf{x},t) = \widehat{\mathbf{S}}(\mathbf{F}(\mathbf{x},t))$, or simply

$$\mathbf{S} = \widehat{\mathbf{S}}(\mathbf{F}). \qquad (4.1)$$

– Terminology: Note the distinction between $\mathbf{S}(\mathbf{x})$ and $\widehat{\mathbf{S}}(\mathbf{F})$. The former is the stress *field* in the body while the latter is the *constitutive response function* for stress. The material is characterized by $\widehat{\mathbf{S}}$.

[2] In a nonlocal theory the stress might, for example, depend on the deformation gradient *and* the gradient of the deformation gradient.

[3] unlike, say, a viscoelastic material which depends on the past history of the deformation.

– Since the Cauchy and Piola stress tensors are related by $\mathbf{T} = \mathrm{J}^{-1}\mathbf{S}\mathbf{F}^T$, knowing $\widehat{\mathbf{S}}$ gives

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F}) \qquad \text{where} \quad \widehat{\mathbf{T}}(\mathbf{F}) = \mathrm{J}^{-1}\widehat{\mathbf{S}}(\mathbf{F})\mathbf{F}^T, \quad \mathrm{J} = \det \mathbf{F}. \tag{4.2}$$

It will be convenient for us to work with the Piola stress tensor.

– If the material is *inhomogeneous* in the reference configuration, the constitutive response function $\widehat{\mathbf{S}}$ will depend explicitly on $\mathbf{x}$: $\mathbf{S} = \widehat{\mathbf{S}}(\mathbf{F}, \mathbf{x})$.

– Third, we assume there is *no energy dissipation* in an elastic material in the following sense: let $W(\mathbf{x}, t)$ denote the *stored energy density*, i.e. the energy stored per unit volume in the reference configuration[4]. The total elastic energy stored in a part of the body is then

$$\int_{\mathcal{D}_\mathrm{R}} W \, dV_x,$$

where $\mathcal{D}_\mathrm{R}$ is the region in the reference configuration occupied by the part under consideration. When we say that an elastic material is dissipation-free we mean that the rate at which external work is done on any part of the body during a quasi-static motion equals the corresponding rate of increase of stored energy[5] provided the fields are smooth:

$$\int_{\partial\mathcal{D}_\mathrm{R}} \mathbf{s} \cdot \mathbf{v} \, dA_x + \int_{\mathcal{D}_\mathrm{R}} \mathbf{b} \cdot \mathbf{v} \, dV_x = \frac{d}{dt} \int_{\mathcal{D}_\mathrm{R}} W \, dV_x. \tag{4.3}$$

This must hold *in all quasi-static motions* for all parts of the body at all times.

– An important consequence of (4.3) can be deduced by combining it with (3.73). Equation (3.73) states that the rate at which the external forces do work equals the rate at which the stresses do work (the stress power). Equation (3.73) together with (4.3) yields

$$\int_{\mathcal{D}_\mathrm{R}} \mathbf{S} \cdot \dot{\mathbf{F}} \, dV_x \; = \; \frac{d}{dt} \int_{\mathcal{D}_\mathrm{R}} W \, dV_x, \tag{4.4}$$

which says that the rate at which the "internal forces" (the stresses) do work equals the rate of increase of the stored energy. This is the sense in which the material is dissipationless.

The preceding equation can be written as

$$\int_{\mathcal{D}_\mathrm{R}} \left(\mathbf{S} \cdot \dot{\mathbf{F}} \; - \dot{W} \right) \, dV_x = 0.$$

[4] The stored energy *per unit mass*, say ψ, is related to the energy per unit reference volume W by $W = \rho_\mathrm{R}\psi$ where ρ_R is the mass density in the reference configuration.

[5] When inertial effects are taken into account, one must include the rate of increase of kinetic energy on the right-hand side of (4.3).

(Recall that the superior dot denotes the derivative with respect to t at a fixed particle $\mathbf{x}$ – the "material time derivative", see Section 2.11.2.) Since this must hold for every part of the body it follows by localization that

$$\mathbf{S}\cdot\dot{\mathbf{F}} \;-\; \dot{W} = 0 \qquad \text{for all } \mathbf{x}\in\mathcal{R}_{\mathrm{R}}. \tag{4.5}$$

– Finally, just as for the stress, it is natural to assume that the stored energy density W at a particle $\mathbf{x}$ at time t depends on the deformation only through the deformation gradient tensor $\mathbf{F}$ at the same particle $\mathbf{x}$ at the same time t, i.e. that there is a constitutive response function $\widehat{W}$ such that $W(\mathbf{x},t) = \widehat{W}(\mathbf{F}(\mathbf{x},t))$ or more simply

$$W = \widehat{W}(\mathbf{F}).$$

Remark: In certain settings elasticity is coupled to some other physical phenomenon. For example mechanical and thermal effects are coupled in thermoelasticity. The free energy function (the counterpart of the strain energy function) describing a thermoelastic material has the form $\widehat{W}(\mathbf{F},\theta)$ with $\theta(\mathbf{x},t)$ being the temperature field. Additional physical principles, in this case the first and second laws of thermodynamics, must be enforced as part of the model. A brief introduction to coupled phenomena can be found in Chapter 9.

– On using $W = \widehat{W}(\mathbf{F})$ and $\mathbf{S} = \widehat{\mathbf{S}}(\mathbf{F})$ in (4.5) we get

$$\widehat{\mathbf{S}}\cdot\dot{\mathbf{F}} - \frac{\partial\widehat{W}}{\partial\mathbf{F}}\cdot\dot{\mathbf{F}} = 0 \qquad\Rightarrow\qquad \left[\widehat{\mathbf{S}}(\mathbf{F}) - \frac{\partial\widehat{W}}{\partial\mathbf{F}}(\mathbf{F})\right]\cdot\dot{\mathbf{F}} = 0. \tag{4.6}$$

This must hold in *every quasi-static motion of the body.* Observe that the terms in the square brackets only involve the deformation gradient tensor $\mathbf{F}$ and not its rate $\dot{\mathbf{F}}$. For a given $\mathbf{F}$, one can always construct a motion with an *arbitrary* $\dot{\mathbf{F}}$ at a particular particle at a particular instant[6]. It follows that (4.6) must hold for all tensors $\dot{\mathbf{F}}$ and therefore that

$$\widehat{\mathbf{S}}(\mathbf{F}) = \frac{\partial\widehat{W}}{\partial\mathbf{F}}(\mathbf{F}). \tag{4.7}$$

[6]We encountered this issue previously in Problem 1.8.4. To illustrate the argument used, consider a one-dimensional continuum. Let $y = y(x,t)$ be a motion with the stretch λ and stretch-rate $\dot{\lambda}$ defined by $\lambda(x,t) = \partial y/\partial x$ and $\dot{\lambda}(x,t) = \partial\lambda/\partial t$. Then the claim is that one can always find a motion $y(x,t)$ in which the values of λ and $\dot{\lambda}$ at some *particular instant* can be arbitrarily and independently prescribed.

To see this, pick and fix an arbitrary instant t_o and let $\lambda_o > 0$ and r_o be any two constants, each chosen arbitrarily and independently of the other. Consider the motion

$$y(x,t) = \lambda_o x \exp\left(\frac{r_o(t-t_o)}{\lambda_o}\right), \tag{i}$$

This tells us that given the constitutive response function $\widehat{\mathbf{S}}$ for the stress can be calculated from the constitutive response function $\widehat{W}$ for the stored energy. We write this less formally as

$$\boxed{\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}},} \qquad \boxed{\mathbf{T} = \mathrm{J}^{-1}\mathbf{S}\mathbf{F}^T = \frac{1}{J}\frac{\partial W}{\partial \mathbf{F}}\,\mathbf{F}^T.} \tag{4.8}$$

We thus conclude that an elastic material is characterized by the constitutive response function $\widehat{W}(\mathbf{F})$ for the stored energy per unit reference volume. It is referred to as the **strain-energy function**.

If the material is inhomogeneous in the reference configuration we would have $W = \widehat{W}(\mathbf{F}, \mathbf{x})$.

4.2.1 An elastic material. Alternative approach.

The preceding analysis hinged on the balance (4.3) between the rate of work and energy, and therefore cannot be used, at least not directly, when there is dissipation. In this subsection we briefly present a modification of the preceding analysis based on the dissipation inequality. While not essential in elasticity, this approach can be used in the study of inelastic materials (and we shall do so in Chapter 9 when we touch on coupled problems).

– **Dissipation inequality:** Let $W(\mathbf{x}, t)$ be the *free energy* per unit reference volume. The total free energy

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and observe that

$$\lambda(x, t_0) = \lambda_o, \qquad \dot{\lambda}(x, t_o) = r_o. \tag{ii}$$

Suppose that $\widehat{\sigma}(\lambda)$ and $\widehat{W}(\lambda)$ are functions such that

$$\Big[\widehat{\sigma}(\lambda(x,t)) - \widehat{W}'(\lambda(x,t))\Big]\,\dot{\lambda}(x,t) = 0, \tag{iii}$$

for all motions $y(x,t)$, all particles x and all instants of time t. Since this holds for all t it necessarily holds at $t = t_o$:

$$\Big[\widehat{\sigma}(\lambda(x,t_o)) - \widehat{W}'(\lambda(x,t_o))\Big]\,\dot{\lambda}(x,t) = 0. \tag{iv}$$

Substituting (*ii*) into (*iv*) yields

$$\Big[\widehat{\sigma}(\lambda_o) - \widehat{W}'(\lambda_o)\Big]\,r_o = 0. \tag{v}$$

This must hold for all r_o. Since λ_o is independent of r_o, it follows that necessarily

$$\widehat{\sigma}(\lambda_o) = \widehat{W}'(\lambda_o),$$

which holds for all $\lambda_o > 0$. For a discussion of this issue in three-dimensions, see Section 3.4 of Gurtin et al. [10].

in a part of the body is then

$$\int_{\mathcal{D}_{\mathrm{R}}} W \, dV_x,$$

where $\mathcal{D}_{\mathrm{R}}$ is the region in the reference configuration occupied by the part under consideration.

The *dissipation inequality* states that the rate of increase of free energy cannot exceed the rate at which external work is done[7]:

$$\int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{s}\cdot\mathbf{v}\, dA_x + \int_{\mathcal{D}_{\mathrm{R}}} \mathbf{b}\cdot\mathbf{v}\, dV_x \;\geq\; \frac{d}{dt}\int_{\mathcal{D}_{\mathrm{R}}} W\, dV_x. \tag{4.9}$$

This must hold *in all quasi-static motions* for all parts of the body at all times. Proceeding as above leads one to the local inequality (Exercise)

$$\mathbf{S}\cdot\dot{\mathbf{F}} \,-\, \dot{W} \geq 0 \qquad \text{for all } \mathbf{x}\in\mathcal{R}_{\mathrm{R}}. \tag{4.10}$$

– **Constitutive equations: primitive form.** We assume that the stress and free energy at particle $\mathbf{x}$ at time t depend only on the deformation of the particles in the immediate neighborhood of that particle at that same instant. This implies that $\mathbf{S}(\mathbf{x},t)$ and $W(\mathbf{x},t)$ depend on the deformation solely through $\mathbf{F}(\mathbf{x},t)$. We are thus led to consider constitutive relations of the form $\mathbf{S}(\mathbf{x},t) = \widehat{\mathbf{S}}(\mathbf{F}(\mathbf{x},t))$ and $W(\mathbf{x},t) = \widehat{W}(\mathbf{F}(\mathbf{x},t))$.

Accordingly, we now assume that the material is characterized by the constitutive equations

$$\mathbf{S} = \widehat{\mathbf{S}}(\mathbf{F}), \qquad W = \widehat{W}(\mathbf{F}). \tag{4.11}$$

This is the primitive form of the constitutive relations.

– **Constitutive equations: simplified (reduced) form.** On using (4.11) in (4.10) we get

$$\widehat{\mathbf{S}}\cdot\dot{\mathbf{F}} - \frac{\partial\widehat{W}}{\partial\mathbf{F}}\cdot\dot{\mathbf{F}} \geq 0 \qquad \Rightarrow \qquad \left[\widehat{\mathbf{S}}(\mathbf{F}) - \frac{\partial\widehat{W}}{\partial\mathbf{F}}(\mathbf{F})\right]\cdot\dot{\mathbf{F}} \geq 0. \tag{4.12}$$

This must hold in *every quasi-static motion of the body*. Observe that the terms in the square brackets only involve the deformation gradient tensor $\mathbf{F}$ and not its rate $\dot{\mathbf{F}}$. The argument used on page 345 can be generalized (and is referred to as the Coleman-Noll argument) to conclude that the terms within the square brackets must vanish:

$$\widehat{\mathbf{S}}(\mathbf{F}) = \frac{\partial\widehat{W}}{\partial\mathbf{F}}(\mathbf{F}). \tag{4.13}$$

We thus conclude that an elastic material is completely characterized by the constitutive response function $\widehat{W}(\mathbf{F})$ for the free energy.

– One can now use (4.13) to show that (4.10), and therefore (4.9), hold with *equality*. This recovers the power - energy *balance* of the preceding section and motivates us to refer to W as the stored energy.

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[7] When inertial effects are taken into account, one must include the rate of increase of kinetic energy on the right-hand side of (4.9).

4.3 Material frame indifference.

Material frame indifference refers to the general idea that physical laws should be independent of the observer. In the particular context of constitutive response, it refers to the requirement that when two observers view a body undergoing some motion, they should perceive no difference in the material's response. A conceptually different requirement is the invariance of a material's response in two rigidly-related motions as viewed by a single observer. The consequences of these two notions, as far as the material response of elastic solids at the continuum scale is concerned, are the same. Our analysis in these notes is based on the latter idea, where we shall require the stored elastic energy of an elastic material to be unaffected by a superposed rigid deformation. The reader may refer to, e.g., Chapter 4.3 of Chadwick [6], Chapter 2 of Steigmann [21], Chapter 20 of Gurtin et al. [10] for a treatment based on invariance with respect to two observers.

– We are concerned with the stored energy density at some fixed particle $\mathbf{x}$ in the body. Its value in the reference configuration is $\widehat{W}(\mathbf{I})$. Now suppose that the body is subjected to a rigid rotation characterized by the proper orthogonal tensor $\mathbf{Q}$. Such a deformation does not distort the body and so we expect that it will not store any additional elastic energy. Accordingly one expects $\widehat{W}(\mathbf{Q}) = \widehat{W}(\mathbf{I})$ for all proper orthogonal tensors $\mathbf{Q}$. It would therefore be natural to require the function $\widehat{W}$ to have this property.

– **Principle of material frame indifference:** More generally, suppose that the body is subjected to a homogeneous deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$. The associated stored energy density is $\widehat{W}(\mathbf{F})$. Now subject this body to a *further* rigid rotation $\mathbf{Q}$. This is equivalent to considering the deformation $\mathbf{y} = \mathbf{QF}\mathbf{x}$. The associated stored energy density is $\widehat{W}(\mathbf{QF})$. We do not expect any additional energy to be stored in the body due to the subsequent rigid rotation, and therefore require $\widehat{W}$ to have the property

$$\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{QF}) \quad \text{for all nonsingular tensors } \mathbf{F} \text{ and proper orthogonal tensors } \mathbf{Q}. \tag{4.14}$$

A strain energy function $\widehat{W}(\mathbf{F})$ that conforms to (4.14) is said to be *frame indifferent* or *objective* (as opposed to subjective).

Problem 4.3.1. Show that the Cauchy and Piola stress response functions $\widehat{\mathbf{T}}(\mathbf{F})$ and $\widehat{\mathbf{S}}(\mathbf{F})$ are frame indifferent if

$$\widehat{\mathbf{T}}(\mathbf{QF}) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T, \qquad \widehat{\mathbf{S}}(\mathbf{QF}) = \mathbf{Q}\widehat{\mathbf{S}}(\mathbf{F}), \tag{4.15}$$

for all nonsingular tensors $\mathbf{F}$ and proper orthogonal tensors $\mathbf{Q}$.

Problem 4.3.2. Consider two deformations $\mathbf{y} = \mathbf{F}\mathbf{x}$ and $\mathbf{y} = \mathbf{Q}\mathbf{F}\mathbf{x}$, and two bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ where the latter basis is obtained by rotating the former by $\mathbf{Q}$: $\mathbf{e}'_i = \mathbf{Q}\mathbf{e}_i$. Let T_{ij} and T'_{ij} be the components of the Cauchy stress tensor in these two bases. Show that material frame indifference is equivalent to the requirement

$$\widehat{T}_{ij}(\mathbf{F}) = \widehat{T}'_{ij}(\mathbf{Q}\mathbf{F}). \tag{4.16}$$

This is precisely the idea we used in the motivational example in Section 4.1 (and leads to the next problem).

Problem 4.3.3. It has been conjectured that the constitutive relation of a certain material has the form

$$\mathbf{T} = \mathbf{T}(\mathbf{E}),$$

where $\mathbf{E}$ is the Green Saint-Venant strain. Show that this is *not* consistent in general with the requirement $(4.15)_1$ (or (4.16)) of material frame indifference (unless it is a scalar multiple of the identity).

– Equation (4.14) imposes a restriction on the allowable functions $\widehat{W}(\mathbf{F})$. We now determine the most general form of $\widehat{W}$ that conforms to (4.14).

Claim: The material frame indifference requirement (4.14) holds if and only if

$$\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{U}) \qquad \text{where } \mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}. \tag{4.17}$$

Proof: First suppose that (4.14) holds. Since it holds for all proper orthogonal tensors $\mathbf{Q}$ it must necessarily hold for the particular choice $\mathbf{Q} = \mathbf{R}^T$ where $\mathbf{R}$ is the rotation in the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$. Thus a necessary condition for (4.14) to hold is obtained by setting $\mathbf{Q} = \mathbf{R}^T$ in it:

$$\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{R}^T\mathbf{F}) = \widehat{W}(\mathbf{R}^T\mathbf{R}\mathbf{U}) = \widehat{W}(\mathbf{U}),$$

where in the last step we used $\mathbf{R}^T\mathbf{R} = \mathbf{I}$. This yields (4.17).

Conversely, suppose that (4.17) holds for all nonsingular $\mathbf{F}$ which we may write as

$$\widehat{W}(\mathbf{F}) = \widehat{W}\big(\sqrt{\mathbf{F}^T\mathbf{F}}\big). \tag{4.18}$$

Since this holds for all nonsingular $\mathbf{F}$ it must hold for the tensor $\mathbf{Q}\mathbf{F}$ where $\mathbf{Q}$ is any proper orthogonal tensor. Replacing $\mathbf{F}$ by $\mathbf{Q}\mathbf{F}$ in the preceding equation yields

$$\widehat{W}(\mathbf{Q}\mathbf{F}) = \widehat{W}\big(\sqrt{(\mathbf{Q}\mathbf{F})^T(\mathbf{Q}\mathbf{F})}\big) = \widehat{W}\big(\sqrt{\mathbf{F}^T\mathbf{Q}^T\mathbf{Q}\mathbf{F}}\big) = \widehat{W}\big(\sqrt{\mathbf{F}^T\mathbf{F}}\big) = \widehat{W}(\mathbf{U}).$$

This together with (4.17) leads to (4.14). $\square$

We therefore conclude that (4.14) *holds if and only if* (4.17) *holds, i.e. if the stored energy depends on the deformation through only the Lagrangian stretch tensor* $\mathbf{U}$.

Problem 4.3.4. Show that the material frame indifference requirements $(4.15)_{1,2}$ for the Cauchy and Piola stress response functions $\widehat{\mathbf{T}}(\mathbf{F})$ and $\widehat{\mathbf{S}}(\mathbf{F})$ hold if and only if

$$\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{R}\widehat{\mathbf{T}}(\mathbf{U})\mathbf{R}^T, \qquad \widehat{\mathbf{S}}(\mathbf{F}) = \mathbf{R}\widehat{\mathbf{S}}(\mathbf{U}), \tag{4.19}$$

where $\mathbf{R}$ and $\mathbf{U}$ are the factors of $\mathbf{F}$ in the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$.

Observe from this that if the constitutive relation we considered in the motivational example in Section 4.1 had been $\mathbf{T} = \mathbf{R}\widehat{\mathbf{T}}(\mathbf{E})\mathbf{R}^T$ instead of $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E})$, it would have been acceptable.

– As noted previously in the discussion surrounding (2.58), there is a one-to-one relation between $\mathbf{U}$ and the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$. Moreover, given $\mathbf{F}$, it is a lot easier to calculate $\mathbf{C}$ than $\mathbf{U}$. Therefore we introduce a function $\overline{W}(\mathbf{C}) = \widehat{W}(\sqrt{\mathbf{C}}) = \widehat{W}(\mathbf{U})$ and thus express the stored energy in the form

$$\widehat{W}(\mathbf{F}) = \overline{W}(\mathbf{C}) \qquad \text{where } \mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{4.20}$$

Since every Lagrangian strain tensor $\mathbf{E}$ has a one-to-one correspondence with the Lagrangian stretch tensor $\mathbf{U}$, it follows that W *can equivalently be written as a function of any Lagrangian strain tensor* $\mathbf{E}$.

By using (4.20), $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and the chain rule, one can write the constitutive relations (4.8) for stress as (Exercise)

$$\boxed{\mathbf{S} = 2\mathbf{F}\,\frac{\partial\overline{W}}{\partial\mathbf{C}}, \qquad \mathbf{T} = \frac{2}{J}\,\mathbf{F}\,\frac{\partial\overline{W}}{\partial\mathbf{C}}\,\mathbf{F}^T.} \tag{4.21}$$

Remark: Observe that the right-hand side of $(4.21)_2$ is a symmetric tensor. Thus the value of the Cauchy stress yielded by the constitutive relation $(4.21)_2$ will be automatically symmetric.

Remark: Observe from (4.21) that $\widehat{\mathbf{S}}(\mathbf{F})$ and $\widehat{\mathbf{T}}(\mathbf{F})$ *cannot* be written in terms of $\mathbf{C}$ (or $\mathbf{U}$ or any Lagrangian strain) alone. They involve the rotational part $\mathbf{R}$ of the deformation gradient tensor as well.

Problem 4.3.5. Using the fact that the second Piola-Kirchhoff stress $\mathbf{S}^{(2)}$ is work conjugate to the Green Saint-Venant strain $\mathbf{E}$, show that the constitutive equation for it is $\mathbf{S}^{(2)} = \partial W(\mathbf{E})/\partial \mathbf{E}$.

4.4 Material symmetry.

– In the discussion above we considered subjecting a body to a rotation *after* having first deformed it, i.e. we were concerned with two deformations $\mathbf{y} = \mathbf{F}\mathbf{x}$ and $\mathbf{y} = \mathbf{Q}\mathbf{F}\mathbf{x}$. What if instead we had rotated the body first *before* deforming it? That is, had we considered two deformations $\mathbf{y} = \mathbf{F}\mathbf{x}$ and $\mathbf{y} = \mathbf{F}\mathbf{Q}\mathbf{x}$, would we have required $\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{F}\mathbf{Q})$? The answer, in general, is "no". It depends on the symmetry of the material as we shall now see.

The strain energy function depends on the reference configuration: If we change the reference configuration, the strain energy *function* W changes. To see this in a simple setting consider a one-dimensional elastic bar. Suppose it has length L in some homogeneously *deformed* configuration and that the stored energy has a certain *value* in this configuration. There is nothing unique about a reference configuration. All that is required is that it be a configuration that the body *can* achieve. Accordingly consider two reference configurations and let L_1 and L_2 be the lengths of the bar in those two configurations. The stretch of the bar from these respective reference configurations (to the same deformed configuration) is $\lambda_1 = L/L_1$ and $\lambda_2 = L/L_2 \neq \lambda_1$. If the material is described by a strain energy function $W(\lambda)$ that does not depend on the choice of reference configuration, then we would conclude that the energy density in the deformed body has the two values $W(\lambda_1)$ and $W(\lambda_2)$ in the deformed configuration. This cannot be since we did not change the deformed configuration and the energy in the deformed body has one definite value. Thus we conclude that the function W must depend on the choice of reference configuration and so we have different strain energy *functions* W_1 and W_2 associated with the two reference configurations. Since W_i represents the energy per unit reference length, the total stored elastic energy in the deformed configuration can be written in the equivalent forms $L_1 W_1(\lambda_1)$ and $L_2 W_2(\lambda_2)$. Since these two values must be equal, it follows that the functions W_1 and W_2 must be such that $L_1 W_1(\lambda_1) = L_2 W_2(\lambda_2)$:

$$\frac{W_2(\lambda_2)}{\lambda_2} = \frac{W_1(\lambda_1)}{\lambda_1}. \tag{4.22}$$

We now generalize this.

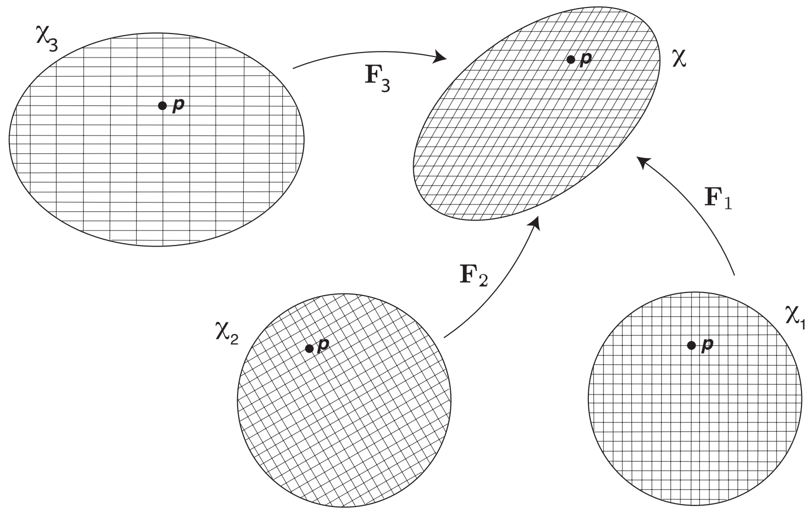

Figure 4.2: A sketch of the regions occupied by a body in a deformed configuration χ and three reference configurations χ_1, χ_2 and χ_3. The lattices are shown merely for motivational purposes where the lattice in χ_1 has been rotated to get χ_2 and stretched to get χ_3. The strain energy *functions* $\widehat{W}_1, \widehat{W}_2$ and $\widehat{W}_3$ with respect to these three reference configurations would be distinct in general.

– The deformation gradient tensor $\mathbf{F}$ depends on the reference configuration but the energy stored in the deformed configuration does not depend on this choice. Therefore, since $\mathbf{F}$ depends on the choice of reference configuration it is necessary that the strain energy function $\widehat{W}$ also depend on the reference configuration (in a suitable way).

This can be readily seen from Figure 4.2 which shows the regions occupied by a body in a deformed configuration χ and three reference configurations χ_1, χ_2 and χ_3. The lattices are shown for motivational purposes only where the lattice in χ_1 has been rotated to get χ_2 and stretched to get χ_3. There are three strain energy functions $\widehat{W}_1(\mathbf{F}_1), \widehat{W}_2(\mathbf{F}_2)$ and $\widehat{W}_3(\mathbf{F}_3)$ associated with these three reference configurations and yet the strain energy in the deformed configuration χ has one definite value. Thus the way in which $\widehat{W}$ depends on $\mathbf{F}$ must be such that the *value* of the stored energy remains unaffected by a change of reference configuration.

– While there is in general a different strain energy function associated with each reference configuration, if we know (a) the strain energy function associated with one such configuration and (b) the gradient of the mapping from it to a second reference configuration, we can calculate the strain energy function associated with the second reference configuration.

Consider a (single) deformed configuration χ and let $\mathbf{F}_1$ and $\mathbf{F}_2$ denote the deformation gradient tensors with respect to two reference configurations χ_1 and χ_2. Let $\widehat{W}_1$ and $\widehat{W}_2$ be the two strain energy functions associated with the two reference configurations. Keep in mind that these functions represent the energy stored per unit reference volume. Consider an infinitesimal part of volume dV_y in the deformed configuration and let the volumes of this part in the respective reference configurations be dV_1 and dV_2. The energy stored in this part can be written in either of the forms $\widehat{W}_1(\mathbf{F}_1)\, dV_1$ or $\widehat{W}_2(\mathbf{F}_2)\, dV_2$. Since the value of this stored energy cannot depend on the choice of reference configuration we must have $\widehat{W}_1(\mathbf{F}_1)\, dV_1 = \widehat{W}_2(\mathbf{F}_2)\, dV_2$. Since $dV_y = \det \mathbf{F}_1\, dV_1 = \det \mathbf{F}_2\, dV_2$ we can write this as

$$\frac{\widehat{W}_2(\mathbf{F}_2)}{\det \mathbf{F}_2} = \frac{\widehat{W}_1(\mathbf{F}_1)}{\det \mathbf{F}_1}\,; \tag{4.23}$$

cf. (4.22)

– Suppose the two reference configurations χ_1 and χ_2 are related by some nonsingular tensor $\mathbf{A}$ in the sense that the deformation gradient tensors $\mathbf{F}_1$ and $\mathbf{F}_2$ are related by $\mathbf{F}_1 = \mathbf{F}_2\mathbf{A}$. Then we can write (4.23) as $\widehat{W}_2(\mathbf{F}_2) = \widehat{W}_1(\mathbf{F}_2\mathbf{A})/\det \mathbf{A}$ since $\det \mathbf{F}_1 = \det(\mathbf{F}_2\mathbf{A}) = \det \mathbf{F}_2 \det \mathbf{A}$. Thus we conclude that

$$\widehat{W}_2(\mathbf{F}) = \frac{1}{\det\ \mathbf{A}}\widehat{W}_1(\mathbf{F}\mathbf{A}) \qquad \text{for all nonsingular } \mathbf{F}, \tag{4.24}$$

where $\mathbf{A}$ is the nonsingular tensor that relates χ_2 to χ_1. This tells us that if we know the strain energy function $\widehat{W}_1$ associated with one reference configuration, and the gradient $\mathbf{A}$ of the mapping from it to another reference configuration, then the strain energy function $\widehat{W}_2$ associated with the second reference configuration can be determined from (4.24).

If the two reference configurations are related by a rotation, i.e. if $\mathbf{A} = \mathbf{Q}$ is proper orthogonal, then (4.24) reduces to

$$\widehat{W}_2(\mathbf{F}) = \widehat{W}_1(\mathbf{F}\mathbf{Q}) \qquad \text{for all nonsingular } \mathbf{F}. \tag{4.25}$$

Material symmetry: We now turn to a discussion of material symmetry where we restrict attention to reference configurations related by a rotation[8]. Consider the two lattices[9] associated with the two reference configurations χ_1 and χ_2 as shown in Figure 4.2. When the rotation $\mathbf{Q}$ that takes $\chi_1 \to \chi_2$ is arbitrary, these latices would be distinct in general. However for certain special rotations $\mathbf{Q}$, such as the 90^o rotation associated with Figure 4.3, the

[8]In a general analysis of material symmetry, one allows for transformations between reference configurations that are more general than proper orthogonal transformations. See the discussion following (4.29). See also Section 4.2.3 of Ogden and Section 31 of Truesdell and Noll.

[9]We refer to lattices for purely motivational purposes.

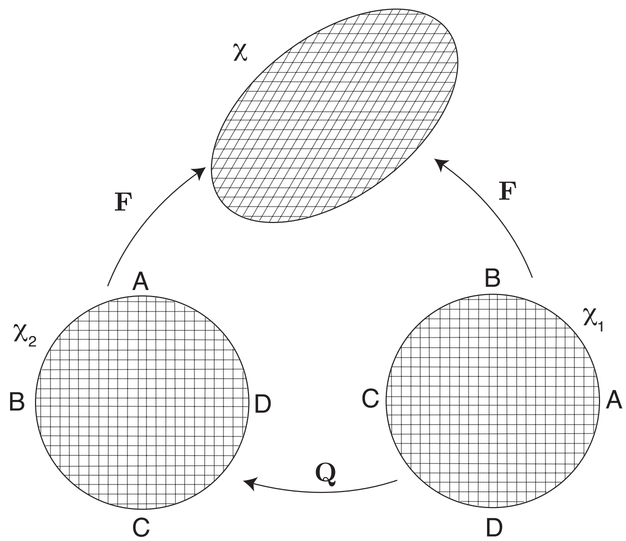

Figure 4.3: A sketch of the regions occupied by a body in a deformed configuration χ and two reference configurations χ_1 and χ_2. Note that the configurations χ_1 here and in Figure 4.2 are identical. The rotation $\mathbf{Q}$ in Figure 4.2 was arbitrary while here it is special. Here, it rotates the underlying square lattice through an angle $\pi/2$. The locations of 4 material points A, B, C, D in the two reference configuration are shown. Note the symmetry between the reference configurations χ_1 and χ_2 even though they are distinct. The particular transformation $\mathbf{Q}$ from $\chi_1 \to \chi_2$ here preserves the symmetry of the material.

lattices coincide. Such a rotation $\mathbf{Q}$ *preserves material symmetry.* (Note that the reference configuration χ_1 is the same in Figures 4.2 and 4.3.)

– Thus it may so happen that two *particular* reference configurations have the same strain energy functions. In that event

$$\widehat{W}_1(\mathbf{F}) = \widehat{W}_2(\mathbf{F}) \qquad \text{for all nonsingular } \mathbf{F}, \tag{4.26}$$

and so from (4.25) and (4.26)

$$\widehat{W}_1(\mathbf{F}) = \widehat{W}_1(\mathbf{FQ}) \text{ for the } \textit{particular} \text{ tensor } \mathbf{Q} \text{ relating those two configurations,} \tag{4.27}$$

and all nonsingular tensors $\mathbf{F}$. A tensor $\mathbf{Q}$ for which (4.27) holds preserves the symmetry of the material in the configuration χ_1. The set of all $\mathbf{Q}$ for which (4.27) holds characterizes the symmetry of the material in the configuration χ_1.

– Accordingly consider a given reference configuration κ with associated strain energy function $\widehat{W}$. Let $\mathcal{G}$ denote the set of all symmetry preserving transformation of κ, i.e. the set of all rotation tensors $\mathbf{Q}$ that take this reference configuration into a configuration with the identical strain energy function:

$$\mathcal{G} = \{\mathbf{Q} : \ \mathbf{Q}\mathbf{Q}^T = \mathbf{I}, \ \det \mathbf{Q} = 1, \ \widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{F}\mathbf{Q}) \text{ for all nonsingular } \mathbf{F}\}.$$

This set $\mathcal{G}$ of transformations is called the *material symmetry group*[10] of the given reference configuration. Thus

$$\boxed{\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{F}\mathbf{Q}) \quad \text{for all } \mathbf{Q} \in \mathcal{G} \text{ and all nonsingular } \mathbf{F}.} \tag{4.28}$$

As an example, if a material has one preferred direction $\mathbf{m}_R$ (as would be the case in the presence of one family of fibers in the referential direction $\mathbf{m}_R$), the material symmetry group will contain all rotations about $\mathbf{m}_R$. We shall explore such materials – said to be *transversely isotropic* – in Chapter 6.

– The "larger" the set $\mathcal{G}$, the greater the symmetry of the reference configuration.

– Note that *symmetry is a property of the material specific to a configuration.* In general, the same body, composed of the same material, will have different symmetries in different configurations. Symmetry transformations are the particular transformations that leave the "material microstructure" invariant.

– *Terminology:* Though symmetry is a property of a material in some configuration, when there is no chance for confusion, it will be convenient (despite being imprecise) to call $\mathcal{G}$ the symmetry group of the material.

– Observe that while the material frame indifference requirement $\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{Q}\mathbf{F})$ holds for *all* rotations $\mathbf{Q}$, the material symmetry requirement $\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{F}\mathbf{Q})$ holds only for the *particular* $\mathbf{Q}$'s that are in the material symmetry group. The rotation $\mathbf{Q}$ in the former is imposed on the deformed configuration, while the rotation $\mathbf{Q}$ in the latter is imposed on the reference configuration.

4.4.1 Material symmetry and frame indifference combined.

We now show that with material frame indifference in hand, a rotation $\mathbf{Q} \in \mathcal{G}$ if and only if

$$\overline{W}(\mathbf{C}) = \overline{W}\big(\mathbf{Q}\mathbf{C}\mathbf{Q}^T\big) \quad \text{for all symmetric positive definite } \mathbf{C}. \tag{4.29}$$

[10]One can readily confirm that if $\mathbf{Q}_1 \in \mathcal{G}$ and $\mathbf{Q}_2 \in \mathcal{G}$ then $\mathbf{Q}_1\mathbf{Q}_2 \in \mathcal{G}$. Moreover if $\mathbf{Q} \in \mathcal{G}$ then $\mathbf{Q}^{-1} \in \mathcal{G}$. In linear algebra, a group is a set of tensors with these two properties and so the set $\mathcal{G}$ is indeed a group in this mathematical sense.

– To show this, pick and fix a symmetry transformation $\mathbf{Q} \in \mathcal{G}$. Then

$$\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{FQ}) \quad \text{for all nonsingular } \mathbf{F}. \tag{4.30}$$

First, since this holds for all nonsingular $\mathbf{F}$ it necessarily holds with $\mathbf{F}$ replaced by $\mathbf{FQ}^T$. This tells us that $\widehat{W}(\mathbf{FQ}^T) = \widehat{W}(\mathbf{FQ}^T\mathbf{Q}) = \widehat{W}(\mathbf{F})$:

$$\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{FQ}^T) \quad \text{for all nonsingular } \mathbf{F}. \tag{4.31}$$

(By comparing (4.31) with (4.28) we see that if $\mathbf{Q} \in \mathcal{G}$ then $\mathbf{Q}^T \in \mathcal{G}$.) Second, we turn to material frame indifference. Since (4.20) holds for all nonsingular $\mathbf{F}$ it too holds with $\mathbf{F}$ replaced by $\mathbf{FQ}^T$. This yields

$$\widehat{W}(\mathbf{FQ}^T) = \overline{W}\big(\mathbf{QCQ}^T\big), \tag{4.32}$$

where on the right-hand side we have used $(\mathbf{FQ}^T)^T\mathbf{FQ}^T = \mathbf{QF}^T\mathbf{FQ}^T = \mathbf{QCQ}^T$. Combining (4.31) and (4.32) gives $\widehat{W}(\mathbf{F}) = \overline{W}\big(\mathbf{QCQ}^T\big)$. We can now replace $\widehat{W}(\mathbf{F})$ in this by $\overline{W}(\mathbf{C})$ because of (4.20) which leads to (4.29).

Conversely if (4.29) holds, then $\mathbf{Q} \in \mathcal{G}$. (Exercise)

– Certain materials possess symmetries under "geometric transformations" that cannot be achieved by deformation. Consider for example a crystal lattice that remains invariant under a *reflection* in a plane perpendicular to a direction $\mathbf{n}_{\mathrm{R}}$ so that the distinction between the lattices before and after reflection cannot be detected. Recall from Problem 1.10 that this reflection is characterized by $\mathbf{Q}_1 = \mathbf{I} - 2\mathbf{n}_{\mathrm{R}} \otimes \mathbf{n}_{\mathrm{R}}$. This tensor is improper orthogonal and since $\det \mathbf{Q}_1 = -1 < 0$ cannot be achieved by deformation. Observe however that $\mathbf{Q}_2 = -\mathbf{Q}_1$ is proper orthogonal (and therefore can be achieved by deformation). Moreover, if (4.29) holds for $\mathbf{Q} = \mathbf{Q}_2$ then it necessarily holds for $\mathbf{Q} = -\mathbf{Q}_2 = \mathbf{Q}_1$ and vice versa. Clearly our choice of the particular tensor $\mathbf{Q}_1$ here is purely motivational: the preceding observation holds for any rotation tensor $\mathbf{Q}$ and reflection tensor $-\mathbf{Q}$. A symmetry group can be extended in this way to accommodate reflection symmetries.

Remark: Observe by specializing Problem 1.11 that the proper orthogonal tensor $\mathbf{Q}_2 = -\mathbf{I} + 2\mathbf{n}_{\mathrm{R}} \otimes \mathbf{n}_{\mathrm{R}} = -\mathbf{Q}_1$ represents a 180^o-*rotation* about the direction $\mathbf{n}_{\mathrm{R}}$. Therefore if mechanical testing cannot detect a difference in elastic properties before and after a 180^o-rotation about $\mathbf{n}_{\mathrm{R}}$, then it necessarily cannot detect a difference in properties before and after a reflection in the plane perpendicular to $\mathbf{n}_{\mathrm{R}}$.

Problem 4.4.1. If $\mathbf{Q} \in \mathcal{G}$, then the stress response function for the Cauchy stress obeys $\widehat{\mathbf{T}}(\mathbf{F}) = \widehat{\mathbf{T}}(\mathbf{FQ})$. In view of the material frame indifference requirement $(4.19)_1$, show that

$$\mathbf{Q}\overline{\mathbf{T}}(\mathbf{C})\mathbf{Q}^T = \overline{\mathbf{T}}(\mathbf{QCQ}^T) \qquad \text{for} \quad \mathbf{Q} \in \mathcal{G}, \tag{4.33}$$

where $\widehat{\mathbf{T}}(\mathbf{F}) = \overline{\mathbf{T}}(\mathbf{C}),\ \mathbf{C} = \mathbf{F}^T\mathbf{F}$.

4.4.2 Isotropic material.

We now turn to "isotropic materials". (As mentioned above, symmetry is a property of a material in some configuration, but when there is no chance for confusion, we use the (inexact) terminology that attributes symmetry to the material. Accordingly what we really mean here is that we are considering a reference configuration in which the material is isotropic.)

Some anisotropic materials will be considered in Chapter 6.

– **Isotropic material:** If the material symmetry group $\mathcal{G}$ contains *all* proper orthogonal tensors, we say the material is isotropic. Thus by (4.29) a material is *isotropic* if

$$\overline{W}(\mathbf{C}) = \overline{W}(\mathbf{QCQ}^T) \quad \text{for all rotations } \mathbf{Q} \text{ and all symmetric positive definite tensors } \mathbf{C}. \tag{4.34}$$

In this event it follows from the result in Problem 1.35 that W has the representation

$$\boxed{W = \widetilde{W}\big(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C})\big)} \tag{4.35}$$

where

$$I_1(\mathbf{C}) = \operatorname{tr}\mathbf{C}, \qquad I_2(\mathbf{C}) = \frac{1}{2}\big[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}(\mathbf{C}^2)\big], \qquad I_3(\mathbf{C}) = \det\,\mathbf{C}, \tag{4.36}$$

are the principal scalar invariants of the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$.

Observe by taking $\mathbf{Q} = \mathbf{R}$ in (4.34) that $\overline{W}(\mathbf{C}) = \overline{W}(\mathbf{B})$ for an isotropic material.

– The constitutive relation for stress can now be specialized to an isotropic material by substituting (4.35) into (4.21), using the chain rule, and recalling

$$\frac{\partial I_1}{\partial \mathbf{C}} = \mathbf{I}, \qquad \frac{\partial I_2}{\partial \mathbf{C}} = I_1\,\mathbf{I} - \mathbf{C} \qquad \frac{\partial I_3}{\partial \mathbf{C}} = J^2\mathbf{C}^{-1}, \tag{4.37}$$

(see equation (1.187) in Problem 1.8.4). This leads to the following constitutive relations for an isotropic elastic material:

$$\left.\begin{aligned} \mathbf{T} &= 2J\frac{\partial\widetilde{W}}{\partial I_3}\mathbf{I} + \frac{2}{J}\left[\frac{\partial\widetilde{W}}{\partial I_1} + I_1\frac{\partial\widetilde{W}}{\partial I_2}\right]\mathbf{B} - \frac{2}{J}\frac{\partial\widetilde{W}}{\partial I_2}\mathbf{B}^2, \\ \mathbf{S} &= 2\mathbf{F}\left[I_3\frac{\partial\widetilde{W}}{\partial I_3}\mathbf{C}^{-1} + \left[\frac{\partial\widetilde{W}}{\partial I_1} + I_1\frac{\partial\widetilde{W}}{\partial I_2}\right]\mathbf{I} - \frac{\partial\widetilde{W}}{\partial I_2}\mathbf{C}\right]. \end{aligned}\right\} \tag{4.38}$$

– In the undeformed configuration where $\mathbf{F} = \mathbf{C} = \mathbf{B} = \mathbf{I}$, the principal scalar invariants have the values $I_1 = 3, I_2 = 3, I_3 = 1$. On setting $\mathbf{F} = \mathbf{B} = \mathbf{I}$ and $I_1 = I_2 = 3, I_3 = J = 1$ in (4.38) we see that the stress in the reference configuration is

$$\mathbf{T} = \mathbf{S} = 2\left[\frac{\partial\widetilde{W}}{\partial I_1} + 2\frac{\partial\widetilde{W}}{\partial I_2} + \frac{\partial\widetilde{W}}{\partial I_3}\right]\mathbf{I} \qquad \text{evaluated at } (I_1, I_2, I_3) = (3, 3, 1). \tag{4.39}$$

Note that this stress is hydrostatic; this is a consequence of the material being isotropic in the reference configuration. If the reference configuration is stress-free, then it is necessary that

$$\frac{\partial\widetilde{W}}{\partial I_1} + 2\frac{\partial\widetilde{W}}{\partial I_2} + \frac{\partial\widetilde{W}}{\partial I_3} = 0 \qquad \text{at} \quad (I_1, I_2, I_3) = (3, 3, 1). \tag{4.40}$$

– Note from $(4.38)_1$ that $\mathbf{T}$ and $\mathbf{B}$ *are coaxial for an isotropic material*, i.e. they have the same principal directions. Therefore we can write

$$\mathbf{T} = \tau_1\,\boldsymbol{\ell}_1 \otimes \boldsymbol{\ell}_1 + \tau_2\,\boldsymbol{\ell}_2 \otimes \boldsymbol{\ell}_2 + \tau_3\,\boldsymbol{\ell}_3 \otimes \boldsymbol{\ell}_3, \tag{4.41}$$

where the τ_i's are the principal Cauchy stresses and the $\boldsymbol{\ell}_i$'s are the principal directions of $\mathbf{B}$ (and $\mathbf{T}$).

– In terms of the principal stretches $\lambda_1, \lambda_2, \lambda_3$ one can write the principal scalar invariants of $\mathbf{C}$ (or $\mathbf{B}$) as

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \qquad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{4.42}$$

It follows from (4.35) and (4.42) that the strain energy function for an isotropic material can be written in the form[11]

$$W = W^*(\lambda_1, \lambda_2, \lambda_3). \tag{4.43}$$

[11] Before imposing material symmetry, we had $W = \overline{W}(\mathbf{C})$. By the spectral representation we know that $\mathbf{C}$ is fully determined by its eigenvalues and eigenvectors, and so we knew at that stage that $W = W(\lambda_1, \lambda_2, \lambda_3, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$. What isotropy says is that W does not depend on the eigenvectors of $\mathbf{C}$.

Since the I_i's remain invariant if any two of the λ's are switched, e.g. $\lambda_1 \leftrightarrow \lambda_2$, the constitutive response function W^* must also remain invariant if any two of its arguments are switched:

$$W^*(\lambda_1, \lambda_2, \lambda_3) = W^*(\lambda_2, \lambda_1, \lambda_3) = W^*(\lambda_1, \lambda_3, \lambda_2). \tag{4.44}$$

– We can now write the constitutive relation for $\mathbf{T}$ by changing $\overline{W}$ to W^* in $(4.21)_2$ and using the chain rule:

$$\mathbf{T} = \frac{2}{J}\mathbf{F}\,\frac{\partial W^*}{\partial \mathbf{C}}\,\mathbf{F}^T = \frac{2}{J}\mathbf{F}\left(\sum_{i=1}^{3}\frac{\partial W^*}{\partial \lambda_i}\,\frac{\partial \lambda_i}{\partial \mathbf{C}}\right)\mathbf{F}^T. \tag{4.45}$$

To simplify this we need an expression for $\partial\lambda_i/\partial\mathbf{C}$. Let λ_i^2 and $\mathbf{r}_i$ be an eigenvalue and corresponding eigenvector of $\mathbf{C}$. One can show by differentiating $\mathbf{C}\mathbf{r}_i = \lambda_i^2\mathbf{r}_i$ (no sum on i) and using the fact that $\mathbf{r}_i$ is a unit vector that

$$\frac{\partial \lambda_i}{\partial \mathbf{C}} = \frac{1}{2\lambda_i}\mathbf{r}_i \otimes \mathbf{r}_i \qquad \text{(no sum on } i\text{)}; \tag{4.46}$$

see Problem 2.25. Substituting (4.46) into (4.45) and simplifying using $\mathbf{F}\mathbf{r}_i = \lambda_i \boldsymbol{\ell}_i$ leads to

$$\mathbf{T} = \sum_{k=1}^{3} \tau_k\, \boldsymbol{\ell}_k \otimes \boldsymbol{\ell}_k \qquad \text{where} \quad \tau_i = \frac{\lambda_i}{J}\frac{\partial W^*}{\partial \lambda_i} \quad \text{(no sum on } i\text{)}. \tag{4.47}$$

It follows from (4.41) and (4.47) that the constitutive relation for the principal Cauchy stresses is

$$\tau_1 = \frac{\lambda_1}{\lambda_1\lambda_2\lambda_3}\,\frac{\partial W^*}{\partial \lambda_1}, \qquad \tau_2 = \frac{\lambda_2}{\lambda_1\lambda_2\lambda_3}\,\frac{\partial W^*}{\partial \lambda_2}, \qquad \tau_3 = \frac{\lambda_3}{\lambda_1\lambda_2\lambda_3}\,\frac{\partial W^*}{\partial \lambda_3}. \tag{4.48}$$

Thus given $\mathbf{F}$, one can find the principal values of $\mathbf{T}$ from (4.48) and the principal directions of $\mathbf{T}$ by finding the principal directions of $\mathbf{B} = \mathbf{F}\mathbf{F}^T$:

– Since the Piola stress tensor $\mathbf{S}$ is not symmetric in general, it may not have principal values. However $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-\mathrm{T}}$, $\mathbf{F}^{-1} = \lambda_i^{-1}\mathbf{r}_i \otimes \boldsymbol{\ell}_i$ and (4.47) enable us to write the constitutive relation for $\mathbf{S}$ as

$$\mathbf{S} = \sum_{k=1}^{3} \sigma_k\, \boldsymbol{\ell}_k \otimes \mathbf{r}_k \qquad \text{where} \quad \sigma_i = \frac{\partial W^*}{\partial \lambda_i}; \tag{4.49}$$

here $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ and $\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3$ are the eigenvectors of the Lagrangian and Eulerian stretch tensors $\mathbf{U}$ and $\mathbf{V}$ respectively.

We will sometimes consider particular deformations where $\mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3$. In this case the rotation tensor $\mathbf{R} = \mathbf{I}$ and so the bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ and $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ coincide and

$$\mathbf{S} = \sigma_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_3 \mathbf{e}_3 \otimes \mathbf{e}_3 \qquad \text{where } \sigma_i = \frac{\partial W^*}{\partial \lambda_i}.$$

When we consider spherically symmetric problems for isotropic materials in Chapter 5, we will find that the bases $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ coincide; and that they are the principal bases for $\mathbf{B}$ and $\mathbf{T}$; and that $[F]$ and $[S]$ are diagonal in these bases.

– In Problem 4.15 you are asked to show for the Biot stress tensor introduced in (3.82) that

$$\mathbf{S}^{(1)} = \sum_{k=1}^{3} \sigma_k \, \mathbf{r}_k \otimes \mathbf{r}_k \qquad \text{where} \quad \sigma_i = \frac{\partial W^*}{\partial \lambda_i} = \frac{\tau_i J}{\lambda_i} \quad \text{(no sum on } i). \tag{4.50}$$

Equation $(4.50)_1$ is a consequence of $\mathbf{S}^{(1)}$ being work conjugate to $\mathbf{U}$ (as established in Problem 3.32.)

Perhaps it is worth remarking that if one is to conduct laboratory experiments to find W^*, it is necessary to carry out experiments that probe various paths of $\lambda_1, \lambda_2, \lambda_3$-space. Carrying out, for example a uniaxial tension test alone, would only probe a single path in this space; see Problem 4.7.

Problem 4.4.2. Blatz and Ko proposed the following strain energy function for the foam rubber material they studied in their experiments:

$$\widetilde{W}(I_1, I_2, I_3) = \frac{\mu}{2}\left(\frac{I_2}{I_3} + 2\sqrt{I_3} - 5\right); \tag{i}$$

here $\mu > 0$ is a material constant. See page 394 for a reference to their paper. Determine the response of this material in uniaxial stress and simple shear. What are the values of the Young's modulus, Poisson's ratio and shear modulus of this material at infinitesimal deformations?

See also Problem 4.7.1 concerning the bending of a block made of a Blatz-Ko material.

Solution: In terms of the principal stretches, we find from (4.42) and (i) that

$$W^*(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu}{2}\big(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} + 2\lambda_1\lambda_2\lambda_3 - 5\big). \tag{ii}$$

The constitutive equation for $\mathbf{T}$ is given by (i) and $(4.38)_1$ to be

$$\mathbf{T} = \frac{\mu}{J^3}\Big[(J^3 - I_2)\mathbf{I} + I_1\mathbf{B} - \mathbf{B}^2\Big], \tag{iii}$$

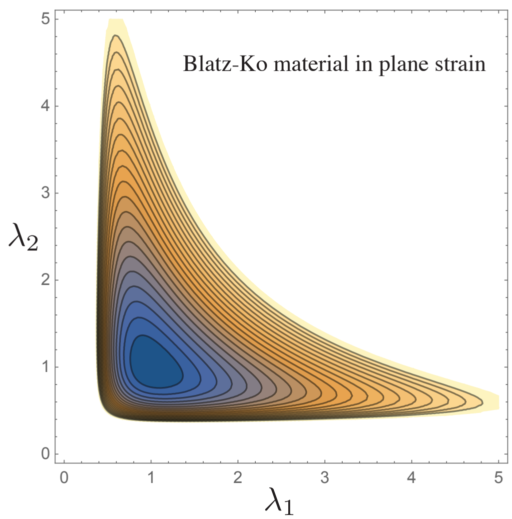

Figure 4.4: Constant energy contours $W^*(\lambda_1, \lambda_2, 1)$ for the Blatz-Ko material in plane strain ($\lambda_3 = 1$). The reference configuration corresponds to the local minimum at $(\lambda_1, \lambda_2) = (1, 1)$.

and from (4.48) and (ii) the principal stresses can be written in terms of the principal stretches as

$$\tau_k = \frac{\lambda_k}{J}\frac{\partial W^*}{\partial \lambda_k} = \mu\Big[1 - \lambda_k^{-2}/J\Big], \qquad J = \lambda_1\lambda_2\lambda_3. \tag{iv}$$

We will find below that the form (iv) of the constitutive relation is more convenient to use when examining uniaxial stress, while it will be more natural to use the form (iii) when considering simple shear.

Consider a state of uniaxial stress in the $\mathbf{e}_1$-direction:

$$\tau_1 = \tau, \qquad \tau_2 = \tau_3 = 0, \tag{v}$$

and assume the deformation to be a homogeneous pure stretch $\mathbf{y} = \mathbf{F}\mathbf{x}$ with

$$\mathbf{F} = \lambda \mathbf{e}_1 \otimes \mathbf{e}_1 + \Lambda(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3). \tag{vi}$$

Our aim is to calculate the longitudinal stretch λ (in the direction of the applied stress) and the transverse stretch Λ.

The principal stretches are $\lambda_1 = \lambda, \lambda_2 = \lambda_3 = \Lambda$ and the Jacobian determinant is

$$J = \lambda_1\lambda_2\lambda_3 = \lambda\Lambda^2. \tag{vii}$$

From (iv) and $\tau_2 = 0$ we get

$$1 - \lambda_2^{-2}/J = 0 \quad \overset{(vii)}{\Rightarrow} \quad 1 - \Lambda^{-2}/(\lambda\Lambda^2) = 0 \qquad \Rightarrow \quad \Lambda = \lambda^{-1/4}. \qquad \square \tag{$viii$}$$

Likewise from (iv) with $k = 1$ we find

$$\tau = \mu\Big[1 - \lambda_1^{-2}/J\Big] \overset{(vii)}{=} \mu\Big[1 - \lambda^{-2}/(\lambda\Lambda^2)\Big] \overset{(viii)}{=} \mu(1 - \lambda^{-5/2}). \qquad \square \tag{ix}$$

This relation between τ and λ is monotonic with $\tau \to -\infty$ as $\lambda \to 0^+$, $\tau = 0$ when $\lambda = 1$, and $\tau \to \mu$ as $\lambda \to \infty$; it is sketched in Figure 4.5.

Observe since $J = \lambda\Lambda^2 \stackrel{(viii)}{=} \lambda^{1/2}$ that $\lambda_1 = J^2$ and $\lambda_2 = J^{-1/2}$. Therefore from (ii) it follows that (in uniaxial stress) $W = \frac{\mu}{2}(J^{-4} + 2J^{-1} + J - 5)$. We see from this that $W \to \infty$ when both $J \to 0^+$ and $J \to \infty$, i.e. at extreme deformations.

For infinitesimal deformations we write the principal stretches as

$$\lambda_1 = \lambda = 1 + \varepsilon_1, \qquad \lambda_2 = \Lambda = 1 + \varepsilon_2, \tag{x}$$

where the principal strains ε_1 and ε_2 are small: $\varepsilon_1 \ll 1, \varepsilon_2 \ll 1$. Substituting $(x)_1$ into (ix) and linearizing gives

$$\tau = \mu\Big(1 - (1+\varepsilon_1)^{-5/2}\Big) \doteq \mu\left(1 - \left(1 - \frac{5}{2}\varepsilon_1\right)\right) = \frac{5}{2}\mu\varepsilon_1 \qquad \Rightarrow \qquad \tau/\varepsilon_1 = \frac{5}{2}\mu, \qquad \square \tag{xi}$$

where we have used the Taylor expansion $(1+\varepsilon)^m = 1 + m\varepsilon + O(\varepsilon^2)$ as $\varepsilon \to 0$. Therefore the Young's modulus of this material is $5\mu/2$. Similarly substituting (x) into $(viii)$ and linearizing gives

$$1 + \varepsilon_2 = (1+\varepsilon_1)^{-1/4} \doteq 1 - \frac{1}{4}\varepsilon_1 \qquad \Rightarrow \qquad -\varepsilon_2/\varepsilon_1 = 1/4, \qquad \square \tag{xii}$$

and so the Poisson's ratio of this material is 0.25.

Alternatively we could have differentiated (ix) with respect to λ and used the fact that the Young's modulus equals $d\tau/d\lambda$ evaluated at $\lambda = 1$. Likewise, the Poisson's ratio is $-d\Lambda/d\lambda$ evaluated at $\lambda = 1$ which can be calculated from $(viii)$.

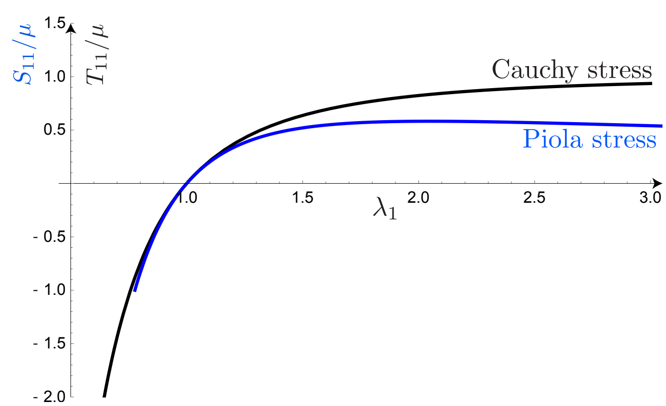

Figure 4.5: Uniaxial stress: Cauchy stress T_{11} (black) and Piola stress S_{11} versus stretch λ_1.

We can find the Piola stress by using $\mathbf{T} = J^{-1}\mathbf{S}\mathbf{F}^T$. However it is easier (and more insightful) to use the following calculation: suppose that the cross-section of the undeformed specimen (normal to the stressing direction) is 1×1. The cross-section of the deformed specimen is then $\Lambda \times \Lambda$. Therefore the force acting on the cross-section can be written as $S_{11} \times 1$ and equivalently as $T_{11} \times \Lambda^2$. Therefore

$$S_{11} = T_{11}\Lambda^2 = \tau\Lambda^2 \stackrel{(viii),(ix)}{=} \mu(\lambda^{-1/2} - \lambda^{-3}). \tag{$xiii$}$$

Observe that the relation between S_{11} and λ is not monotonic. As λ increase, the stress S_{11} first increases, then reaches a maximum value at $\lambda = 6^{2/5}$ and then decreases with $S_{11} \to 0$ as $\lambda \to \infty$; it is sketched in Figure 4.5.

Now consider a simple shear deformation with shearing direction $\mathbf{e}_1$ and glide plane normal $\mathbf{e}_2$:

$$y_1 = x_1 + kx_2, \qquad y_2 = x_2, \qquad y_3 = x_3. \tag{xiv}$$

The deformation gradient tensor, left Cauchy-Green deformation tensor and its square are

$$\begin{aligned}
&\mathbf{F} = \mathbf{I} + k\,\mathbf{e}_1 \otimes \mathbf{e}_2,\\
&\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{I} + k^2\,\mathbf{e}_1 \otimes \mathbf{e}_1 + k(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1),\\
&\mathbf{B}^2 = \mathbf{I} + (3k^2 + k^4)\,\mathbf{e}_1 \otimes \mathbf{e}_1 + k^2\,\mathbf{e}_2 \otimes \mathbf{e}_2 + (2k + k^3)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1),
\end{aligned} \tag{xv}$$

and so the principal scalar invariants of $\mathbf{B}$ are

$$I_1 = \operatorname{tr}\mathbf{B} = 3 + k^2, \qquad I_2 = \frac{1}{2}\big[(\operatorname{tr}\mathbf{B})^2 - \operatorname{tr}(\mathbf{B}^2)\big] = 3 + k^2 \qquad I_3 = \det\mathbf{B} = 1. \tag{xvi}$$

Substituting (*xv*) and (*xvi*) into the constitutive relation (*iii*) and simplifying gives

$$T_{12} = \mu k, \qquad T_{22} = -\mu k^2, \qquad T_{11} = T_{33} = T_{13} = T_{23} = 0.$$

Observe that the relation between the shear stress T_{12} and the amount of shear k is linear for all deformations. The shear modulus is μ. Observe also that the normal stress $T_{22} \neq 0$ (in contrast to the linear theory for infinitesimal deformations). For small k this term is $O(k^2)$ and so is an order of magnitude smaller than the shear stress. For large k however this is no longer true.

4.5 Materials with Internal Constraints.

Thus far in this chapter we have assumed that the body under consideration can undergo *any* deformation at all (provided it is subjected to suitable body forces and surface tractions). Sometimes it is convenient, and not a bad approximation, to idealize the body such that it is permitted to *only* undergo motions of a certain restricted class. For example a *rigid body* can only undergo rigid motions, i.e. motions in which

$$\mathbf{F}^T(\mathbf{x}, t)\mathbf{F}(\mathbf{x}, t) = \mathbf{I} \qquad \text{for all } \mathbf{x} \text{ and } t;$$

an *incompressible body* can only undergo isochoric (volume-preserving) motions, i.e. motions in which

$$\det\mathbf{F}(\mathbf{x}, t) = 1 \qquad \text{for all } \mathbf{x} \text{ and } t;$$

a body that is inextensible in a certain (referential) direction $\mathbf{m}_\mathrm{R}$ can only undergo motions in which

$$|\mathbf{F}(\mathbf{x},t)\mathbf{m}_\mathrm{R}| = 1 \qquad \text{for all } \mathbf{x} \text{ and } t.$$

All of these idealizations constrain the set of possible deformation gradient tensors. Note that this is part of modeling the material's constitutive behavior.

These constraints (and several others) can be described by equations of the form

$$\widehat{\phi}(\mathbf{F}(\mathbf{x},t)) = 0 \qquad \text{for all } \mathbf{x} \text{ and } t. \tag{4.51}$$

The aforementioned constraints of rigidity, incompressibility and inextensibility correspond to

$$\widehat{\phi}(\mathbf{F}) = \mathbf{F}^T\mathbf{F} - \mathbf{I}, \qquad \widehat{\phi}(\mathbf{F}) = \det\mathbf{F} - 1, \qquad \widehat{\phi}(\mathbf{F}) = \mathbf{F}\mathbf{m}_\mathrm{R}\cdot\mathbf{F}\mathbf{m}_\mathrm{R} - 1, \tag{4.52}$$

respectively.

We now turn to the *stress* in a constrained body. In order to explain the basic idea, consider as an example a spherical body composed of an incompressible isotropic material. It is subjected to a uniform radial pressure p on its boundary. Since the geometry, the material and the loading are all spherically symmetric, let us restrict attention to spherically symmetric deformations. Thus the body must remain spherical when p is applied. However, due to incompressibility, the radius of the spherical body, and in fact the radius of every spherical surface within the body, cannot change. Therefore: (a) irrespective of the value of p, incompressibility implies that the deformation must be the trivial one, $\widehat{\mathbf{y}}(\mathbf{x},t) = \mathbf{x}$, so that $\mathbf{F}(\mathbf{x},t) = \mathbf{I}$ for all p. (b) The stress on the other hand would certainly depend on the value of the applied pressure p and will change as the value of p changes. (c) Thus the stress $\mathbf{T}$ does not vanish (and can have different values depending on the value of p) though $\mathbf{F} = \mathbf{I}$ (for all values of p). (d) Observe that the pressure does no work since the boundary does not displace and so the energy stored in the body does not increase as p increases.

Observation (c) implies that the stress is *not* completely determined by the deformation gradient $\mathbf{F}$, or equivalently, *different stress fields can correspond to the same deformation.* This contradicts our earlier assumption (4.1), (4.2) that the stress is completely determined by the deformation gradient. We must therefore modify this assumption when considering a constrained body. We choose to do this by allowing a part of the stress to be determined by the deformation and the other to be indeterminate (as far as the constitutive relation is concerned). Motivated by observation (d) we determine the additional part of the stress by requiring it to do no work in all deformations conforming to the constraint.

Mathematically, we now assume that

$$\mathbf{S} = \widehat{\mathbf{S}}(\mathbf{F}) + \mathbf{N}, \tag{4.53}$$

where $\widehat{\mathbf{S}}(\mathbf{F})$ is the constitutively determined part of $\mathbf{S}$, and $\mathbf{N}$ is the part that arises as a reaction to the constraint. We further assume that the *reactive stress* $\mathbf{N}$ does no work in the sense that

$$\mathbf{N} \cdot \dot{\mathbf{F}} = 0. \tag{4.54}$$

The strain energy function on the other hand continues to have the form

$$W = \widehat{W}(\mathbf{F}),$$

and the rate of working of the stress is related to the rate of increase of stored energy by

$$\mathbf{S} \cdot \dot{\mathbf{F}} = \dot{W}.$$

Our immediate goal is to determine the form of $\mathbf{N}$. If (4.54) held for all $\dot{\mathbf{F}}$ then we would have $\mathbf{N} = \mathbf{0}$. However it only holds in all allowable motions (i.e. all motions consistent with the constraint). To determine the restriction placed on $\dot{\mathbf{F}}$ by the constraint (4.51), we note that (4.51) holds at all points $\mathbf{x}$ in the body and at all times t. Differentiating it with respect to t gives $\dot{\phi} = 0$ whence

$$\frac{\partial \widehat{\phi}}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} = 0, \tag{4.55}$$

where $\partial \widehat{\phi}/\partial \mathbf{F}$ is the tensor with cartesian components $\partial \widehat{\phi}/\partial F_{ij}$. Thus (4.54) must hold for all $\dot{\mathbf{F}}$ that conform to (4.55).

Problem 1.52 describes the following algebraic result: if[12] $\mathbf{A}_1 \cdot \mathbf{X} = 0$ for all tensors $\mathbf{X}$ for which $\mathbf{A}_2 \cdot \mathbf{X} = 0$, then there is a scalar q such that $\mathbf{A}_2 = -q\mathbf{A}_1$. Using this result with $\mathbf{X} = \dot{\mathbf{F}}, \mathbf{A}_2 = \mathbf{N}$ and $\mathbf{A}_1 = \partial \widehat{\phi}/\partial \mathbf{F}$, leads to[13]

$$\mathbf{N} = -q \, \frac{\partial \widehat{\phi}}{\partial \mathbf{F}}. \tag{4.56}$$

The stress field $\mathbf{N}(\mathbf{x})$ that arises in reaction to the constraint is referred to as the **reactive (or reaction) stress**. It should be noted that $q(\mathbf{x})$ is a scalar *field* and in general is not a constant.

[12] i.e. if $\mathbf{A}_1$ is orthogonal to all tensors $\mathbf{X}$ that are orthogonal to $\mathbf{A}_2$,

[13] Many authors use the symbol p instead of q. We shall reserve p to denote the applied pressure on a body, e.g. in an inflated tube.

The constitutive equation for the Piola stress $\mathbf{S}$ for a constrained material is therefore taken to be

$$\mathbf{S} = \frac{\partial \widehat{W}}{\partial \mathbf{F}} - q\,\frac{\partial \widehat{\phi}}{\partial \mathbf{F}}. \tag{4.57}$$

The corresponding relation for the Cauchy stress tensor $\mathbf{T}$ is

$$\mathbf{T} = \frac{1}{J}\,\frac{\partial \widehat{W}}{\partial \mathbf{F}}\,\mathbf{F}^T - \frac{q}{J}\,\frac{\partial \widehat{\phi}}{\partial \mathbf{F}}\,\mathbf{F}^T. \tag{4.58}$$

Note that the theory now involves an additional scalar field $q(\mathbf{x})$, but we also have an additional scalar field equation $\widehat{\phi}(\mathbf{F}(\mathbf{x})) = 0$ at our disposal. Observe also that (4.57) can be written as

$$\mathbf{S} = \frac{\partial}{\partial \mathbf{F}}\left(\widehat{W} - q\,\widehat{\phi}\right), \tag{4.59}$$

from which we see that we have, essentially, added the constraint ϕ to W using a Lagrange multiplier q.

As an example, consider an *incompressible body* in which case

$$\widehat{\phi}(\mathbf{F}) = \det \mathbf{F} - 1.$$

Differentiating this with respect to $\mathbf{F}$ using the formula (1.208) from Problem 1.47 gives

$$\frac{\partial \widehat{\phi}}{\partial \mathbf{F}} = (\det \mathbf{F})\,\mathbf{F}^{-T} \overset{J=1}{=} \mathbf{F}^{-T}. \tag{4.60}$$

Thus (4.57), (4.58) and (4.60) yield

$$\mathbf{S} = \frac{\partial \widehat{W}}{\partial \mathbf{F}} - q\,\mathbf{F}^{-T}, \qquad \mathbf{T} = \frac{\partial \widehat{W}}{\partial \mathbf{F}}\,\mathbf{F}^T - q\,\mathbf{I}. \tag{4.61}$$

Observe that the part of the Cauchy stress arising in reaction to the constraint is hydrostatic.

Exercise: In Problem 4.22 you are asked to derive the constitutive relations for $\mathbf{S}$ and $\mathbf{T}$ in the presence of the inextensibility constraint $|\mathbf{F}\mathbf{m}_{\mathrm{R}}| = 1$ and to physically interpret the reactive part of the Cauchy stress.

Frame indifference: It is natural to require (4.51) to be frame indifferent. That is, if a deformation gradient tensor $\mathbf{F}$ obeys the constraint (4.51), a subsequent rigid rotation should not lead to a violation of the constraint. This requires $\phi(\mathbf{F}) = \phi(\mathbf{QF})$ for all nonsingular tensors $\mathbf{F}$ and all rotations $\mathbf{Q}$. The earlier discussion in Section 4.3 can be readily adapted to the present context to show that (4.51) is frame indifferent if and only if the constraint can be expressed in the form

$$\phi(\mathbf{U}) = 0 \tag{4.62}$$

where $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$. Exercise: Verify that the constraints (4.52) can be written in this way.

Problem 4.5.1. Show that the reactive stress to be added to the Biot stress is $q\,\partial\phi/\partial\mathbf{U}$.

Material symmetry: Some care must be taken when analyzing material symmetry. Previously we said that a proper orthogonal tensor $\mathbf{Q}$ was in the material symmetry group if $W(\mathbf{F}) = W(\mathbf{FQ})$ for all nonsingular $\mathbf{F}$. Here however we must limit attention to those deformation gradient tensors that obey the constraint $\phi(\mathbf{F}) = 0$. Thus a symmetry transformation $\mathbf{Q}$ must be such that *both* $W(\mathbf{F}) = W(\mathbf{FQ})$ *and* $\phi(\mathbf{F}) = \phi(\mathbf{FQ}) = 0$ hold. Material symmetry must therefore be compatible with the constraint.

First, consider an incompressible material where $\phi(\mathbf{F}) = \det\mathbf{F} - 1$. Since $\phi(\mathbf{FQ}) = \det(\mathbf{FQ}) - 1 = \det\mathbf{F} - 1 = \phi(\mathbf{F})$ for all proper orthogonal $\mathbf{Q}$ we see that the constraint imposes no further restrictions on material symmetry. Thus for example an isotropic incompressible material is simply one that obeys $W(\mathbf{QCQ}^T) = W(\mathbf{C})$ for all orthogonal $\mathbf{Q}$.

Second, consider a material that is inextensible in the direction $\mathbf{m}_{\mathrm{R}}$ so that $\phi(\mathbf{F}) = \mathbf{Fm}_{\mathrm{R}} \cdot \mathbf{Fm}_{\mathrm{R}} - 1$. If $\mathbf{Q}$ is a rotation about $\mathbf{m}_{\mathrm{R}}$ then $\mathbf{Qm}_{\mathrm{R}} = \mathbf{m}_{\mathrm{R}}$ and so $\phi(\mathbf{FQ}) = \mathbf{FQm}_{\mathrm{R}} \cdot \mathbf{FQm}_{\mathrm{R}} - 1 = \mathbf{Fm}_{\mathrm{R}} \cdot \mathbf{Fm}_{\mathrm{R}} - 1 = \phi(\mathbf{F})$ for such a $\mathbf{Q}$. Thus an inextensible material would be transversely isotropic with respect to the direction $\mathbf{m}_{\mathrm{R}}$ if $W(\mathbf{QCQ}^T) = W(\mathbf{C})$ for all rotations about $\mathbf{m}_{\mathrm{R}}$. (In fact, we will see in Chapter 6 that $I_4 = \mathbf{Fm}_{\mathrm{R}} \cdot \mathbf{Fm}_{\mathrm{R}} = \mathbf{Cm}_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}}$ is one of the invariants for such a material; see also page 395.) On the other hand an isotropic material cannot be inextensible since $\phi(\mathbf{FQ}) \neq \phi(\mathbf{F})$ for *all* rotations $\mathbf{Q}$.

The reader is referred to Section 6.3 of Steigmann for more details.

In the case of an isotropic incompressible material the analysis proceeds as for an unconstrained material. In particular, one finds that the strain energy function $\widehat{W}(\mathbf{F})$ depends on the deformation only through the principal scalar invariants of $\mathbf{C}$. However, since $I_3(\mathbf{C}) = \det\, \mathbf{C} = (\det\, \mathbf{F})^2 = 1$ due to incompressibility, there are only 2 nontrivial invariants and the energy takes the form $\widetilde{W}(I_1, I_2)$. The stress tensors $\mathbf{T}$ and $\mathbf{S}$ are now found to be related to the deformation through

$$\mathbf{T} = -q\,\mathbf{I} \;+\; 2\left[\frac{\partial\widetilde{W}}{\partial I_1} + I_1\frac{\partial\widetilde{W}}{\partial I_2}\right]\mathbf{B} \;-\; 2\frac{\partial\widetilde{W}}{\partial I_2}\mathbf{B}^2, \tag{4.63}$$

$$\mathbf{S} = -q\,\mathbf{F}^{-T} \;+\; 2\left[\frac{\partial\widetilde{W}}{\partial I_1} + I_1\frac{\partial\widetilde{W}}{\partial I_2}\right]\mathbf{F} \;-\; 2\frac{\partial\widetilde{W}}{\partial I_2}\mathbf{B}\mathbf{F}. \tag{4.64}$$

If the strain energy function is expressed in terms of the principal stretches,

$$W = W^*(\lambda_1, \lambda_2, \lambda_3), \qquad \lambda_1\lambda_2\lambda_3 = 1, \tag{4.65}$$

with W^* being invariant if any two of its arguments are switched,

$$W^*(\lambda_1, \lambda_2, \lambda_3) = W^*(\lambda_2, \lambda_1, \lambda_3) = W^*(\lambda_1, \lambda_3, \lambda_2) = \dots\ , \tag{4.66}$$

then the principal Cauchy stress components can be written as

$$\tau_i = \lambda_i\,\frac{\partial W^*}{\partial\lambda_i} - q, \qquad i = 1, 2, 3, \qquad (\text{no sum on } i). \tag{4.67}$$

A strain energy function $W(\mathbf{F})$ for an incompressible material is only defined on the set of all nonsingular tensors with $\det\mathbf{F} = 1$. In view of the constraint $\det\mathbf{F} = 1$, one has to explain what one means by the term $\partial W/\partial\mathbf{F}$ that enters into the constitutive equation for stress. The usual approach is to consider the following function $W^o(\mathbf{F})$ defined on the set of *all* nonsingular tensors with positive determinant:

$$W^o(\mathbf{F}) = W\left(\frac{\mathbf{F}}{(\det\mathbf{F})^{1/3}}\right). \tag{4.68}$$

Observe that the tensor argument of W on the right-hand side of (4.68) has determinant one even when the determinant of $\mathbf{F}$ is not unity. Moreover, note that $W(\mathbf{F}) = W^o(\mathbf{F})$ on the subset of tensors with $\det\mathbf{F} = 1$. Thus $W^o(\mathbf{F})$ is an extension of $W(\mathbf{F})$ to the larger set of all nonsingular tensors with positive determinant. Then by $\partial W/\partial\mathbf{F}$ we mean $\partial W^o/\partial\mathbf{F}$.

Exercise: The function $W^*(\lambda_1, \lambda_2, \lambda_3)$ is defined for all positive λs subject to the constraint $\lambda_1\lambda_2\lambda_3 = 1$. Extend W^* to the set of all positive λs (not subjected to the constraint) and use this to define the partial derivative $\partial W^*/\partial\lambda_i$.

4.6 Response of Isotropic Elastic Materials.

In this section we examine the response of an isotropic elastic material in *uniaxial tension, simple shear and plane stress biaxial stretch*. We keep the strain energy function, $W(I_1, I_2)$ or $W(I_1, I_2, I_3)$ as the case may be, general, and so the results hold for any isotropic elastic material. In Section 4.7 we consider some particular constitutive relations, but before doing so, in Section 4.6.3, we will make some brief remarks on restrictions one might impose on the strain energy function for physical and mathematical reasons.

All of the deformations we consider in this section are homogeneous in that the deformation gradient tensor is uniform throughout the body. Therefore[14] the stress field $\mathbf{S}(\mathbf{x})$ (resp. $\mathbf{T}(\mathbf{y})$) is also uniform and does not depend on $\mathbf{x}$ (resp. $\mathbf{y}$). The equilibrium equation without body forces, $\mathrm{Div}\,\mathbf{S} = \mathbf{o}$ (resp. $\mathrm{div}\,\mathbf{T} = \mathbf{o}$) therefore holds automatically.

It is convenient to record again the constitutive relations. For an (unconstrained) isotropic elastic material we have

$$\left.\begin{aligned} \mathbf{T} &= 2JW_3\,\mathbf{I} + \frac{2}{J}\left[W_1 + I_1W_2\right]\mathbf{B} - \frac{2}{J}W_2\mathbf{B}^2, \\ \mathbf{S} &= 2I_3W_3\,\mathbf{F}^{-T} + 2\left[W_1 + I_1W_2\right]\mathbf{F} - 2W_2\mathbf{B}\mathbf{F}, \end{aligned}\right\} \tag{4.69}$$

and for an incompressible isotropic elastic material we have

$$\begin{aligned} \mathbf{T} &= -q\,\mathbf{I} + 2\left[W_1 + I_1W_2\right]\mathbf{B} - 2W_2\mathbf{B}^2, \\ \mathbf{S} &= -q\,\mathbf{F}^{-T} + 2\left[W_1 + I_1W_2\right]\mathbf{F} - 2W_2\mathbf{B}\mathbf{F}, \end{aligned} \tag{4.70}$$

where we have set

$$W_\alpha = \frac{\partial W}{\partial I_\alpha}, \qquad \alpha = 1, 2. \tag{4.71}$$

Terminology: An isotropic material whose constitutive behavior is described by (4.69) is frequently referred to as a compressible material. However a compressible material (i.e. one that is not incompressible) may involve some other constraint, for example an inextensibility constraint, in which case its constitutive relation would not be (4.69) even though it is compressible. In order to avoid any confusion we shall speak of (4.69) as describing an **unconstrained** isotropic elastic material (rather than a compressible material).

[14] In the case of an incompressible material it is assumed that $q(\mathbf{x})$ is constant.

4.6.1 Incompressible isotropic materials.

Uniaxial stress.

Consider a state of uniaxial stress in the $\mathbf{e}_1$-direction. The Cauchy stress tensor is

$$\mathbf{T} = \tau\, \mathbf{e}_1 \otimes \mathbf{e}_1, \tag{4.72}$$

and we assume the deformation to be a homogeneous pure stretch $\mathbf{y} = \mathbf{F}\mathbf{x}$ with

$$\mathbf{F} = \lambda \mathbf{e}_1 \otimes \mathbf{e}_1 + \Lambda(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3). \tag{4.73}$$

The constant parameter λ is the longitudinal stretch (in the direction of T_{11}) and Λ is the transverse stretch (in the direction perpendicular to T_{11}). We have assumed[15] that $\lambda_2 = \lambda_3$. The principal stretches are

$$\lambda_1 = \lambda, \qquad \lambda_2 = \lambda_3 = \Lambda, \tag{4.74}$$

and the tensors $\mathbf{B}$ and $\mathbf{B}^2$ are

$$\mathbf{B} = \lambda_1^2\, \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2^2\, \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3^2\, \mathbf{e}_3 \otimes \mathbf{e}_3, \qquad \mathbf{B}^2 = \lambda_1^4\, \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2^4\, \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3^4\, \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{4.75}$$

Our aim is to calculate the normal stress τ and the transverse stretch Λ in terms of the longitudinal stretch λ.

Incompressibility requires $\lambda_1\lambda_2\lambda_3 = 1$ which on using (4.74) reads $\lambda\Lambda^2 = 1$. Therefore the transverse stretch Λ and longitudinal stretch λ are related by

$$\Lambda = \lambda^{-1/2}. \tag{4.76}$$

The principal scalar invariants associated with this deformation are

$$\left.\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \overset{(4.74)}{=} \lambda^2 + 2\Lambda^2 \overset{(4.76)}{=} \lambda^2 + 2\lambda^{-1}, \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \overset{(4.74)}{=} 2\lambda^2\Lambda^2 + \Lambda^4 \overset{(4.76)}{=} 2\lambda + \lambda^{-2}. \end{aligned}\right\} \tag{4.77}$$

We now turn to the constitutive relation $(4.70)_1$ for stress, keeping in mind that it involves the reaction pressure q. We first determine q by making use of the fact that $T_{22} = T_{33} = 0$; thereafter we calculate T_{11}. Substituting (4.75) into $(4.70)_1$ gives

$$T_{22} = -q + 2(W_1 + I_1 W_2)\lambda_2^2 - 2W_2\lambda_2^4 = 0,$$

[15] Re-examine this analysis without making the a priori assumption $\lambda_3 = \lambda_2$. See Problem 4.30.

which can be solved for q:

$$q = 2\lambda^{-1}W_1 + 2(\lambda + \lambda^{-2})W_2, \tag{4.78}$$

where we have made use of (4.74), (4.76) and (4.77) to eliminate I_1 and λ_2 in favor of λ. The normal stress $T_{11} = \tau$ is now found by substituting (4.75), (4.77) and (4.78) into $(4.70)_1$, which after simplification leads to

$$\tau = T_{11} = -q + 2(W_1 + I_1W_2)\lambda^2 - 2W_2\lambda^4 = 2(W_1 + \lambda^{-1}W_2)(\lambda^2 - \lambda^{-1}). \tag{4.79}$$

This describes the *stress-stretch response in uniaxial stress*. In both (4.78) and (4.79) the derivatives W_1 and W_2 of the strain energy function are evaluated at the values of the invariants given by (4.77), i.e. at $I_1 = \lambda^2 + 2\lambda^{-1}$, $I_2 = 2\lambda + \lambda^{-2}$.

We can now calculate the components of the Piola stress tensor by using the formula $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$. However, just as in Problem 4.4.2, it is illuminating to do so by physical reasoning instead. Suppose the cross-section of the body normal to the axis of stressing has dimensions 1×1 in the reference configuration. In the deformed configuration its dimensions are $\lambda_2 \times \lambda_3$. Thus the areas of this cross-section in the undeformed and deformed configurations are 1 and $\lambda_2\lambda_3$ respectively. Therefore the axial force on this cross-section can be written in the equivalent forms $S_{11} \times 1$ and $T_{11} \times \lambda_2\lambda_3$. Thus $S_{11} = T_{11}\lambda_2\lambda_3 = \tau\Lambda^2$:

$$\sigma = S_{11} = 2(W_1 + \lambda^{-1}W_2)(\lambda - \lambda^{-2}), \tag{4.80}$$

with all other stress components S_{ij} being zero.

Let $w(\lambda)$ be the restriction of the strain energy function W to uniaxial stress:

$$w(\lambda) := W(I_1, I_2)\Big|_{I_1=\lambda^2+2\lambda^{-1},\ I_2=2\lambda+\lambda^{-2}}, \qquad \lambda > 0. \tag{4.81}$$

It is readily seen by differentiating (4.81) with respect to λ and using (4.80) that

$$\sigma = w'(\lambda). \tag{4.82}$$

This is in fact a consequence of $\mathbf{S} \cdot \dot{\mathbf{F}} = \dot{W}$ which when specialized to the present setting reads $S_{11}\dot{\lambda} = \dot{w} = w'(\lambda)\,\dot{\lambda}$ thus resulting in (4.82).

Finally we linearize these results for an infinitesimal deformation. The normal strain components ε_1 and ε_2 of the infinitesimal strain tensor $\boldsymbol{\epsilon}$ are related to the stretches by

$$\lambda = \lambda_1 = 1 + \varepsilon_1, \qquad \Lambda = \lambda_2 = 1 + \varepsilon_2. \tag{4.83}$$

First consider the relation between the transverse stretch Λ and the axial stretch λ. Substituting (4.83) into (4.76) and approximating the result for small ε_1 gives

$$1+\varepsilon_2=(1+\varepsilon_1)^{-1/2}\approx 1-\frac{1}{2}\varepsilon_1+\ldots \qquad \Rightarrow \qquad -\frac{\varepsilon_2}{\varepsilon_1}\ \doteq\ \frac{1}{2}.$$

This shows that for all incompressible isotropic elastic materials the Poisson's ratio ν at infinitesimal deformations is

$$\nu=\frac{1}{2}.$$

Next we linearize the stress-stretch relation (4.80) (or equivalently (4.79)). First consider the term $\lambda-\lambda^{-2}$. It can be linearized as follows:

$$\lambda-\lambda^{-2}=(1+\varepsilon_1)-(1+\varepsilon_1)^{-2}\approx(1+\varepsilon_1)-(1-2\varepsilon_1+\ldots)=3\varepsilon_1.$$

Since this term is order $\mathrm{O}(\varepsilon)$, and it multiplies the remaining terms on the right hand side of (4.80), we need only approximate those other terms to $\mathrm{O}(1)$. So we set $\lambda=1$ and write the remaining term as $2(W_1+W_2)$ keeping in mind that the W_i's are now evaluated at $I_1=I_2=3$ (which is what we get by setting $\lambda=1$ in (4.77)). We are thus led to the linear stress-strain relation

$$\sigma\doteq 6(W_1+W_2)\Big|_{I_1=I_2=3}\varepsilon_1.$$

The Young's modulus of any incompressible isotropic elastic material at infinitesimal deformations is therefore

$$E:=6(W_1+W_2)\Big|_{I_1=I_2=3}. \tag{4.84}$$

Alternatively the Poisson's ratio ν and Young's modulus E can be calculated directly from their definitions, i.e. by differentiating (4.76) and (4.80) and evaluating the results in the undeformed state:

$$\nu:=-\left.\frac{d\Lambda}{d\lambda}\right|_{\lambda=1}, \qquad E:=\left.\frac{d\sigma}{d\lambda}\right|_{\lambda=1}.$$

Simple shear.

Now consider a simple shear deformation of a unit cube with shearing direction $\mathbf{e}_1$ and glide plane normal $\mathbf{e}_2$:

$$y_1=x_1+kx_2, \qquad y_2=x_2, \qquad y_3=x_3. \tag{4.85}$$

One finds that $\det \mathbf{F} = 1$ and so the deformation is automatically volume preserving. The left Cauchy-Green deformation tensor and its square are

$$\begin{aligned} &\mathbf{B} = \mathbf{I} + k^2\, \mathbf{e}_1 \otimes \mathbf{e}_1 + k(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \\ &\mathbf{B}^2 = \mathbf{I} + (3k^2 + k^4)\, \mathbf{e}_1 \otimes \mathbf{e}_1 + k^2\, \mathbf{e}_2 \otimes \mathbf{e}_2 + (2k + k^3)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), . \end{aligned} \tag{4.86}$$

and so the principal scalar invariants of $\mathbf{B}$ are

$$I_1 = \operatorname{tr} \mathbf{B} = 3 + k^2, \qquad I_2 = \frac{1}{2}\big[(\operatorname{tr} \mathbf{B})^2 - \operatorname{tr}(\mathbf{B}^2)\big] = 3 + k^2. \tag{4.87}$$

We now turn to the constitutive relation $(4.70)_1$. Substituting (4.86) and (4.87) into $(4.70)_1$ gives the shear stress T_{12} to be

$$T_{12} = 2k(W_1 + W_2)\Big|_{I_1 = 3+k^2, I_2 = 3+k^2}. \tag{4.88}$$

This is the relation between the shear stress T_{12} and the amount of shear k. The remaining shear stress components are readily seen to vanish $T_{23} = T_{31} = 0$.

Turning to the normal components of stress, we first note that they involve the reaction pressure q which we cannot determine unless we know the stress on one pair of faces of the cube. Suppose that the faces perpendicular to the $\mathbf{e}_3$-direction are traction-free: $T_{33} = 0$. Substituting (4.86) into $(4.70)_1$ gives T_{33} and setting it equal to zero yields

$$T_{33} = -q + 2(W_1 + I_1 W_2) - 2W_2 = 0.$$

This can be solved for q, which after using (4.87), can be written as

$$q = 2W_1 + 2W_2(2 + k^2). \tag{4.89}$$

The normal stress components T_{11} and T_{22} can now be found from $(4.70)_1$, (4.86) , (4.87) and (4.89) to be

$$T_{11} = 2k^2 W_1, \qquad T_{22} = -2k^2 W_2. \tag{4.90}$$

Observe from this that in general, *normal stresses are needed in order to maintain a simple shear deformation*. This is a feature of finite deformations and is in contrast to the linearized theory where the shear stress T_{12} is the only nonzero stress. This is not unexpected since in Section 2.6.1 we noticed that the normal strain component E_{22} of the Green Saint-Venant strain did not vanish in simple shear. The existence of non-zero normal stresses in simple shear is sometimes called the *Poynting effect*.

It is interesting to observe from (4.88) and (4.90) that

$$T_{11} - T_{22} = kT_{12},$$

which is a relation between the stress components that does not involve W. It therefore holds for all incompressible isotropic materials and is called a *universal relation.*

Problem 4.6.1. Show that the preceding universal relation holds even if we did not take $T_{33} = 0$.

Next we define the restriction of the strain energy function W to simple shear by

$$w(k) \coloneqq W(I_1, I_2)\Big|_{I_1=3+k^2,\ I_2=3+k^2}. \tag{4.91}$$

Note that w here is different to the function w introduced in (4.81). Differentiating (4.91) with respect to k and using (4.88) yields

$$T_{12} = w'(k).$$

Again, this is a consequence of $\mathbf{S} \cdot \dot{\mathbf{F}} = \dot{W}$ specialized to the present setting.

Finally consider an infinitesimal deformation (4.85) with $|k| \ll 1$. Linearizing (4.88) for small $|k|$ gives

$$T_{12} = 2k(W_1 + W_2)\Big|_{I_1=I_2=3} + \mathrm{O}(k^2).$$

Thus we have the linear stress-strain relation $T_{12} = \mu k$ where

$$\mu \coloneqq 2(W_1 + W_2)\Big|_{I_1=I_2=3} \tag{4.92}$$

is the *shear modulus* of linear elasticity. Observe from (4.90) that for infinitesimal amounts of shear

$$T_{11} = \mathrm{O}(k^2), \qquad T_{22} = \mathrm{O}(k^2),$$

which is why these stress components are neglected in the linearized theory. Note from (4.84) and (4.92) that for any incompressible isotropic elastic material the Young's modulus and shear modulus are related by $E = 3\mu$.

Biaxial stretch in plane stress.

Rivlin and Saunders [20] carried out biaxial stress-stretch experiments on thin sheets of rubber and so we turn to such states of deformation and stress next. Consider a thin square sheet of dimension $1 \times 1 \times h$ in the reference configuration. The coordinate axes are aligned with the edges of the sheet with the x_3-axis being perpendicular to the square faces. The sheet is subjected to a pure stretch

$$y_1 = \lambda_1 x_1, \qquad y_2 = \lambda_2 x_2, \qquad y_3 = \lambda_3 x_3, \tag{4.93}$$

with the top and bottom faces of the sheet being traction-free:

$$T_{31} = T_{32} = T_{33} = 0 \qquad \text{for } x_3 = \pm h/2.$$

In view of incompressibility,

$$\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}, \tag{4.94}$$

and so the principal scalar invariants take the form

$$\left.\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \overset{(4.94)}{=} \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}, \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \overset{(4.94)}{=} \lambda_1^2\lambda_2^2 + \lambda_1^{-2} + \lambda_2^{-2}. \end{aligned}\right\} \tag{4.95}$$

The left Cauchy-Green tensor and its square associated with the deformation (4.93) and (4.94) are

$$\begin{aligned} \mathbf{B} &= \lambda_1^2 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2^2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_1^{-2}\lambda_2^{-2} \mathbf{e}_3 \otimes \mathbf{e}_3, \\ \mathbf{B}^2 &= \lambda_1^4 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2^4 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_1^{-4}\lambda_2^{-4} \mathbf{e}_3 \otimes \mathbf{e}_3. \end{aligned} \tag{4.96}$$

It follows from $(4.70)_1$ and (4.96) that the shear stress components vanish throughout the sheet and so the traction-free boundary condition on the shear stress components, $T_{31} = T_{32} = 0$ for $x_3 = \pm h/2$, holds automatically. Calculating T_{33} from $(4.70)_1$, (4.95) and (4.96) and setting the result equal to zero allows one to solve for the reaction pressure q. This leads to

$$q = 2\lambda_1^{-2}\lambda_2^{-2}\, W_1 \; - 2\lambda_1^2\lambda_2^2\, W_2\,. \tag{4.97}$$

The in-plane normal stress components are now found from $(4.70)_1$, (4.95) and (4.96), together with (4.97), to be

$$T_{11} = 2(\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2})(W_1 + \lambda_2^2 W_2), \qquad T_{22} = 2(\lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2})(W_1 + \lambda_1^2 W_2). \tag{4.98}$$

The corresponding Piola stresses are readily found by noting that the dimensions of the sheet in the deformed configuration are $\lambda_1, \lambda_2, h\lambda_3$. Thus the force on a face perpendicular to the x_1-axis can be written as $T_{11} \times \mathsf{deformed\ area} = T_{11}\lambda_2 h\lambda_3$ and equivalently as $S_{11} \times \mathsf{undeformed\ area} = S_{11}h$. Likewise the force on a face perpendicular to the x_2-axis is $T_{22}\lambda_1 h\lambda_3$ and equivalently $S_{22}h$. On equating these one has $T_{11}\lambda_2 h\lambda_3 = S_{11}h$ and $T_{22}\lambda_1 h\lambda_3 = S_{22}h$ whence $S_{11} = T_{11}\lambda_1^{-1}$ and $S_{22} = T_{22}\lambda_2^{-1}$. Solving (4.98) for W_1 and W_2 and using $S_{11} = T_{11}\lambda_1^{-1}$ and $S_{22} = T_{22}\lambda_2^{-1}$ leads to

$$\begin{aligned} W_1 &= \frac{1}{2(\lambda_1^2 - \lambda_2^2)}\left[\frac{\lambda_1^3 S_{11}}{\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2}} - \frac{\lambda_2^3 S_{22}}{\lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2}}\right], \\ W_2 &= \frac{1}{2(\lambda_2^2 - \lambda_1^2)}\left[\frac{\lambda_1 S_{11}}{\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2}} - \frac{\lambda_2 S_{22}}{\lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2}}\right]. \end{aligned} \tag{4.99}$$

The energy $w(\lambda_1, \lambda_2)$ that is the restriction of $W(I_1, I_2)$ to the class of deformations at hand is

$$w(\lambda_1, \lambda_2) := \widetilde{W}(I_1, I_2)\Big|_{I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2},\ I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^{-2} + \lambda_2^{-2}}.$$

One can readily verify that

$$S_{11} = \frac{\partial w}{\partial \lambda_1}, \qquad S_{22} = \frac{\partial w}{\partial \lambda_2}. \tag{4.100}$$

Rivlin and Saunders [20] carried out biaxial stress-stretch experiments on thin rubber sheets. They varied the stretches λ_1 and λ_2, keeping I_1 fixed and allowing I_2 to vary, where I_1 and I_2 are given by (4.95). The experiments were repeated for different fixed values of I_1. They measured the values of S_{11} and S_{22} and then used (4.99) to determine W_1 and W_2. They also carried out experiments in which I_2 was fixed and I_1 was varied. In this way they determined $W_1(I_1, I_2)$ and $W_2(I_1, I_2)$ along various straight lines $I_1 = \mathsf{constant}$ and $I_2 = \mathsf{constant}$ in the I_1, I_2-plane. This information was then used to determine the strain energy function $W(I_1, I_2)$ describing the particular material they were testing.

4.6.2 Unconstrained isotropic materials.

Uniaxial stress.

Consider a state of uniaxial stress in the $\mathbf{e}_1$-direction. The Cauchy stress tensor is again given by (4.72), the deformation by (4.73), the principal stretches by (4.74), and the tensors $\mathbf{B}$ and $\mathbf{B}^2$ by (4.75). Our aim is to calculate the normal stress T and transverse stretch Λ in terms of the longitudinal stretch λ.

In the case of an incompressible material we used the incompressibility constraint to determine Λ. Here we will use the condition $T_{22} = 0$ to determine Λ. In the incompressible case the equation $T_{22} = 0$ was used to determine the reaction pressure q which is absent for an unconstrained material.

The principal scalar invariants associated with the deformation at hand are

$$\left.\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \overset{(4.74)}{=} \lambda^2 + 2\Lambda^2, \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \overset{(4.74)}{=} 2\lambda^2\Lambda^2 + \Lambda^4, \\ I_3 &= J^2 = \lambda_1^2\lambda_2^2\lambda_3^2 \overset{(4.74)}{=} \lambda^2\Lambda^4. \end{aligned}\right\} \tag{4.101}$$

We can now calculate the stress from the constitutive relation $(4.69)_1$. For T_{11} we get

$$T_{11} = 2\lambda\Lambda^2 W_3 + \frac{2}{\lambda\Lambda^2}\big(W_1 + (\lambda^2 + 2\Lambda^2)W_2\big)\lambda^2 - \frac{2}{\lambda\Lambda^2}W_2\lambda^4, \tag{4.102}$$

where the derivatives of W appearing in (4.102) are evaluated at the values of the invariants given in (4.101). This is an equation of the form $T_{11} = T_{11}(\lambda, \Lambda)$. Similarly we calculate T_{22}, and then set the result to zero:

$$T_{22} = 2\lambda\Lambda^2 W_3 + \frac{2}{\lambda\Lambda^2}\big(W_1 + (\lambda^2 + 2\Lambda^2)W_2\big)\Lambda^2 - \frac{2}{\lambda\Lambda^2}W_2\Lambda^4 = 0. \tag{4.103}$$

Equation (4.103) is a nonlinear algebraic equation of the form $f(\lambda, \Lambda) = 0$. If (in principle) (4.103) can be solved for Λ, one has a relation of the form $\Lambda = \Lambda(\lambda)$ for finding the transverse stretch in terms of the longitudinal stretch. This is now substituted into (4.102) to obtain the stress T_{11} as a function of the stretch λ.

Remark: If we had carried out this analysis using the representation $W^*(\lambda_1, \lambda_2, \lambda_3)$ of the strain energy function together with the associated constitutive relation (4.48), the transverse stretch and normal stress are given by the respective equations

$$\left.\frac{\partial W^*}{\partial \lambda_2}\right|_{\lambda_1=\lambda, \lambda_2=\lambda_3=\Lambda} = 0, \qquad T_{11} = \frac{1}{\Lambda^2}\left.\frac{\partial W^*}{\partial \lambda_1}\right|_{\lambda_1=\lambda, \lambda_2=\lambda_3=\Lambda}.$$

Simple shear.

The tensors $\mathbf{B}$ and $\mathbf{B}^2$ again have the components given in (4.86) and the principal scalar invariants are

$$I_1 = 3 + k^2, \qquad I_2 = 3 + k^2, \qquad I_3 = 1. \tag{4.104}$$

Substituting these into $(4.69)_1$ and simplifying leads to

$$T_{12} = 2k(W_1 + W_2)\Big|_{I_1=3+k^2,\ I_2=3+k^2,\ I_3=1} \tag{4.105}$$

where we have kept in mind that the strain energy function here is a function of all three invariants: $W = W(I_1, I_2, I_3)$. Again, the normal components of stress are nonzero in general.

4.6.3 Restrictions on the strain energy function.

A nonlinearly elastic material is characterized by its strain energy function $W(\mathbf{F})$. Typically, W is determined by using (micro-mechanical reasoning to motivate) some functional form that is then fitted to experimental measurements. Since some level of judgement is used in coming up with the functional form, and since experimental data is necessarily limited to a certain finite (even if large) number of tests, it is important to make sure that a proposed constitutive model is not fundamentally flawed in some way.

There are two types of restrictions that one might consider imposing on $W(\mathbf{F})$ in order to address this issue. The first ensures that the response predicted by the constitutive model is "physically reasonable". For example in simple shear, the shear stress τ and amount of shear k are related by $\tau = 2k(\widetilde{W}_1 + \widetilde{W}_2)$. This by itself does not ensure, for example, that a positive shear stress $\tau > 0$ is needed to deform the body by a positive amount of shear $k > 0$. That would require $\widetilde{W}$ to obey the inequality

$$\left(\frac{\partial \widetilde{W}}{\partial I_1} + \frac{\partial \widetilde{W}}{\partial I_2}\right)\Bigg|_{I_1=I_2=3+k^2, I_3=1} > 0 \qquad \text{for all } k.$$

Similarly in uniaxial stress, equations (4.102) and (4.103) do not ensure that a tensile stress ($T_{11} > 0$) is required to elongate the body ($\lambda_1 > 1$). This would require $\widetilde{W}$ to obey a second inequality. One must of course be careful in deciding what is "reasonable". For example in some situations one might require the stress-stretch relation in uniaxial stress to be monotonically increasing for all stretches. There are however certain situations where this is not the case and the stress-stretch relation is monotonically increasing for only certain ranges of stretch. We shall encounter such a problem in Section 5.6.

The second type of restriction stems from mathematical considerations. For example, without certain requirements on W it is possible that boundary-value problems may have no solution. Or the solutions that exist maybe unstable. Again, one must be careful in what one imposes. Consider for example the strong ellipticity condition described below that is related to a certain notion of stability. When one models phase transformations in solids, it is known that the constitutive model has to violate this condition at certain deformation gradients (but not others). This does not mean that the strong ellipticity condition should be abandoned entirely; only that it not be required at *all* deformation gradients. It is also worth mentioning that one does not always expect uniqueness of solutions to boundary value problems in the nonlinear theory. If we did, we would not be able to study instabilities such as buckling. Thus uniqueness is not something one would insist on in general.

In this sub-section we list a few restrictions on the strain energy function W that have been suggested in the literature. Checking whether the specific strain energy functions presented in the next section obey these conditions is left as an exercise. See also Problems 4.6, 4.29, 4.6.2 and 4.6.3 .

Baker-Ericksen inequalities. Consider an *isotropic* material subjected to a pure homogeneous deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ where $\mathbf{F} = \lambda_1\mathbf{e}_1\otimes\mathbf{e}_1 + \lambda_2\mathbf{e}_2\otimes\mathbf{e}_2 + \lambda_3\mathbf{e}_3\otimes\mathbf{e}_3$. Let the accompanying Cauchy stress be $\mathbf{T} = \tau_1\mathbf{e}_1\otimes\mathbf{e}_1 + \tau_2\mathbf{e}_2\otimes\mathbf{e}_2 + \tau_3\mathbf{e}_3\otimes\mathbf{e}_3$. Baker and Ericksen [3] suggested that if the principal stretch λ_i is larger than the principal stretch λ_j, then one would expect the principal Cauchy stress τ_i to be larger than the principal Cauchy stress τ_j, i.e. that $\lambda_i > \lambda_j$ implies $\tau_i > \tau_j$. This requires

$$(\tau_i - \tau_j)(\lambda_i - \lambda_j) > 0 \qquad \text{whenever} \quad \lambda_i \neq \lambda_j, \tag{4.106}$$

which, by using the constitutive relation $\tau_i = \lambda_i J^{-1}\partial W/\partial\lambda_i$ can be written as

$$\frac{\lambda_i\,\partial W/\partial\lambda_i - \lambda_j\,\partial W/\partial\lambda_j}{\lambda_i - \lambda_j} > 0 \qquad \text{provided} \quad \lambda_i \neq \lambda_j; \tag{4.107}$$

the usual rules of indicial notion have been suspended in (4.107). Problem 4.30 illustrates one consequence of this, namely that in a uniaxial tensile stress state $\mathbf{T} = \tau\mathbf{e}_1\otimes\mathbf{e}_1$ with $\tau > 0$, the Baker-Ericksen inequalities hold if and only if the principal stretches obey $\lambda_1 > \lambda_2 = \lambda_3 > 0$.

Problem 4.6.2. Determine the restrictions imposed by the Baker-Ericksen inequalities on an (incompressible) Mooney-Rivlin material,

$$W = \frac{\mu}{2}\Big[\alpha(I_1 - 3) + (1-\alpha)(I_2 - 3)\Big],$$

μ and α being material parameters.

Solution: We first write the Mooney-Rivlin strain energy function in the equivalent form

$$W = \frac{\mu}{2}\Big[\alpha(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + (1-\alpha)(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3)\Big], \tag{i}$$

and then use the constitutive equation (4.67) for the principal Cauchy stresses to get

$$\tau_1 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - q = \mu\big[\alpha\lambda_1^2 - (1-\alpha)\lambda_1^{-2}\big] - q, \tag{ii}$$

$$\tau_2 = \lambda_2 \frac{\partial W}{\partial \lambda_2} - q = \mu\big[\alpha\lambda_2^2 - (1-\alpha)\lambda_2^{-2}\big] - q. \tag{iii}$$

Subtracting (iii) from (ii) gives

$$\tau_1 - \tau_2 = \mu\big[\alpha(\lambda_1^2 - \lambda_2^2) - (1-\alpha)(\lambda_1^{-2} - \lambda_2^{-2}\big] = \mu(\lambda_1^2 - \lambda_2^2)\big[\alpha + (1-\alpha)\lambda_3^2\big]$$

where we used $\lambda_1\lambda_2\lambda_3 = 1$ in getting to the second equality. Therefore

$$\frac{\tau_1 - \tau_2}{\lambda_1 - \lambda_2} = \mu(\lambda_1 + \lambda_2)\big[\alpha + (1-\alpha)\lambda_3^2\big], \qquad \lambda_1 \neq \lambda_2.$$

The Baker-Ericksen inequalities require $(\tau_1 - \tau_2)/(\lambda_1 - \lambda_2) > 0$ provided $\lambda_1 \neq \lambda_2$ which in the present case reduces to

$$\mu\alpha + \mu(1-\alpha)\lambda_3^2 > 0 \tag{iv}$$

since the principal stretches are positive. Equation (iv) must hold for all $\lambda_3 > 0$.

First suppose that $\alpha = 1$. In this particular case, (iv) holds if and only if $\mu > 0$:

$$\mu > 0, \qquad \alpha = 1. \tag{v}$$

Next suppose that $\alpha \neq 1$. For (iv) to hold when $\lambda_3 \to \infty$ we must have

$$\mu(1-\alpha) > 0 \qquad \Leftrightarrow \qquad \mu > \mu\alpha, \tag{vi}$$

whereas for it to hold when $\lambda_3 \to 0$ we must have

$$\mu\alpha > 0. \tag{vii}$$

Equations (vi) and (vii) together require

$$\mu > \mu\alpha > 0.$$

Therefore we must have $\mu > 0$. With this in hand, dividing by μ gives $1 > \alpha > 0$. Thus it is necessary that

$$\mu > 0, \qquad 0 < \alpha < 1. \tag{$viii$}$$

Combining $(viii)$ with (v)

$$\mu > 0, \qquad 0 < \alpha \leq 1. \qquad \square \tag{ix}$$

Conversely when the inequalities (ix) hold so does (iv). Thus the Baker-Ericksen inequalities hold if and only if (ix) holds.

Monotonicity: Consider again a pure homogeneous deformation of an isotropic material. Let $\mathbf{S} = \sigma_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_3\mathbf{e}_3 \otimes \mathbf{e}_3$ be the associated Piola stress. If one requires each stress component σ_i (which is effectively the ith component of force) to be an increasing function of the corresponding stretch λ_i, it then follows from the constitutive relation $\sigma_i = \partial W/\partial\lambda_i$ that one must have

$$\frac{\partial^2 W}{\partial \lambda_i^2} > 0. \tag{4.108}$$

This describes a certain type of convexity of $W(\lambda_1, \lambda_2, \lambda_3)$.

Convexity. Convexity plays a central role in proving the existence of solutions (to minimization problems in the Calculus of Variations). There are various notions of convexity such as quasiconvexity, polyconvexity and rank-one convexity, but this is a topic that is beyond the scope of these notes. The interested reader is referred to, for example, Antman [2], Ball [5] Marsden and Hughes [15] and Steigmann [21]. It is shown in some of these references that the usual notion of convexity of $W(\mathbf{F})$ leads to various difficulties including incompatibility with material frame indifference, preclusion of buckling and the violation of the growth condition $W \to \infty$ as $\det \mathbf{F} \to 0$.

Consider the special case of a homogeneous deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ of a homogeneous elastic body whose entire boundary is subjected to a prescribed Piola traction ("dead loading"). If $\mathbf{F}$ is a local minimizer of $W(\mathbf{F})$ one must have

$$\mathbb{A}_{ijk\ell}(\mathbf{A})\, H_{ij} H_{k\ell} > 0 \tag{4.109}$$

for all tensors $\mathbf{H} \neq \mathbf{0}, |\mathbf{H}| \ll 1$, where

$$\mathbb{A}_{ijk\ell}(\mathbf{F}) := \frac{\partial^2 W}{\partial F_{ij} \partial F_{k\ell}}(\mathbf{F}), \tag{4.110}$$

are the components of the **elasticity tensor** $\mathbb{A}$. If one limits attention to (a) isotropic materials, (b) symmetric deformation gradient tensors $\mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3$, and ($c$) perturbations $\mathbf{H}$ that are symmetric and coaxial with $\mathbf{F}$ (i.e. $\mathbf{H}$ that have the same principal directions as $\mathbf{F}$), the inequality (4.109) leads to the requirement that the Hessian matrix with elements

$$\frac{\partial^2 W^*}{\partial \lambda_i \partial \lambda_j} \qquad \text{be positive definite.} \tag{4.111}$$

We shall work out the details of this in the context of a particular problem in Section 5.3. Additional inequality restrictions on the $\partial W/\partial\lambda_i$'s are needed when the perturbation $\mathbf{H}$ is not coaxial with $\mathbf{F}$. For these, as well as the modification of (4.111) for incompressible materials, see Ogden [18].

Strong ellipticity (material stability): The basic question underlying strong ellipticity (material stability) is: when one perturbs a homogeneous equilibrium configuration, do the perturbations remain bounded at all times? To illustrate this issue, consider a one-dimensional setting. The deformation $y(x,t)$, stretch $\lambda(x,t)$, particle speed $v(x,t)$ and stress $\sigma(x,t)$ obey the equations

$$\lambda = y_x, \qquad v = y_t, \qquad \sigma = W_\lambda, \qquad \sigma_x = \rho_{\mathrm{R}} v_t,$$

where the subscripts x, t and λ denote partial differentiation, and the mass density ρ_{R} in the reference configuration is positive. Consider a homogeneous equilibrium configuration

$$y(x) = \lambda_0 x, \qquad \lambda(x) = \lambda_0, \qquad v(x) = 0, \qquad \sigma(x) = \sigma_0 = W_\lambda(\lambda_0),$$

where $\lambda_0 > 0$ is a constant. Now consider an infinitesimal perturbation $u(x,t)$ of this equilibrium deformation: $y(x,t) = \lambda_0 x + u(x,t)$ where $|u_x| \ll 1$. The associated stretch, particle speed and stress are

$$\lambda = y_x = \lambda_0 + u_x, \quad v = y_t = u_t, \quad \sigma = W_\lambda(\lambda_0 + u_x) \doteq W_\lambda(\lambda_0) + W_{\lambda\lambda}(\lambda_0) u_x = \sigma_0 + W_{\lambda\lambda}(\lambda_0) u_x.$$

Substituting these expressions into the equation of motion $\sigma_x = \rho_{\mathrm{R}} v_t$ gives

$$\alpha\, u_{xx} = \rho_{\mathrm{R}} u_{tt} \quad \text{where} \quad \alpha := W_{\lambda\lambda}(\lambda_0).$$

In order to study the behavior of the perturbation $u(x,t)$ we must examine the solutions of this linear partial differential equation. Consider solutions of the form $u(x,t) = a \exp ik(x - ct)$ which describe propagating harmonic waves with wave number k and propagation speed c. Observe that if c is imaginary, say $c = \pm i\gamma$, the temporal term reads $\exp(\pm k\gamma t)$ and becomes unbounded as $t \to \infty$. Thus we shall say that material stability prevails when c is real and nonzero. Substituting $u(x,t) = a \exp ik(x - ct)$ into $\alpha u_{xx} = \rho_{\mathrm{R}} u_{tt}$ gives

$$\rho_{\mathrm{R}} c^2 = \alpha,$$

whence c is real and nonzero when $\alpha > 0$. Therefore we say that W is strongly elliptic *at the stretch* λ_0 when

$$W_{\lambda\lambda}(\lambda_0) > 0.$$

Observe that when $W_{\lambda\lambda}(\lambda_0) > 0$ the partial differential equation $\alpha u_{xx} = \rho_{\mathrm{R}} u_{tt}$ is hyperbolic; the energy $W(\lambda)$ is convex at $\lambda = \lambda_0$; and the slope $W_\lambda(\lambda)$ of the stress-stretch curve is positive at $\lambda = \lambda_0$. If the inequality $W_{\lambda\lambda}(\lambda_0) > 0$ holds for all $\lambda_0 > 0$ one says that the *material* is strongly elliptic.

We now turn to the three-dimensional setting. Consider a pure homogeneous equilibrium deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ of a homogenous but not necessarily isotropic elastic body, and superpose on it an infinitesimal perturbation $\mathbf{u}(\mathbf{x},t)$. Thus we consider a motion $\mathbf{y} = \mathbf{F}\mathbf{x} + \mathbf{u}(\mathbf{x},t)$

where $|\nabla \mathbf{u}| \ll 1$. On linearizing the equations of motion $\text{Div}\,(\partial W/\partial \mathbf{F}) = \rho_R \ddot{\mathbf{y}}$ we arrive at the following system of linear partial differential equations

$$\mathbb{A}_{pqrs}\frac{\partial^2 u_r}{\partial x_q \partial x_s} = \rho_{\mathrm{R}} \ddot{u}_p \qquad \text{where} \quad \mathbb{A}_{pqrs}(\mathbf{F}) := \frac{\partial^2 W(\mathbf{F})}{\partial F_{pq} \partial F_{rs}}. \tag{4.112}$$

Observe that $\mathbb{A}_{pqrs} = \mathbb{A}_{rspq}$. Suppose the perturbed motion $\mathbf{u}(\mathbf{x}, t)$ describes a plane harmonic wave propagating in the direction $\mathbf{n}$, with wave speed c, wave number k and particle motion in the direction $\mathbf{a}$. Then $\mathbf{u}(\mathbf{x}, t) = \mathbf{a}\, \exp[ik(\mathbf{x}\cdot\mathbf{n} - ct)]$. If the wave speed is imaginary (or complex), the perturbation $\mathbf{u}$ involves a term that grows exponentially with time t and so the homogeneous deformation would be unstable. Thus we say that **material stability**[16] holds provided the wave speed c is real and nonzero (for all $\mathbf{a}$ and $\mathbf{n}$). Substituting $\mathbf{u}(\mathbf{x}, t) = \mathbf{a}\, \exp[ik(\mathbf{x}\cdot\mathbf{n} - ct)]$ into (4.112) leads to

$$\mathbb{A}_{pqrs} a_r n_q n_s = \rho_R c^2 a_p.$$

We can write this as

$$\big[\mathbf{A}(\mathbf{n}) - \rho_R c^2\, \mathbf{I}\big]\mathbf{a} = \mathbf{o} \tag{4.113}$$

in terms of the **acoustic tensor** $\mathbf{A}(\mathbf{n})$ whose components are defined by

$$A_{pr} := \mathbb{A}_{pqrs} n_q n_s. \tag{4.114}$$

Since $\mathbb{A}_{pqrs} = \mathbb{A}_{rspq}$ it follows that $\mathbf{A}$ is symmetric. From (4.113) we see that $\rho_R c^2$ are the eigenvalues of $\mathbf{A}$ and so for c to be real and nonzero the acoustic tensor must be positive definite. This requires that the elasticity tensor $\mathbb{A}(\mathbf{F})$ obey the inequality

$$\mathbb{A}_{pqrs}(\mathbf{F}) a_p n_q a_r n_s > 0 \qquad \text{for all unit vectors } \mathbf{a} \text{ and } \mathbf{n}. \tag{4.115}$$

When $W(\mathbf{F})$ satisfies (4.115) at some $\mathbf{F}$, we say that W is **strongly elliptic**[17] at that $\mathbf{F}$. Equation (4.115) is equivalent to the acoustic tensor $\mathbf{A}(\mathbf{n})$ being positive definite for all directions $\mathbf{n}$.

Observe that the inequality (4.115) can be written equivalently as

$$\frac{d^2}{d\xi^2} W(\mathbf{F} + \xi\, \mathbf{a}\otimes\mathbf{n})\Big|_{\xi=0} > 0 \qquad \text{for all unit vectors } \mathbf{a} \text{ and } \mathbf{n}. \tag{4.116}$$

[16] One refers to this as *material stability* because no boundary or initial conditions are involved in this notion (in contrast to other notions of stability).

[17] Knowles and Sternberg [14] have shown that the strain energy function W that describes a material capable of undergoing solid-solid phase transitions cannot be strongly elliptic at *all* $\mathbf{F}$, though it would be strongly elliptic at certain $\mathbf{F}$ in general.

Viewed in this way, strong ellipticity can be thought of as requiring W to be strictly convex at $\mathbf{F}$ along certain paths in deformation gradient tensor space. If the strict inequality in (4.115) (or (4.116)) is replaced by $\geq$, the resulting inequality is the **Legendre-Hadamard condition** of the Calculus of Variations which in turn is equivalent to the so-called **rank-one convexity**[18] condition (given that we have implicitly assumed W to be twice continuously differentiable on the set of all tensors with positive determinant); see Ball [5].

Problem 4.6.3. Consider a "compressible neo-Hookean" material[19]

$$W(I_1, I_2, I_3) = \frac{\mu}{2}(I_1 - 3) + h(J), \qquad J = \sqrt{I_3}. \tag{i}$$

Determine the conditions under which strong ellipticity prevails at a given deformation.

Solution: Differentiating $I_1 = \operatorname{tr} \mathbf{F}^T \mathbf{F}$ gives

$$\frac{\partial I_1}{\partial F_{pq}} = 2F_{pq}, \qquad \frac{\partial^2 I_1}{\partial F_{pq} \partial F_{rs}} = 2\delta_{pr}\delta_{qs}, \tag{iv}$$

and using (1.206) and (1.207) (page 104) to calculate $\partial J/\partial F_{ij}$ and $\partial F^{-1}_{ij}/\partial F_{k\ell}$ leads to

$$\frac{\partial J}{\partial F_{pq}} = J F^{-1}_{qp}, \qquad \frac{\partial^2 J}{\partial F_{pq}\partial F_{rs}} = J F^{-1}_{sr} F^{-1}_{qp} - J F^{-1}_{qr} F^{-1}_{sp}. \tag{v}$$

We may now differentiate (i) to get

$$\frac{\partial W}{\partial F_{pq}} = \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial F_{pq}} + \frac{\partial W}{\partial J}\frac{\partial J}{\partial F_{pq}} \stackrel{(i)}{=} \frac{\mu}{2}\frac{\partial I_1}{\partial F_{pq}} + h'(J)\frac{\partial J}{\partial F_{pq}} = \mu F_{pq} + J h'(J) F^{-1}_{qp}, \tag{vi}$$

$$\mathbb{A}_{pqrs} = \frac{\partial^2 W}{\partial F_{pq}\partial F_{rs}} = \mu \delta_{pr}\delta_{qs} + \big(J h'(J)\big)' J F^{-1}_{sr} F^{-1}_{qp} - J h'(J) F^{-1}_{qr} F^{-1}_{sp}. \tag{vii}$$

The components of the acoustic tensor $\mathbf{A}$ are defined by (4.114). For the particular material at hand we find from (vii) that

$$\mathbf{A} = \mu \mathbf{I} + J^2 h''(J)\,(\mathbf{F}^{-T}\mathbf{n} \otimes \mathbf{F}^{-T}\mathbf{n}). \tag{$viii$}$$

The eigenvalues of $\mathbf{A}$ are

$$\mu, \quad \mu, \quad \mu + J^2 h''(J)\,(\mathbf{F}^{-T}\mathbf{n} \cdot \mathbf{F}^{-T}\mathbf{n}) = \mu + J^2 h''(J) \mathbf{C}^{-1}\mathbf{n} \cdot \mathbf{n}. \tag{ix}$$

Strong ellipticity requires $\mathbf{A}$ to be positive definite and therefore the three eigenvalues of $\mathbf{A}$ to be positive for all unit vectors $\mathbf{n}$. Taking $\mathbf{n}$ to be in turn the three principal directions of the Lagrangian stretch tells us that necessary for the eigenvalues to be positive is

$$\mu > 0, \qquad \mu + \lambda_i^{-2} J^2 h''(J) > 0 \quad \text{for } i = 1, 2, 3. \qquad \square \tag{x}$$

---

[18] Note that $\mathbf{a} \otimes \mathbf{n}$ is a rank one tensor.

[19] This is a special case of the Hadamard material given in Problem 4.33.

Conversely when (x) holds it is readily seen that the eigenvalues (ix) are positive for all unit vectors $\mathbf{n}$ and so (x) is necessary and sufficient for strong ellipticity at a particular deformation. Exercise: If the strain energy function (i) is to be strongly elliptic *at all deformations* what restriction does this impose on h?

Problem 4.6.4. Show for an isotropic material that strong ellipticity implies both the Baker-Ericksen inequalities (4.106) and the monotonicity inequality (4.108). (Problem 4.31)

For an incompressible material, according to Problem 4.28, the strong ellipticity condition is

$$\mathbb{A}_{ijk\ell}a_ib_ja_kb_\ell > 0 \qquad \text{for all unit vectors } \mathbf{a} \text{ and } \mathbf{b} \text{ with } \mathbf{a}\cdot\mathbf{F}^{-T}\mathbf{b} = 0, \tag{4.117}$$

where the restriction $\mathbf{a}\cdot\mathbf{F}^{-T}\mathbf{b} = 0$ arises from the requirement that the motions be isochoric. This can be written equivalently as

$$\mathbb{B}_{ijk\ell}a_in_ja_kn_\ell > 0 \qquad \text{for all unit vectors } \mathbf{a} \text{ and } \mathbf{n} \text{ with } \mathbf{a}\cdot\mathbf{n} = 0 \tag{4.118}$$

where

$$\mathbb{B}_{abcd}(\mathbf{F}) = \mathbb{A}_{apcq}(\mathbf{F})F_{bp}F_{dq}. \tag{4.119}$$

For an incompressible isotropic material, the components of $\mathbb{B}$ in the principal basis $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ are given in Problem 6.1.8 of Ogden [17] where his tensor $\mathcal{A}_0^1$ is related to our tensor $\mathbb{B}$ by $\mathcal{A}^1_{0ijk\ell} = \mathbb{B}_{ji\ell k}$ (see his equations (6.1.14), (6.1.29) and keep in mind that his tensor $\mathbf{S}$ is the transpose of the Piola stress tensor). With the summation convention suspended,

$$\mathbb{B}_{iijj} = \lambda_i\lambda_j\frac{\partial^2 W}{\partial\lambda_i\partial\lambda_j}, \tag{4.120}$$

$$\mathbb{B}_{jiji} = \frac{\lambda_i^2\left(\lambda_i\frac{\partial W}{\partial\lambda_i} - \lambda_j\frac{\partial W}{\partial\lambda_j}\right)}{\lambda_i^2 - \lambda_j^2}, \qquad i \neq j,\ \lambda_i \neq \lambda_j, \tag{4.121}$$

$$\mathbb{B}_{jiij} = \mathbb{B}_{ijji} = \mathbb{B}_{jiji} - \lambda_i\frac{\partial W}{\partial\lambda_i}, \qquad i \neq j. \tag{4.122}$$

If $\lambda_i = \lambda_j$, (4.121) is replaced by

$$\mathbb{B}_{jiji} = \frac{1}{2}\left(\mathbb{B}_{iiii} - \mathbb{B}_{iijj} + \lambda_i\frac{\partial W}{\partial\lambda_i}\right), \qquad i \neq j,\ \lambda_i = \lambda_j. \tag{4.123}$$

Growth conditions. A deformation is "extreme" if (at least) one of the principal stretches tends to 0 or ∞. It is reasonable to require (for a compressible material) that the strain energy density associated with an extreme deformation be infinite:

$$W(\mathbf{F}) \to \infty \qquad \text{when } \det\mathbf{F} \to 0 \text{ or } \infty. \tag{4.124}$$

The reader is referred to Section 2, Chapter XIII of Antman [2] for a detailed discussion of growth conditions and their implications. The way in which W grows at extreme deformations plays an important role when considering the existence of certain types of discontinuous deformation fields such as those associated with fracture or cavitation. In particular, Ball [4] shows that the growth condition

$$\frac{W(\mathbf{F})}{|\mathbf{F}|^n} \to \infty \qquad \text{as} \quad |\mathbf{F}| \to \infty \tag{4.125}$$

precludes the possibility of cavitation (where $n = 2$ or 3 is the dimension of the space). Therefore in order to study cavitation one must consider strain energy functions for which this condition fails; see Section 5.4.

4.7 Some Models of Isotropic Elastic Materials.

In order to describe the detailed response of a particular isotropic elastic material one needs to know the specific strain energy function $W(I_1, I_2, I_3)$ that characterizes that material. Determining explicit forms for W must be done using laboratory experiments (together with micro-mechanical modeling when possible).

The data from the early experiments of Treloar [23, 24] continue to be important in modeling rubberlike materials. The reader is encouraged to read the classic paper by Rivlin and Saunders [20] which describes some of their experiments carried out to determine the specific W for a certain rubber-like material.

One can find a great many[20] strain energy functions in the literature. In this section we give a few specific examples of particularly simple strain energy functions W. We do not discuss how these forms were developed. Our intention is to simply give a flavor for some explicit examples. In order to determine the response according to each of these W's one simply substitutes them into the formulae we derived in Sections 4.6.1 and 4.6.2.

The Gent, 1-term Ogden and Fung models that we consider below, see (4.137), (4.143) and (4.147), each involves a single constitutive parameter J_m, n and β respectively (in addition to the shear modulus μ). According to Table 11.1 of Goriely [8] , the Gent material provides a reasonable model for elastomers when $20 < J_m < 200$ and for soft biological tissues when

[20] Holzapfel [11] makes this point (tongue-in-cheek) by saying "... new specific forms [of W] are published on a daily basis."

$1/3 < J_m < 5/2$. For the 1-term Ogden material the appropriate parameter ranges are $n \approx 3$ (elastomers) and $n \geq 9$ (soft biological tissues). For the Fung material, $3 < \beta < 20$ provides a reasonable model for soft biological tissues.

In Figure 4.6 we show the stress-stretch responses in uniaxial stress according to several of these models. In the left-hand figure we plot the Cauchy and Piola stresses versus stretch for a neo-Hookean material. In the right-hand figure we plot the Piola stress versus stretch for neo-Hookean, 1-term Ogden, Gent and Fung materials.

1. **A compressible inviscid fluid:** Consider an elastic material characterized by the strain energy function
$$W = W(I_3).$$
Substituting this into (4.69) yields the following constitutive relation for the Cauchy stress:
$$\mathbf{T} = -p\,\mathbf{I} \qquad \text{where} \quad p = -2JW'(I_3),\ I_3 = J^2.$$
Observe that for this class of materials, the stress tensor is hydrostatic in *every* deformation. Therefore the strain energy function $W = W(I_3)$ describes an inviscid fluid. The constitutive relation for $\mathbf{T}$ can be written in the form familiar in fluid mechanics by replacing J with the mass density ρ using
$$\rho = \rho_{\mathrm{R}}/J,$$
where ρ_{R} is the mass density in the reference configuration, and replacing $W(I_3)$ by the function $\psi(\rho)$ defined by
$$\psi(\rho) := \frac{1}{\rho_{\mathrm{R}}} W(I_3), \qquad I_3 = J^2 = (\rho_{\mathrm{R}}/\rho)^2.$$
One can then show that the constitutive relation above can be written as
$$\mathbf{T} = -p\,\mathbf{I} \qquad \text{where} \quad p = \rho^2\,\psi'(\rho).$$

2. **Generalized neo-Hookean model. (Incompressible):** A generalized neo-Hookean material is described by a strain energy function of the form
$$W(I_1, I_2) = W(I_1) \qquad \text{for } I_1 \geq 3. \tag{4.126}$$

Many constitutive models for incompressible isotropic rubber-like materials, including the neo-Hookean, Arruda-Boyce and Gent models, are special cases of the generalized neo-Hookean model. Substituting (4.126) into $(4.70)_1$ leads to the constitutive relation

$$\mathbf{T} = -q\,\mathbf{I} \;+\; 2W'(I_1)\,\mathbf{B}, \qquad I_1 = \operatorname{tr}\mathbf{B}. \tag{4.127}$$

In *uniaxial stress*, the stress-stretch relations for the Cauchy stress and the Piola stress are found from (4.79) and (4.80) respectively:

$$T = 2W'(I_1)(\lambda^2 - \lambda^{-1}) \qquad \text{where } I_1 = \lambda^2 + 2\lambda^{-1}, \tag{4.128}$$

$$S = 2W'(I_1)(\lambda - \lambda^{-2}) \qquad \text{where } I_1 = \lambda^2 + 2\lambda^{-1}. \tag{4.129}$$

The relation between the transverse stretch Λ and the longitudinal stretch λ is $\Lambda = \lambda^{-1/2}$.

In **simple shear** one finds from (4.88) and (4.126) that the shear stress T_{12} is related to the amount of shear k by

$$T_{12} = 2W'(I_1)\,k, \qquad I_1 = 3 + k^2, \tag{4.130}$$

and from(4.90) that the normal stresses are

$$T_{11} = 2k^2W'(I_1), \qquad T_{22} = T_{33} = 0, \qquad I_1 = 3 + k^2, \tag{4.131}$$

having assumed that $T_{33} = 0$. The fact that setting $T_{33} = 0$ automatically implies $T_{22} = 0$ is a peculiarity of *all* generalized neo-Hookean materials. For a more general W, one finds that $T_{22} \neq 0$ in general; see for example the response of a Mooney-Rivlin material described later in this section. If the shear stress T_{12} is to be > 0 when the amount of shear k is > 0, (4.130) shows that we must have $W' > 0$. Thus it is reasonable to require the constitutive function $W(I_1)$ for a generalized neo-Hookean material to obey

$$W'(I_1) > 0 \qquad \text{for } I_1 \geq 3. \tag{4.132}$$

From the linearized expression (4.92) we see that the infinitesimal shear modulus of the material is $\mu = 2W'(3)$.

3. **Neo-Hookean model. (Incompressible):** A neo-Hookean material is characterized by the strain energy function

$$W(I_1, I_2) = \frac{\mu}{2}(I_1 - 3), \qquad \mu > 0, \tag{4.133}$$

where μ is a material constant. This is a special case of a generalized neo-Hookean material. The neo-Hookean model can in fact be derived from a simple statistical thermodynamic analysis of a polymer chain. The interested reader can refer to Chapter 3.1 of Treloar [24] (or Chapter 9 of Volume II).

Substituting (4.133) into (4.127) leads to the constitutive relation

$$\mathbf{T} = -q\,\mathbf{I} \;+\; \mu\mathbf{B}. \tag{4.134}$$

The responses in *uniaxial stress* and *simple shear* are immediately found by specializing (4.128), (4.129) and (4.130):

$$T = \mu(\lambda^2 - \lambda^{-1}), \qquad S = \mu(\lambda - \lambda^{-2}), \qquad T_{12} = \mu k.$$

See Figure 4.6.

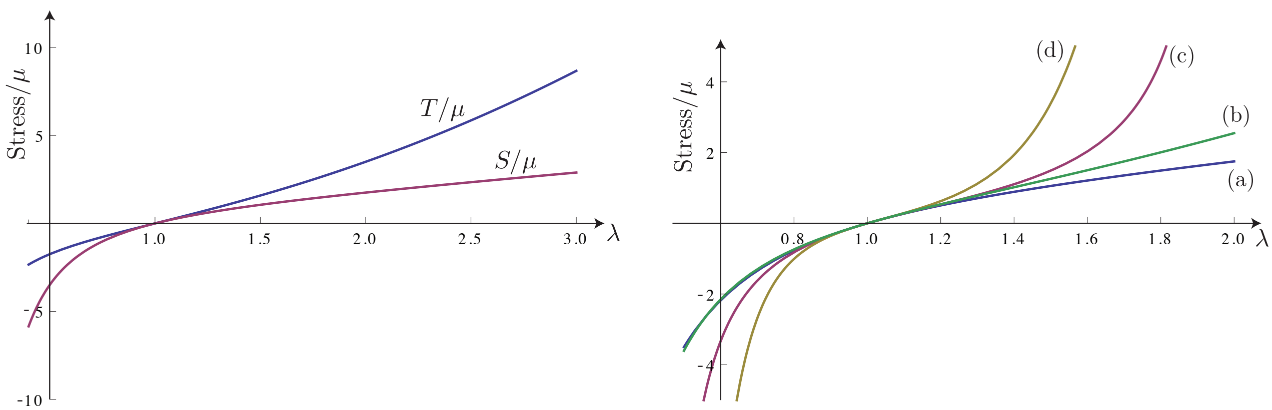

Figure 4.6: Stretch-stretch response in uniaxial stress. Left: Cauchy and Piola stress for neo-Hookean material. Right: Piola stress for (a) neo-Hookean, (b) 1-term Ogden material with $n_1 = 3, \mu_1 = 2\mu/n_1$, (c) Gent material with $J_m = 2$, and (d) Fung material with $Q(\mathbf{B}) = \beta(I_1 - 3), \alpha = \mu/(2\beta), \beta = 2$.

4. **Mooney-Rivlin model. (Incompressible):** A Mooney-Rivlin material is characterized by the strain energy function[21]

$$W(I_1, I_2) = \frac{\mu}{2}\Big[\alpha(I_1 - 3) + (1-\alpha)(I_2 - 3)\Big], \qquad \mu > 0,\; 0 < \alpha < 1, \tag{4.135}$$

where μ and α are material constants. Due to the presence of the term I_2 this is *not* a special case of a generalized neo-Hookean material when $\alpha \neq 1$, though it specializes to

[21] The strict inequalities on α are required by the Baker-Eriksen inequalities.

a neo-Hookean material for $\alpha = 1$. Substituting (4.135) into $(4.70)_1$ gives the following constitutive relation for $\mathbf{T}$,

$$\mathbf{T} = -q\,\mathbf{I} \;+\; \mu\alpha\mathbf{B} + \mu(1-\alpha)\big[I_1\,\mathbf{B} \;-\; \mathbf{B}^2\big]. \qquad (4.136)$$

The response of this material in various settings can be readily studied as above.

We shall simply make one observation here. Note from (4.127) that for a generalized neo-Hookean material, the term involving $\mathbf{B}^2$ in the general constitutive equation $(4.70)_1$ drops out while we see from (4.136) that this is not so for the Mooney-Rivlin material. To see one consequence of this consider a simple shear deformation with shearing direction $\mathbf{e}_1$ and glide plane normal $\mathbf{e}_2$, and again suppose that the boundary conditions give $T_{33} = 0$. The response in simple shear can be calculated from (4.88) and one is led to

$$T_{12} = \mu k, \qquad T_{11} = \mu\alpha k^2 \qquad T_{22} = -\mu(1-\alpha)k^2.$$

Observe that $T_{22} \neq 0$ when $0 < \alpha < 1$ which implies that one needs to apply a normal stress on the glide plane in order to maintain a simple shear deformation. This is in contrast to the behavior of a generalized neo-Hookean materials where $T_{22} = 0$; see (4.131).

Reference:

M. Mooney, A theory of large elastic deformation, *Journal of Applied Physics*, volume 11 (1940), pp. 582-592.

R.S. Rivlin, Some applications of elasticity theory to rubber engineering. Original paper 1948. See reprint in *Collected Papers of R.S. Rivlin*, edited by G.I. Barenblatt and D.D. Joseph, Springer, 1997.

5. **Gent Model. Limited Extensibility. (Incompressible):**

Rubber-like materials are composed of a network of freely-jointed randomly oriented polymer chains. As the stress increases in uniaxial tension, initially, most of the deformation arises due to the unfolding of the polymer chains and the slope of the corresponding stress-stretch curve is relatively small. As the polymer chains orient themselves in the pulling direction, the slope begins to increase rapidly until eventually when the polymer chains are “all” oriented in the axial direction, any further increase in stress requires the chains themselves to stretch. Gent modeled this by limiting the extensibility of polymer chains so that the stress tends to infinity as the stretch approaches a certain finite critical value. The strain energy function proposed by Gent is the particular generalized neo-Hookean material

$$W = W(I_1) = -\frac{\mu}{2}\, J_m \ln\left(1 - \frac{I_1 - 3}{J_m}\right), \qquad \mu > 0,\ J_m > 0, \tag{4.137}$$

where μ and J_m are positive material constants. Since the argument of the logarithm must be positive, the principal invariant I_1 cannot exceed $3 + J_m$:

$$I_1 < 3 + J_m. \tag{4.138}$$

Exercise: Show that in the limit $J_m \to \infty$, the Gent model reduces to the neo-Hookean model (4.133).

Substituting the particular form (4.137) of W into the general constitutive equation $(4.70)_1$ leads to

$$\mathbf{T} = -q\mathbf{I} + \frac{\mu J_m}{3 + J_m - I_1}\, \mathbf{B}. \tag{4.139}$$

In *uniaxial stress* the principal invariant I_1 is

$$I_1 = \lambda^2 + 2\lambda^{-1}, \tag{4.140}$$

having used $\lambda_2 = \lambda_3 = \lambda^{-1/2}$ and $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. Since I_1 cannot exceed $3 + J_m$ we must have

$$\lambda^2 + 2\lambda^{-1} < 3 + J_m \qquad \Rightarrow \qquad \lambda^3 - (3 + J_m)\lambda + 2 < 0.$$

The cubic equation $\lambda^3 - (3 + J_m)\lambda + 2 = 0$ has three real roots, two of which, λ_{min} and λ_{max}, are positive with $0 < \lambda_{min} < 1 < \lambda_{max}$. This implies that the stretch λ must lie in the interval $\lambda_{min} < \lambda < \lambda_{max}$ and in particular cannot exceed λ_{max} . The stress-stretch relation in uniaxial stress is found by substituting (4.137) into (4.128):

$$T_{11} = \left(\lambda^2 - \lambda^{-1}\right)\left(\frac{\mu J_m}{3 + J_m - \lambda^2 - 2\lambda^{-1}}\right), \qquad \lambda_{min} < \lambda < \lambda_{max}. \tag{4.141}$$

Note that $T_{11} \to +\infty$ as $\lambda \to \lambda_{max}$ (and $T_{11} \to -\infty$ as $\lambda \to \lambda_{min}$). The corresponding Piola stress $S = T_{11}\lambda^{-1}$ is plotted as a function of stretch in Figure 4.6.

Reference: A.N. Gent, A new constitutive relation for rubber, *Rubber Chemistry and Technology*, 69(1996), pp. 59-61

6. **Ogden model. (Incompressible):** An N-term incompressible Ogden material is characterized by the strain energy function

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{N} \frac{\mu_i}{n_i}\,(\lambda_1^{n_i} + \lambda_2^{n_i} + \lambda_3^{n_i} - 3), \tag{4.142}$$

where $N, \mu_1, \dots \mu_N, n_i \dots n_N$ are material constants such that

$$\mu_i n_i > 0 \qquad \text{for each } i.$$

The material constants n_i need not be integers. Figure 4.6 shows a graph of the Piola stress versus stretch in uniaxial stress for the one-term Ogden model

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{2\mu}{n^2}\,(\lambda_1^n + \lambda_2^n + \lambda_3^n - 3), \qquad \mu > 0,\ n > 0. \tag{4.143}$$

The strain energy function (4.143) reduces to the neo-Hookean form when $n = 2$ and yields the Varga model when $n = 1$:

$$W = \mu_1(\lambda_1 + \lambda_2 + \lambda_3 - 3). \tag{4.144}$$

Valanis and Landel suggested that the strain energy function for an isotropic incompressible elastic material should have the form

$$W(\lambda_1, \lambda_2, \lambda_3) = w(\lambda_1) + w(\lambda_2) + w(\lambda_3) \tag{4.145}$$

for some suitable function $w(\cdot)$. Certainly the Ogden, neo-Hookean, Mooney-Rivlin and Varga models are of this form.

Reference: R. W. Ogden, Large deformation isotropic elasticity: On the correlation of theory and experiment for incompressible rubberlike solids, *Proceedings of the Royal Society of London. Series A*, Vol. 326, (1972), issue 1567, pp. 565-584.

7. **Arruda-Boyce model. (Incompressible):**

   By using statistical mechanical arguments applied to a cubic representative volume element with eight polymer chains, Arruda and Boyce developed the following particular generalized neo-Hookean material model:

$$W(I_1) = c_1\left[\beta\mathcal{L}(\beta) - \ln\left(\frac{\sinh\beta}{\beta}\right)\right], \tag{i}$$

   where

$$\beta = \mathcal{L}^{-1}\left(\sqrt{\frac{I_1}{3c_2}}\right), \tag{ii}$$

   c_1 and c_2 being constant material parameters. The function $\mathcal{L}^{-1}$ in (ii) is the inverse of the Langevin function

$$\mathcal{L}(x) = \coth x - \frac{1}{x}, \qquad -\infty < x < \infty. \tag{iii}$$

The Langevin function (iii) is monotonically increasing with

$$\mathcal{L}(x) \to \pm 1 \qquad \text{as } x \to \pm\infty.$$

Therefore $\mathcal{L}^{-1}(x)$ is only defined for $-1 < x < 1$ and so according to (ii)

$$I_1 < 3c_2.$$

Thus, like the Gent model, the extensibility of this material is limited.

Reference: E. M. Arruda and M.C. Boyce, A three-dimensional model for the large stretch behavior of rubber elastic materials, *J. Mech. Phys. Solids*, 41, 1993, pp. 389-412.

8. **Fung model for soft biological tissue.**

The mechanical response of soft biological tissue is dominated by its fibrous constituents: collagen and elastin. At small strains, the collagen fibers are unstretched and the mechanical response is almost entirely due to the soft, isotropic elastin. As the load increases, the collagen fibers straighten-out and align with the direction of loading. This leads to a rapid increase in the stiffness, as well as to anisotropic material behavior due to the preferred direction induced by the alignment of collagen fibers. Both of these effects can be modeled by the strain energy function

$$W = W(\mathbf{C}) = \alpha\Big(\mathrm{e}^{Q(\mathbf{C})} - 1\Big), \tag{4.146}$$

where α is a material constant and $Q(\mathbf{C})$ is a scalar-valued function of the right Cauchy Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. The exponential term leads to rapid stiffening of the material. Different functional forms of Q have been considered in the literature, a general quadratic form being the most common. By suitably choosing the form of $Q(\mathbf{C})$, material anisotropy can be built in.

Figure 4.6 shows a graph of the Piola stress versus stretch in uniaxial stress for the particular (isotropic) Fung model

$$W = \frac{\mu}{2\beta}\Big(\mathrm{e}^{\beta(I_1-3)} - 1\Big). \tag{4.147}$$

Observe that in the limit $\beta \to 0$, the strain energy function (4.147) reduces to the neo-Hookean model.

Reference: Review article by J. D. Humphrey, Continuum biomechanics of soft biological tissues, *Proceeding of the Royal Society: Series A*, Vol. 459, 2003, pp. 3 - 46.

9. **Blatz-Ko Model. (Unconstrained):** In Problem 4.4.2 (page 360) we gave the expression for the strain energy function for a Blatz-Ko material and examined its response in uniaxial stress and simple shear. This constitutive model was proposed by Blatz and Ko based on their experiments on a foam rubber.

   Reference: R. J. Blatz and W.L. Ko, Application of finite elastic theory to the deformation of rubbery materials, *Transactions of the Society of Rheology*, volume 6 (1962), pp. 223-251 .

10. **Ogden model. (Unconstrained):** The Ogden strain energy function[22] for unconstrained materials is

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{M} a_i \phi(\alpha_i) + \sum_{i=1}^{N} b_i \psi(\beta_i) + h(J), \qquad J = \lambda_1\lambda_2\lambda_3,$$

   where

$$\phi(\alpha) = \lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha - 3, \qquad \psi(\beta) = (\lambda_1\lambda_2)^\beta + (\lambda_2\lambda_3)^\beta + (\lambda_3\lambda_1)^\beta - 3$$

   and

$$a_i > 0,\ b_i > 0, \quad \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_M \geq 1, \quad \beta_1 \geq \beta_2 \geq \ldots \geq \beta_M \geq 1.$$

   Reference: R. W. Ogden, Large deformation isotropic elasticity - On the correlation of theory and experiment for compressible rubberlike solids, *Proceedings of the Royal Society of London. Series A*, Vol. 328, Issue 1575, (1972), pp. 567-583.

11. **Some incompressible anisotropic material models:**

   As we shall see in Chapter 6, the strain energy function $W(\mathbf{C})$ characterizing an anisotropic elastic material with one preferred direction (one family of fibers) involves, in addition to the invariants I_1, I_2 and I_3, two other (pseudo) invariants I_4 and I_5. If the material has two preferred directions, its strain energy function involves three additional invariants I_6, I_7 and I_8.

   An example of such a constitutive model that has been proposed by Holzapfel et. al. [12] as a model for soft biological tissues is

$$W(I_1, I_4, I_6) = \frac{\mu_1}{2}(I_1 - 3) + \frac{1}{2}\frac{\mu_4}{k_4}\Big[\exp[k_4(I_4 - 1)^2] - 1\Big] + \frac{1}{2}\frac{\mu_6}{k_6}\Big[\exp[k_6(I_6 - 1)^2] - 1\Big], \quad (i)$$

[22] The original form proposed by Ogden does not include the functions ψ. For the modified form shown here, see for example the paper by Ball [5].

where $\mu_1, \mu_4, \mu_6, k_1, k_4, k_6$ are material constants and

$$I_4 = \mathbf{C}\mathbf{m}_R \cdot \mathbf{m}_R, \qquad I_6 = \mathbf{C}\mathbf{m}'_R \cdot \mathbf{m}'_R. \tag{ii}$$

Here the unit vectors $\mathbf{m}_R$ and $\mathbf{m}'_R$ denote the preferred directions in the reference configuration. Observe that the term involving I_1 in (i) has the neo-Hookean form while the terms involving I_4, I_6 are of the Fung form. If $I_4 - 1$ and $I_6 - 1$ are small, (i) can be replaced by

$$W = \frac{\mu_1}{2}(I_1 - 3) + \frac{\mu_4}{2}(I_4 - 1)^2 + \frac{\mu_6}{2}(I_6 - 1)^2. \tag{iii}$$

The special case of (iii) corresponding to $\mu_1 = \mu, \mu_4 = \mu_6 = \beta\mu$ is referred to as the **standard fiber reinforcing model**,

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{\mu\beta}{2}\Big[(I_4 - 1)^2 + (I_6 - 1)^2\Big], \qquad \mu > 0, \beta > 0; \tag{iv}$$

see Goriely [8].

Problem 4.7.1. Recall the bending of a block, the kinematics of which was analyzed in Problem 2.5.4 with the equilibrium equations and boundary conditions examined in Problem 3.7.1. Suppose that the material is composed a Blatz-Ko material. Use this added information to complete the solution to that problem.

Solution: Recall from Problems 2.5.4 and 3.7.1 that

$$\mathbf{F} = \lambda_1 \mathbf{e}_r \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_\theta \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_z \otimes \mathbf{e}_3, \tag{i}$$

$$\mathbf{S} = \sigma_1 \mathbf{e}_r \otimes \mathbf{e}_1 + \sigma_2 \mathbf{e}_\theta \otimes \mathbf{e}_2 + \sigma_3 \mathbf{e}_z \otimes \mathbf{e}_3, \tag{ii}$$

$$\lambda_1 = r'(x_1), \qquad \lambda_2 = \alpha r(x_1), \qquad \lambda_3 = \Lambda, \tag{iii}$$

where we have set

$$\alpha = \beta/B.$$

Moreover

$$\sigma_1(x_1) = \widehat{\sigma}_1(\lambda_1, \lambda_2, \lambda_3), \qquad \sigma_2(x_1) = \widehat{\sigma}_2(\lambda_1, \lambda_2, \lambda_3), \qquad \sigma_3(x_1) = \widehat{\sigma}_3(\lambda_1, \lambda_2, \lambda_3), \tag{iv}$$

and equilibrium required

$$\frac{d\sigma_1}{dx_1} - \alpha\sigma_2 = 0 \qquad \text{for} \quad -A \le x_1 \le A. \tag{v}$$

In the absence of an explicit constitutive relation we were unable to proceed further.

Now that we are told that the block is composed of a Blatz-Ko material, we have

$$\sigma_1 = \mu\big(\lambda_2\lambda_3 - \lambda_1^{-3}\big), \qquad \sigma_2 = \mu\big(\lambda_1\lambda_3 - \lambda_2^{-3}\big), \tag{vi}$$

which upon using (iii) gives

$$\sigma_1 = \mu\left(\alpha\Lambda r(x_1) - \frac{1}{[r'(x_1)]^3}\right), \qquad \sigma_2 = \mu\left(\Lambda r'(x_1) - \frac{1}{[\alpha r(x_1)]^3}\right). \tag{vii}$$

Substituting (vii) into the equilibrium equation (v) and simplifying leads to (the nonlinear ordinary differential equation)

$$\frac{r''}{(r')^4} + \frac{1}{3r^3\alpha^2} = 0 \qquad \text{for} \quad -A \le x_1 \le A. \tag{viii}$$

Integrating once gives

$$r' = \frac{\alpha r\sqrt{3}}{\sqrt{c_1 r^2 - 1}} \qquad \text{for} \quad -A \le x_1 \le A, \tag{ix}$$

and integrating again yields

$$x_1 = \frac{1}{\alpha\sqrt{3}}\Big[\sqrt{c_1 r^2(x_1) - 1} - \tan^{-1}\sqrt{c_1 r^2(x_1) - 1} + c_2\Big]. \qquad \text{for} \quad -A \le x_1 \le A, \tag{x}$$

The boundary conditions

$$\sigma_1(\pm A) = 0, \qquad 2C\int_{-A}^{A} r(x_1)\sigma_2(x_1)dx_1 = m,$$

are now to be used to determine the unknown constants c_1, c_2 and $\alpha\,(= \beta/B)$.

4.8 Linearized elasticity.

In the preceding chapters on kinematics and stress, we specialized the general theory to the case where the displacement gradient tensor $\mathbf{H} = \nabla\mathbf{u} = \mathbf{F} - \mathbf{I}$ was infinitesimal: $|\mathbf{H}| \ll 1$. In particular, we found that *all* of the general strain measures $\mathbf{E}(\mathbf{U})$ and $\mathcal{E}(\mathbf{V})$ defined in (2.67) and (2.70) reduce to the *infinitesimal strain tensor* $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T), \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4.148}$$

We also found that the right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ yielded

$$\mathbf{C} = \mathbf{I} + 2\boldsymbol{\varepsilon} + \mathrm{O}(|\mathbf{H}|^2). \tag{4.149}$$

In the case of stress, the Cauchy and Piola stress tensors coincided to leading order and we denoted this common *stress* by the symmetric tensor $\boldsymbol{\sigma}$.

We now approximate the general frame-indifferent constitutive relationship for an elastic solid to this case where the displacement gradient tensor is infinitesimal. It is convenient to start with the expression $(4.21)_1$ for the Piola stress:

$$\mathbf{S} = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}(\mathbf{C}). \tag{4.150}$$

In the reference configuration we have $\mathbf{F} = \mathbf{I}, \mathbf{C} = \mathbf{I}$. Substituting this into the right hand side of (4.150) gives the stress in the reference configuration which we denote by $\overset{o}{\boldsymbol{\sigma}}$:

$$\overset{o}{\boldsymbol{\sigma}} = 2\left.\frac{\partial W}{\partial \mathbf{C}}\right|_{\mathbf{C}=\mathbf{I}}. \tag{4.151}$$

This is called the *residual stress.*

We now approximate (4.150) by using $\mathbf{F} = \mathbf{I} + \mathbf{H}$, $\mathbf{C} = \mathbf{I} + 2\boldsymbol{\varepsilon}$ and carrying out a Taylor expansion of its right hand side:

$$\begin{aligned}
\sigma_{ij} = 2F_{ik}\frac{\partial W}{\partial C_{kj}}(\mathbf{C}) \quad &= \quad 2\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\left.\frac{\partial W}{\partial C_{kj}}\right|_{\mathbf{C}=\mathbf{I}+2\boldsymbol{\varepsilon}}, \\
&= \quad 2\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\left[\left.\frac{\partial W}{\partial C_{kj}}\right|_{\mathbf{C}=\mathbf{I}} + \left.\frac{\partial^2 W}{\partial C_{kj}\partial C_{pq}}\right|_{\mathbf{C}=\mathbf{I}} 2\varepsilon_{pq} + \mathrm{O}(|\mathbf{H}|^2)\right], \\
&\overset{(4.151)}{=} \quad \left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\left[\overset{o}{\sigma}_{kj} + \mathbb{C}_{kjpq}\varepsilon_{pq} + O(|\mathbf{H}|^2)\right], \\
&= \quad \overset{o}{\sigma}_{ij} + \overset{o}{\sigma}_{kj}\frac{\partial u_i}{\partial x_k} + \mathbb{C}_{ijpq}\varepsilon_{pq} + \mathrm{O}(|\mathbf{H}|^2),
\end{aligned}$$

where we have introduced

$$\mathbb{C}_{ijk\ell} := 4\left.\frac{\partial^2 W}{\partial C_{ij}C_{k\ell}}\right|_{\mathbf{C}=\mathbf{I}}. \tag{4.152}$$

The constitutive relation for stress in the linearized theory is thus

$$\sigma_{ij} = \overset{o}{\sigma}_{ij} + \overset{o}{\sigma}_{kj}\frac{\partial u_i}{\partial x_k} + \mathbb{C}_{ijk\ell}\varepsilon_{k\ell}. \tag{4.153}$$

Observe that the second term on the right-hand side involves the displacement gradient tensor (which can be written in terms of the infinitesimal strain *and* rotation tensors using $\nabla\mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$).

The 4-tensor $\mathbb{C}$ is known as the **elasticity tensor**. Note that it does not depend on the deformation since the right-hand side of (4.152) is evaluated at $\mathbf{C} = \mathbf{I}$. Its components

$\mathbb{C}_{ijk\ell}$ represent the various elastic moduli of the material. The elasticity tensor has $3^4 = 81$ components but not all of them are independent. Observe from (4.152) that

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{k\ell ij}, \qquad \mathbb{C}_{ijk\ell} = \mathbb{C}_{jik\ell}, \qquad \mathbb{C}_{ijk\ell} = \mathbb{C}_{ij\ell k}, \tag{4.154}$$

which implies that $\mathbb{C}$ has 21 independent components. Therefore the most general (anisotropic) elastic material has 21 elastic constants.

From hereon we shall take the residual stress to vanish so that

$$\left.\frac{\partial W}{\partial \mathbf{C}}\right|_{\mathbf{C}=\mathbf{I}} = \mathbf{0}. \tag{4.155}$$

The constitutive relation for stress is then

$$\boxed{\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}, \qquad \sigma_{ij} = \mathbb{C}_{ijk\ell}\varepsilon_{k\ell}.} \tag{4.156}$$

For completeness we note the corresponding approximation for the strain energy function W. A Taylor expansion of $W(\mathbf{C})$ about $\mathbf{C} = \mathbf{I}$ is readily shown to lead to

$$\boxed{W(\boldsymbol{\varepsilon}) = \frac{1}{2}\mathbb{C}_{ijk\ell}\varepsilon_{i,j}\varepsilon_{k,l}.} \tag{4.157}$$

Observe from (4.156) and (4.157) that

$$\boxed{\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}(\boldsymbol{\varepsilon}), \qquad \mathbb{C}_{ijk\ell} = \frac{\partial^2 W}{\partial \varepsilon_{ij}\varepsilon_{k\ell}}(\boldsymbol{\varepsilon}).} \tag{4.158}$$

Material symmetry in the linearized theory. Following the discussion in Section 4.4, the *material symmetry group* $\mathcal{G}$ is the collection of proper orthogonal transformations that preserves the symmetry of the material in the sense that

$$W(\boldsymbol{\varepsilon}) = W(\mathbf{Q}\boldsymbol{\varepsilon}\mathbf{Q}^T) \qquad \text{for each } \mathbf{Q} \in \mathcal{G} \tag{4.159}$$

and all symmetric tensors $\boldsymbol{\varepsilon}$; see (4.29). Note that the tensor $\mathbf{Q}$ (that operates on the reference configuration prior to deforming the body) need not be infinitesimal. In view of (4.157),

$$\begin{aligned} W(\mathbf{Q}\boldsymbol{\varepsilon}\mathbf{Q}^T) &= \tfrac{1}{2}\mathbb{C}_{pqrs}(\mathbf{Q}\boldsymbol{\varepsilon}\mathbf{Q}^T)_{pq}(\mathbf{Q}\boldsymbol{\varepsilon}\mathbf{Q}^T)_{rs} = \\ &= \tfrac{1}{2}\mathbb{C}_{pqrs}(Q_{pi}\varepsilon_{ij}Q_{qj})(Q_{rk}\varepsilon_{k\ell}Q_{s\ell}) = \\ &= \tfrac{1}{2}Q_{pi}Q_{qj}Q_{rk}Q_{s\ell}\mathbb{C}_{pqrs}\varepsilon_{ij}\varepsilon_{k\ell}. \end{aligned}$$

Therefore if $\mathbf{Q} \in \mathcal{G}$ it follows from this and (4.159) that

$$\frac{1}{2}\mathbb{C}_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell} = \frac{1}{2}Q_{pi}Q_{qj}Q_{rk}Q_{s\ell}\mathbb{C}_{pqrs}\varepsilon_{ij}\varepsilon_{k\ell},$$

which must hold for all strains $\boldsymbol{\varepsilon}$ whence

$$\mathbb{C}_{ijk\ell} = Q_{pi}Q_{qj}Q_{rk}Q_{s\ell}\mathbb{C}_{pqrs} \qquad \text{for each } \mathbf{Q} \in \mathcal{G}. \tag{4.160}$$

Recall that in general, the components $\mathbb{C}'_{ijk\ell}$ and $\mathbb{C}_{ijk\ell}$ of a 4-tensor $\mathbb{C}$ in two bases $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are related by $\mathbb{C}'_{ijk\ell} = Q_{pi}Q_{qj}Q_{rk}Q_{s\ell}\mathbb{C}_{pqrs}$ where the proper orthogonal matrix $[Q]$ relates the two bases by $\mathbf{e}'_i = Q_{ij}\mathbf{e}_j$. Therefore (4.160) tells us that $\mathbb{C}'_{ijk\ell} = \mathbb{C}_{ijk\ell}$ if the bases are related by a symmetry transformation, i.e. the components of $\mathbb{C}$ in two bases related by a symmetry transformation are identical.

For an *isotropic material*, $\mathcal{G}$ contains all proper orthogonal transformations and therefore, as in the finite deformation theory, we may conclude that W depends on the strain $\boldsymbol{\varepsilon}$ only through its scalar invariants. It is convenient to choose the invariants

$$i_i = \operatorname{tr}\boldsymbol{\varepsilon}, \qquad i_2 = \operatorname{tr}\boldsymbol{\varepsilon}^2, \qquad i_3 = \operatorname{tr}\boldsymbol{\varepsilon}^3.$$

Thus for an isotropic material we have $W(i_1, i_2, i_3)$. However according to (4.157) W is a quadratic (polynomial) function of strain. Therefore it cannot depend on i_3 and it must depend linearly on i_2 and i_1^2:

$$W = \mu\, i_2 + \frac{\lambda}{2}\, i_1^2 = \mu\operatorname{tr}(\boldsymbol{\varepsilon}^2) + \frac{\lambda}{2}\,(\operatorname{tr}\boldsymbol{\varepsilon})^2 = \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}\,\varepsilon_{ii}\varepsilon_{jj}, \tag{4.161}$$

where μ and λ are material parameters. (Caution: λ here is not the stretch.) It is illuminating to write (4.161) in the form

$$W = \mu\,\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} + \frac{\lambda}{2}\,(\operatorname{tr}\boldsymbol{\varepsilon})^2, \tag{4.162}$$

which can be compared with the Saint-Venant Kirchhoff model (Problem 4.2) and the Hencky model (Problem 4.9) both of which are used at finite strain.

It follows from (4.161) and $(4.158)_1$ that the constitutive relation for stress is

$$\boldsymbol{\sigma} = 2\mu\,\boldsymbol{\varepsilon} + \lambda\operatorname{tr}\boldsymbol{\varepsilon}\,\mathbf{I}, \qquad \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\,\delta_{ij}, \tag{4.163}$$

and from $(4.158)_2$ that

$$\mathbb{C}_{ijkl} = \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \lambda\delta_{ij}\delta_{k\ell}. \tag{4.164}$$

Observe that an isotropic elastic material is described by 2 elastic moduli. The two moduli λ and μ are known as the Lamé constants. The more familiar material constants shear modulus G, bulk modulus κ, Young's modulus E and Poisson's ratio ν can be expressed in terms of λ and μ:

$$G = \mu, \qquad \kappa = \lambda + \frac{2}{3}\mu, \qquad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{4.165}$$

Remark: Linearization of the strain energy function $W = W(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}))$ of the nonlinear theory shows, after a lengthy calculation, that the material parameters λ and μ can be expressed as

$$\mu = 2\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)\Big|_{I_1=I_2=3, I_3=1},$$

$$\begin{aligned}\lambda = 4\Bigg(\frac{\partial W}{\partial I_2} + \frac{\partial W}{\partial I_3} \quad &+ \quad \frac{\partial^2 W}{\partial I_1^2} + 4\frac{\partial^2 W}{\partial I_2^2} + \frac{\partial^2 W}{\partial I_3^2} + 4\frac{\partial^2 W}{\partial I_1 \partial I_2} + \\ &+ \quad 2\frac{\partial^2 W}{\partial I_1 \partial I_3} + 4\frac{\partial^2 W}{\partial I_2 \partial I_3}\Bigg)\Big|_{I_1=I_2=3, I_3=1}.\end{aligned}$$

Expressions for λ and μ in terms of the derivatives of $W^*(\lambda_1, \lambda_2, \lambda_3)$ can be found in Problem 4.20.

For an incompressible material one has the constraint

$$\operatorname{tr} \boldsymbol{\varepsilon} = 0, \qquad \varepsilon_{kk} = 0, \tag{4.166}$$

and a reactive stress $-q\mathbf{I}$ must be added to the expression for stress:

$$\sigma_{ij} = \mathbb{C}_{ijk\ell}\varepsilon_{k\ell} - q\delta_{ij}.$$

If the material is isotropic, this reduces to

$$\boldsymbol{\sigma} = 2\mu\,\boldsymbol{\varepsilon} - q\,\mathbf{I}, \qquad \sigma_{ij} = 2\mu\varepsilon_{ij} - q\,\delta_{ij}. \tag{4.167}$$

Note that (4.167) can be formally obtained from (4.163) by letting $\operatorname{tr}\boldsymbol{\varepsilon} \to 0$ and $\lambda \to \infty$ with $\lambda \operatorname{tr}\boldsymbol{\varepsilon}$ held constant. Observe that taking this limit in the expression (4.161) for the strain energy yields

$$W = \mu\varepsilon_{ij}\varepsilon_{ij} - \frac{1}{2}q\varepsilon_{ii} \overset{(4.166)}{=} \mu\varepsilon_{ij}\varepsilon_{ij},$$

where in the first expression q plays the role of a Lagrange multiplier associated with the constraint $\varepsilon_{ii} = 0$.

Exercise: Problem 4.20.

4.9 Exercises.

Response of an elastic material.

Problem 4.1. Carry out the calculations described in Problem 4.1.1 and show that $\mathbf{T}(\mathbf{B})$ must necessarily have the form

$$\mathbf{T}(\mathbf{B}) = \beta_2\mathbf{B}^2 + \beta_1\mathbf{B} + \beta_0\mathbf{I}, \qquad (i)$$

where the β_j's are functions of the principal scalar invariants of $\mathbf{B}$. This restricted form of $\mathbf{T}(\mathbf{B})$ implies the material is isotropic. (The result of Problem 1.38 will be useful.)

Solution: Suppose that $\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{T}(\mathbf{B})$ where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. Then, for any nonsingular $\mathbf{F}$ and proper orthogonal $\mathbf{Q}$ we have

$$\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{T}\big(\mathbf{F}\mathbf{F}^T\big) = \mathbf{T}(\mathbf{B}), \qquad (ii)$$

$$\widehat{\mathbf{T}}(\mathbf{Q}\mathbf{F}) = \mathbf{T}\big((\mathbf{Q}\mathbf{F})(\mathbf{Q}\mathbf{F})^T\big) = \mathbf{T}\big(\mathbf{Q}\mathbf{F}\mathbf{F}^T\mathbf{Q}^T\big) = \mathbf{T}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T). \qquad (iii)$$

Substituting (ii) and (iii) into the requirement $(4.15)_1$ of frame indifference gives

$$\mathbf{T}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T) = \mathbf{Q}\mathbf{T}(\mathbf{B})\mathbf{Q}^T. \qquad (iv)$$

Since (iv) must hold for all proper orthogonal $\mathbf{Q}$ and symmetric positive definite $\mathbf{B}$, the desired result (i) follows from Problem 1.38 (page 104).

Problem 4.2. (Based on Holzapfel) The *Saint-Venant Kirchhoff model* of an isotropic unconstrained material is described by the strain energy function

$$W = \frac{\alpha}{2}(\operatorname{tr}\mathbf{E})^2 + \mu\,(\mathbf{E}\cdot\mathbf{E}) = \frac{\alpha}{2}(\operatorname{tr}\mathbf{E})^2 + \mu\operatorname{tr}(\mathbf{E}^2), \qquad (i)$$

where $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ is the Green Saint-Venant strain tensor and $\mu > 0$ and $\alpha > 0$ are material constants. (The symbol λ is usually used for the parameter α, but since we use λ for stretch, we are using α instead.)

(a) Does this strain energy function W obey the growth conditions (4.124)? i.e. does $W \to \infty$ as $J \to \infty$ and $J \to 0^+$?

(b) Derive the constitutive law associated with (i) relating the Green Saint-Venant strain tensor $\mathbf{E}$ to the second Piola-Kirchhoff stress tensor $\mathbf{S}^{(2)}$. (Recall that $\mathbf{S}^{(2)}$ is work-conjugate to the Green Saint-Venant strain.)

(c) Consider a uniaxial *deformation* $\mathbf{y} = \mathbf{F}\mathbf{x}$ where $\mathbf{F} = \mathbf{I} + (\lambda - 1)\mathbf{e}_1 \otimes \mathbf{e}_1$. $(c1)$: Calculate the stress-stretch relation between S_{11} and λ where S_{11} is the 1,1 component of the Piola stress tensor $\mathbf{S}$; $(c2)$: show that this relation loses monotonicity in compression at the stretch $\lambda = \sqrt{1/3}$; and $(c3)$ that $S_{11} \to 0$ at extreme contraction $\lambda \to 0^+$.

(d) Show for the modified Saint-Venant Kirchhoff model

$$W = \frac{\kappa}{2}(\ln J)^2 + \mu \operatorname{tr}(\mathbf{E}^2), \tag{ii}$$

where $\mu > 0$ and κ are material constants, that $S_{11} \to -\infty$ as $\lambda \to 0^+$ in a uniaxial deformation.

Solution:

(a) Consider a pure hydrostatic deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ where $\mathbf{F} = \lambda\mathbf{I}$. For this $\mathbf{F}$, we have

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\lambda^2 - 1)\mathbf{I}, \qquad \operatorname{tr}\mathbf{E} = \frac{3}{2}(\lambda^2 - 1), \qquad \operatorname{tr}\mathbf{E}^2 = \frac{3}{4}(\lambda^2 - 1)^2. \tag{iii}$$

and so from (iii) and (i),

$$W = \frac{3}{8}(3\alpha + 2\mu)(\lambda^2 - 1)^2.$$

Thus

$$W \to \infty \quad \text{when} \quad J = \lambda^3 \to \infty, \qquad W \to \frac{3}{8}(3\alpha + 2\mu) \quad \text{when} \quad J = \lambda^3 \to 0^+. \qquad \square$$

Thus W does not tend to ∞ as $J \to 0^+$.

(b) In Section 3.8.1 we showed that the second Piola-Kirchhoff stress tensor $\mathbf{S}^{(2)}$ is work conjugate to the Green Saint-Venant strain tensor $\mathbf{E}^{(2)}$ which for convenience we shall write as $\mathbf{E}$. Therefore from (3.80) and (4.5),

$$\mathbf{S}^{(2)} \cdot \dot{\mathbf{E}} = \dot{W}. \tag{iv}$$

From material frame indifference we know that $W = \overline{W}(\mathbf{C})$. Since $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ it follows that the strain energy function can be expressed as a function of $\mathbf{E} : W = W(\mathbf{E})$. From this and (iv),

$$\left(\mathbf{S}^{(2)} - \frac{\partial W}{\partial \mathbf{E}}\right) \cdot \dot{\mathbf{E}} = 0 \qquad \Rightarrow \qquad \mathbf{S}^{(2)} = \frac{\partial W}{\partial \mathbf{E}}, \tag{v}$$

where in getting to $(v)_2$ we have used the fact that $(v)_1$ holds for all $\dot{\mathbf{E}}$ and the term within the parentheses is independent of $\dot{\mathbf{E}}$. Since (as can be shown, e.g., using components in a basis),

$$\frac{\partial}{\partial \mathbf{E}}\big(\operatorname{tr}\mathbf{E}\big)^2 = (2\operatorname{tr}\mathbf{E})\mathbf{I}, \qquad \frac{\partial}{\partial \mathbf{E}}\operatorname{tr}(\mathbf{E}^2) = 2\mathbf{E}, \tag{vi}$$

it follows from $(v)_2$ and (vi) that for the particular material (i),

$$\mathbf{S}^{(2)} = 2\mu\mathbf{E} + \alpha(\operatorname{tr}\mathbf{E})\,\mathbf{I}. \qquad \square \tag{vii}$$

(c) Since $\mathbf{F} = \lambda\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3$, we have

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{\lambda^2 - 1}{2}\mathbf{e}_1 \otimes \mathbf{e}_1. \tag{viii}$$

Therefore from (vii) and $(viii)$,

$$S^{(2)}_{11} = \frac{1}{2}(\alpha + 2\mu)(\lambda^2 - 1).$$

However we are asked to find the $1,1$ component of the Piola stress tensor $\mathbf{S}$. Since $\mathbf{S} = \mathbf{F}\mathbf{S}^{(2)}$ it is readily shown that

$$S_{11} = \lambda S^{(2)}_{11} = \frac{1}{2}(\alpha + 2\mu)(\lambda^3 - \lambda). \qquad \square \tag{ix}$$

By differentiating (ix),

$$\frac{dS_{11}}{d\lambda} < 0 \quad \text{for } 0 < \lambda < \frac{1}{\sqrt{3}}, \qquad \frac{dS_{11}}{d\lambda} = 0 \quad \text{for } \lambda = \frac{1}{\sqrt{3}}, \qquad \frac{dS_{11}}{d\lambda} > 0 \quad \text{for } \lambda > \frac{1}{\sqrt{3}}. \qquad \square$$

From (ix), $S_{11} \to 0$ when $\lambda \to 0^+$.

(d) We need to first calculate $\partial J/\partial \mathbf{E}$. Since $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ and $\det \mathbf{C} = J^2$,

$$\frac{\partial J}{\partial \mathbf{E}} = 2\frac{\partial J}{\partial \mathbf{C}} = \frac{\partial}{\partial \mathbf{C}}(J^2) = \frac{\partial}{\partial \mathbf{C}}(\det \mathbf{C}) \overset{(1.208)}{=} (\det\, \mathbf{C})\, \mathbf{C}^{-\mathrm{T}} = J^2\, \mathbf{C}^{-1}. \tag{x}$$

It follows that

$$\mathbf{S}^{(2)} \overset{(v)}{=} \frac{\partial W}{\partial \mathbf{E}} \overset{(ii)}{=} \kappa\, \frac{\ln J}{J}\, \frac{\partial J}{\partial \mathbf{E}} + \mu \frac{\partial}{\partial \mathbf{E}}(\mathrm{tr}\, \mathbf{E}^2) \overset{(vi),(x)}{=} \kappa\, J \ln \mathrm{J}\, \mathbf{C}^{-1} + 2\mu \mathbf{E}. \qquad \square \tag{xi}$$

Therefore for a uniaxial deformation $\mathbf{F} = \lambda \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3$ this yields

$$S^{(2)}_{11} = 2\mu E_{11} + \kappa\lambda \ln \lambda C^{-1}_{11} = \mu(\lambda^2 - 1) + \kappa \frac{\ln \lambda}{\lambda}.$$

Since $\mathbf{S}^{(2)}$ is related to the Piola stress tensor by $\mathbf{S} = \mathbf{F}\mathbf{S}^{(2)}$, it follows that $S_{11} = \lambda S^{(2)}_{11}$. Thus

$$S_{11} = \lambda S^{(2)}_{11} = \mu(\lambda^3 - \lambda) + \kappa \ln \lambda$$

Since $\ln \lambda \to -\infty$ as $\lambda \to 0^+$ it follows that $S_{11} \to -\infty$ as $\lambda \to 0^+$.

Problem 4.3. Consider an unconstrained isotropic elastic material characterized by the strain energy function

$$W(\mathbf{E}) = \mu \mathbf{E} \cdot \mathbf{E} + \frac{\alpha}{2}\big(\mathrm{tr}\, \mathbf{E}\big)^2$$

where $\mathbf{E} = \ln \mathbf{U}$ is the Hencky strain tensor, and μ and α are material constants. (This is identical to the strain energy function in Problem 4.9.)

(a) Express the given strain energy function in terms of the principal stretches.

(b) Consider a hydrostatic deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$, $\mathbf{F} = \lambda \mathbf{I}$. Calculate the magnitude τ of the corresponding hydrostatic stress $\mathbf{T} = \tau \mathbf{I}$. Plot τ versus J. Calculate the bulk modulus of this material at infinitesimal deformations.

(c) Consider a uniaxial stress $\mathbf{T} = \tau\, \mathbf{e}_1 \otimes \mathbf{e}_1$ with corresponding homogeneous deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ where $\mathbf{F} = \lambda\, \mathbf{e}_1 \otimes \mathbf{e}_1 + \Lambda(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3)$. Calculate Λ in terms of λ. Calculate and plot the relations between τ and σ versus λ (where $\sigma = S_{11}$). Determine the Young's modulus and Poisson's ratio of the material at infinitesimal deformations.

(d) Calculate and plot the relation between the shear stress and the amount of shear in a simple shear deformation.

Solution: The various conventions of indicial notation are suspended in this solution.

(a) Since $\mathbf{E} = \sum_{i=1}^{3} \ln \lambda_i (\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i)$, in a principal basis for $\mathbf{E}$ we have $E_{ii} = \ln \lambda_i$ (no sum on i) and $E_{ij} = 0$ for $i \neq j$. Thus

$$\mathbf{E} \cdot \mathbf{E} = E_{ij}E_{ij} = E_{11}^2 + E_{22}^2 + E_{33}^2 = (\ln \lambda_1)^2 + (\ln \lambda_2)^2 + (\ln \lambda_3)^2,$$

$$\operatorname{tr} \mathbf{E} = E_{11} + E_{22} + E_{33} = \ln \lambda_1 + \ln \lambda_2 + \ln \lambda_3 = \ln \lambda_1 \lambda_2 \lambda_3,$$

and so the given strain energy function can be expressed as

$$W^*(\lambda_1, \lambda_2, \lambda_3) = \mu\Big[(\ln \lambda_1)^2 + (\ln \lambda_2)^2 + (\ln \lambda_3)^2\Big] + \frac{\alpha}{2}\Big[\ln(\lambda_1\lambda_2\lambda_3)\Big]^2. \qquad \square \qquad (i)$$

From (i) we obtain

$$\sigma_k = \frac{\partial W^*}{\partial \lambda_k} = \frac{2\mu \, \ln \lambda_k + \alpha \ln J}{\lambda_k}, \qquad \tau_k = \sigma_k \frac{\lambda_k}{J} = \frac{2\mu \, \ln \lambda_k + \alpha \ln J}{J}. \qquad (ii)$$

(b) In a hydrostatic deformation we have $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda$, $J = \lambda^3$ and $\tau_1 = \tau_2 = \tau_3 =: \tau$. Thus $(ii)_2$ gives

$$\tau = \frac{2\mu \, \ln \lambda + \alpha \ln J}{J} = \frac{2\mu \, \ln J^{1/3} + \alpha \ln J}{J} = \left(\frac{2}{3}\mu + \alpha\right) \frac{\ln J}{J} = (2\mu + 3\alpha)\, \frac{\ln \lambda}{\lambda^3}. \qquad \square \qquad (iii)$$

In an infinitesimal deformation we have $\lambda = 1 + \varepsilon$ where $|\varepsilon| \ll 1$. Thus $J = (1+\varepsilon)^3 \doteq 1 + 3\varepsilon$ and so (iii) can be approximated as

$$\tau \doteq (2\mu + 3\alpha)\, \frac{\ln(1+\varepsilon)}{1 + 3\varepsilon} \doteq (2\mu + 3\alpha)\, \varepsilon.$$

Thus the bulk modulus at infinitesimal deformations is

$$\kappa := \frac{\frac{1}{3}(\tau_1 + \tau_2 + \tau_3)}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} = \frac{\tau}{3\varepsilon} = \frac{2}{3}\mu + \alpha. \qquad \square$$

Observe that we can now write (iii) as

$$\tau = \kappa \, \frac{\ln J}{J}. \qquad \square$$

This has been plotted in Figure 4.7.

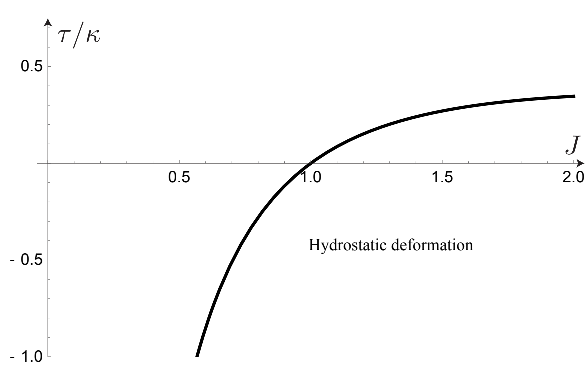

Figure 4.7: Hydrostatic deformation: Stress τ/κ versus dilatation J.

(c) In uniaxial stress we have $\tau_1 = \tau, \tau_2 = \tau_3 = 0$, $\lambda_1 = \lambda, \lambda_2 = \lambda_3 = \Lambda$ and $J = \lambda\Lambda^2$. We calculate τ_2 from $(ii)_2$ and set it equal to zero:

$$\tau_2 = \frac{2\mu\,\ln\lambda_2 + \alpha\ln J}{J} = \frac{2\mu\,\ln\Lambda + \alpha\ln(\lambda\Lambda^2)}{J} = \frac{(2\mu+2\alpha)\,\ln\Lambda + \alpha\ln\lambda}{J} = 0$$

which gives

$$(2\mu+2\alpha)\,\ln\Lambda + \alpha\ln\lambda = 0 \qquad \Rightarrow \qquad \Lambda^{2(\mu+\alpha)}\lambda^{\alpha} = 1 \qquad \Rightarrow \qquad \Lambda = \lambda^{-\frac{\alpha}{2\mu+2\alpha}} \qquad \square. \qquad (iv)$$

In an infinitesimal deformation we have $\lambda = 1+\varepsilon_1$, $\Lambda = 1+\varepsilon_2$, $|\varepsilon_1|, |\varepsilon_2| \ll 1$. Thus we can approximate (iv) as

$$1+\varepsilon_2 = (1+\varepsilon_1)^{-\frac{\alpha}{2\mu+2\alpha}} \doteq 1 - \frac{\alpha}{2\mu+2\alpha}\varepsilon_1 \qquad \Rightarrow \qquad \frac{\varepsilon_2}{\varepsilon_1} = -\frac{\alpha}{2\mu+2\alpha}.$$

The Poisson's ratio is thus

$$\nu := -\frac{\varepsilon_2}{\varepsilon_1} = \frac{\alpha}{2\mu+2\alpha}. \qquad \square \qquad (v)$$

We could also have derived (v) by differentiating (iv) and using:

$$\nu := -\left.\frac{d\Lambda}{d\lambda}\right|_{\lambda=1}.$$

Observe because of (v) that (iv) can be written as

$$\Lambda = \lambda^{-\nu}.$$

Furthermore, in uniaxial stress we have $J = \lambda\Lambda^2 = \lambda^{\frac{\mu}{\mu+\alpha}}$ and so

$$\ln J = \frac{\mu}{\mu+\alpha}\ln\lambda.$$

We now calculate τ_1 from $(ii)_2$:

$$\tau_1 = \frac{2\mu\,\ln\lambda_1 + \alpha\ln J}{J} = \frac{2\mu\,\ln\lambda + [\alpha\mu/(\mu+\alpha)]\ln\lambda}{J} = \mu\left(\frac{2\mu+3\alpha}{\mu+\alpha}\right)\frac{\ln\lambda}{\lambda^{\frac{\mu}{\mu+\alpha}}}. \qquad \square \qquad (vi)$$

For an infinitesimal deformation we have $\ln\lambda = \ln(1+\varepsilon_1) \doteq \varepsilon_1$ and so we can approximate (vi) as

$$\tau_1 = \mu\left(\frac{2\mu+3\alpha}{\mu+\alpha}\right)\frac{\ln(1+\varepsilon_1)}{(1+\varepsilon_1)^{\frac{\mu}{\mu+\alpha}}} \doteq \mu\left(\frac{2\mu+3\alpha}{\mu+\alpha}\right)\varepsilon_1.$$

The Young's modulus is therefore

$$E := \frac{\tau_1}{\varepsilon_1} = \mu\left(\frac{2\mu+3\alpha}{\mu+\alpha}\right). \qquad \square$$

We could alternatively derived this expression for E by differentiating (vi) and using:

$$E := \left.\frac{d\tau_1}{d\lambda}\right|_{\lambda=1}.$$

Observe that we can now write (vi) as

$$\tau_1 = E\frac{\ln\lambda}{\lambda^{1-2\nu}}. \qquad \square$$

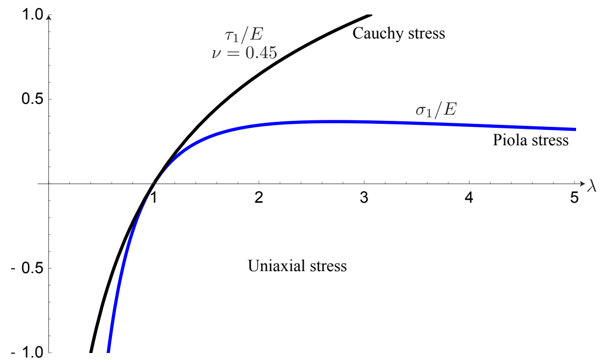

Figure 4.8: Uniaxial stress: Cauchy stress τ_1/E and Piola stress σ_1/E versus stretch λ. The curve for the Cauchy stress has been drawn for $\nu = 0.45$.

The corresponding Piola stress is

$$\sigma_1 = \tau_1 \Lambda^2 = E\frac{\ln\lambda}{\lambda}. \qquad \square$$

(d) From Section 4.6.2, the relation between the shear stress τ and amount of shear k in simple shear (for an isotropic unconstrained material) is

$$\tau = w'(k) \tag{vii}$$

where

$$w(k) := W^*(\lambda_1(k), \lambda_2(k), \lambda_3(k)). \tag{viii}$$

The principal stretches in simple shear were found in Problem 2.5.2 to be

$$\lambda_1(k) = \frac{\sqrt{k^2+4}+k}{2}, \qquad \lambda_2(k) = \frac{\sqrt{k^2+4}-k}{2}, \qquad \lambda_3(k) = 1. \tag{ix}$$

From (vii) and $(viii)$ we obtain

$$\tau = \frac{\partial W^*}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial k} + \frac{\partial W^*}{\partial \lambda_2}\frac{\partial \lambda_2}{\partial k} + \frac{\partial W^*}{\partial \lambda_3}\frac{\partial \lambda_3}{\partial k}, \tag{x}$$

and from (ix),

$$\frac{\partial \lambda_1}{\partial k} = \frac{\lambda_1}{\sqrt{k^2+4}}, \qquad \frac{\partial \lambda_2}{\partial k} = -\frac{\lambda_2}{\sqrt{k^2+4}}, \qquad \frac{\partial \lambda_3}{\partial k} = 0. \tag{xi}$$

Therefore (x) reduces to

$$\tau = \frac{1}{\sqrt{k^2+4}}\left[\lambda_1\frac{\partial W^*}{\partial \lambda_1} - \lambda_2\frac{\partial W^*}{\partial \lambda_2}\right]. \tag{xii}$$

Now consider the particular material (i). Differentiating W^* with respect to λ_k gives

$$\frac{\partial W^*}{\partial \lambda_k} = 2\mu\frac{\ln\lambda_k}{\lambda_k} + \alpha\frac{\ln(\lambda_1\lambda_2\lambda_3)}{\lambda_k}, \qquad k = 1,2,3, \qquad \text{(no sum on } k\text{)}, \tag{xiii}$$

which simplifies in simple shear, since $\lambda_1\lambda_2\lambda_3 = 1$, to

$$\frac{\partial W^*}{\partial \lambda_k} = 2\mu\frac{\ln\lambda_k}{\lambda_k} \qquad \text{(no sum on } k\text{)}. \tag{xiv}$$

Therefore from (xii) and (xiv) we obtain the relation between the shear stress τ and the amount of shear k in simple shear for the material (i):

$$\tau = \frac{2\mu}{\sqrt{k^2+4}} \ln(\lambda_1/\lambda_2) = \frac{2\mu}{\sqrt{k^2+4}} \ln\left[\frac{\sqrt{k^2+4}+k}{\sqrt{k^2+4}-k}\right] = \frac{4\mu}{\sqrt{k^2+4}} \ln\left[\frac{\sqrt{k^2+4}+k}{2}\right]. \quad \square \qquad (xv)$$

This relation is plotted in Figure 4.9. For small amounts of shear, $|k| \ll 1$, equation (xv) approximates to

$$\tau \doteq 2\mu \ \ln[1+k/2] \doteq \mu k$$

and so the shear modulus at infinitesimal deformations is μ.

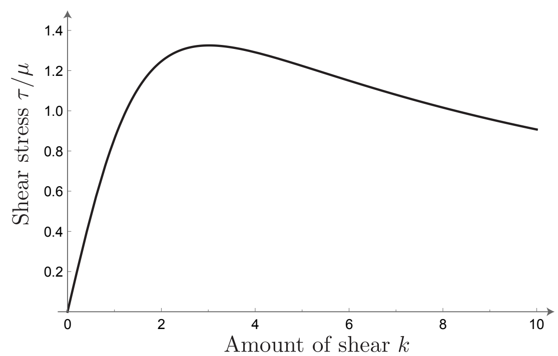

Figure 4.9: Shear stress versus the amount of shear for the material (i).

Problem 4.4. (Based on Chadwick) The constitutive relation for a certain class of foam rubbers has the form

$$\mathbf{T} = \frac{1}{J^3}\Big[\big[f(J) - \beta I_2\big]\mathbf{I} + \beta I_1 \mathbf{B} - \beta \mathbf{B}^2\Big] \qquad (i)$$

where β is a constitutive parameter and $f(J)$ is a constitutive function.

(a) A uniaxial stress experiment is carried out in order to determine the function $f(J)$. Let λ be the stretch in the direction of the applied stress. During the experiment λ and the transverse stretch Λ are measured. A plot of Λ versus λ on a logarithmic scale is found to be a straight line with slope $-\nu$ where ν is a positive constant. Deduce the form of the function f.

(b) Calculate the Cauchy stress - stretch relation in uniaxial stress and from it determine the Young's modulus of the material.

Solution:

(a) Suppose that the stress is applied in the x_1-direction and that the deformation gradient tensor is

$$\mathbf{F} = \lambda \mathbf{e}_1 \otimes \mathbf{e}_1 + \Lambda(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3). \qquad (ii)$$

Then

$$J = \det \mathbf{F} = \lambda\Lambda^2. \qquad (iii)$$

Keep in mind that foam rubber is compressible and so we do *not* require $J = 1$. It follows from (ii) that

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \lambda^2 \mathbf{e}_1 \otimes \mathbf{e}_1 + \Lambda^2(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \qquad (iv)$$

$$\mathbf{B}^2 = \lambda^4 \mathbf{e}_1 \otimes \mathbf{e}_1 + \Lambda^4(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \qquad (v)$$

$$I_1 = \operatorname{tr} \mathbf{B} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda^2 + 2\Lambda^2, \qquad (vi)$$

$$I_2 = \frac{1}{2}\big[(\operatorname{tr} \mathbf{B})^2 - \operatorname{tr} \mathbf{B}^2\big] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 = 2\lambda^2\Lambda^2 + \Lambda^4. \qquad (vii)$$

We can calculate the transverse stress T_{22} by substituting $(iv), (v), (vi)$ and (vii) into the constitutive relation (i). This leads to $T_{22} = J^{-3}\big[f(J) - \beta\lambda^2\Lambda^2\big]$. Since we have a state of uniaxial stress in the x_1-direction, T_{22} must vanish:

$$T_{22} = \frac{1}{J^3}\big[f(J) - \beta\lambda^2\Lambda^2\big] = 0 \qquad \Rightarrow \qquad f(J) = \beta\lambda^2\Lambda^2. \qquad (viii)$$

While this gives $f(J)$ in a state of uniaxial stress, in order to obtain $f(J)$ in a form that can be used in *all* experiments, we have to express the right hand side of $(viii)$ in terms of J alone. We are told that the experiments give $\ln \Lambda = -\nu \ln \lambda + \ln c$, or equivalently $\Lambda = c\lambda^{-\nu}$. In the undeformed configuration we have $\lambda = \Lambda = 1$ which implies that $c = 1$. Thus

$$\Lambda = \lambda^{-\nu}. \qquad (ix)$$

We now solve (iii) and (ix) for λ and Λ in terms of J. This leads to

$$\lambda = J^{1/(1-2\nu)}, \qquad \Lambda = J^{-\nu/(1-2\nu)}. \qquad (x)$$

Substituting (x) into $(viii)$ gives

$$f(J) = \beta J^{\frac{2(1-\nu)}{1-2\nu}}. \qquad \square \qquad (xi)$$

(b) Substituting $(iv), (v), (vi)$ and (vii) into the constitutive relation (i) and calculating the axial stress T_{11} gives

$$T_{11} = \frac{1}{J^3}\big[f(J) - \beta\Lambda^4\big] \overset{(x),(xi)}{=} \frac{1}{\lambda^{3(1-2\nu)}}\big[\beta\lambda^{2-2\nu} - \beta\lambda^{-4\nu}\big] = \beta\big[\lambda^{-1+4\nu} - \lambda^{-3+2\nu}\big]. \qquad \square \qquad (xii)$$

The Young's modulus is defined as

$$E := \left.\frac{d}{d\lambda}T_{11}\right|_{\lambda=1} = \beta\Big[(-1+4\nu)\lambda^{-2+4\nu} - (-3+2\nu)\lambda^{-4+2\nu}\Big]_{\lambda=1} = \beta\Big[(-1+4\nu) - (-3+2\nu)\Big] = 2(1+\nu)\beta. \ \square$$

Alternatively, define the normal strain ε by $\lambda = 1 + \varepsilon$ so that (xii) gives

$$T_{11} = \beta\big[(1+\varepsilon)^{-1+4\nu} - (1+\varepsilon)^{-3+2\nu}\big].$$

Approximating this for small ε using $(1+\varepsilon)^n = 1 + n\varepsilon \ldots$ leads to

$$T_{11} = \beta\big[(1 + (-1+4\nu)\varepsilon) - (1 + (-3+2\nu)\varepsilon) + \ldots\big] \approx 2(1+\nu)\beta\varepsilon.$$

The Young's modulus is the coefficient of ε:

$$E = 2(1+\nu)\beta. \qquad \square$$

Problem 4.5. (Spencer) Consider a body composed of a neo-Hookean material. In a reference configuration it occupies the unit cube $0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1$ and undergoes the deformation

$$y_1 = \lambda x_1 + k x_2, \qquad y_2 = \lambda^{-1} x_2, \qquad y_3 = x_3. \tag{o}$$

(a) Sketch the region occupied by the body in the deformed configuration noting the lengths of the edges.

(b) Calculate the components of the Cauchy and Piola stress tensors.

(c) Suppose the faces $x_3 = 0$ and $x_3 = 1$ are known to be traction-free. Simplify your answer to part (b).

(d) Calculate the force that must be applied to the face (that in the reference configuration corresponded to) $x_2 = 1$.

(e) Determine the (true) Cauchy traction that must be applied on the face (that in the reference configuration corresponded to) $x_1 = 1$.

Solution:

(a) We see from (o) that particles do not displace in the x_3-direction. Moreover, the u_1 and u_2 displacement components do not depend on x_3. Thus this deformation is planar (in the x_1, x_2-plane) meaning every section $x_3 =$ constant deforms identically and in-plane. Thus, in sketching the body we can simply look at the x_1, x_2-plane. Consider the four points O, A, B and C. In the reference configuration they have coordinates $(x_1, x_2, x_3) = (0,0,0), (1,0,0), (1,1,0)$ and $(0,1,0)$. Substituting this into (o) gives the coordinates of the points O', A', B' and C' in the deformed configuration $(y_1, y_2, y_3) = (0,0,0), (\lambda,0,0), (\lambda+k, \lambda^{-1}, 0)$ and $(k, \lambda^{-1}, 0)$. Figure 4.10 shows a view of $\mathcal{R}$ and $\mathcal{R}_*$ looking down the x_3-axis.

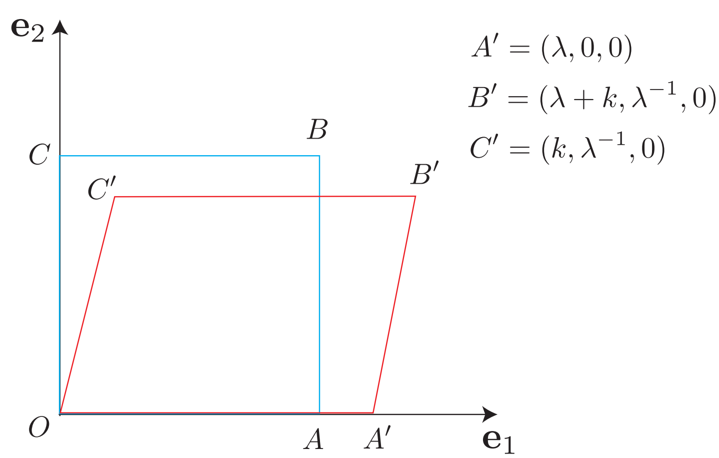

Figure 4.10: Side view of cube that has been biaxially stretched and sheared.

The lengths of the edges are

$$|OA'| = |B'C'| = \lambda, \qquad |A'B'| = |C'O| = \sqrt{k^2 + \lambda^{-2}} \tag{i}$$

(b) Differentiating (o) gives the components $F_{ij} = \partial y_i/\partial x_j$ of the deformation gradient tensor. The components of the left Cauchy-Green tensor can then be calculated from $[B] = [F][F]^T$:

$$[F] = \begin{pmatrix} \lambda & k & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad [B] = [F][F]^T = \begin{pmatrix} \lambda^2 + k^2 & k\lambda^{-1} & 0 \\ k\lambda^{-1} & \lambda^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{ii}$$

Note that

$$J = \det \mathbf{F} = 1, \tag{iii}$$

so that the incompressibility of the material places no restrictions on the parameters λ and k.

The components of the Cauchy stress tensor are given by the neo-Hookean constitutive relation $\mathbf{T} = -p\mathbf{I} + \mu\mathbf{B}$:

$$[T] = \begin{pmatrix} -p + \mu(\lambda^2 + k^2) & \mu k\lambda^{-1} & 0 \\ \mu k\lambda^{-1} & -p + \mu\lambda^{-2} & 0 \\ 0 & 0 & -p + \mu \end{pmatrix}. \tag{iv}$$

The corresponding components of the Piola stress tensor can be found from $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$ after noting that

$$[F]^{-1} = \begin{pmatrix} \lambda^{-1} & -k & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{v}$$

$$[S] = J[T][F]^{-T} = \begin{pmatrix} \lambda^{-1}T_{11} - kT_{12} & \lambda T_{12} & 0 \\ \lambda^{-1}T_{12} - kT_{22} & \lambda T_{22} & 0 \\ 0 & 0 & T_{33} \end{pmatrix}, \tag{vi}$$

where the T_{ij}'s are given in (iv).

(c) The traction components on the plane $x_3 = 1$ are T_{13}, T_{23} and T_{33}. The first two vanish automatically. The requirement $T_{33} = 0$ implies using (iv) that $p = \mu$. Using this in (iv) leads to

$$[T] = \begin{pmatrix} \mu(\lambda^2 + k^2 - 1) & \mu k\lambda^{-1} & 0 \\ \mu k\lambda^{-1} & \mu(\lambda^{-2} - 1) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{vii}$$

Substituting (vii) into (vi)

$$[S] = \begin{pmatrix} \mu(\lambda - \lambda^{-1}) & \mu k & 0 \\ \mu k & \mu(\lambda^{-1} - \lambda) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{viii}$$

(d) On the surface $x_2 = 1$ (whose edge is BC in the figure)

$$n_1^o = 0, \qquad n_2^o = 1, \qquad n_3^o = 0,$$

and so from $\mathbf{s} = \mathbf{S}\mathbf{n}_0$

$$s_1 = S_{12} = \mu k, \qquad s_2 = S_{22} = \mu(\lambda^{-1} - \lambda), \qquad s_3 = S_{32} = 0.$$

Thus the force can be calculated from $\mathsf{force} = \mathbf{s} \cdot A_x$ where for this surface $A_x = 1$. Thus

$$\text{force} = \mu k \, \mathbf{e}_1 + \mu(\lambda^{-1} - \lambda) \, \mathbf{e}_2.$$

Alternatively consider the current configuration. On surface $y_1 = \lambda^{-1}$ (whose edge is $B'C'$ in the figure)

$$n_1 = 0, \qquad n_2 = 1, \qquad n_3 = 0,$$

and so from $\mathbf{t} = \mathbf{T}\mathbf{n}$

$$t_1 = T_{12} = \mu k \lambda^{-1}, \qquad t_2 = T_{22} = \mu(\lambda^{-2} - 1), \qquad t_3 = T_{32} = 0.$$

Thus the force can be calculated from $\mathsf{force} = \mathbf{t} \cdot A_y$ where for this surface $A_y = \lambda$. This leads to (of course the same result

$$\text{force} = \mu k \mathbf{e}_1 + \mu(\lambda^{-1} - \lambda) \mathbf{e}_2.$$

(e) Finally consider the surface $x_1 = 1$ (whose edge is AB in the figure) on which

$$n_1^o = 1, \qquad n_2^o = 0, \qquad n_3^o = 0.$$

From $\mathbf{s} = \mathbf{S}\mathbf{n}_0$ we get

$$s_1 = S_{11} = \mu(\lambda - \lambda^{-1}), \qquad s_2 = S_{21} = \mu k, \qquad s_3 = S_{31} = 0.$$

The (true) Cauchy traction $\mathbf{t}$ and the Piola traction $\mathbf{s}$ are related by $\mathbf{t}.A_y = \mathbf{s}.A_x$, where for the surface under consideration here, $A_x = 1$ and $A_y = \sqrt{k^2 + \lambda^{-2}}$. Thus

$$\mathbf{t} = \frac{A_x}{A_y} \, \mathbf{s} = \frac{1}{\sqrt{k^2 + \lambda^{-2}}} \, \Big[\mu(\lambda - \lambda^{-1}) \mathbf{e}_1 + \mu k \mathbf{e}_2 \Big].$$

Alternatively one can calculate $\mathbf{t}$ using $\mathbf{t} = \mathbf{T}\mathbf{n}$ where $\mathbf{n}$ is the outward unit normal to the deformed image of $x_1 = 1$ (whose edge is $A'B'$ in the figure).

Problem 4.6. (Ball) Consider an incompressible isotropic elastic material characterized by the strain energy function

$$W = \alpha(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \beta(\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} - 3), \qquad (i)$$

where α and β are material constants.

(a) What restrictions (if any) do the Baker-Ericksen inequalities (4.106) impose on α and β?

(b) Assume that the restrictions determined in (a) hold. Consider a state of uniaxial stress in the x_3-direction. Show that the graph of the Piola stress S_{33} versus λ_3 is not everywhere increasing if

$$\beta^2 > 64\alpha(2\alpha + \beta). \tag{ii}$$

(This is related to the phenomenon of necking, in which a material can have an instability in tension leading to a greater extension and thinner deformed cross-section.)

Solution:

(a) The principal Cauchy stress components associated with (i) are given by

$$\left.\begin{aligned} T_1 &\overset{(4.67)}{=} \lambda_1 W_1 - q \overset{(i)}{=} (2\alpha\lambda_1^2 - \beta\lambda_1^{-1}) - q, \\ T_2 &\overset{(4.67)}{=} \lambda_2 W_2 - q \overset{(i)}{=} (2\alpha\lambda_2^2 - \beta\lambda_2^{-1}) - q, \\ T_3 &\overset{(4.67)}{=} \lambda_3 W_3 - q \overset{(i)}{=} (2\alpha\lambda_3^2 - \beta\lambda_3^{-1}) - q. \end{aligned}\right\} \tag{iii}$$

Therefore

$$T_1 - T_2 = 2\alpha(\lambda_1^2 - \lambda_2^2) - \beta(\lambda_1^{-1} - \lambda_2^{-1}) = \Big[2\alpha(\lambda_1 + \lambda_2) + \beta\lambda_1^{-1}\lambda_2^{-1}\Big](\lambda_1 - \lambda_2),$$

and (when $\lambda_1 \neq \lambda_2$),

$$\frac{T_1 - T_2}{\lambda_1 - \lambda_2} = \Big[2\alpha(\lambda_1 + \lambda_2) + \beta\lambda_1^{-1}\lambda_2^{-1}\Big].$$

The Baker-Ericksen inequalities requires this to be positive:

$$2\alpha(\lambda_1 + \lambda_2) + \beta\lambda_1^{-1}\lambda_2^{-1} > 0. \tag{iva}$$

This must hold for all λ_1 and λ_2. For it to hold when $\lambda_1\lambda_2 \to 0$ one must have $\beta > 0$, and when $\lambda_1 \to \infty$ we need $\alpha > 0$. Thus it is necessary that

$$\alpha > 0, \qquad \beta > 0. \tag{ivb}$$

Conversely, when (ivb) holds, it is seen immediately that (iva) holds (since the principal stretches are positive). Thus the inequalities (ivb) are necessary and sufficient for the Baker-Ericksen inequalities to hold.

(b). We now consider a state of uniaxial stress in the x_3-direction. Assume $\lambda_1 = \lambda_2$ so that by incompressibility

$$\lambda_1\lambda_2\lambda_3 = 1 \qquad \Rightarrow \qquad \lambda_1 = \lambda_2 = \lambda_3^{-1/2}. \tag{v}$$

By setting $T_1 = 0$ we find the reaction pressure q and then use that value of q to calculate T_3. From (iii),

$$T_1 = \lambda_1 W_1 - q = (2\alpha\lambda_1^2 - \beta\lambda_1^{-1}) - q \overset{(v)}{=} (2\alpha\lambda_3^{-1} - \beta\lambda_3^{1/2}) - q = 0 \quad \Rightarrow \quad q = 2\alpha\lambda_3^{-1} - \beta\lambda_3^{1/2},$$

and so

$$T_3 = \lambda_3 W_3 - q = (2\alpha\lambda_3^2 - \beta\lambda_3^{-1}) - q = (2\alpha\lambda_3^2 - \beta\lambda_3^{-1}) - 2\alpha\lambda_3^{-1} + \beta\lambda_3^{1/2}.$$

The corresponding Piola stress tensor component $S_{33} = S_3$ is related to T_3 by $S_3 = T_3\lambda_1\lambda_2$ (how?) and so

$$S_3 = T_3\lambda_1\lambda_2 = T_3\lambda_3^{-1} = 2\alpha\lambda_3 - (2\alpha + \beta)\lambda_3^{-2} + \beta\lambda_3^{-1/2}. \tag{vi}$$

The slope of the λ_3, S_3-curve is

$$\frac{dS_3}{d\lambda_3} = 2\alpha + 2(2\alpha+\beta)\lambda_3^{-3} - \frac{1}{2}\beta\lambda_3^{-3/2}. \qquad (vii)$$

Observe from (vii) and (iv) that

$$\frac{dS_3}{d\lambda_3}(1) = 6\alpha + \frac{3}{2}\beta > 0, \qquad \frac{dS_3}{d\lambda_3} \to 2\alpha > 0 \text{ as } \lambda_3 \to \infty, \qquad \frac{dS_3}{d\lambda_3} \to 2(2\alpha+\beta) > 0 \text{ as } \lambda_3 \to 0$$

We can write (vii) as

$$\frac{dS_3}{d\lambda_3} = 2(2\alpha+\beta)\xi^2 - \frac{1}{2}\beta\xi + 2\alpha$$

having set $\xi = \lambda_3^{-3/2}$. If $dS_3/d\lambda_3 = 0$ at some λ_3 then

$$2(2\alpha+\beta)\xi^2 - \frac{1}{2}\beta\xi + 2\alpha = 0. \qquad (viii)$$

For there to be a real value of ξ at which $(viii)$ holds the discriminant of this quadratic equation must be positive:

$$\left(\frac{1}{2}\beta\right)^2 - 4(2\alpha+\beta)(2\alpha) > 0 \qquad \Rightarrow \qquad \beta^2 > 64\alpha(2\alpha+\beta). \qquad (ix)$$

For this root to correspond to a real value of the *stretch*, this root must be positive. Keeping (iv) in mind, the coefficients of ξ^2 and ξ^0 in the quadratic equation $(viii)$ are positive, while the coefficient of ξ is negative. This guarantees that (both real) roots of $(viii)$ are positive (when (ix) holds).

Problem 4.7. For a particular isotropic, incompressible, elastic material, the stress-stretch relation (for the Cauchy stress) in uniaxial stress is

$$\tau = \mu(\lambda^2 - \lambda^{-1}), \qquad (i)$$

where $\mu > 0$ is a material parameter. Note that $\tau \to -\infty$ when $\lambda \to 0^+$ and $\tau \to \infty$ when $\lambda \to \infty$.

Determine *two* strain energy functions $W(I_1, I_2)$ that yield this same stress-stretch relation in uniaxial stress. For each W, calculate and plot the corresponding relation between the shear stress and amount of shear in simple shear.

Solution: In uniaxial stress we set $\lambda_1 = \lambda, \lambda_2 = \lambda_3 = \lambda^{-1/2}$ to get

$$\tau_1 = \tau = \lambda_1 \frac{\partial W^*}{\partial \lambda_1}\bigg|_{\lambda_1=\lambda,\lambda_2=\lambda_3=\lambda^{-1/2}} - q, \qquad \tau_2 = 0 = \lambda_2 \frac{\partial W^*}{\partial \lambda_2}\bigg|_{\lambda_1=\lambda,\lambda_2=\lambda_3=\lambda^{-1/2}} - q, \qquad (ii)$$

which by eliminating q gives

$$\tau = \left(\lambda_1 \frac{\partial W^*}{\partial \lambda_1} - \lambda_2 \frac{\partial W^*}{\partial \lambda_2}\right)\bigg|_{\lambda_1=\lambda,\lambda_2=\lambda_3=\lambda^{-1/2}}. \qquad (iii)$$

On setting

$$w(\lambda) := W^*(\lambda_1, \lambda_2, \lambda_3)\Big|_{\lambda_1=\lambda,\lambda_2=\lambda_3=\lambda^{-1/2}} \qquad (iv)$$

we can write (iii) as

$$\tau = \lambda w'(\lambda) \tag{v}$$

which together with (i) gives

$$\lambda w'(\lambda) = \mu(\lambda^2 - \lambda^{-1}). \tag{vi}$$

Integrating and taking $w(1) = 0$ yields

$$w(\lambda) = \frac{\mu}{2}(\lambda^2 + 2\lambda^{-1} - 3). \tag{vii}$$

Therefore from (iv) and (vii),

$$W^*(\lambda_1, \lambda_2, \lambda_3)\Big|_{\lambda_1=\lambda, \lambda_2=\lambda_3=\lambda^{-1/2}} = \frac{\mu}{2}(\lambda^2 + 2\lambda^{-1} - 3). \tag{$viii$}$$

Since

$$I_1 = \lambda^2 + 2\lambda^{-1}, \qquad I_2 = 2\lambda + \lambda^{-2}, \tag{ix}$$

we can write $(viii)$ equivalently as

$$\widetilde{W}(I_1, I_2)\Big|_{I_1=\lambda^2+2\lambda^{-1},\, I_2=2\lambda+\lambda^{-2}} = \frac{\mu}{2}(\lambda^2 + 2\lambda^{-1} - 3). \tag{x}$$

We see immediately from (ix) and (x) that

$$\widetilde{W}(I_1, I_2) = \frac{\mu}{2}(I_1 - 3) \tag{xi}$$

is consistent with the given information.

To determine additional forms of W, we aim to add a term to (xi) that vanishes identically in isochoric uniaxial tension but not in general. To this end we eliminate λ from (ix) to get

$$18 I_1 I_2 + I_1^2 I_2^2 - 4I_1^3 - 4I_2^3 - 27 = 0.$$

I obtained this using MATHEMATICA. (This relation between the invariants holds in isochoric uniaxial tension but not in general.) Therefore a second (family) of strain energy functions consistent with the given information is

$$\widetilde{W}(I_1, I_2) = \frac{\mu}{2}(I_1 - 3) + f(18 I_1 I_2 + I_1^2 I_2^2 - 4I_1^3 - 4I_2^3 - 27) \qquad \square$$

for a function f with $f(0) = 0$; f is arbitrary other than for the restrictions demanded by Section 4.6.3.

In simple shear,

$$I_1 = I_2 = 3 + k^2$$

and so

$$\overline{w}(k) := W^*(I_1, I_2)\Big|_{I_1=I_2=3+k^2} = \frac{\mu}{2}k^2 + f\big((4 + k^2)k^6\big)$$

and so the relation between the shear stress τ_{12} and the amount of shear k is

$$\tau_{12} = \overline{w}\,'(k) = \mu k + 8k^5(k^2 + 3)f'\big((4 + k^2)k^6\big). \qquad \square$$

Figure 4.11 shows plots of the response in simple shear for material (xi), and material (xii) with $f(I) = \mu I$.

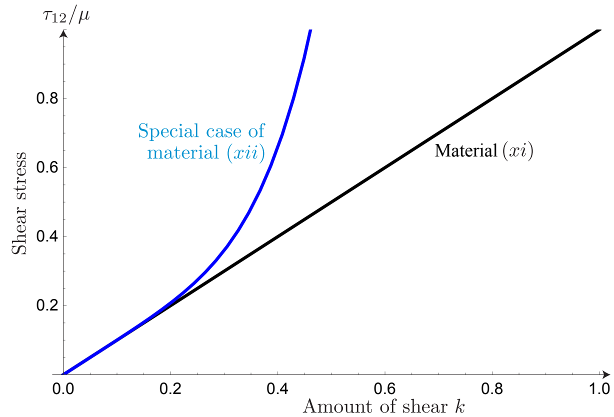

Figure 4.11: Response in simple shear. Black: $\tau_{12} = \mu k$ (material (xi)). Blue: $\tau_{12} = \mu k + 8\mu k^5(k^2 + 3)$ (case $f(I) = \mu I$ of material (xii)).

Problem 4.8. Can you construct an explicit example of a strain energy function for an isotropic *unconstrained* material that has a Poisson's ratio (at infinitesimal deformations) equal to 1/2 even though the material is compressible at finite deformations? Ensure that the energy and stress in the reference configuration vanish; that the Baker-Ericksen inequalities hold; that $W \to \infty$ when $J \to 0^+$ and $J \to \infty$; and the Legendre Hadamard condition (page 384) at $\mathbf{F} = \mathbf{I}$ holds. If you are able to construct such a W, determine and sketch a graph of the associated pressure-volume relation in pure dilatation.

Problem 4.9. (Based on Anand [1]) Consider an unconstrained isotropic elastic material characterized by the strain energy function

$$W(\mathbf{E}) = \mu\,\mathbf{E} \cdot \mathbf{E} + \frac{\alpha}{2}\big(\mathrm{tr}\,\mathbf{E}\big)^2 \qquad (i)$$

where

$$\mathbf{E} = \ln \mathbf{U} \qquad (ii)$$

is the Hencky (logarithmic) strain tensor, and μ and α are material constants.

(a) Show that this is identical to the strain energy function (i) in Problem 4.3.

(b) Under what conditions does this material satisfy the Baker-Ericksen inequalities?

(c) Determine the linearized form of W at infinitesimal deformations and thus interpret μ and α.

(d) Calculate and sketch the relation between τ_1 and λ_1 in a uniaxial *deformation* $\lambda_2 = \lambda_3 = 1$.

(e) Consider a so-called pure shear deformation $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}, \lambda_3 = 1$. Calculate and sketch the relation between τ_1 and λ.

(f) Does this strain energy function W satisfy the convexity condition (4.111)? Is it strongly elliptic?

Some general considerations.

Problem 4.10. Show that an elastic material is isotropic if and only if

$$\mathbf{TB} = \mathbf{BT} \tag{i}$$

where $\mathbf{T}$ is the Cauchy stress tensor and $\mathbf{B}$ is the left Cauchy-Green deformation tensor.

Problem 4.11. If an elastic material is isotropic, show that

$$\mathbf{S}^T\mathbf{R} = \mathbf{R}^T\mathbf{S}, \tag{i}$$

and hence that the Biot stress tensor can be written (in this case) as

$$\mathbf{S}^{(1)} = \mathbf{S}^T\mathbf{R}. \tag{ii}$$

Solution: Recall that the Piola stress tensor $\mathbf{S}$ and the Cauchy stress tensor $\mathbf{T}$ are related by

$$\mathbf{T} = \frac{1}{J}\mathbf{SF}^T = \frac{1}{J}\mathbf{FS}^T. \tag{iii}$$

If the material is isotropic, $\mathbf{T}$ and $\mathbf{V}$ are coaxial (see remark above (4.41)), and so by Problem 1.22,

$$\mathbf{TV} = \mathbf{VT}. \tag{iv}$$

Substituting (iii) into (iv) and using the polar decomposition $\mathbf{F} = \mathbf{VR}$ gives

$$\mathbf{FS}^T\mathbf{V} = \mathbf{VSF}^T \qquad \Rightarrow \qquad \mathbf{VRS}^T\mathbf{V} = \mathbf{VSR}^T\mathbf{V} \qquad \Rightarrow \qquad \mathbf{RS}^T = \mathbf{SR}^T.$$

Multiplying this by $\mathbf{R}^T$ from the front and by $\mathbf{R}$ from the back and using $\mathbf{RR}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$ leads to the desired result:

$$\mathbf{S}^T\mathbf{R} = \mathbf{R}^T\mathbf{S}. \qquad \square$$

The general definition of the Biot stress tensor is

$$\mathbf{S}^{(1)} = \frac{1}{2}(\mathbf{S}^T\mathbf{R} + \mathbf{R}^T\mathbf{S}).$$

So when (i) holds, (ii) follows immediately.

Problem 4.12. (*Cauchy elasticity*) See Problem 4.13 for an important observation. An elastic material that is not hyperelastic is called a Cauchy elastic material. One cannot associate a strain energy function $W(\mathbf{F})$ with a Cauchy elastic material. The Cauchy stress in such a material is related to the deformation through

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F}), \tag{i}$$

where the stress response function $\widehat{\mathbf{T}}$ is defined for all tensors with positive determinant. Explain why material frame indifference requires

$$\widehat{\mathbf{T}}(\mathbf{QF}) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T, \tag{ii}$$

for all proper orthogonal tensors $\mathbf{Q}$ and all tensors $\mathbf{F}$ with positive determinant. Explain also why the material symmetry group $\mathcal{G}$ for such a material is defined to be the set of all proper orthogonal tensors $\mathbf{Q}$ for which

$$\widehat{\mathbf{T}}(\mathbf{F}) = \widehat{\mathbf{T}}(\mathbf{FQ}), \qquad \mathbf{Q} \in \mathcal{G}, \tag{iii}$$

for all tensors $\mathbf{F}$ with positive determinant .

(a) Show that (ii) holds if and only if

$$\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{R}\widehat{\mathbf{T}}(\mathbf{U})\mathbf{R}^T. \tag{v}$$

(b) Assume that (ii) holds. Show that (iii) holds for a particular $\mathbf{Q} \in \mathcal{G}$ if and only if

$$\widehat{\mathbf{T}}(\mathbf{QFQ}^T) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T, \tag{iv}$$

for all tensors $\mathbf{F}$ with positive determinant.

(c) Show that a material is isotropic if and only if

$$\widehat{\mathbf{T}}(\mathbf{QVQ}^T) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{V})\mathbf{Q}^T, \tag{vi}$$

for all proper orthogonal tensors $\mathbf{Q}$.

(d) If the material is isotropic show that

$$\widehat{\mathbf{T}}(\mathbf{V})\mathbf{V} = \mathbf{V}\widehat{\mathbf{T}}(\mathbf{V}). \tag{vii}$$

Solution:

(d) Let λ and $\boldsymbol{\ell}$ be an eigenvalue and normalized eigenvector of $\mathbf{V}$:

$$\mathbf{V}\boldsymbol{\ell} = \lambda\boldsymbol{\ell}.$$

Pick $\mathbf{Q}$ to be the rotation through an angle π about the axis $\boldsymbol{\ell}$. Then from Problem 1.11,

$$\mathbf{Q} = 2\boldsymbol{\ell} \otimes \boldsymbol{\ell} - \mathbf{I}, \qquad \mathbf{Q}\boldsymbol{\ell} = \boldsymbol{\ell}.$$

One can readily verify for this $\mathbf{Q}$ that

$$\mathbf{QVQ}^T = \mathbf{V},$$

and therefore from (vi) that

$$\widehat{\mathbf{T}}(\mathbf{V}) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{V})\mathbf{Q}^T \qquad \Rightarrow \quad \widehat{\mathbf{T}}(\mathbf{V})\mathbf{Q} = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{V}).$$

Consequently

$$\mathbf{Q}\widehat{\mathbf{T}}(\mathbf{V})\boldsymbol{\ell} = \widehat{\mathbf{T}}(\mathbf{V})\mathbf{Q}\boldsymbol{\ell} = \widehat{\mathbf{T}}(\mathbf{V})\boldsymbol{\ell}$$

and therefore $\widehat{\mathbf{T}}(\mathbf{V})\boldsymbol{\ell}$ is an eigenvector of $\mathbf{Q}$. But $\mathbf{Q}$ is a rotation and therefore it has only one eigenvector and it is its axis of rotation $\boldsymbol{\ell}$ (see Problem 1.57). Thus $\widehat{\mathbf{T}}(\mathbf{V})\boldsymbol{\ell}$ has to be parallel to $\boldsymbol{\ell}$:

$$\widehat{\mathbf{T}}(\mathbf{V})\boldsymbol{\ell} = \tau\boldsymbol{\ell}$$

for some scalar τ. This shows that $\boldsymbol{\ell}$ is an eigenvector of $\widehat{\mathbf{T}}(\mathbf{V})$. Thus all three eigenvectors of $\mathbf{V}$ are eigenvectors of $\widehat{\mathbf{T}}(\mathbf{V})$ and so $\widehat{\mathbf{T}}(\mathbf{V})$ and $\mathbf{V}$ are coaxial. Therefore by Problem 1.22,

$$\widehat{\mathbf{T}}(\mathbf{V})\mathbf{V} = \mathbf{V}\widehat{\mathbf{T}}(\mathbf{V}).$$

Problem 4.13. (*Cauchy elasticity*) For a Cauchy elastic material, the stress response function $\mathbf{T}(\mathbf{F})$ is not derivable from a strain energy function $W(\mathbf{F})$. This is in contrast to a Green elastic (or hyperelastic) material where it is. The present example, taken from Carroll[23], concerns a particular Cauchy elastic material undergoing a particular loading cycle. You are asked to demonstrate that the total work done in this cycle is negative even though it is closed, i.e. despite the initial and final configurations of the body being the same. Therefore the material is a source of energy and can be used to construct a perpetual motion machine! This casts a shadow on Cauchy elastic materials (unless they are in fact Green elastic).

Consider a body that occupies a unit cube in a reference configuration with its edges parallel to the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. It undergoes a (quasi-static), spatially homogeneous, motion of the following plane strain, biaxial form:

$$y_1 = \lambda_1(t)x_1, \qquad y_2 = \lambda_2(t)x_2, \qquad y_3 = x_3, \tag{i}$$

over some time interval. According to $(4.19)_2$, frame indifference requires the constitutive relation to have the form

$$\mathbf{S} = \mathbf{R}\,\widehat{\mathbf{S}}(\mathbf{U}). \tag{ii}$$

and the material is isotropic when $\widehat{\mathbf{S}}(\mathbf{Q}\mathbf{U}\mathbf{Q}^T) = \mathbf{Q}\widehat{\mathbf{S}}(\mathbf{U})\mathbf{Q}^T$ for all rotations $\mathbf{Q}$. Here we consider the particular isotropic material

$$\widehat{\mathbf{S}}(\mathbf{U}) = 2\mu\ln\mathbf{U} + \beta\,[\mathrm{tr}\,(\ln\mathbf{U})]\,\mathbf{I}, \qquad \mu > 0,\ \beta > 0, \tag{iii}$$

$\ln\mathbf{U}$ being the Lagrangian logarithmic (Hencky) strain and μ and β material constants. One can show that it obeys the Baker-Ericksen inequalities for stretches $\lambda_i \geq 1$.

Suppose that the body undergoes the particular quasi-static loading history depicted on the λ_1, λ_2-plane in Figure 4.12. It is comprised of 4 stages, and the body returns eventually to its original undeformed configuration $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

Stage I: $\lambda_1(t)$ increases from 1 to $a\,(>1)$ while λ_2 remains constant at $\lambda_2(t) = 1$.

Stage II: $\lambda_2(t)$ increases from 1 to $b\,(>1)$ while λ_1 remains constant at $\lambda_1(t) = a$.

Stage III: $\lambda_1(t)$ decreases from $a\,(>1)$ to 1 while λ_2 remains constant at $\lambda_2(t) = b$.

Stage IV: $\lambda_2(t)$ decreases from $b\,(>1)$ to 1 while λ_1 remains constant at $\lambda_1(t) = 1$.

The deformed shape of the body, at the beginning and end of each loading stage, is depicted in Figure 4.13.

[23] M. M. Carroll, Must elastic materials be hyperelastic?, *Mathematics and Mechanics of Solids*, Volume 14, 2009, pp. 369–376.

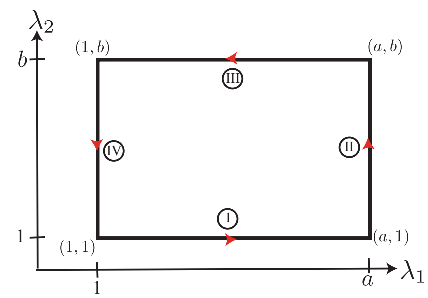

Figure 4.12: Plane strain ($\lambda_3 = 1$) loading path in the λ_1, λ_2-plane, starting and ending at $(\lambda_1, \lambda_2) = (1, 1)$.

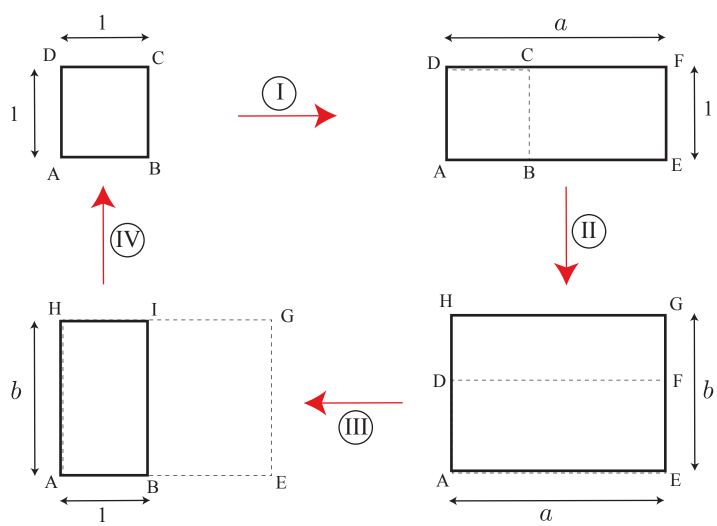

Figure 4.13: Deformed shape of the body (in the $\mathbf{e}_1, \mathbf{e}_2$-plane) at the beginning and end of each loading step.

Calculate the total work done in this cyclic loading. If $b > a$ show that this work is negative!

Solution: For a deformation

$$y_1 = \lambda_1(t)x_1, \qquad y_2 = \lambda_2(t)x_2, \qquad y_3 = \lambda_3(t)x_3,$$

we have

$$\mathbf{F} = \mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i, \qquad \mathbf{R} = \mathbf{I}, \tag{iv}$$

and so

$$\ln \mathbf{U} = \sum_{i=1}^{3} \ln(\lambda_i) \mathbf{e}_i \otimes \mathbf{e}_i.$$

Therefore the constitutive relation $(ii), (iii)$ yields

$$\mathbf{S} = \sum_{i=1}^{3} \sigma_i(\lambda_1, \lambda_2, \lambda_3) \mathbf{e}_i \otimes \mathbf{e}_i \quad \text{where} \quad \sigma_i(\lambda_1, \lambda_2, \lambda_3) = 2\mu \ln(\lambda_i) + \beta \ln(\lambda_1 \lambda_2 \lambda_3). \tag{v}$$

The principal Cauchy stresses are $\tau_i = \sigma_i \lambda_i / J$ and the Baker-Ericksen inequalities

$$(\tau_1 - \tau_2)/(\lambda_1 - \lambda_2) > 0 \qquad \text{for } \lambda_1 \neq \lambda_2.$$

can be shown to hold for $\lambda_i \geq 1$ (exercise).

From (iv) and $(v)_1$ the stress power is

$$\mathbf{S} \cdot \dot{\mathbf{F}} = \sum_{i=1}^{3} \sigma_i(\lambda_1, \lambda_2, \lambda_3) \dot{\lambda}_i. \tag{vi}$$

The total rate of working in general would be the volume integral of $\mathbf{S} \cdot \dot{\mathbf{F}}$ over the body in the reference configuration, and the work would be the time integral of this. In our present case, since all fields are uniform over the body and the volume of the body in the reference configuration is unity, the total work done during some time interval $[t_1, t_2]$ is simply the time integral of $\mathbf{S} \cdot \dot{\mathbf{F}}$:

$$\mathbb{W}_{t_1 \to t_2} = \int_{t_1}^{t_2} \mathbf{S} \cdot \dot{\mathbf{F}} \, dt = \int_{t_1}^{t_2} \left[\sum_{i=1}^{3} \sigma_i(\lambda_1, \lambda_2, \lambda_3) \, \dot{\lambda}_i \right] dt. \tag{vii}$$

For the plane strain deformation (i) with $\lambda_3 = 1$ we can write $(v)_2$ as

$$\sigma_1 = \alpha \ln \lambda_1 + \beta \ln \lambda_2, \qquad \sigma_2 = \alpha \ln \lambda_2 + \beta \ln \lambda_1, \qquad \sigma_3 = \beta \ln \lambda_1 \lambda_2,$$

where

$$\alpha = 2\mu + \beta > 0.$$

We now calculate the work done during each stage:

Stage I: $\lambda_1(t)$ increases from 1 to a, $\lambda_2(t) = 1$:

$$W_I = \int_1^a \sigma_1(\lambda_1, 1) d\lambda_1 = \int_1^a \alpha \lambda_1 \, d\lambda_1 = \alpha(a \ln a - a + 1).$$

Stage II: $\lambda_2(t)$ increases from 1 to b, $\lambda_1(t) = a$:

$$W_{II} = \int_1^b \sigma_2(a, \lambda_2) d\lambda_2 = \int_1^b \Big[\alpha \lambda_2 + \beta \ln a \Big] d\lambda_2 = \alpha(b \ln b - b + 1) + \beta(b-1) \ln a.$$

Stage III: $\lambda_1(t)$ decreases from a to 1, $\lambda_2(t) = b$:

$$W_{III} = \int_a^1 \sigma_1(\lambda_1, b) d\lambda_1 = \int_a^1 \Big[\alpha \lambda_1 + \beta \ln b \Big] d\lambda_1 = \alpha(-a \ln a + a - 1) + \beta(1-a) \ln b.$$

Stage IV: $\lambda_2(t)$ decreases from b to 1, $\lambda_1(t) = 1$:

$$W_{IV} = \int_b^1 \sigma_2(1, \lambda_2) d\lambda_2 = \int_b^1 \alpha \lambda_2 \, d\lambda_2 = \alpha(-b \ln b + b - 1).$$

Therefore the total work done over the loading cycle is

$$W_{total} = W_I + W_{II} + W_{III} + W_{IV} = \beta \Big[(b-1) \ln a - (a-1) \ln b \Big].$$

Observe (since $\ln x/(x-1)$ is monotonically decreasing for $x > 1$) that

$$W_{total} < 0 \quad \text{for} \quad b > a,$$

(and $W_{total} > 0$ for $a > b$).

Remark: If one is given the strain energy function $W(\lambda_1, \lambda_2, \lambda_3)$, to find the Piola stress components σ_i one simply differentiates W: $\sigma_i = \partial W/\partial \lambda_i$. On the other hand if one is given the three functions $\sigma_i(\lambda_1, \lambda_2, \lambda_3)$, in order to find the associated strain energy function W (if one exists), one must integrate

$$\frac{\partial W}{\partial \lambda_1} = \sigma_1(\lambda_1, \lambda_2, \lambda_3), \qquad \frac{\partial W}{\partial \lambda_2} = \sigma_2(\lambda_1, \lambda_2, \lambda_3), \qquad \frac{\partial W}{\partial \lambda_3} = \sigma_3(\lambda_1, \lambda_2, \lambda_3).$$

This is a set of 3 equations for the 1 unknown W and so it is, in general, an overdetermined system of equations. It can be solved only if the $\sigma_i(\lambda_1, \lambda_2, \lambda_3)$s satisfy certain consistency (compatibility) conditions. A necessary condition for the solvability of the preceding set of equations can be obtained by eliminating W from them. This can be achieved by first calculating

$$\frac{\partial \sigma_j}{\partial \lambda_i} = \frac{\partial}{\partial \lambda_i}\left(\frac{\partial W}{\partial \lambda_j}\right) = \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}, \qquad\qquad \frac{\partial \sigma_i}{\partial \lambda_j} = \frac{\partial}{\partial \lambda_j}\left(\frac{\partial W}{\partial \lambda_i}\right) = \frac{\partial^2 W}{\partial \lambda_j \partial \lambda_i},$$

which yields the necessary conditions

$$\frac{\partial \sigma_i}{\partial \lambda_j} = \frac{\partial \sigma_j}{\partial \lambda_i}.$$

It is easy to verify that (v) does not satisfy these conditions and so there is no strain energy function W associated with this material. It is Cauchy elastic and not Green elastic.

Problem 4.14. For an isotropic material one can express the Cauchy stress tensor $\mathbf{T}$ as

$$\mathbf{T} = \sum_{i=1}^{3} \tau_i \boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i, \tag{i}$$

where τ_1, τ_2, τ_3 are the principal Cauchy stresses and $\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3$ are the principal directions of both $\mathbf{T}$ and the Eulerian stretch tensor $\mathbf{V}$. Show that the Piola stress tensor $\mathbf{S}$ and the Biot stress tensor $\mathbf{S}^{(1)}$ (defined in (3.82)) can be written in the respective forms

$$\mathbf{S} = \sum_{i=1}^{3} \frac{J\tau_i}{\lambda_i} \boldsymbol{\ell}_i \otimes \mathbf{r}_i, \qquad \mathbf{S}^{(1)} = \sum_{i=1}^{3} \frac{J\tau_i}{\lambda_i} \mathbf{r}_i \otimes \mathbf{r}_i. \tag{4.168}$$

Observe that this, together with (4.48), yields (4.49) and (4.50).

Solution: The summation convention is suspended in this solution. From (2.56),

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \boldsymbol{\ell}_i \otimes \mathbf{r}_i, \qquad \mathbf{R} = \sum_{i=1}^{3} \boldsymbol{\ell}_i \otimes \mathbf{r}_i, \tag{ii}$$

and therefore

$$\mathbf{F}^{-1} = \sum_{i=1}^{3} (1/\lambda_i)\, \mathbf{r}_i \otimes \boldsymbol{\ell}_i \qquad \text{and} \qquad \mathbf{F}^{-T} = \sum_{i=1}^{3} (1/\lambda_i)\, \boldsymbol{\ell}_i \otimes \mathbf{r}_i. \tag{iii}$$

From (i), $(iii)_2$ and $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$

$$\begin{aligned}
\mathbf{S} &= J\left(\sum_{i=1}^{3}\tau_i \boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i\right)\left(\sum_{j=1}^{3}(1/\lambda_j)\,\boldsymbol{\ell}_j \otimes \mathbf{r}_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}(J\tau_i/\lambda_j)(\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i)(\,\boldsymbol{\ell}_j \otimes \mathbf{r}_j) = \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}(J\tau_i/\lambda_j)(\boldsymbol{\ell}_i \cdot \boldsymbol{\ell}_j)(\boldsymbol{\ell}_i \otimes \mathbf{r}_j) = \sum_{i=1}^{3}\sum_{j=1}^{3}(J\tau_i/\lambda_j)\delta_{ij}(\boldsymbol{\ell}_i \otimes \mathbf{r}_j) = \\
&= \sum_{i=1}^{3}(J\tau_i/\lambda_i)(\boldsymbol{\ell}_i \otimes \mathbf{r}_i), \qquad \square \qquad\qquad (iv)
\end{aligned}$$

where in the last step we used the substitution rule. This establishes $(4.168)_1$.

Likewise from $(ii)_2$, and (iv),

$$\mathbf{S}^T\mathbf{R} = \left(\sum_{i=1}^{3}(J\tau_i/\lambda_i)(\mathbf{r}_i \otimes \boldsymbol{\ell}_i)\right)\left(\sum_{j=1}^{3}\boldsymbol{\ell}_j \otimes \mathbf{r}_j\right) = \sum_{i=1}^{3}(J\tau_i/\lambda_i)(\mathbf{r}_i \otimes \mathbf{r}_i), \qquad (v)$$

and

$$\mathbf{R}^T\mathbf{S} = \left(\sum_{j=1}^{3}\mathbf{r}_j \otimes \boldsymbol{\ell}_j\right)\left(\sum_{i=1}^{3}(J\tau_i/\lambda_i)(\boldsymbol{\ell}_i \otimes \mathbf{r}_i)\right) = \sum_{i=1}^{3}(J\tau_i/\lambda_i)(\mathbf{r}_i \otimes \mathbf{r}_i), \qquad (vi)$$

and so the Biot stress tensor (for an isotropic material) can be expressed as

$$\mathbf{S}^{(1)} = \frac{1}{2}(\mathbf{S}^T\mathbf{R} + \mathbf{R}^T\mathbf{S}) \overset{(v),(vi)}{=} \mathbf{S}^T\mathbf{R} = \sum_{i=1}^{3}(J\tau_i/\lambda_i)(\mathbf{r}_i \otimes \mathbf{r}_i). \qquad \square$$

This establishes $(4.168)_2$.

Note from (v) and (vi) that $\mathbf{S}^T\mathbf{R} = \mathbf{R}^T\mathbf{S}$ (for an isotropic material) which is the result of Problem 4.11.

Problem 4.15. (*Biot stress*) It was shown in Problem 3.32 that the Biot stress

$$\mathbf{S}^{(1)} = \frac{1}{2}\left(\mathbf{S}^T\mathbf{R} + \mathbf{R}^T\mathbf{S}\right), \qquad (4.169)$$

is work conjugate to the Lagrangian stretch tensor $\mathbf{U}$:

$$\mathbf{S}^{(1)} \cdot \dot{\mathbf{U}} = \mathbf{S} \cdot \dot{\mathbf{F}}. \qquad (4.170)$$

(a) Show that $\mathbf{S}^{(1)}$ obeys the constitutive relation

$$\mathbf{S}^{(1)} = \frac{\partial \widehat{W}}{\partial \mathbf{U}}(\mathbf{U}). \qquad (4.171)$$

(b) For an isotropic elastic material show that the principal components of the Biot stress obey

$$S^{(1)}_k = \frac{\partial W^*}{\partial \lambda_k}(\lambda_1, \lambda_2, \lambda_3), \qquad k = 1, 2, 3, \qquad (4.172)$$

with associated principal directions $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ (the eigenvectors of $\mathbf{U}$), and therefore that

$$\mathbf{S}^{(1)} = \sum_{k=1}^{3}\frac{\partial W^*}{\partial \lambda_k}\,\mathbf{r}_k \otimes \mathbf{r}_k. \qquad (4.173)$$

(c) When the material is subjected to the constitutive constraint $\phi(\mathbf{U}) = 0$, show that

$$\mathbf{S}^{(1)} = \widehat{\mathbf{S}}^{(1)}(\mathbf{U}) - q\frac{\partial \phi}{\partial \mathbf{U}} \tag{4.174}$$

(d) For an incompressible isotropic material show that the principal components of the Biot stresses are

$$S_k^{(1)} = \frac{\partial W^*}{\partial \lambda_k} - q\lambda_k^{-1}, \qquad k = 1, 2, 3. \tag{4.175}$$

Solution:

(a) From (4.19), material frame difference requires $\widehat{\mathbf{S}}(\mathbf{F}) = \mathbf{R}\widehat{\mathbf{S}}(\mathbf{U})$. Substituting this into the expression (4.169) for the Biot stress gives

$$\widehat{\mathbf{S}}^{(1)} \stackrel{(4.19)}{=} \frac{1}{2}\Big(\widehat{\mathbf{S}}^T(\mathbf{U})\mathbf{R}^T\mathbf{R} + \mathbf{R}^T\mathbf{R}\widehat{\mathbf{S}}(\mathbf{U})\Big) = \frac{1}{2}\Big(\widehat{\mathbf{S}}^T(\mathbf{U}) + \widehat{\mathbf{S}}(\mathbf{U})\Big),$$

and so $\widehat{\mathbf{S}}^{(1)}$ is only a function of $\mathbf{U}$:

$$\mathbf{S}^{(1)} = \mathbf{S}^{(1)}(\mathbf{U}). \tag{i}$$

Since the stress $\mathbf{S}^{(1)}$ is work conjugate to the strain $\mathbf{U} - \mathbf{I}$ we have $\mathbf{S}^{(1)} \cdot \dot{\mathbf{U}} = \dot{W}$. Moreover, by material frame indifference $W = W(\mathbf{U})$. Therefore

$$\mathbf{S}^{(1)} \cdot \dot{\mathbf{U}} = \dot{W} = \frac{\partial W}{\partial \mathbf{U}} \cdot \dot{\mathbf{U}} \qquad \Rightarrow \qquad \left(\mathbf{S}^{(1)}(\mathbf{U}) - \frac{\partial W}{\partial \mathbf{U}}(\mathbf{U})\right) \cdot \dot{\mathbf{U}} = 0. \tag{ii}$$

Since this must hold for all $\dot{\mathbf{U}}$ and the terms inside the parenthesis are independent of $\dot{\mathbf{U}}$, they must vanish. This yields (4.171).

(b) For an isotropic material we see from (4.168) (page 421) that the principal directions of the Biot stress are $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ and the principal stresses are

$$S_k^{(1)} = \frac{J\tau_k}{\lambda_k} \stackrel{(4.48)}{=} \frac{\partial W^*}{\partial \lambda_k}. \tag{iii}$$

This establishes (4.172) and (4.173).

(c) Now consider a frame independent constraint described by $\phi(\mathbf{U}) = 0$ (see (4.62)). This implies that $\partial\phi/\partial\mathbf{U} \cdot \dot{\mathbf{U}} = 0$. We assume the stress to be expressible as $\mathbf{S}^{(1)} = \mathbf{S}^{(1)}(\mathbf{U}) + \mathbf{N}$ where $\mathbf{N}$ is the part of the stress that is not determined by the deformation and arises in reaction to the constraint. We assume the part $\mathbf{N}$ is workless which, since the stress power is given by $\mathbf{S}^{(1)} \cdot \dot{\mathbf{U}}$, requires $\mathbf{N} \cdot \dot{\mathbf{U}} = 0$. Therefore we need

$$\mathbf{N} \cdot \dot{\mathbf{U}} = 0 \quad \text{for all } \dot{\mathbf{U}} \text{ such that } \quad \frac{\partial \phi}{\partial \mathbf{U}} \cdot \dot{\mathbf{U}} = 0,$$

which by the same argument as in Section 4.5 implies that $\mathbf{N}$ is a scalar multiple of $\partial\phi/\partial\mathbf{U}$. This leads to (4.174).

(d) For an incompressible material the reaction stress to be added to the Piola stress tensor is $-q\mathbf{F}^{-T}$ it follows by substituting this into (4.169) that the reaction stress to be added to the Biot stress is

$$= \frac{1}{2}\Big(\big(-q\mathbf{F}^{-T}\big)^T\mathbf{R} + \mathbf{R}^T\big(-q\mathbf{F}^{-T}\big)\Big) = -\frac{q}{2}\Big(\big(\mathbf{R}\mathbf{U}^{-1}\big)^T\mathbf{R} + \mathbf{R}^T\big(\mathbf{R}\mathbf{U}^{-1}\big)\Big) = -q\mathbf{U}^{-1}.$$

Therefore from this and (4.171),

$$\mathbf{S}^{(1)} = \frac{\partial W}{\partial \mathbf{U}} - q\mathbf{U}^{-1}.$$

When the material is isotropic we can write this using (4.173) as

$$\mathbf{S}^{(1)} = \sum_{k=1}^{3} \frac{\partial W^*}{\partial \lambda_k} \mathbf{r}_k \otimes \mathbf{r}_k - q \sum_{k=1}^{3} \lambda_k^{-1} \mathbf{r}_k \otimes \mathbf{r}_k \tag{4.176}$$

which yields (4.172).

Problem 4.16. In Problem 2.44 we decomposed the deformation gradient tensor multiplicatively into the product of the hydrostatic tensor $J^{1/3}\mathbf{I}$ that described the volume change, and a second tensor $\overline{\mathbf{F}}$ that characterized the shape change:

$$\overline{\mathbf{F}} = J^{-1/3}\mathbf{F}, \qquad J = \det \mathbf{F}. \tag{i}$$

In this problem you are to decompose the constitutive relation for the Cauchy stress into a part due to the volume change and a part due to the shape change.

The modified left Cauchy-Green tensor $\overline{\mathbf{B}}$ associated with $\overline{\mathbf{F}}$, and its scalar invariants $\overline{I}_1, \overline{I}_2$ and $\overline{I}_3$, were shown in Problem 2.44 to obey

$$\overline{\mathbf{B}} = J^{-2/3}\mathbf{B}, \qquad \overline{I}_1 = J^{-2/3} I_1, \qquad \overline{I}_2 = J^{-4/3} I_2, \qquad \overline{I}_3 = 1. \tag{ii}$$

Here I_1, I_2 and $I_3 = J^2$ are the principal scalar invariants of $\mathbf{B}$. Show that there is a one-to-one relation between $\{I_1, I_2, J\}$ and $\{\overline{I}_1, \overline{I}_2, J\}$ and therefore that the strain energy function for an unconstrained isotropic elastic material can be written as

$$W = \mathcal{W}(\overline{I}_1, \overline{I}_2, J). \tag{iii}$$

Note that $\overline{\mathbf{F}}, \overline{\mathbf{B}}, \overline{I}_1$ and $\overline{I}_2$ are associated with the shape change. Show that the constitutive relation for the Cauchy stress can be expressed as

$$\mathbf{T} = \frac{2}{J}\left[-\frac{1}{3}\left(\overline{I}_1\mathcal{W}_1 + 2\overline{I}_2\mathcal{W}_2\right)\mathbf{I} + \left(\mathcal{W}_1 + \overline{I}_1\mathcal{W}_2\right)\overline{\mathbf{B}} - \mathcal{W}_2\overline{\mathbf{B}}^2\right] + \frac{\partial \mathcal{W}}{\partial J}\mathbf{I} \tag{iv}$$

where we have written

$$\mathcal{W}_\alpha := \frac{\partial \mathcal{W}}{\partial \overline{I}_\alpha}, \qquad \alpha = 1, 2.$$

If $\mathcal{W}$ is separable into a part that depends on $\overline{I}_1, \overline{I}_2$ only plus a part that depends on J only, i.e. $\mathcal{W}(\overline{I}_1, \overline{I}_2, J) = f(\overline{I}_1, \overline{I}_2) + g(J)$, then (iv) provides an additive decomposition of the constitutive relation for $\mathbf{T}$ into a part determined by $\overline{\mathbf{F}}$ and a part determined by J.

Problem 4.17. (Ericksen's theorem[24] on universal deformations for an unconstrained material. The situation for incompressible materials is rather different; see Problem 4.18 for an example.) In the absence of

[24] J.L. Ericksen, Deformations possible in every compressible, isotropic, perfectly elastic material. Journal of Mathematics and Physics, 34 (1955) 126–128.

body forces, a deformation $\mathbf{y}(\mathbf{x})$ obeys the equilibrium equations

$$\frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial F_{ij}}(\mathbf{F}(\mathbf{x})\right) = 0 \qquad \text{for all } \mathbf{x} \in \mathcal{R}_R \tag{i}$$

where $\mathbf{F}(\mathbf{x}) = \nabla\mathbf{y}(\mathbf{x})$. Suppose that a *particular* deformation $\mathbf{y}(\mathbf{x})$ is possible in *every* homogeneous, isotropic, unconstrained material (in the absence of body forces). Such a deformation $\mathbf{y}(\mathbf{x})$ obeys (i) for every choice of W and is called a *universal deformation.* Show that it is necessary and sufficient that a universal deformation be a homogeneous deformation.

Remark: Suppose one plans to perform experiments on a body in order to determine its constitutive relation, and suppose that these tests involve subjecting the body to certain deformations. Since we don't know W a priori, it is possible that the material at hand cannot in fact sustain the deformations to be applied. This theorem says that a homogeneous deformation is the only deformation that every (homogeneous, isotropic, unconstrained) material can sustain.

Solution: Since $\mathbf{F}(\mathbf{x})$ is constant in a homogeneous deformation, we see immediately that (i) holds for such a deformation no matter what the material W. In order to prove the converse, i.e. that the deformation must necessarily be homogeneous, we must show that $F_{ij,k}(\mathbf{x}) = 0$ on $\mathcal{R}_R$.

In the calculations to follow we keep the particular (as yet unknown) deformation $\mathbf{y}(\mathbf{x})$ *fixed* and vary the constitutive function W; we do this since we know that this particular deformation satisfies (i) for every W. Since (i) holds for all $W(I_1, I_2, I_3)$, it must necessarily hold for the choice $W = f(I_1)$. Then (i) specializes to

$$\frac{\partial}{\partial x_j}\left(f'(I_1)F_{ij}\right) = 0 \qquad \Rightarrow \qquad f'(I_1)F_{ij,j} + 2f''(I_1)F_{ij}F_{pq}F_{pq,j} = 0. \tag{ii}$$

First consider $f(I_1) = I_1$. Then (ii) reduces to

$$F_{ij,j}(\mathbf{x}) = 0 \qquad \text{for all } \mathbf{x} \in \mathcal{R}_R. \tag{iii}$$

(Remark: Since we can write this as $F_{ij,j} = y_{i,jj} = \nabla^2 y_i = 0$ it follows from a theorem concerning solutions of Laplace's equation that $y_i \in C^\infty(\mathcal{R}_R)$.) Now take $f(I_1) = I_1^2$ and keep in mind that equation (iii) still holds since we are continuing to consider the same deformation as above. Thus (ii) now simplifies to

$$F_{ij}F_{pq}F_{pq,j} = 0. \tag{iv}$$

Multiplying this by F^{-1}_{si} gives

$$F_{pq}F_{pq,j} = 0. \tag{v}$$

Differentiating (v) with respect to x_j

$$F_{pq,j}F_{pq,j} + F_{pq}F_{pq,jj} = 0 \qquad \Rightarrow \qquad F_{pq,j}F_{pq,j} + F_{pq}y_{p,jjq} = 0 \qquad \Rightarrow \qquad F_{pq,j}F_{pq,j} = 0, \tag{vi}$$

where in getting to the last expression we set $y_{p,jjq} = 0$ since $y_{p,jj} = 0$ from the line below (iii). The left-hand side of $(vi)_3$ involves the sum of the squares of $F_{pq,j}$ and therefore we must have

$$F_{pq,j}(\mathbf{x}) = 0 \qquad \text{for all } \mathbf{x} \in \mathcal{R}_R.$$

Therefore $\mathbf{F}(\mathbf{x})$ is necessarily constant on $\mathcal{R}_R$ and so $\mathbf{y}(\mathbf{x})$ is a homogeneous deformation.

Problem 4.18. (Related to Ericksen's problem on universal deformations. See Problem 4.17 for universal deformations in an unconstrained material.)

Consider a body composed of an isotropic *incompressible* elastic material that is in equilibrium with no body forces. Traction is prescribed on $\partial\mathcal{R}$ and this leads to a deformation $\mathbf{y} = \mathbf{y}(\mathbf{x})$ for which the associated principal scalar invariants $I_1(\mathbf{B})$ and $I_2(\mathbf{B})$ are constants independent of $\mathbf{x}$. Does this imply that the deformation is homogeneous?

Solution:

Reference: M. Singh and A. C. Pipkin, A note on Ericksen's problem, Zeitschrift fur angewandte Mathematik und Physik (ZAMP), 16(1965), pp. 706-709.

The answer is no. Here is the counter example given by Singh and Pipkin. (Fosdick previously gave the counter example $B = 0$.)

Consider the following deformation in cylindrical polar coordinates:

$$r = \widehat{r}(R,\Theta,Z) = AR, \qquad \theta = \widehat{\theta}(R,\Theta,Z) = B\ln R + C\Theta, \qquad z = \widehat{z}(R,\Theta,Z) = Z/(A^2C), \tag{i}$$

where A, B and C are *constants*. From the general formula (2.79), the associated left Cauchy-Green tensor is

$$\mathbf{B} = A^2\mathbf{e}_r\otimes\mathbf{e}_r + A^2K^2\mathbf{e}_\theta\otimes\mathbf{e}_\theta + A^{-4}C^{-2}\mathbf{e}_z\otimes\mathbf{e}_z + A^2B(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r), \tag{ii}$$

where $K^2 = B^2 + C^2$. Clearly the invariants of $\mathbf{B}$ will only depend on (the constants) A, B, C, so they are constants.

Therefore (i) is an inhomogeneous deformation with constant principal scalar invariants. We must verify that the deformation is isochoric and that it can be in equilibrium without body forces.

First, since the material is incompressible we must verify that the deformation is locally volume preserving:

$$\det\mathbf{B} = B_{zz}(B_{rr}B_{\theta\theta} - B_{r\theta}^2) = A^{-4}C^{-2}(A^4K^2 - A^4B^2) = 1. \tag{iii}$$

Second we confirm that the body is in equilibrium with no body forces. From (ii),

$$\mathbf{B}^2 = (B_{rr}^2 + B_{r\theta}^2)\mathbf{e}_r\otimes\mathbf{e}_r + (B_{\theta\theta}^2 + B_{r\theta}^2)\mathbf{e}_\theta\otimes\mathbf{e}_\theta + B_{zz}^2\mathbf{e}_z\otimes\mathbf{e}_z + B_{r\theta}(B_{rr} + B_{\theta\theta})(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r) \tag{iv}$$

Substituting (ii) and (iv) into the constitutive relation

$$\mathbf{T} = -q\mathbf{I} + \varphi_1\mathbf{B} + \varphi_2\mathbf{B}^2 \tag{v}$$

(where the φ's are functions of I_1 and I_2) gives

$$\begin{aligned}
T_{rr} &= -q + \varphi_1 A^2 + \varphi_2(A^4 + A^4B^2),\\
T_{\theta\theta} &= -q + \varphi_1 A^2K^2 + \varphi_2(A^4K^4 + A^4B^2),\\
T_{zz} &= -q + \varphi_1 A^{-4}C^{-2} + \varphi_2 A^{-8}C^{-4},\\
T_{r\theta} &= T_{\theta r} = \varphi_1 A^2B + \varphi_2 A^2B(A^2 + A^2K^2),\\
T_{\theta z} &= T_{zr} = 0.
\end{aligned} \tag{vi}$$

Note that the reaction pressure need not be constant: $q = q(r, \theta, z)$. On substituting these stresses into the equilibrium equations $\operatorname{div} \mathbf{T} = \mathbf{o}$ given in cylindrical polar coordinates in (3.95) we get

$$\frac{\partial q}{\partial r} = \frac{1}{r}(T_{rr} - T_{\theta\theta}), \qquad \frac{\partial q}{\partial \theta} = 2T_{r\theta}, \qquad \frac{\partial q}{\partial z} = 0. \tag{vii}$$

Solving (vii) yields

$$q(r, \theta, z) = (T_{rr} - T_{\theta\theta}) \ln r + 2\theta T_{r\theta} + q_0. \tag{viii}$$

Thus when $q(r, \theta, z)$ is given by $(viii)$, the stress field (vi) is in equilibrium with no body forces.

Problem 4.19. *Energy-Momentum Tensor*

The tensor

$$\mathbf{P} = W(\mathbf{F})\,\mathbf{I} - \mathbf{F}^T\mathbf{S} \tag{4.177}$$

is known as the **Energy-Momentum Tensor**. Consider a homogeneous elastic body that is in equilibrium (with no body forces).

(a) Show that

$$\operatorname{Div} \mathbf{P} = \mathbf{o} \qquad \text{at all } \mathbf{x} \in \mathcal{R}_R. \tag{i}$$

(b) Suppose that the body contains a cavity in its interior. Let $\mathcal{S}$ be an arbitrary closed surface in the body that encloses the cavity. Show that the value of the integral

$$\int_{\mathcal{S}} \mathbf{P}\mathbf{n}_R \, dA_x \tag{ii}$$

is the same for all such surfaces. (*Remark:* This result underlies the path-independent nature of the famous J-integral of fracture mechanics.)

(c) If the material is isotropic, show that

$$\mathbf{P} = \mathbf{P}^T. \tag{iii}$$

Solution:

(a) In order to establish (i) we work with components in a fixed cartesian basis:

$$\begin{aligned}
\frac{\partial P_{ij}}{\partial x_j} &= \frac{\partial}{\partial x_j}\big(W\delta_{ij} - F_{ki}S_{kj}\big) = \frac{\partial W}{\partial x_i} - \frac{\partial}{\partial x_j}\big(F_{ki}S_{kj}\big) = \frac{\partial W}{\partial F_{pq}}\frac{\partial F_{pq}}{\partial x_i} - \frac{\partial F_{ki}}{\partial x_j}S_{kj} - \frac{\partial S_{kj}}{\partial x_j}F_{ki} = \\
&\overset{(4.7),(3.63)}{=} S_{pq}\frac{\partial F_{pq}}{\partial x_i} - \frac{\partial F_{ki}}{\partial x_j}S_{kj} \overset{(2.22)}{=} S_{pq}\frac{\partial^2 y_p}{\partial x_i \partial x_q} - S_{kj}\frac{\partial^2 y_k}{\partial x_j \partial x_i} = \\
&\overset{(*)}{=} S_{pq}\frac{\partial^2 y_p}{\partial x_i \partial x_q} - S_{pq}\frac{\partial^2 y_p}{\partial x_q \partial x_i} \overset{(**)}{=} S_{pq}\frac{\partial^2 y_p}{\partial x_i \partial x_q} - S_{pq}\frac{\partial^2 y_p}{\partial x_i \partial x_q} = 0.
\end{aligned}$$

In step (*) we simply changed repeated subscripts, and in step (**) we changed the order of partial differentiation.

(b) First consider any subregion D of the body that does *not* include the cavity. It follows from the divergence theorem and (i) that

$$\int_{\partial D} \mathbf{P}\mathbf{n}_R \, dA_x = \mathbf{o}. \tag{iv}$$

Let $\partial\mathcal{R}_0$ be the the boundary of the cavity. Now choose for D the region between the closed surface $\mathcal{S}$ and the closed surface $\partial\mathcal{R}_0$. Then $\partial D = \partial\mathcal{S} \cup \partial\mathcal{R}_0$ and so (iv) gives

$$\mathbf{o} = \int_{\partial\mathcal{S}} \mathbf{P}\mathbf{n}_R \, dA_x + \int_{\partial\mathcal{R}_0} \mathbf{P}\mathbf{n}_R \, dA_x$$

(where $\mathbf{n}_R$ points inwards on $\partial\mathcal{R}_0$ and outwards on $\partial\mathcal{S}$. Thus we have

$$\int_{\partial\mathcal{S}} \mathbf{P}\mathbf{n}_R \, dA_x = -\int_{\partial\mathcal{R}_0} \mathbf{P}\mathbf{n}_R \, dA_x.$$

This holds for any closed surface $\mathcal{S}$ that encloses the cavity while the right-hand side is independent of $\mathcal{S}$. This establishes the result.

(c) Multiplying (operating on) the constitutive relation (4.38) for $\mathbf{S}$ by $\mathbf{F}^T$ from the front gives an equation of the form

$$\mathbf{F}^T\mathbf{S} = c_1\mathbf{I} + c_2\mathbf{C} + c_3\mathbf{C}^2$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Since $\mathbf{C}$ is symmetric it follows that $\mathbf{F}^T\mathbf{S}$ is symmetric and therefore by (4.177), so is $\mathbf{P}$.

Problem 4.20. In terms of the strain energy function $W^*(\lambda_1, \lambda_2, \lambda_3)$ for an isotropic unconstrained material, show that

$$\left.\frac{\partial^2 W^*}{\partial\lambda_i^2}\right|_{\lambda_1=\lambda_2=\lambda_3=1} = \lambda + 2\mu = \kappa + \frac{4}{3}\mu, \qquad \left.\frac{\partial^2 W^*}{\partial\lambda_i\partial\lambda_j}\right|_{\lambda_1=\lambda_2=\lambda_3=1} = \lambda = \kappa - \frac{2}{3}\mu \qquad i \neq j,$$

where λ is a Lamè constant and κ and μ are the bulk and shear moduli respectively at infinitesimal deformations.

Problem 4.21. Show that the constitutive relation $(4.38)_1$ for an isotropic unconstrained material can be written equivalently as

$$\mathbf{T} = \frac{2}{J}\left(I_2\frac{\partial\widetilde{W}}{\partial I_2} + I_3\frac{\partial\widetilde{W}}{\partial I_3}\right)\mathbf{I} \; + \; \frac{2}{J}\frac{\partial\widetilde{W}}{\partial I_1}\mathbf{B} \; - \; 2J\frac{\partial\widetilde{W}}{\partial I_2}\mathbf{B}^{-1}. \tag{4.178}$$

Materials with internal constraints.

Problem 4.22.

(a) Consider a body that is inextensible in a direction $\mathbf{m}_R$ (in the reference configuration). It may, for example, involve a family of very stiff fibers in that direction. Determine the corresponding reactive stress that needs to be added to the constitutively determined part of the stress for both the Cauchy stress and the Piola stress. Physically interpret the reactive part of the Cauchy stress.

(b) Now reconsider Problem 2.3 where the body involved two families of inextensible fibers in directions $\mathbf{m}_R^+$ and $\mathbf{m}_R^-$ and the material was incompressible. Write down the reaction stress that needs to be added to the constitutively determined part of the Cauchy stress.

Solution:

(a) The inextensibility constraint is described by $\phi(\mathbf{F}) = 0$ where

$$\phi(\mathbf{F}) = |\mathbf{F}\mathbf{m}_R|^2 - 1 = \mathbf{F}\mathbf{m}_R \cdot \mathbf{F}\mathbf{m}_R - 1. \qquad (i)$$

To calculate $\partial\phi/\partial\mathbf{F}$ we shall work with cartesian components:

$$\frac{\partial\phi}{\partial F_{pq}} = \frac{\partial}{\partial F_{pq}}(F_{ij}m_j^R F_{ik}m_k^R) = 2\delta_{ip}\delta_{jq}m_j^R F_{ik}m_k^R = 2F_{pk}m_k^R m_q^R, \qquad (ii)$$

i.e.

$$\frac{\partial\phi}{\partial\mathbf{F}} = 2\mathbf{F}\mathbf{m}_R \otimes \mathbf{m}_R. \qquad (iii)$$

Therefore the reaction stress to be added to the Piola stress is

$$\mathbf{N} = 2q\mathbf{F}\mathbf{m}_R \otimes \mathbf{m}_R. \qquad \square \qquad (iv)$$

Using $\mathbf{T} = J^{-1}\mathbf{S}\mathbf{F}^T$, the reaction stress to be added to the Cauchy stress is

$$= J^{-1}\mathbf{N}\mathbf{F}^T = 2\frac{q}{J}(\mathbf{F}\mathbf{m}_R \otimes \mathbf{m}_R)\mathbf{F}^T = 2\frac{q}{J}(\mathbf{F}\mathbf{m}_R \otimes \mathbf{F}\mathbf{m}_R) = 2\frac{q}{J}\mathbf{m} \otimes \mathbf{m}, \qquad (v)$$

where we used the identity $(\mathbf{a}\otimes\mathbf{b})\mathbf{A} = \mathbf{a}\otimes\mathbf{A}^T\mathbf{b}$ and set $\mathbf{m} = \mathbf{F}\mathbf{m}_R/|\mathbf{F}\mathbf{m}_R| = \mathbf{F}\mathbf{m}_R$. Note that $\mathbf{m}$ is the fiber direction in the deformed configuration and therefore we see from (v) that the reactive part of the Cauchy stress is a uniaxial stress in the direction of the deformed fiber. (The factors 2 and J can be absorbed into q in (v).)

(b) Note that the constraints

$$\phi_1(\mathbf{F}) = \det\,\mathbf{F} - 1, \qquad \phi_2(\mathbf{F}) = |\mathbf{F}\mathbf{m}_{\rm R}^+| - 1, \qquad \phi_3(\mathbf{F}) = |\mathbf{F}\mathbf{m}_{\rm R}^-| - 1$$

have to be enforced individually (and not as a single constraint $\phi(\mathbf{F}) = \det\,\mathbf{F} - 1 + |\mathbf{F}\mathbf{m}_{\rm R}^+| - 1 + |\mathbf{F}\mathbf{m}_{\rm R}^-| - 1$. Why?) There is a reaction stress associated with each constraint and each of them must be added to the constitutively determined part of the stress. Thus to the Cauchy stress we should add

$$q_1\mathbf{I} \;+\; q_2\mathbf{F}\mathbf{m}_{\rm R}^+ \otimes \mathbf{F}\mathbf{m}_{\rm R}^+ \;+\; q_3\mathbf{F}\mathbf{m}_{\rm R}^- \otimes \mathbf{F}\mathbf{m}_{\rm R}^-. \qquad \square$$

Problem 4.23. Reconsider the problem considered previously in Chapter 2 and illustrated in Figure 4.14. The material is incompressible and there are two families of inextensible fibers. In the reference configuration the fiber directions $\mathbf{m}_1$ and $\mathbf{m}_2$ are

$$\mathbf{m}_1 = \cos\Theta\,\mathbf{e}_1 + \sin\Theta\,\mathbf{e}_2, \qquad \mathbf{m}_2 = \cos\Theta\,\mathbf{e}_1 - \sin\Theta\,\mathbf{e}_2, \qquad 0 < \Theta < \pi/2. \qquad (i)$$

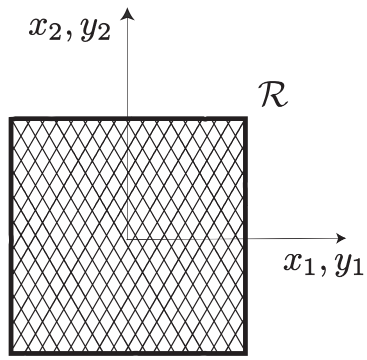

Figure 4.14: Region occupied (in a reference configuration) by an incompressible rectangular block with two families of inextensible fibers.

The two faces perpendicular to the y_3-axis are traction-free. The four faces perpendicular to the y_1- and y_2-axes are free of shear traction. Normal tractions are applied on these faces leading to the homogeneous stress state

$$\mathbf{T} = T_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + T_{22}\mathbf{e}_2 \otimes \mathbf{e}_2, \qquad (ii)$$

in the body. Assume the resulting deformation gradient tensor to have the form

$$\mathbf{F} = \lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3\mathbf{e}_3 \otimes \mathbf{e}_3. \qquad (iii)$$

The constitutive relation for this material is such that the stress is *only* due to the reaction stresses, i.e. assume that

$$\mathbf{T} = q_0\,\mathbf{I} + q_1\mathbf{F}\mathbf{m}_1 \otimes \mathbf{F}\mathbf{m}_1 + q_2\mathbf{F}\mathbf{m}_2 \otimes \mathbf{F}\mathbf{m}_2, \qquad (iv)$$

for some q_0, q_1, q_2. Given T_{11}, T_{22} and Θ, calculate λ_1, λ_2 and λ_3. Discuss your results.

Problem 4.24. Here we consider a body subjected to the following kinematic constraint: let the unit vector $\mathbf{m}_{\mathrm{R}}$ denote a direction in the reference configuration, and suppose that the area of any plane normal to $\mathbf{m}_{\mathrm{R}}$ cannot change. (Though the body is treated as a homogeneous continuum, it might, for example, be a solid that has a family of stiff parallel planes aligned normal to the direction $\mathbf{m}_{\mathrm{R}}$.) Determine the corresponding reaction stress.

Problem 4.25. Ericksen has suggested that certain elastic crystals obey the kinematic constraint

$$\mathrm{tr}\,\mathbf{C} = 3.$$

(a) Determine the associated reaction stress that should be added to the Cauchy stress. (b) Can an Ericksen material be isotropic? (See the discussion preceding (4.63).) If so, write down the constitutive relation between $\mathbf{T}$ and $\mathbf{B}$ for an isotropic Ericksen material.

(c) Show that for infinitesimal deformations Ericksen's constraint is equivalent to the incompressibility constraint. (d) Show for finite plane strain deformations that the only deformation that simultaneously satisfies the incompressibility constraint and Ericksen's constraint is a rigid deformation. (e) Is this true for all finite deformations?

Reference: J. L. Ericksen, Constitutive theory for some constrained elastic crystals, *International Journal of Solids and Structures*, Vol. 22, 1986, pp. 951-964.

Solution:

(a) Differentiating $\phi(\mathbf{F}) = \mathrm{tr}\,\mathbf{C} - 3 = \mathrm{tr}\,(\mathbf{F}^T\mathbf{F}) - 3$ with respect to $\mathbf{F}$ gives

$$\frac{\partial \phi}{\partial \mathbf{F}} = 2\mathbf{F}.$$

Therefore the reaction stress to be added to the Piola stress tensor is $q\mathbf{F}$. Since the Cauchy stress is related to the Piola stress by $\mathbf{T} = J^{-1}\mathbf{S}\mathbf{F}^T$ it follows that the reaction stress to be added to the Cauchy stress is

$$= q\mathbf{F}\mathbf{F}^T = q\mathbf{B} \qquad \square$$

where we have absorbed the scalar field J into q.

(b) Since $\phi(\mathbf{F}) = \mathrm{tr}\,\mathbf{C} - 3 = \mathrm{tr}\,\mathbf{B} - 3 = \mathrm{tr}\,\mathbf{F}\mathbf{F}^T - 3$ it follows that $\phi(\mathbf{F}) = \phi(\mathbf{F}\mathbf{Q})$ for all rotations $\mathbf{Q}$. Therefore this constraint, $\mathrm{tr}\,\mathbf{C} = 3$, places no restrictions on material symmetry. Thus such a material can be isotropic.

For an isotropic Ericksen material, since $I_1(\mathbf{C}) = 3$, the strain energy function depends only on the other two invariants: $W = \widetilde{W}(I_1, I_3)$ and so in place of (4.38) we get

$$\mathbf{T} = 2J\frac{\partial \widetilde{W}}{\partial I_3}\mathbf{I} \ + \ q\,\mathbf{B} \ - \ \frac{2}{J}\frac{\partial \widetilde{W}}{\partial I_2}\mathbf{B}^2,$$

having absorbed all scalar terms multiplying $\mathbf{B}$ into q.

(c) Let $\mathbf{H} = \nabla \mathbf{u}$ be the displacement gradient tensor so that $\mathbf{F} = \mathbf{I} + \mathbf{H}$. Thus

$$\det \mathbf{F} = \det(\mathbf{I} + \mathbf{H}) = 1 + \mathrm{tr}\,\mathbf{H} + \mathrm{O}(|\mathbf{H}|^2),$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = (\mathbf{I} + \mathbf{H})^T(\mathbf{I} + \mathbf{H}) = \mathbf{I} + \mathbf{H} + \mathbf{H}^T + \mathrm{O}(|\mathbf{H}|^2),$$

and so

$$\det \mathbf{F} - 1 = \mathrm{tr}\,\mathbf{H} + \mathrm{O}(|\mathbf{H}|^2) \qquad \mathrm{tr}\,\mathbf{C} - 3 = 2\mathrm{tr}\,\mathbf{H} + \mathrm{O}(|\mathbf{H}|^2).$$

Therefore in the case of infinitesimal deformations, the incompressibility constraint $\det \mathbf{F} = 1$ and the Ericksen constraint $\mathrm{tr}\,\mathbf{C} = 3$ both reduce to $\mathrm{tr}\,\mathbf{H} = 0$.

(d) The Ericksen constraint together with incompressibility requires

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 3, \qquad \lambda_1\lambda_2\lambda_3 = 1.$$

In plane strain, where $\lambda_3 = 1$, these specialize to

$$\lambda_1^2 + \lambda_2^2 = 2, \qquad \lambda_1\lambda_2 = 1.$$

Multiply first equation with λ_1^2 and use second equation to get

$$\lambda_1^4 + 1 = 2\lambda_1^2 \qquad \Rightarrow \qquad (\lambda_1^2 - 1)^2 = 0 \qquad \Rightarrow \qquad \lambda_1 = 1,$$

and so $\lambda_2 = 1$ as well from incompressibility. So all principal stretches are unity and so $\mathbf{C} = \mathbf{I}$ and the deformation is rigid.

(e) The Ericksen constraint together with incompressibility in a general deformation requires

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 3, \qquad \lambda_1^2\lambda_2^2\lambda_3^2 = 1.$$

Given I_1 and I_2, the triplet of equations

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = I_1, \qquad \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 = I_2, \qquad \lambda_1^2\lambda_2^2\lambda_3^2 = 1,$$

can be solved for real positive λ's if and only if (I_1, I_2) lies in a certain region of the I_1, I_2-plane. K.N. Sawyers, *Journal of Elasticity*, Volume 7, Number 1, 1977, pp. 99-102, has shown that this is the region between the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined parametrically by

$$\mathcal{C}_1: \qquad I_1 = 2\xi + 1/\xi^2, \quad I_2 = \xi^2 + 2/\xi, \quad 0 < \xi \le 3,$$

$$\mathcal{C}_2: \qquad I_1 = 2\xi + 1/\xi^2, \quad I_2 = \xi^2 + 2/\xi, \quad 3 \le \xi < \infty.$$

These curves lie in the quadrant $I_1 \ge 3, I_2 \ge 3$ and intersect at $(I_1, I_2) = (3, 3)$. So in particular, if $I_1 = 3$ then one must have $I_2 = 3$ (and vice versa) and the roots of the triplet of equations are $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Thus $\mathbf{C} = \mathbf{I}$ and the deformation is rigid.

Problem 4.26. (Chadwick) The only deformations that a particular body can undergo are those that preserve the angle between pairs of material fibers that, in the reference configuration, lie in the directions $\mathbf{m}_{\mathrm{R}}$ and $\mathbf{n}_{\mathrm{R}}$. Determine the associated reactive stress to be added to the Cauchy stress tensor.

Solution: Since the angle between these fibers in the deformed and undeformed configurations are the same, we have

$$\frac{\mathbf{F}\mathbf{m}_{\mathrm{R}}}{|\mathbf{F}\mathbf{m}_{\mathrm{R}}|} \cdot \frac{\mathbf{F}\mathbf{n}_{\mathrm{R}}}{|\mathbf{F}\mathbf{n}_{\mathrm{R}}|} = \mathbf{m}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}}. \tag{i}$$

Therefore, we can characterize this constraint by

$$\phi(\mathbf{F}) = \mathbf{F}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{F}\mathbf{n}_{\mathrm{R}} - (\mathbf{m}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}})|\mathbf{F}\mathbf{m}_{\mathrm{R}}|\,|\mathbf{F}\mathbf{n}_{\mathrm{R}}| = 0. \tag{ii}$$

Since we want to calculate $\partial\phi/\partial\mathbf{F}$ it is convenient to first show that (exercise)

$$\frac{\partial}{\partial \mathbf{F}}(\mathbf{F}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{F}\mathbf{n}_{\mathrm{R}}) = \mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{n}_{\mathrm{R}} + \mathbf{F}\mathbf{n}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}}, \qquad \frac{\partial}{\partial \mathbf{F}}|\mathbf{F}\mathbf{m}_{\mathrm{R}}| = \frac{\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}}}{|\mathbf{F}\mathbf{m}_{\mathrm{R}}|}. \tag{iii}$$

Thus, differentiating (ii) gives

$$\frac{\partial\phi}{\partial\mathbf{F}} = \mathbf{F}\mathbf{n}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}} + \mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{n}_{\mathrm{R}} - (\mathbf{m}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}})\frac{|\mathbf{F}\mathbf{n}_{\mathrm{R}}|}{|\mathbf{F}\mathbf{m}_{\mathrm{R}}|}\,\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}} - (\mathbf{m}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}})\frac{|\mathbf{F}\mathbf{m}_{\mathrm{R}}|}{|\mathbf{F}\mathbf{n}_{\mathrm{R}}|}\,\mathbf{F}\mathbf{n}_{\mathrm{R}} \otimes \mathbf{n}_{\mathrm{R}}. \tag{iv}$$

This can be simplified by using $\phi = 0$:

$$\frac{\partial \phi}{\partial \mathbf{F}} = \mathbf{F}\mathbf{n}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}} + \mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{n}_{\mathrm{R}} - (\mathbf{F}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{F}\mathbf{n}_{\mathrm{R}})\, \frac{\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}}}{|\mathbf{F}\mathbf{m}_{\mathrm{R}}|^2} - (\mathbf{F}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{F}\mathbf{n}_{\mathrm{R}})\, \frac{\mathbf{F}\mathbf{n}_{\mathrm{R}} \otimes \mathbf{n}_{\mathrm{R}}}{|\mathbf{F}\mathbf{n}_{\mathrm{R}}|^2}.$$

The reaction stress to be added to the Cauchy stress is a scalar multiple of

$$\frac{\partial \phi}{\partial \mathbf{F}}\mathbf{F}^T = \mathbf{F}\mathbf{n}_{\mathrm{R}} \otimes \mathbf{F}\mathbf{m}_{\mathrm{R}} + \mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{F}\mathbf{n}_{\mathrm{R}} - (\mathbf{F}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{F}\mathbf{n}_{\mathrm{R}})\, \frac{\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{F}\mathbf{m}_{\mathrm{R}}}{|\mathbf{F}\mathbf{m}_{\mathrm{R}}|^2} - (\mathbf{F}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{F}\mathbf{n}_{\mathrm{R}})\, \frac{\mathbf{F}\mathbf{n}_{\mathrm{R}} \otimes \mathbf{F}\mathbf{n}_{\mathrm{R}}}{|\mathbf{F}\mathbf{n}_{\mathrm{R}}|^2}. \qquad (v)$$

Denoting the directions of these material fibers in the deformed configuration by

$$\mathbf{m} = \frac{\mathbf{F}\mathbf{m}_{\mathrm{R}}}{|\mathbf{F}\mathbf{m}_{\mathrm{R}}|}, \qquad \mathbf{n} = \frac{\mathbf{F}\mathbf{n}_{\mathrm{R}}}{|\mathbf{F}\mathbf{n}_{\mathrm{R}}|}, \qquad (vi)$$

we can write (v) as

$$\frac{\partial \phi}{\partial \mathbf{F}}\mathbf{F}^T = |\mathbf{F}\mathbf{m}_{\mathrm{R}}||\mathbf{F}\mathbf{n}_{\mathrm{R}}|\, \Big[\, \mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n} - (\mathbf{m} \cdot \mathbf{n})\, \mathbf{m} \otimes \mathbf{m} - (\mathbf{m} \cdot \mathbf{n})\, \mathbf{n} \otimes \mathbf{n} \Big]. \qquad (vii)$$

The scalar factor $|\mathbf{F}\mathbf{m}_{\mathrm{R}}||\mathbf{F}\mathbf{n}_{\mathrm{R}}|$ can be absorbed into the scalar factor multiplying this and so we can write the reactive stress (for the Cauchy stress tensor) as

$$q\Big[\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n} - (\mathbf{m} \cdot \mathbf{n})\, \big[\mathbf{m} \otimes \mathbf{m} + \mathbf{n} \otimes \mathbf{n}\big]\Big]. \qquad \square$$

Restrictions on W.

Problem 4.27. (*Strong ellipticity.*) Consider a homogeneous body that is in equilibrium under the deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ where $\mathbf{F}$ is a constant tensor with positive determinant. Consider the (time-dependent) motion

$$\mathbf{y}(\mathbf{x}, t) = \mathbf{F}\mathbf{x} + \mathbf{u}(\mathbf{x}, t) \qquad \text{where} \quad |\nabla \mathbf{u}| \ll 1. \qquad (i)$$

This is a small perturbation superposed on the given homogeneous deformation. Show that the equation of motion $\operatorname{Div} \mathbf{S} = \rho_{\mathrm{R}} \ddot{\mathbf{y}}$ when linearized about the deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ reads

$$\mathbb{A}_{pqrs} \frac{\partial^2 u_r}{\partial x_q \partial x_s} = \rho_{\mathrm{R}} \ddot{u}_p \qquad \text{where} \quad \mathbb{A}_{pqrs}(\mathbf{F}) := \frac{\partial^2 W(\mathbf{F})}{\partial F_{pq} \partial F_{ra}}. \qquad (ii)$$

Suppose that the motion $\mathbf{u}(\mathbf{x}, t)$ is a plane harmonic wave propagating in a direction $\mathbf{n}$ with wave speed c and wave number k, the particle motion being in a direction $\mathbf{a}$:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{a}\; \mathrm{e}^{ik(\mathbf{n} \cdot \mathbf{x} - ct)}. \qquad (iii)$$

Here i (does not denote an integer but rather) is the unit imaginary number ($i^2 = -1$); $\mathbf{a}$ and $\mathbf{n}$ are constant unit vectors; and the scalars c and k are constants.

Show using (ii) that the wave speed c is given by

$$\rho_{\mathrm{R}} c^2 = \mathbb{A}_{pqrs} a_p n_q a_r n_s,$$

and therefore that material stability requires the strong ellipticity condition (4.115) to hold for all directions $\mathbf{a}$ and $\mathbf{n}$.

See Problem 4.28 for the case of an incompressible material.

Solution: It follows from (i) that

$$\nabla \mathbf{y} = \mathbf{F} + \nabla \mathbf{u}, \qquad \mathbf{v} = \dot{\mathbf{u}}, \qquad \dot{\mathbf{v}} = \ddot{\mathbf{u}}. \tag{v}$$

The associated stress, approximated for small $|\nabla \mathbf{u}|$, is

$$\mathbf{S}(\nabla \mathbf{y}) = \left.\frac{\partial W}{\partial \mathbf{F}}\right|_{\nabla \mathbf{y}=\mathbf{F}+\nabla \mathbf{u}} = \left.\frac{\partial W}{\partial \mathbf{F}}\right|_{\nabla \mathbf{y}=\mathbf{F}} + \mathbb{A}(\mathbf{F})\nabla \mathbf{u} + \text{h.o.t.} = \mathbf{S}(\mathbf{F}) + \mathbb{A}(\mathbf{F})\nabla \mathbf{u} + \text{h.o.t.}, \tag{vi}$$

where

$$\mathbb{A}_{pqrs}(\mathbf{F}) = \frac{\partial^2 W(\mathbf{F})}{\partial F_{pq} \partial F_{rs}} \qquad \text{and} \qquad \big(\mathbb{A}(\mathbf{F})\nabla \mathbf{u}\big)_{pq} = \mathbb{A}_{pqrs}(\mathbf{F}) \frac{\partial u_r}{\partial x_s}. \tag{vii}$$

Substituting $(v)_3$ and (vi) into the equation of motion

$$\frac{\partial S_{pq}}{\partial x_q} = \rho_R \dot{v}_p$$

gives

$$\mathbb{A}_{pqrs} \frac{\partial^2 u_r}{\partial x_q \partial x_s} = \rho_R \ddot{u}_p. \tag{$viii$}$$

Now consider the motion

$$u_r = a_r \ \mathrm{e}^{ik(\mathbf{n}\cdot\mathbf{x} - ct)} \tag{ix}$$

where i is the unit imaginary number, $\mathbf{a}$ and $\mathbf{n}$ are constant unit vectors and the scalar c is a constant. This gives

$$\frac{\partial u_r}{\partial x_q} = ika_r n_q \ \mathrm{e}^{ik(\mathbf{n}\cdot\mathbf{x} - ct)}, \qquad \frac{\partial^2 u_r}{\partial x_q \partial x_s} = -k^2 a_r n_q n_s \ \mathrm{e}^{ik(\mathbf{n}\cdot\mathbf{x} - ct)}, \qquad \ddot{u}_p = -k^2 c^2 a_p \ \mathrm{e}^{ik(\mathbf{n}\cdot\mathbf{x} - ct)}. \tag{x}$$

Substituting this into $(viii)$ gives

$$\mathbb{A}_{pqrs} a_r n_q n_s = \rho_R c^2 a_p. \tag{xi}$$

Multiplying by a_p gives

$$\mathbb{A}_{pqrs} a_p a_r n_q n_s = \rho_R c^2. \tag{xii}$$

For the wave speed c to be real (and non-zero) one must have

$$\mathbb{A}_{pqrs} a_p n_q a_r n_s > 0. \tag{$xiii$}$$

Remark: The 2-tensor $\mathbf{A}(\mathbf{n})$, known as the *acoustic tensor*, is defined as the tensor with cartesian components

$$A_{ik} := \mathbb{A}_{ijk\ell} n_j n_\ell. \tag{xiv}$$

Since $\mathbb{A}$ has the symmetry $\mathbb{A}_{ijk\ell} = \mathbb{A}_{k\ell ij}$, see $(ii)_2$, it follows that $\mathbf{A}$ is symmetric. Moreover, equation (xi) can now be written as

$$\mathbf{A}\mathbf{a} = \rho_R c^2 \mathbf{a}$$

which is the eigenvalue problem for $\mathbf{A}$. Strong ellipticity is therefore equivalent to the positive definiteness of the acoustic tensor in which event its three real eigenvalues must be positive.

Problem 4.28. (*Strong ellipticity for incompressible material.*)

By carrying out calculations analogous to those in Problem 4.27, derive the conditions for strong ellipticity for an incompressible elastic material. The calculations in Chapter 5.7.2 will be helpful.

Solution: In Chapter 5.7.2 we develop the equations governing an equilibrium deformation $\mathbf{y} = \mathbf{F}\mathbf{z} + \mathbf{u}(\mathbf{x})$ which is superposed on the homogeneous deformation $\mathbf{x} = \mathbf{F}\mathbf{z}$ for an incompressible material. It is straightforward to include inertial effects in that analysis and this leads to the following equation of motion

$$\frac{\partial \Sigma_{pq}}{\partial x_q} = \rho \ddot{u}_p, \qquad (i)$$

where

$$\Sigma_{pq} = \mathbb{B}_{pqrs} H_{rs} + q_0 H_{qp} - \widetilde{q}\delta_{pq}, \qquad H_{pq} = \frac{\partial u_p}{\partial x_q}, \qquad (ii)$$

$$\mathbb{A}_{pqrs}(\mathbf{F}) = \frac{\partial^2 W(\mathbf{F})}{\partial F_{pq} \partial F_{rs}}, \qquad \mathbb{B}_{abcd}(\mathbf{F}) = \mathbb{A}_{apcq}(\mathbf{F}) F_{bp} F_{dq}, \qquad (iii)$$

with incompressibility requiring

$$H_{pp} = \frac{\partial u_p}{\partial x_p} = 0. \qquad (iv)$$

We consider a perturbation of the form of a plane harmonic wave and so take

$$u_p(\mathbf{x}, t) = a_p \ \mathrm{e}^{ik(\mathbf{n}\cdot\mathbf{x} - ct)}, \qquad \widetilde{q} = ikQ \ \mathrm{e}^{ik(\mathbf{n}\cdot\mathbf{x} - ct)}, \qquad (v)$$

where $\widetilde{q}$ is the perturbation of the reactive pressure field. Substituting $(v)_1$ into $(ii)_2$ gives

$$H_{pq} = ika_p n_q \ \mathrm{e}^{ik(\mathbf{n}\cdot\mathbf{x} - ct)}, \qquad (vi)$$

and so the incompressibility equation (iv) yields

$$H_{pp} = 0 \qquad \Rightarrow \qquad \mathbf{a}\cdot\mathbf{n} = 0. \qquad (vii)$$

Substituting (v) into (ii) gives

$$\Sigma_{pq} = [ik\mathbb{B}_{pqrs} a_r n_s + ikq_0 a_q n_p - ikQ\delta_{pq}\,]\, \mathrm{e}^{ik(\mathbf{n}\cdot\mathbf{x} - ct)}, \qquad (viii)$$

and therefore

$$\frac{\partial \Sigma_{pq}}{\partial x_q} = \left[-k^2 \mathbb{B}_{pqrs} a_r n_s n_q + k^2 Q n_p\,\right] \mathrm{e}^{ik(\mathbf{n}\cdot\mathbf{x} - ct)}, \qquad (ix)$$

where the term involving q_0 has dropped out because of (iv). Substituting (ix) and $(v)_1$ into the equation of motion (i) gives

$$-k^2 \mathbb{B}_{pqrs} a_r n_s n_q + k^2 Q n_p = -\rho k^2 a_p c^2, \qquad (x)$$

that on multiplying by a_p and using (vii) leads to

$$\mathbb{B}_{pqrs} a_r a_p n_s n_q = \rho c^2. \qquad (xi)$$

Material stability requires c to be real, and taking it to be nonzero as well, we are led to the strong ellipticity condition

$$\mathbb{B}_{pqrs} a_p n_q a_r n_s > 0 \qquad \text{for all vectors } \mathbf{a} \text{ and } \mathbf{n} \text{ with } \mathbf{a}\cdot\mathbf{n} = 0. \qquad (4.179)$$

Remark: The inequality (4.180) can be written equivalently using (*iii*) and (*vii*) as

$$\mathbb{A}_{ijk\ell} a_i b_j a_k b_\ell > 0 \qquad \text{for all vectors } \mathbf{a} \text{ and } \mathbf{b} \text{ with } \mathbf{a} \cdot \mathbf{F}^{-T}\mathbf{b} = 0. \tag{4.180}$$

Problem 4.29. Consider an (isotropic, incompressible) generalized neo-Hookean material characterized by the strain energy function $W(I_1)$.

(a) Determine the restrictions imposed on W by the Baker-Ericksen inequalities.

(b) Find necessary and sufficient conditions for strong ellipticity.

Problem 4.30. *Baker-Ericksen:* Consider a uniaxial tensile stress state

$$\mathbf{T} = T\mathbf{e}_3 \otimes \mathbf{e}_3, \qquad T > 0,$$

in an isotropic elastic material. Show that the Baker-Ericksen inequalities (4.106) hold if and only if the principal stretches obey

$$\lambda_3 > \lambda_1 = \lambda_2 > 0. \tag{o}$$

Solution:

Reference: S. Marzano, An interpretation of Baker-Ericksen inequalities in uniaxial deformation and stress, *Meccanica*, volume 18, 1983 , pp. 233-235.

We show that BE implies (o). The proof of the converse is left as an exercise.

We first show that if the principal Cauchy stress components $T_1 = T_2$, then the BE inequalities imply $\lambda_1 = \lambda_2$. The constitutive relation for an isotropic elastic material can be written in the form

$$\mathbf{T} = \beta_0 \mathbf{I} + \beta_1 \mathbf{B} + \beta_{-1}\mathbf{B}^{-1}.$$

Therefore in a principal basis for $\mathbf{T}$ and $\mathbf{B}$,

$$T_1 = \beta_0 + \beta_1\lambda_1^2 + \beta_{-1}\lambda_1^{-2}, \qquad T_2 = \beta_0 + \beta_1\lambda_2^2 + \beta_{-1}\lambda_2^{-2}, \qquad T_3 = \beta_0 + \beta_1\lambda_3^2 + \beta_{-1}\lambda_3^{-2}, \tag{i}$$

and so

$$T_1 - T_2 = \Big[\beta_1 - \beta_{-1}\lambda_1^{-2}\lambda_2^{-2}\Big](\lambda_1^2 - \lambda_2^2). \tag{ii}$$

Therefore the BE inequalities require

$$\beta_1 - \beta_{-1}\lambda_1^{-2}\lambda_2^{-2} > 0, \qquad \lambda_1 \neq \lambda_2. \tag{iii}$$

Now suppose that $T_1 = T_2$. Then from (ii)

$$\Big[\beta_1 - \beta_{-1}\lambda_1^{-2}\lambda_2^{-2}\Big](\lambda_1^2 - \lambda_2^2) = 0. \tag{iv}$$

If $\lambda_1 \neq \lambda_2$, then (iv) leads to $\beta_1 - \beta_{-1}\lambda_1^{-2}\lambda_2^{-2} = 0$. This contradicts (iii). Therefore λ_1 and λ_2 cannot be distinct and (iv) implies $\lambda_1 = \lambda_2$. $\square$

Return to the constitutive relation (i) with $T_3 = T$ and $T_1 = 0$:

$$0 = \beta_0 + \beta_1\lambda_1^2 + \beta_{-1}\lambda_1^{-2}, \qquad T = \beta_0 + \beta_1\lambda_3^2 + \beta_{-1}\lambda_3^{-2}.$$

Subtracting the first from the second gives

$$T = \beta_1(\lambda_3^2 - \lambda_1^2) + \beta_{-1}(\lambda_3^{-2} - \lambda_1^{-2}) = (\beta_1 - \beta_{-1}\lambda_1^{-2}\lambda_3^{-2})(\lambda_3^2 - \lambda_1^2).$$

We are told that $T > 0$, and we know that the BE implies $\beta_1 - \beta_{-1}\lambda_1^{-2}\lambda_3^{-2} > 0$. It follows from the preceding equation that

$$\lambda_3 > \lambda_1.$$

Problem 4.31. *Strong ellipticity:* The notion of strong ellipticity was introduced previously in (4.110), (4.115).

Suppose the material is isotropic: $W(\mathbf{F}) = W(\lambda_1, \lambda_2, \lambda_3)$. By taking

$$\mathbf{F} = \lambda_1\, \mathbf{e}_1 \otimes \mathbf{e}_1 \ + \lambda_2\, \mathbf{e}_2 \otimes \mathbf{e}_2 \ + \lambda_3\, \mathbf{e}_3 \otimes \mathbf{e}_3, \quad \mathbf{a} = \mathbf{e}_1, \quad \mathbf{b} = \mathbf{e}_2,$$

in the strong ellipticity condition show that the Baker-Ericksen inequalities are implied by strong ellipticity. By taking

$$\mathbf{F} = \lambda_1\, \mathbf{e}_1 \otimes \mathbf{e}_1 \ + \lambda_2\, \mathbf{e}_2 \otimes \mathbf{e}_2 \ + \lambda_3\, \mathbf{e}_3 \otimes \mathbf{e}_3, \quad \mathbf{a} = \mathbf{e}_1, \quad \mathbf{b} = \mathbf{e}_1,$$

in the strong ellipticity condition show that

$$\frac{\partial^2 W}{\partial \lambda_1^2} > 0 \qquad \Rightarrow \qquad S_{11} \text{ is an increasing function of } \lambda_1.$$

It will be useful to note that by replacing $\mathbf{b}$ by $\mathbf{F}^T\mathbf{b}$ in the strong ellipticity inequality (4.115), it can be written equivalently as

$$\mathbb{B}_{ijk\ell}a_ib_ja_kb_\ell > 0 \qquad \text{for all vectors } \mathbf{a} \text{ and } \mathbf{b} \tag{4.181}$$

where

$$\mathbb{B}_{ijk\ell}(\mathbf{F}) = \mathbb{A}_{ipkq}(\mathbf{F})F_{jp}F_{\ell q}. \tag{4.182}$$

For an isotropic unconstrained material the components of $\mathbb{B}$ in the principal basis $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ are given in Problem 6.1.7 of Ogden [17] keeping in mind that his $\mathcal{A}^1_{0ijk} = \mathbb{B}_{ji\ell k}$:

$$\left.\begin{aligned} &\mathbb{B}_{iijj} = \lambda_j \frac{\partial \tau_i}{\partial \lambda_j} + (1 - \delta_{ij})\tau_i, \\ &\mathbb{B}_{jiji} = \frac{\lambda_i^2\,(\tau_i - \tau_j)}{\lambda_i^2 - \lambda_j^2}, \qquad i \neq j,\ \lambda_i \neq \lambda_j, \\ &\mathbb{B}_{jiij} = \mathbb{B}_{ijji} = \mathbb{B}_{jiji} - \tau_i, \qquad i \neq j, \end{aligned}\right\} \quad \text{where} \quad \tau_i = \frac{\lambda_i}{J}\frac{\partial W}{\partial \lambda_i}. \tag{4.183}$$

Solution: On taking $\mathbf{a} = \mathbf{e}_1$ and $\mathbf{b} = \mathbf{e}_2$, the inequality (4.181) specializes to

$$\mathbb{B}_{ijk\ell}\delta_{1i}\delta_{2j}\delta_{1k}\delta_{2\ell} = \mathbb{B}_{1212} \overset{(4.183)}{=} \frac{\lambda_2^2\,(\tau_2 - \tau_1)}{\lambda_2^2 - \lambda_1^2} > 0 \qquad \Rightarrow \qquad \frac{\tau_2 - \tau_1}{\lambda_2 - \lambda_1} > 0,$$

where in getting to the second inequality we used $\lambda_1 > 0, \lambda_2 > 0$. This is one of the Baker-Ericksen inequalities.

On taking $\mathbf{a} = \mathbf{e}_1$ and $\mathbf{b} = \mathbf{e}_1$, the inequality (4.181) specializes to

$$\mathbb{B}_{ijk\ell}\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1\ell} = \mathbb{B}_{1111} \overset{(4.183)}{=} \lambda_1 \frac{\partial \tau_1}{\partial \lambda_1} \overset{(4.183)}{=} \frac{\lambda_1}{\lambda_2\lambda_3}\frac{\partial^2 W}{\partial \lambda_1^2} > 0 \qquad \Rightarrow \qquad \frac{\partial^2 W}{\partial \lambda_1^2} > 0,$$

which is the monotonicity inequality (4.108).

Problem 4.32. The components of the elasticity tensor $\mathbb{A}$ are defined by

$$\mathbb{A}_{ijk\ell}(\mathbf{F}) = \frac{\partial^2 W(\mathbf{F})}{\partial F_{ij}\partial F_{k\ell}}.$$

Calculate these components for an unconstrained isotropic material characterized by the strain energy function

$$W(\mathbf{F}) = f(I_1) + g(J) \qquad \text{where} \quad I_1 = \mathbf{F}\cdot\mathbf{F},\ J = \det \mathbf{F}.$$

Linearize your answer for infinitesimal deformations and identify the Lamé constants.

Problem 4.33. Determine necessary and sufficient conditions for strong ellipticity of the "compressible Mooney-Rivlin" material (also known as a Hadamard material)

$$W(I_1, I_2, I_3) = \frac{\mu}{2}\Big[\alpha(I_1 - 3) + (1-\alpha)(I_2 - 3)\Big] + h(I_3).$$

Problem 4.34. This problem asks you to construct from first principles the one-dimensional counterparts of the theory developed in Chapters 2, 3 and 4 for a *deformable elastic string.* When a body is modeled as a string, its cross-section is invisible and the body is treated as a one-dimensional object. Assume that the string lies in a plane.

Part A: Kinematics. A string occupies a curve $\mathcal{R}_\mathrm{R}$ in a reference configuration as illustrated in the left-hand figure in Figure 4.15. The position vector of a particle in this configuration is denoted by $\mathbf{x}(x), 0 \leq x \leq L_\mathrm{R}$, where x is arc length along this curve and L_R is the corresponding length of the string. A *particle* can be identified (i.e. labeled) by its coordinate x in this configuration. Thus, rather than saying "the particle located at $\mathbf{x}(x)$ in the reference configuration" we can simply say "the particle x". Since the choice of reference configuration is arbitrary, provided only that it be a configuration the body *can* occupy, the string could, for example, be straight and horizontal in this configuration in which case $\mathbf{x}(x) = x\,\mathbf{e}_1$.

At time t during a motion, the position vector of particle x is $\mathbf{y}(x,t)$ and the string occupies the curve $\mathcal{R}_t$; see the right-hand figure in Figure 4.15. Time is not important in this part of the problem, all calculations being carried out at the same instant t. Thus we will not keep referring to time.

Let $\boldsymbol{\ell}_{\mathrm{R}}$ and $\boldsymbol{\ell}$ be unit vectors tangent to the string in the reference and current configurations, and let x and s be arc lengths along the string in these respective configurations. A material fiber in the reference and current configurations can then be expressed as

$$d\mathbf{x} = dx\,\boldsymbol{\ell}_{\mathrm{R}}, \qquad d\mathbf{y} = ds\,\boldsymbol{\ell}; \tag{ia}$$

see Figure 4.15. The stretch λ of the fiber, being the ratio between the deformed and undeformed lengths ds and dx, is given through

$$ds = \lambda dx. \tag{iia}$$

$(a1)$ Derive expressions for $\boldsymbol{\ell}_{\mathrm{R}}, \boldsymbol{\ell}, \lambda$ and s in terms of $\mathbf{x}(x), \mathbf{y}(x,t)$ and their derivatives.

$(a2)$ In terms of only the stretch λ and the unit vectors[25] $\boldsymbol{\ell}_R$ and $\boldsymbol{\ell}$, determine tensors $\mathbf{F}, \mathbf{U}, \mathbf{R}$ and $\mathbf{V}$ with the following properties: $\mathbf{F}$ takes an undeformed material fiber $d\mathbf{x}$ into its deformed image $d\mathbf{y}$; $\mathbf{U}$ stretches $d\mathbf{x}$ by λ without rotating it; and $\mathbf{R}$ rotates a fiber $d\mathbf{x}$ without stretching it.

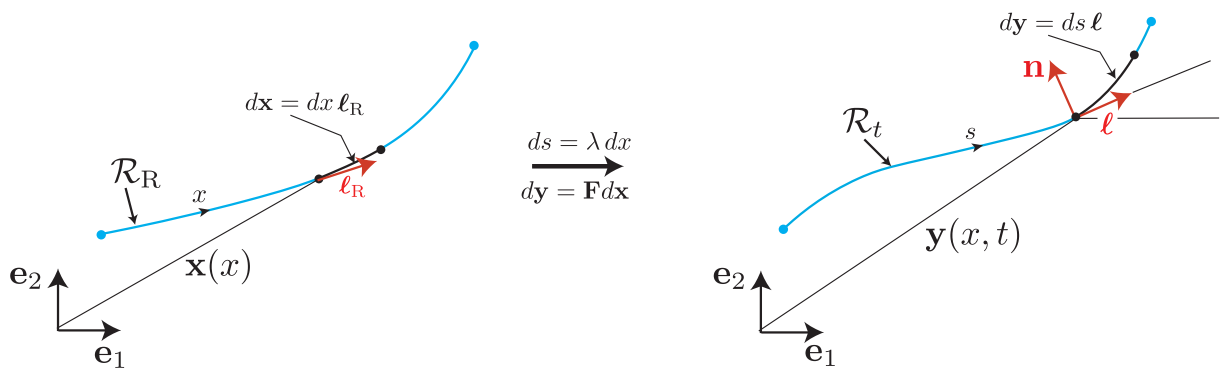

Figure 4.15: An elastic string in a reference configuration (left) and at time t (right). In the reference configuration an infinitesimal material fiber $d\mathbf{x}$ has length dx and direction $\boldsymbol{\ell}_{\mathrm{R}}$. Its image $d\mathbf{y}$ in the current configuration has length $ds = \lambda\, dx$ and direction $\boldsymbol{\ell}$.

Part B: Forces. Equilibrium. Now focus attention on the string in the current configuration at some fixed instant t, and identify points along the string by the arc length s. The string is subjected to a distributed (body) force per unit *deformed* length $\mathbf{b}(s,t)$. This induces an internal contact force field within the string, i.e. if we make a hypothetical cut through the string at some s, the part of the string on one side of the cut applies a force on the part on the other side (and vice versa) due to contact at s. Do *not* assume the contact force to be tangent to the string. Assume that the string, in both the undeformed and current configurations, as well as all forces, lie in the same plane. Neglect inertial effects and enforce force and moment equilibrium for an arbitrary (finite not infinitesimal) part of the string, and from them derive the associated equilibrium field equations that the contact force must obey.

Part C: Rate of working. Energy. Constitutive relation. Thus far did not say anything about the constitutive relation of the string.

[25]Since we are studying a one-dimensional body, we are only permitting the respective unit vectors $\boldsymbol{\ell}_R$ and $\boldsymbol{\ell}$ along the body.

(c1) Derive the relation between stretch rate $\dot{\lambda}$ and velocity gradient $\mathbf{v}'$ where the dot denotes the derivative with respect to t at fixed x; the prime denotes the derivative with respect to x at fixed t; and $\mathbf{v} = \dot{\mathbf{y}}$ is particle velocity.

(c2) Calculate the rate at which the external forces acting on a (finite not infinitesimal) part of the string do work.

(c3) If the material is dissipation-free, the work done is stored in the string. Let $\mathcal{W}$ be the energy stored per unit *deformed* length of the string. Write down the elastic power identity, i.e. the equation that describes the balance between the rate of work and the rate of change of the stored energy for a part of the string. Localize your result. You may find it useful to introduce the energy stored per unit *reference* length, say W. (How is W related to $\mathcal{W}$?)

(c4) Finally, if the string is elastic in the sense that there is a constitutive function $\widetilde{W}(\lambda)$ such that $W(x,t) = \widetilde{W}(\lambda(x,t))$, derive the constitutive relation for τ. (Here $\mathbf{t} = \tau\boldsymbol{\ell}$.)

Solution:

<u>*Part A*</u>:

($a1$) The particles x and $x + dx$ are located at $\mathbf{x}(x)$ and $\mathbf{x}(x + dx)$ in the reference configuration. The infinitesimal material fiber connecting them is

$$d\mathbf{x} = \mathbf{x}(x + dx) - \mathbf{x}(x) \doteq \mathbf{x}'(x)dx; \tag{iiia}$$

a prime denotes the derivative with respect to x (and if the function depends on both x and t it is the derivative with respect to x at fixed t). Note from ($iiia$) that the vector $\mathbf{x}'(x)$ is parallel to the fiber $d\mathbf{x}$ and is therefore tangent to the string in the reference configuration. On comparing ($iiia$) with $(ia)_1$, i.e. $d\mathbf{x} = \boldsymbol{\ell}_{\mathrm{R}}\, dx = \mathbf{x}'(x)dx$, we conclude that

$$\boldsymbol{\ell}_{\mathrm{R}} = \mathbf{x}'(x). \qquad \square \tag{iva}$$

Therefore from the preceding two equations,

$$d\mathbf{x} = dx\, \boldsymbol{\ell}_R. \tag{va}$$

– In the deformed configuration the two particles x and $x + dx$ are located at $\mathbf{y}(x,t)$ and $\mathbf{y}(x + dx, t)$ and thus the fiber under consideration is

$$d\mathbf{y} = \mathbf{y}(x + dx, t) - \mathbf{y}(x,t) \doteq \mathbf{y}'(x,t)\, dx. \tag{via}$$

Note from (va) that the vector $\mathbf{y}'(x,t)$ is parallel to the fiber $d\mathbf{y}$ and is therefore tangent to the string in the deformed configuration. On combining $(ia)_2, (iia)$ and (va) we get $d\mathbf{y} = \boldsymbol{\ell}\, ds = \lambda\, \boldsymbol{\ell}\, dx = \mathbf{y}'(x,t)\, dx$ and so we conclude that

$$\mathbf{y}' = \lambda\boldsymbol{\ell}. \tag{viia}$$

Therefore from the preceding two equations,

$$d\mathbf{y} = \lambda\, dx\, \boldsymbol{\ell}. \tag{viiia}$$

Taking the magnitude of both sides of $(viia)$ gives

$$\lambda = |\mathbf{y}'|, \qquad \square \tag{ixa}$$

and from $(viia)$ and (ixa) it follows that

$$\boldsymbol{\ell} = \frac{\mathbf{y}'}{|\mathbf{y}'|} = \frac{\mathbf{y}'}{\lambda}. \qquad \square \tag{xa}$$

– The arc length along the deformed fiber can be found by integrating $ds = \lambda\, dx$:

$$s = s(x,t) := \int_0^x \lambda(\xi,t)\, d\xi \overset{(viia)}{=} \int_0^x \big|\mathbf{y}'(\xi,t)\big|\, d\xi, \qquad \square \tag{xia}$$

(where ξ is a dummy variable).

$(a2)$ The tensor $\mathbf{F}$ takes $d\mathbf{x} \mapsto d\mathbf{y}$: from (va) we have $dx = d\mathbf{x} \cdot \boldsymbol{\ell}_R$ and so

$$d\mathbf{y} \overset{(viiia)}{=} \lambda\, dx\, \boldsymbol{\ell} = \lambda\, (d\mathbf{x} \cdot \boldsymbol{\ell}_R)\, \boldsymbol{\ell} = \lambda\, (\boldsymbol{\ell} \otimes \boldsymbol{\ell}_R)\, d\mathbf{x} = \mathbf{F} d\mathbf{x}$$

where

$$\mathbf{F} := \lambda \boldsymbol{\ell} \otimes \boldsymbol{\ell}_R. \qquad \square \tag{xa}$$

– The tensor $\mathbf{U}$ stretches $d\mathbf{x}$ without rotating it: therefore $\mathbf{U}d\mathbf{x} = \lambda d\mathbf{x}$ or equivalently $\mathbf{U}\boldsymbol{\ell}_R = \lambda \boldsymbol{\ell}_R$. Therefore by inspection

$$\mathbf{U} = \lambda \boldsymbol{\ell}_{\mathrm{R}} \otimes \boldsymbol{\ell}_{\mathrm{R}}.$$

Observe that $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$.

– The tensor $\mathbf{R}$ rotates $d\mathbf{x}$ without stretching it: therefore $\mathbf{R}d\mathbf{x} = d\mathbf{y}/\lambda$ or equivalently $\mathbf{R}d\boldsymbol{\ell}_R = d\boldsymbol{\ell}$, and so by inspection

$$\mathbf{R} = \boldsymbol{\ell} \otimes \boldsymbol{\ell}_{\mathrm{R}}.$$

Observe that $\mathbf{F} = \mathbf{R}\mathbf{U}$.

<u>Part B</u>: We are now concerned with the current configuration and we identify different points along the string by the arc length s in that configuration. Again, all calculations will be carried out at a fixed instant t and so we will not keep referring to the time.

Make a hypothetical cut through a point s on the string as depicted in the left-hand figure in Figure 4.16. There are two sides to the cut. The unit tangent vector $\boldsymbol{\ell}$ points out of the string at one (call this the negative side) and into the string at the other (call this the positive side). Then we let $\mathbf{t}^+(s,t)$ be the force applied by the positive side on the negative side (at s) and $\mathbf{t}^-(s,t)$ the force applied by the negative side on the positive side. This is illustrated in the figure. (Recall that the direction of $\boldsymbol{\ell}$ is always in the direction of increasing s.)

Now consider a free body diagram of an arbitrary segment $[s_1, s_2]$ of the string as illustrated in the right-hand figure in Figure 4.16. The tangent vector $\boldsymbol{\ell}(s_2,t)$ points out of the string and so the contact force acting on the part of the string in the free body at s_2 is $\mathbf{t}^+(s_2,t)$. At the other end the tangent vector $\boldsymbol{\ell}(s_1,t)$

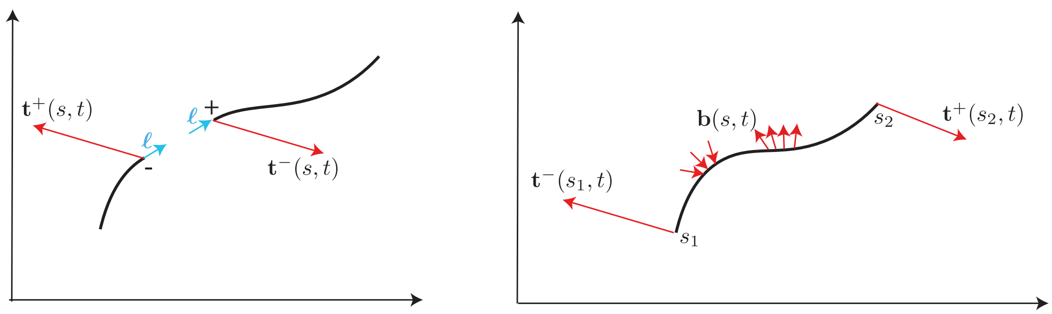

Figure 4.16: Left: A hypothetical cut has been made through the string at some point s. There are two sides to the cut. The unit tangent vector $\boldsymbol{\ell}$ points out of the string at one (call that the negative side) and into the string at the other (the positive side). The force applied by the positive side on the negative side is denoted by $\mathbf{t}^+(s,t)$ and the force applied by the negative side on the positive side is $\mathbf{t}^-(s,t)$. Right: Free body diagram of the segment $s_1 \leq s \leq s_2$ of the string.

points into the string and so the contact force acting on the free body at s_1 is $\mathbf{t}^-(s_1,t)$. Therefore *force balance* of this free body requires

$$\mathbf{t}^+(s_2,t) + \mathbf{t}^-(s_1,t) + \int_{s_1}^{s_2} \mathbf{b}(s,t)\, ds = \mathbf{o}, \qquad (ib)$$

where $\mathbf{b}\, ds$ is the body force acting on an infinitesimal segment of the string. We now derive some consequences of this.

Pick an arbitrary point $s \in (s_1, s_2)$ and consider the limits $s_1 \to s^-$ and $s_2 \to s^+$ with s fixed. Then, assuming all terms in (ib) to be continuous, it yields

$$\mathbf{t}^+(s,t) + \mathbf{t}^-(s,t) = \mathbf{o} \qquad \Rightarrow \qquad \mathbf{t}^+(s,t) = -\mathbf{t}^-(s,t). \qquad (iib)$$

This says that the force applied by the positive side on the negative side is equal in magnitude and opposite in direction to the force applied by the negative side on the positive side. It is convenient therefore to introduce

$$\mathbf{t}(s,t) := \mathbf{t}^+(s,t) = -\mathbf{t}^-(s,t).$$

We can now write (ib) as

$$\mathbf{t}(s_2,t) - \mathbf{t}(s_1,t) + \int_{s_1}^{s_2} \mathbf{b}(s,t)\, ds = \mathbf{o},$$

which in turn can be written as

$$\int_{s_1}^{s_2} \frac{\partial}{\partial s}\big[\mathbf{t}(s,t)\big]\, ds + \int_{s_1}^{s_2} \mathbf{b}\, ds = \mathbf{o} \qquad \Rightarrow \qquad \int_{s_1}^{s_2} \Big[\frac{\partial \mathbf{t}}{\partial s} + \mathbf{b}\Big] ds = \mathbf{o}.$$

Since this integral vanishes for all $[s_1, s_2]$, the integrand must vanish and so we are led to the equilibrium field equation

$$\frac{\partial \mathbf{t}}{\partial s} + \mathbf{b} = \mathbf{o}. \qquad \square \qquad (iiib)$$

This must hold for all s and all t. (Note from (ixa) that s lies in the range $s(0,t) \le s \le s(L_R, t)$).

Moment balance of the free body requires

$$\mathbf{y}_2 \times \mathbf{t}_2 - \mathbf{y}_1 \times \mathbf{t}_1 + \int_{s_1}^{s_2} \mathbf{y} \times \mathbf{b}\, ds = \mathbf{o}, \qquad (ivb)$$

where $\mathbf{y}_\alpha$ and $\mathbf{t}_\alpha$, $\alpha = 1, 2$, are the position vector and contact force respectively at s_α. Equation (ivb) can be written as

$$\int_{s_1}^{s_2} \frac{\partial}{\partial s}\big[\mathbf{y} \times \mathbf{t}\big]\, ds + \int_{s_1}^{s_2} \mathbf{y} \times \mathbf{b}\, ds = \mathbf{o}.$$

Expanding the first term yields

$$\int_{s_1}^{s_2} \left[\frac{\partial \mathbf{y}}{\partial s} \times \mathbf{t} + \mathbf{y} \times \left(\frac{\partial \mathbf{t}}{\partial s} + \mathbf{b}\right)\right] ds = \mathbf{o},$$

and using $(iiib)$ leads to

$$\int_{s_1}^{s_2} \left[\frac{\partial \mathbf{y}}{\partial s} \times \mathbf{t}\right] ds = \mathbf{o}.$$

Since this holds for all $[s_1, s_2]$ the integrand must vanish and so

$$\frac{\partial \mathbf{y}}{\partial s} \times \mathbf{t} = \mathbf{o}.$$

However (iia) and $(viia)$ imply

$$\frac{\partial \mathbf{y}}{\partial s} = \frac{\partial \mathbf{y}}{\partial x}\,\frac{\partial x}{\partial s} \overset{(iia)}{=} \frac{1}{\lambda}\mathbf{y}' \overset{(viia)}{=} \boldsymbol{\ell},$$

and so we can write this moment balance equation as

$$\boldsymbol{\ell} \times \mathbf{t} = \mathbf{o}. \qquad (vb)$$

Equation (vb) tells us that $\mathbf{t}$ is parallel to $\boldsymbol{\ell}$ and so we can write

$$\mathbf{t} = \tau\, \boldsymbol{\ell}. \qquad \square \qquad (vib)$$

This is the field equation associated with moment balance. Observe that it has told us that the force $\mathbf{t}$ in the string is tangent to the (deformed) string at each point, τ being the tension in the string.

Remark: We can use the result (vib) from moment balance to further simplify the force equilibrium equation $(iiib)$ as follows:

$$\frac{\partial \tau}{\partial s}\boldsymbol{\ell} + \tau \frac{\partial \boldsymbol{\ell}}{\partial s} + \mathbf{b} = \mathbf{o}. \qquad \square \qquad (viib)$$

The unit vector $\mathbf{n}$ normal to the deformed string (obtained by rotating $\boldsymbol{\ell}$ through a CCW angle $\pi/2$) and the tangent vector $\boldsymbol{\ell}$ can be expressed as

$$\boldsymbol{\ell} = \cos\phi\, \mathbf{e}_1 + \sin\phi\, \mathbf{e}_2, \qquad \mathbf{n} = -\sin\phi\, \mathbf{e}_1 + \cos\phi\, \mathbf{e}_2, \qquad (viiib)$$

where $\phi(s,t)$ is the angle the deformed string makes with the $\mathbf{e}_1$-direction. Therefore $\partial\boldsymbol{\ell}/\partial s = (\partial\phi/\partial s)\,\mathbf{n}$. Thus $(viib)$ can be written as

$$\frac{\partial \tau}{\partial s}\boldsymbol{\ell} + \tau \frac{\partial \phi}{\partial s}\mathbf{n} + \mathbf{b} = \mathbf{o}. \qquad (ixb)$$

The two components of this vector equilibrium equation in the $\boldsymbol{\ell}$ and $\mathbf{n}$ directions are obtained by taking its scalar product of (ixb) with $\boldsymbol{\ell}$ and $\mathbf{n}$:

$$\frac{\partial \tau}{\partial s} + \mathbf{b}\cdot\boldsymbol{\ell} = 0, \qquad \tau\frac{\partial \phi}{\partial s} + \mathbf{b}\cdot\mathbf{n} = 0. \qquad \square \qquad (xb)$$

Remark: Alternatively, to determine the equilibrium equations in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions we use $(viiib)$ in (ixb) to get

$$\left(\frac{\partial}{\partial s}(\tau\cos\phi)+b_1\right)\mathbf{e}_1+\left(\frac{\partial}{\partial s}(\tau\sin\phi)+b_2\right)\mathbf{e}_2=\mathbf{o}, \qquad (xib)$$

where $\mathbf{b}=b_1\mathbf{e}_1+b_2\mathbf{e}_2$. The vanishing of each component in (xib) can be readily derived directly by enforcing force equilibrium in the x_1- and x_2-directions for an infinitesimal part of the string.

Remark: The second equilibrium equation $(xb)_2$ can be written in classical form in terms of the curvature $\kappa := \partial\phi/\partial s$ as

$$\tau\kappa = p,$$

where $p=-\mathbf{b}\cdot\mathbf{n}$ is the normal pressure on the string. This is effectively the so-called Young-Laplace relation between "surface tension" τ, curvature κ and pressure p.

<u>Part C</u>:

$(c1)$ The velocity of a particle is

$$\mathbf{v} := \dot{\mathbf{y}}, \qquad (ic)$$

where, as mentioned in the problem statement, the dot denotes the derivative with respect to t at fixed x. Differentiating $\mathbf{y}'=\lambda\,\boldsymbol{\ell}$ (see (via)) with respect to t at fixed x gives

$$\dot{\mathbf{y}}'=\dot{\lambda}\boldsymbol{\ell}+\lambda\dot{\boldsymbol{\ell}} \qquad \Rightarrow \qquad \mathbf{v}'=\dot{\lambda}\boldsymbol{\ell}+\lambda\dot{\boldsymbol{\ell}}. \qquad (iic)$$

Taking the scalar product of this equation with $\boldsymbol{\ell}$ gives

$$\boldsymbol{\ell}\cdot\mathbf{v}'=\dot{\lambda}\boldsymbol{\ell}\cdot\boldsymbol{\ell}+\lambda\boldsymbol{\ell}\cdot\dot{\boldsymbol{\ell}} \qquad \Rightarrow \quad \dot{\lambda}=\boldsymbol{\ell}\cdot\mathbf{v}', \qquad \square \qquad (iiic)$$

having used $\boldsymbol{\ell}\cdot\dot{\boldsymbol{\ell}}=0$ which follows from differentiating $\boldsymbol{\ell}\cdot\boldsymbol{\ell}=1$. Observe that $(iiic)$ says that the rate at which the stretch increases equals the velocity gradient tangent to the string (which is in fact $\partial\mathbf{v}/\partial s$).

Remark: We can interpret (iic) as follows. On differentiating $(viiib)$ with respect to t at fixed x we get

$$\dot{\boldsymbol{\ell}}=\dot{\phi}\mathbf{n},$$

where ϕ is the angle the deformed string makes with the $\mathbf{e}_1$-direction. The velocity of particle $x+dx$ relative to particle x is $\mathbf{v}(x+dx,t)-\mathbf{v}(x,t)=\mathbf{v}'\,dx$. Combining this with (iic) yields

$$\mathbf{v}(x+dx,t)-\mathbf{v}(x,t)=dx\,\big(\dot{\lambda}\boldsymbol{\ell}+\lambda\dot{\boldsymbol{\ell}}\big)=\dot{\overline{ds}}\,\boldsymbol{\ell}+ds\,\dot{\boldsymbol{\ell}}=\dot{\overline{ds}}\,\boldsymbol{\ell}+ds\,\dot{\phi}\,\mathbf{n}.$$

where we have used $ds=\lambda dx$. Therefore the velocity of particle $x+dx$ relative to particle x has two components, one due to stretching at the rate $\dot{\overline{ds}}$ in the fiber direction $\boldsymbol{\ell}$ and the other in the direction $\mathbf{n}$ normal to the fiber due to rotation at the angular rate $\dot{\phi}$.

$(c2)$ The *rate of working* of the external forces acting on a part of the string is

$$\begin{aligned}
&= \mathbf{t}_2\cdot\mathbf{v}_2-\mathbf{t}_1\cdot\mathbf{v}_1+\int_{s_1}^{s_2}\mathbf{b}\cdot\mathbf{v}\,ds=\int_{s_1}^{s_2}\left[\frac{\partial}{\partial s}(\mathbf{t}\cdot\mathbf{v})\;+\mathbf{b}\cdot\mathbf{v}\right]ds=\\
&= \int_{s_1}^{s_2}\left[\frac{\partial\mathbf{t}}{\partial s}\cdot\mathbf{v}+\mathbf{t}\cdot\frac{\partial\mathbf{v}}{\partial s}+\mathbf{b}\cdot\mathbf{v}\right]ds=\int_{s_1}^{s_2}\left[\mathbf{t}\cdot\frac{\partial\mathbf{v}}{\partial s}+\left(\frac{\partial\mathbf{t}}{\partial s}+\mathbf{b}\right)\cdot\mathbf{v}\right]ds=\\
&\overset{(iiib)}{=}\int_{s_1}^{s_2}\mathbf{t}\cdot\frac{\partial\mathbf{v}}{\partial s}\,ds\overset{(*)}{=}\int_{s_1}^{s_2}\tau\boldsymbol{\ell}\cdot\frac{1}{\lambda}\frac{\partial\mathbf{v}}{\partial x}\,ds=\int_{s_1}^{s_2}\frac{\tau}{\lambda}\,\boldsymbol{\ell}\cdot\mathbf{v}'\,ds\overset{(**)}{=}\int_{s_1}^{s_2}\tau\frac{\dot{\lambda}}{\lambda}\,ds, \qquad \square \qquad (ivc)
\end{aligned}$$

where we have used (vib) and $ds=\lambda\,dx$ in step $(*)$ and $(iiic)$ in step $(**)$.

($c3$) Since $\mathcal{W}$ is the stored energy per unit deformed length, the energy stored in an infinitesimal part of the string is $\mathcal{W}\,ds$. Therefore the total stored energy is the integral of $\mathcal{W}$ with respect to s from s_1 to s_2, and the rate of change of this energy is its time derivative:

$$\frac{d}{dt}\int_{s_1}^{s_2} \mathcal{W}(s,t)\,ds. \tag{vc}$$

Since the material is dissipation-free we can equate the rate of work (ivc) to the rate of increase of stored energy (vc) to obtain

$$\int_{s_1}^{s_2} \tau\frac{\dot\lambda}{\lambda}\,ds = \frac{d}{dt}\int_{s_1}^{s_2} \mathcal{W}\,ds. \tag{vic}$$

Time, which was unimportant in the previously parts of the problem, is significant here. It is important to keep in mind that by a *part* of the string we mean a segment of the string that involves the same particles at all times. Even though the string occupies different curves in space at different times, a part of the string always involves the same particles. Thus the part of the string between particles x_1 and x_2 can be identified with the interval $[x_1, x_2]$.

The arc lengths associated with the two ends of a part $[x_1, x_2]$ of the string are

$$s_1(t) = s(x_1,t), \qquad s_2(t) = s(x_2,t), \tag{viic}$$

where $s = s(x,t)$ is given by (ixa). Observe that $s_1(t)$ and $s_2(t)$ are *time dependent.* Consequently with more detail, (vic) reads

$$\int_{s_1(t)}^{s_2(t)} \tau\frac{\dot\lambda}{\lambda}\,ds = \frac{d}{dt}\int_{s_1(t)}^{s_2(t)} \mathcal{W}\,ds. \tag{viiic}$$

When we evaluate the right-hand side of ($viiic$) we have to account for the time dependence of both the integrand and the limits of integration.

In order to localize ($viiic$) we want to write this as a single integral that vanishes, and for this we must take the time derivative into the integral on the right-hand side. Since the limits are time dependent we must use Leibniz' rule. Alternatively to avoid this we can map $[s_1(t), s_2(t)]$ into the corresponding (fixed) domain $[x_1, x_2]$ in the reference configuration using $ds = \lambda\,dx$. Then ($viiic$) reads

$$\int_{x_1}^{x_2} \tau\frac{\dot\lambda}{\lambda}\,\lambda\,dx = \frac{d}{dt}\int_{x_1}^{x_2} \mathcal{W}\,\lambda\,dx. \tag{viiic}$$

Recall that $\mathcal{W}$ is the energy per unit deformed length. It is related to the energy per unit undeformed length, W, by $\mathcal{W}ds = W dx$, i.e.

$$\mathcal{W}\,\lambda = W. \tag{ixc}$$

It is important to note that W is not the energy in the undeformed string. It is obtained by dividing the energy in (an infinitesimal part of) the *deformed string* by its undeformed length. We can now write ($viiic$) as

$$\int_{x_1}^{x_2} \tau\dot\lambda\,dx = \frac{d}{dt}\int_{x_1}^{x_2} W\,dx.$$

Since we are concerned with a fixed set of particles of the string, x_1 and x_2 are time independent and therefore we can take the d/dt derivative inside the integral on the right hand side:

$$\int_{x_1}^{x_2} \tau\dot\lambda\,dx = \int_{x_1}^{x_2} \dot W\,dx \qquad \Rightarrow \qquad \int_{x_1}^{x_2} (\tau\dot\lambda - \dot W)\,dx = 0. \tag{xiiic}$$

This holds for every part of the string, and so by localization the integrand must vanish:

$$\tau\dot{\lambda} = \dot{W}. \qquad \square \tag{xivc}$$

($c4$) Finally, when $W = \widetilde{W}(\lambda)$, equation ($xivc$) gives

$$\tau\,\dot{\lambda} = \frac{d\widetilde{W}}{d\lambda}\dot{\lambda} \qquad \Rightarrow \qquad \tau = \frac{d\widetilde{W}}{d\lambda}(\lambda). \qquad \square \tag{xvc}$$

In *summary*, the three scalar equations $(xb)_1, (xb)_2$ and (xvc) are to be solved for the three unknown scalar fields $\lambda(x,t), \phi(s,t)$ and $\tau(s,t)$.

Remark: Any function of s and t can be expressed as a function of x and t by using $s = s(x,t)$. Thus for example we can write the unknown angle as $\widehat{\phi}(x,t) = \phi(s(x,t),t)$ and the unknown tension as $\widehat{\tau}(x,t) = \tau(s(x,t),t)$ where we have simply substituted $s(x,t)$ for s. In this way we can consider the three unknown functions λ, ϕ, τ to be functions of x and t that hold for $0 \leq x \leq L_{\rm R}$ and $t \geq 0$.

Exercise: Let $\mathbf{b}_{\rm R}$ denote the distributed body force per unit undeformed length. Show that the equilibrium equation ($iiib$) can be written as

$$\frac{\partial \widehat{\mathbf{t}}}{\partial x} + \mathbf{b}_R = \mathbf{o}, \qquad 0 \leq x \leq L_{\rm R}$$

where $\widehat{\mathbf{t}} = \mathbf{t}(s(x,t),t)$.

Problem 4.35. *Axisymmetric deformation of an elastic membrane.* (See Problem 4.34 for an analogous problem for an elastic string.) In an unstressed reference configuration an elastic membrane[26] is a circular cylinder of radius R and length L. The Z-axis coincides with the axis of the cylinder and the origin is at one end. The membrane is subjected to a circumferentially uniform but axially varying internal pressure $p(Z)$ per unit deformed area; it acts in a direction perpendicular to the deformed membrane. Formulate the complete theory for the kinematic and force fields in the membrane. Do so by working from first principles, addressing the kinematics, balance laws of equilibrium and constitutive principles. (A different way in which to approach this problem is by taking the limit as the thickness tends to zero of the corresponding problem for a thick-walled tube. Do NOT take this approach here but you may want to try that approach as an exercise.)

A boundary-value problem.

Problem 4.36. In this and the preceding two chapters we studied the three foundational pillars of the subject, viz. kinematics of deformation, stress and equilibrium, and constitutive relations. We focused on each individually, more or less in isolation of the other two. Now that we have these three pillars in hand, we are in a position to combine and draw upon them in order to solve complete boundary-value problems. In

[26]In the membrane model of an elastic body, its thickness is invisible and the membrane is treated as a two-dimensional object. Moreover, a membrane has no bending stiffness and so the internal forces are assumed to be tangential to the deformed membrane.

the next chapter we shall consider several such problems, but the focus there will be on examining inherently nonlinear phenomena. Here we shall solve one problem, solely to illustrate how one might combine many of the concepts and tools we have learnt thus far.

We are told here that the block is composed of a neo-Hookean material.

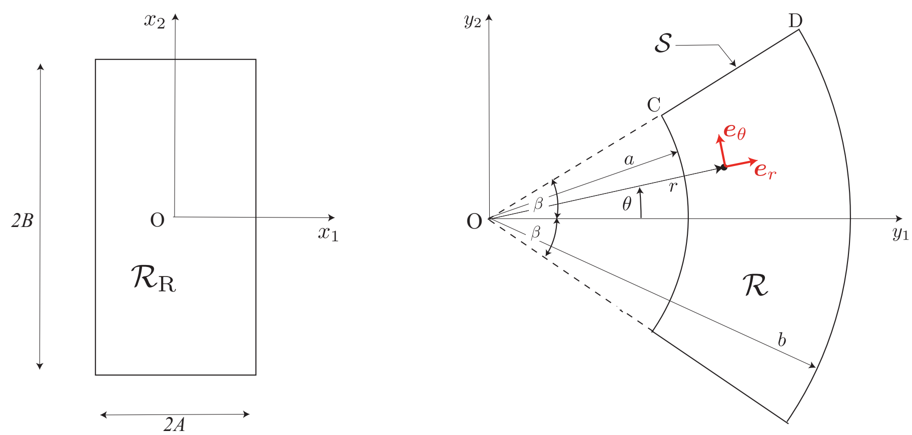

Figure 4.17: A $2A \times 2B \times 2C$ rectangular block is bent as described in Problem 2.5.4 . The extent of bending is measured by the angle 2β. The dimensions A, B, C, the bending angle β and the shear modulus μ are known.

The kinematics of bending a $2A \times 2B \times 2C$ rectangular block was studied in Problem 2.5.4. The deformation we considered there mapped vertical and horizontal straight lines in the reference configuration into, respectively, arcs of circles and radial lines in the deformed configuration. It was described by

$$y_1 = r(x_1)\,\cos\theta(x_2), \qquad y_2 = r(x_1)\,\sin\theta(x_2), \qquad y_3 = x_3, \tag{i}$$

with

$$r(x_1) > 0, \qquad r'(x_1) > 0, \qquad \theta'(x_1) > 0, \qquad \theta(x_2) = -\theta(-x_2). \tag{ii}$$

Moreover, we showed that the principal stretches and the left Cauchy-Green tensor were

$$\lambda_1 = r'(x_1), \qquad \lambda_2 = r(x_1)\theta'(x_2), \qquad \lambda_3 = 1, \tag{iii}$$

$$\mathbf{B} = \lambda_1^2\,\mathbf{e}_r \otimes \mathbf{e}_r + \lambda_2^2\,\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_z \otimes \mathbf{e}_z, \tag{iv}$$

where (r, θ, z) are cylindrical polar coordinates in the deformed configuration and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is the associated basis.

The two curved surfaces ($r = a, -\beta \le \theta \le \beta$ and $r = b, -\beta \le \theta \le \beta$) are traction-free where 2β denotes the angle subtended by the two planar inclined faces of the block in the deformed configuration as depicted in Figure 4.17 – it is a measure of the "amount" of bending. Certain tractions are applied on the two planar inclined faces ($\theta = \pm\beta, a \le r \le b$) such that the resultant force on each face is zero and the resultant moment is $\pm m\,\mathbf{e}_z$.

We want to work in terms of the Cauchy stress and the deformed configuration and to (a) determine the radii a and b of the deformed block (or at least derive two algebraic equations in which the only unknowns are a and b); and (b) calculate an expression for the bending moment m. The dimensions A, B, C, the bending angle β and the shear modulus μ are known.

In the course of solving this (and any) boundary-value problem one must satisfy all of the field equations and boundary conditions. Observe that boundary conditions have been prescribed pointwise at each point on the curved surfaces, but on the flat surfaces only the stress resultants have been given. Therefore, since boundary conditions have not been prescribed at each point on the flat surfaces, the information given above does not fully formulate the boundary-value problem at hand. There will be many elastostatic fields satisfying the given information. More on boundary conditions will be said in Chapter 5.

Solution: We have to enforce the kinematic requirements:

$$\theta(\pm B) = \pm\beta, \qquad r(A) = b, \qquad r(-A) = a; \tag{v}$$

the traction boundary conditions on the two curved surfaces:

$$T_{rr}(b,\theta) = T_{\theta r}(b,\theta) = T_{zr}(b,\theta) = 0, \qquad T_{rr}(a,\theta) = T_{\theta r}(a,\theta) = T_{zr}(a,\theta) = 0 \qquad \text{for } -\beta \le \theta \le \beta; \tag{vi}$$

and the vanishing of the resultant force on the two flat surfaces:

$$\int_a^b T_{r\theta}(r,\pm\beta)\,dr = \int_a^b T_{\theta\theta}(r,\pm\beta)\,dr = \int_a^b T_{z\theta}(r,\pm\beta)\,dr = 0. \tag{vii}$$

In addition, since we intend to enforce the equilibrium equations for the Cauchy stress field in the deformed configuration, we must express $\mathbf{T}$ as a function of r, θ, z. Since we will use the constitutive equation to determine $\mathbf{T}$ in terms of the principal stretches, this means the principal stretches must also be expressed as functions of r, θ, z. Note that the expressions given in (iii) of the principal stretches are *not* in this form.

(a) A neo-Hookean material is incompressible and the deformation must therefore obey the field equation $\det \mathbf{F}(\mathbf{x}) = 1$. Thus by (iii),

$$\det \mathbf{F} = \lambda_1\lambda_2\lambda_3 = r(x_1)r'(x_1)\theta'(x_2) = 1.$$

By separating variables we conclude that

$$r(x_1)r'(x_1) = \frac{1}{\theta'(x_2)} = c_1 \text{ (constant)},$$

which when integrated yields

$$r(x_1) = \sqrt{2c_1x_1 + c_3}, \qquad \theta(x_2) = \frac{x_2}{c_1} + c_2, \tag{$viii$}$$

where c_1, c_2 and c_3 are constants. According to $(ii)_4$ the function $\theta(x_2)$ is odd whence

$$c_2 = 0. \tag{ix}$$

Since the total angle subtended by the flat faces of the deformed block is 2β, we have the kinematic requirement $(v)_1$ and so $(viii)_2$ gives

$$c_1 = B/\beta. \tag{x}$$

Next, it follows from $(v)_2, (viii)_1$ and (x) that

$$c_3 = b^2 - \frac{2BA}{\beta}. \tag{xi}$$

Substituting these values of the constants c_1, c_2 and c_3 back into $(viii)$ gives

$$r(x_1) = \sqrt{b^2 - \frac{2B}{\Lambda\beta}(A - x_1)}, \qquad \theta(x_2) = \beta\frac{x_2}{B}. \tag{xii}$$

Observe from (iii) and (xii) that the principal stretches are now seen to be functions of x_1 only:

$$\lambda_1 = r'(x_1), \qquad \lambda_2 = \frac{\beta}{B}\,r(x_1), \qquad \lambda_3 = 1. \tag{xiii}$$

Also, from $(v)_3$ and $(xii)_1$ we get the relation

$$b^2 - a^2 = 4\frac{AB}{\beta}, \tag{xiv}$$

that involves a and b as the only unknowns. (Note that (xiv) says that the cross-sectional area $4AB$ of the rectangle in Figure 4.17 equals the area $\beta b^2 - \beta a^2$ of the annular sector in the deformed configuration.)

The principal stretches can now be expressed in terms of r using $(xiii)_2$ and incompressibility:

$$\lambda_1 = \frac{B}{\beta r}, \qquad \lambda_2 = \frac{\beta r}{B}, \qquad \lambda_3 = 1. \tag{xv}$$

It follows from $(iv), (xv)$ and the constitutive relation $\mathbf{T} = \mu\mathbf{B} - q\mathbf{I}$ for a neo-Hookean material that

$$T_{rr}(r) = \mu\lambda_1^2 - q = \mu\frac{B^2}{\beta^2 r^2} - q, \qquad T_{\theta\theta} = \mu\lambda_2^2 - q = \mu\frac{\beta^2 r^2}{B^2} - q, \qquad T_{zz}(r) = \mu - q, \tag{xvi}$$

with all shear stress components being zero. Here $q = q(r, \theta, z)$ is the reactive pressure field. Substituting (xvi) into the second and third of the general equilibrium equations (3.95) in cylindrical polar coordinates gives

$$\frac{\partial q}{\partial \theta} = 0, \qquad \frac{\partial q}{\partial z} = 0,$$

and so we conclude the q does not depend on θ and z. The first equilibrium equation now reduces to

$$\frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0. \tag{xvii}$$

Substituting (xvi) into $(xvii)$ leads to

$$\frac{dq}{dr} = -\mu\frac{B^2}{\beta^2 r^3} - \mu\frac{\beta^2 r}{B^2},$$

that when integrated gives

$$q(r) = \mu\frac{B^2}{2\beta^2 r^2} - \mu\frac{\beta^2 r^2}{2B^2} + c_0 \tag{xviii}$$

where c_0 is a constant of integration (to be determined). Substituting $(xviii)$ into $(xvi)_1$ gives the radial normal stress to be

$$T_{rr} = \mu\frac{B^2}{2\beta^2 r^2} + \mu\frac{\beta^2 r^2}{2B^2} - c_0. \tag{xix}$$

The boundary conditions $T_{rr}(a) = 0$ and $T_{rr}(b) = 0$ now yield

$$c_0 = \mu \frac{B^2}{2\beta^2 a^2} + \mu \frac{\beta^2 a^2}{2B^2}, \qquad (xx)$$

and

$$\frac{B^2}{\beta^2 a^2} + \frac{\beta^2 a^2}{B^2} = \frac{B^2}{\beta^2 b^2} + \frac{\beta^2 b^2}{B^2} \qquad \Rightarrow \qquad ab = B^2/\beta^2. \qquad (xxi)$$

The boundary conditions in (vi) involving the shear stress components hold trivially since the shear stresses vanish at all points in the body.

Equations (xiv) and (xxi) are two algebraic equations for determining a and b. They can be solved to find

$$a = \left[\frac{B}{\beta^2} \sqrt{4A^2\beta^2 + B^2} - \frac{2AB}{\beta} \right]^{1/2}, \qquad b = \left[\frac{B}{\beta^2} \sqrt{4A^2\beta^2 + B^2} + \frac{2AB}{\beta} \right]^{1/2}. \qquad \square \qquad (xxii)$$

To calculate the resultant force on a flat face we first note that since the shear stresses vanish everywhere in the body there is no resultant force on such a face in the r and z directions. To calculate the force in the θ direction we can substitute $(xviii)$ and (xx) into $(xvi)_2$ and then integrate $T_{\theta\theta}$ with respect to r from $r = a$ to $r = b$. However one can calculate this more easily as follows: the equilibrium equation $(xvii)$ can be written as $T_{\theta\theta} = \frac{d}{dr}(rT_{rr})$ which when integrated from $r = a$ to $r = b$ gives

$$\int_a^b T_{\theta\theta}\, dr = bT_{rr}(b) - aT_{rr}(a) = 0 \qquad \square$$

since $T_{rr}(b) = T_{rr}(a) = 0$ by the traction-free boundary condition on $r = a$ and $r = b$. Thus the resultant force on each flat face is zero.

(c) The moment of the traction on an inclined face is given by

$$m = \int_a^b 2CrT_{\theta\theta}\, dr = \int_a^b 2C\mu \left[\frac{3}{2} \frac{\beta^2 r^3}{B^2} - \frac{B^2}{2\beta^2 r} - \frac{B^2}{2\beta^2 a^2} r - \frac{\beta^2 a^2}{2B^2} r \right] dr.$$

where $2C$ is the length of the block into the page. Integrating this gives the bending moment

$$m = 2C\mu \left[\frac{3}{8} \frac{\beta^2}{B^2}(b^4 - a^4) - \frac{B^2}{2\beta^2} \ln(b/a) - \left(\frac{B^2}{\beta^2 a^2} + \frac{\beta^2 a^2}{B^2} \right) \frac{(b^2 - a^2)}{4} \right] \qquad \square$$

where a and b are given by $(xxii)$.

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Chapter 5

Some Nonlinear Effects: Illustrative Examples

Nonlinearity can lead to phenomena that are not seen in the linearized theory, phenomena that may even be totally unexpected and counterintuitive. In this chapter we describe some examples of this, but first we summarize the basic field equations and make a few remarks on boundary conditions.

5.1 Summary and boundary conditions.

5.1.1 Field equations.

The strain energy function $W(\mathbf{F})$ characterizing the material is given. The region $\mathcal{R}_{\mathrm{R}}$ occupied by the body in the reference configuration is known and the body force field $\mathbf{b}_{\mathrm{R}}(\mathbf{x})$ is prescribed on $\mathcal{R}_{\mathrm{R}}$. (It might vanish.) The stress field $\mathbf{S}(\mathbf{x})$, deformation gradient tensor field $\mathbf{F}(\mathbf{x})$ and deformation field $\mathbf{y}(\mathbf{x})$ must satisfy the following equations at each point $\mathbf{x} \in \mathcal{R}_{\mathrm{R}}$:

$$\mathrm{Div}\,\mathbf{S} + \mathbf{b}_{\mathrm{R}} = \mathbf{o}, \qquad \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}, \qquad \mathbf{F} = \mathrm{Grad}\,\mathbf{y}. \tag{5.1}$$

(Question: What about moment equilibrium?)

– Observe that the equilibrium equations

$$\frac{\partial S_{ij}}{\partial x_j} + b_i^R = 0, \tag{5.2}$$

comprise a set of three scalar partial differential equations involving the 9 unknown stress components. Thus the equilibrium equations by themselves are *not* sufficient for determining the stress field $S_{ij}(\mathbf{x})$ in the body. Stated differently, in general there will be many stress fields that satisfy the equilibrium equations and traction boundary conditions.

– When the constitutive equations

$$S_{ij} = \frac{\partial W}{\partial F_{ij}} \tag{5.3}$$

are also taken into account, one has an additional set of 9 scalar equations but they involve the 9 components of the deformation gradient tensor field $F_{ij}(\mathbf{x})$.

– Finally, the kinematic equations

$$F_{ij} = \frac{\partial y_i}{\partial x_j} \tag{5.4}$$

provide a set of 9 more scalar equations. They involve the 3 components of the deformation $y_i(\mathbf{x})$.

– Thus taken together, the system (5.2), (5.3), (5.4) comprises a set of 21 $(= 3+9+9)$ scalar equations for the 21 $(= 9+9+3)$ unknown scalar fields $S_{ij}(\mathbf{x}), F_{ij}(\mathbf{x}), y_i(\mathbf{x})$.

– Given the deformation field $\mathbf{y}(\mathbf{x})$, one can calculate the 9 components of the deformation gradient field using (5.4). However given a tensor field $\mathbf{F}(\mathbf{x})$ with positive determinant, the equations

$$\frac{\partial y_i}{\partial x_j} = F_{ij} \tag{5.5}$$

for finding $y_i(\mathbf{x})$ comprise a set of 9 equations for the three fields $y_i(\mathbf{x})$. For an arbitrary $\mathbf{F}(\mathbf{x})$, this would be an over-determined set of equations. If a solution $y_i(\mathbf{x})$ is to exist, the nine fields $F_{ij}(\mathbf{x})$ must be suitably restricted. These are known as the **compatibility conditions** and were looked at previously in Problem 2.27.

– For a homogeneous material[1], substituting (5.3) into (5.2) and using (5.4) leads to

$$\mathbb{A}_{ijk\ell}(\mathbf{F}) \frac{\partial^2 y_k}{\partial x_j \partial x_\ell} + b_i^R = 0, \tag{5.6}$$

where

$$\mathbb{A}_{ijk\ell}(\mathbf{F}) = \frac{\partial^2 W(\mathbf{F})}{\partial F_{ij} \partial F_{k\ell}} \tag{5.7}$$

[1] For an inhomogeneous material the strain energy function would depend on the particle and would have the form $W(\mathbf{F}, \mathbf{x})$. Thus the strain energy would depend on $\mathbf{x}$ both through $\mathbf{F}(\mathbf{x})$ and explicitly.

are the components of the 4-tensor $\mathbb{A}$ that we encountered previously when looking at strong ellipticity.

Equation (5.6) is a set of 3 scalar second-order partial differential equations involving the 3 deformation component fields $y_i(\mathbf{x})$.

5.1.2 Boundary conditions

Let $\partial\mathcal{R}_R$ be the boundary of the region $\mathcal{R}_R$ occupied by the body in a reference configuration. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be complementary parts of $\partial\mathcal{R}_R$ so that $\partial\mathcal{R}_R = \mathcal{S}_1 \cup \mathcal{S}_2$. Then the boundary conditions associated with the **mixed boundary-value problem** of elastostatics are

$$\mathbf{y}(\mathbf{x}) = \widehat{\mathbf{y}}(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathcal{S}_1, \qquad\qquad \mathbf{S}\mathbf{n}_R = \widehat{\mathbf{s}}(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathcal{S}_2, \tag{5.8}$$

where $\widehat{\mathbf{y}}(\mathbf{x})$ and $\widehat{\mathbf{s}}(\mathbf{x})$ are the given deformation and traction on $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Observe that one vector quantity $\mathbf{y}$ or $\mathbf{s}$, or equivalently *three* scalar quantities y_1, y_2, y_3 or s_1, s_2, s_3 are prescribed at (almost[2]) every point on the boundary; recall that (5.6) is a set of *three* scalar second-order partial differential equations.

– In the special case where the deformation is prescribed on the entire boundary, i.e. $\mathbf{y}(\mathbf{x}) = \widehat{\mathbf{y}}(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathcal{R}_R$, one has a deformation boundary-value problem. When the deformation is prescribed on $\partial\mathcal{R}_R$, the displacement is also known on $\partial\mathcal{R}_R$ since $\mathbf{u}(\mathbf{x}) = \widehat{\mathbf{u}}(\mathbf{x}) := \widehat{\mathbf{y}}(\mathbf{x}) - \mathbf{x}$ on $\partial\mathcal{R}_R$. Thus this is equivalently a **displacement boundary-value problem.**

– In the complementary case where the traction is prescribed on the entire boundary, $\mathbf{S}\mathbf{n}_R = \widehat{\mathbf{s}}(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathcal{R}_R$, one has a **traction boundary-value problem**. This loading is referred to as **dead loading**. Observe that in dead loading, the prescribed (Piola) traction $\widehat{\mathbf{s}}(\mathbf{x})$ does not depend on the deformation itself. Since the body is in equilibrium, it is necessary that $\widehat{\mathbf{s}}(\mathbf{x})$ (and $\mathbf{b}_R(\mathbf{x})$) be such that

$$\int_{\partial\mathcal{R}_R} \widehat{\mathbf{s}}(\mathbf{x})\, dA_x + \int_{\partial\mathcal{R}} \mathbf{b}_R(\mathbf{x})\, dV_x = 0. \tag{5.9}$$

Question: what about moment balance?

– **Configuration dependent traction boundary condition.** Let $\mathcal{R}$ be the region occupied by the body in the deformed configuration and let $\partial\mathcal{R}$ be its boundary. Suppose, as an example, that a pressure $-p\mathbf{n}$ is applied on $\partial\mathcal{R}$ where the pressure p is a force per unit

[2]The exception is at corners on the boundary (if any).

deformed area. The associated boundary condition is $\mathbf{Tn} = -p\mathbf{n}$ for $\mathbf{y} \in \partial\mathcal{R}$. By using Nanson's formula $\mathbf{n}dA_y = J\mathbf{F}^{-T}\mathbf{n}_R dA_x$ and $\mathbf{Tn}dA_y = \mathbf{Sn}_R dA_x$, this boundary condition can be written as

$$\mathbf{Sn}_R = -pJ\mathbf{F}^{-T}\mathbf{n}_R \qquad \text{for all } \mathbf{x} \in \partial\mathcal{R}_R. \tag{5.10}$$

Observe that the (Piola) traction on $\partial\mathcal{R}_R$ for this loading, the right-hand side of (5.10), depends on the deformation through $J\mathbf{F}^{-T}$. It is *not* dead loading.

5.2 Example (1): Torsion of a circular cylinder.

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You are encouraged to read the paper by Rivlin and Saunders [4] where they describe various experiments on rubber including, in Section 15, the torsion of a circular cylinder.

The region $\mathcal{R}_{\rm R}$ occupied by the body in a reference configuration is a solid circular cylinder of radius A and length L. It is composed of an (incompressible, isotropic) generalized neo-Hookean material. This cylindrical body is welded onto two rigid plates at its ends. The plate at $x_3 = 0$ is held fixed while that at $x_3 = L$ is rotated through an angle αL about the axis of the cylinder. Here we have in mind rectangular cartesian coordinates with the origin at the center of the fixed end and the x_3-axis along the centerline of the cylinder. The curved lateral boundary is traction-free.

Let (R, Θ, Z) and (r, θ, z) be the cylindrical polar coordinates of a particle in the reference and deformed configurations respectively:

$$\left.\begin{array}{lll} x_1 = R\cos\Theta, & x_2 = R\sin\Theta, & x_3 = Z, \\ y_1 = r\cos\theta, & y_2 = r\sin\theta, & y_3 = z. \end{array}\right\} \qquad (i)$$

The associated basis vectors are $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ respectively. A general deformation that takes $(R, \Theta, Z) \to (r, \theta, z)$ can be characterized by

$$r = \widehat{r}(R, \Theta, Z), \qquad \theta = \widehat{\theta}(R, \Theta, Z), \qquad z = \widehat{z}(R, \Theta, Z). \qquad (ii)$$

Here, since the end $Z = 0$ is held fixed, the boundary condition there requires

$$\widehat{r}(R, \Theta, 0) = R, \qquad \widehat{\theta}(R, \Theta, 0) = \Theta, \qquad \widehat{z}(R, \Theta, 0) = 0. \qquad (iii)$$

Since the other end $Z = L$ is rigidly rotated through an angle αL about the Z-axis, the boundary condition there necessitates

$$\widehat{r}(R, \Theta, L) = R, \qquad \widehat{\theta}(R, \Theta, L) = \Theta + \alpha L, \qquad \widehat{z}(R, \Theta, L) = L. \qquad (iv)$$

The curved lateral boundary of the body is traction-free.

Motivated by the boundary conditions (iii) and (iv), we make the following ansatz[3] that the deformation of the body is given by

$$r = \widehat{r}(R, \Theta, Z) = R, \qquad \theta = \widehat{\theta}(R, \Theta, Z) = \Theta + \alpha Z, \qquad z = \widehat{z}(R, \Theta, Z) = Z, \qquad (v)$$

where α is the given angle of twist per unit length. Note that the region $\mathcal{R}$ that the body occupies in the deformed configuration is also a circular cylinder[4] of radius A and length L.

The deformation (v) satisfies the boundary conditions (iii) and (iv) automatically. In this deformation, the cross-section at $x_3 = Z$ rotates rigidly through an angle αZ about

[3] If, based on this assumed form of the deformation, we can satisfy all of the field equations and boundary conditions, then we certainly have a solution of the problem though we have no assurance that it is unique. This approach to solving problems in solid mechanics is referred to as the "semi-inverse method".

[4] This illustrates why we must not confuse a configuration of a body with the region it occupies. In the present setting, the regions $\mathcal{R}_R$ and $\mathcal{R}$ occupied by the body in the undeformed and deformed configurations are identical, though they correspond to different configurations of the body. See Chapter 1 of Volume II for a careful discussion of this issue.

the x_3-axis. Though each cross-section rotates rigidly, different cross-sections rotate through different angles and so the deformation is not purely a rigid rotation. Consider the cross-sections at $x_3 = Z$ and $x_3 = Z + \Delta Z$. If we draw an infinitesimal square on the surface of the undeformed cylinder touching these two sections, the deformation at hand will rotate the square rigidly through an angle αZ (along with the section at $x_3 = Z$), and the additional rotation $\alpha \Delta Z$ (of the section at $x_3 = Z + \Delta Z$) will shear the square in the circumferential direction $\mathbf{e}_\theta$ (the normal to the shearing plane being $\mathbf{e}_z$).

We want to inquire whether (a) the assumed deformation (v) is possible, i.e. whether the stress field associated with (v) obeys the equilibrium equations and the traction-free boundary condition on $r = A$; and if it is possible, (b) to calculate the loading that must be applied on the ends of the cylinder in order to maintain this deformation. Body forces will be ignored.

Substituting the deformation (v) into the formula (2.77) for the deformation gradient tensor in polar coordinates yields

$$\mathbf{F} = \mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_z \otimes \mathbf{e}_Z + \gamma \mathbf{e}_\theta \otimes \mathbf{e}_Z, \tag{vi}$$

where we have set

$$\gamma(r) := r\alpha. \tag{vii}$$

The expression (vi) for the deformation gradient tensor can be written in the illuminating form

$$\mathbf{F} = (\mathbf{I} + \gamma \mathbf{e}_\theta \otimes \mathbf{e}_z)(\mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_z \otimes \mathbf{e}_Z). \tag{$viii$}$$

Observe that the second factor in $(viii)$ is the rotation tensor that carries the basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ into the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. The first factor is a simple shear with shearing direction $\mathbf{e}_\theta$, glide plane normal $\mathbf{e}_z$ and amount of shear $\gamma = r\alpha$.

The left Cauchy-Green tensor corresponding to (vi) is

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{I} + \gamma^2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \gamma(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta). \tag{ix}$$

It can be readily verified that $\det \mathbf{B} = 1$ and therefore that

$$\det\ \mathbf{F} = 1. \tag{x}$$

Thus the torsional deformation (v) is automatically locally volume preserving (isochoric) and incompressibility does not impose any restrictions on it. From (ix) we get

$$I_1(\mathbf{B}) = \operatorname{tr}\mathbf{B} = 3 + \gamma^2, \qquad I_2(\mathbf{B}) = \frac{1}{2}\left[(\operatorname{tr}\mathbf{B})^2 - \operatorname{tr}\mathbf{B}^2\right] = 3 + \gamma^2. \tag{xi}$$

Exercise: Calculate the components of $\mathbf{B}$ in rectangular cartesian coordinates by first writing (i) and (v) as

$$\left.\begin{aligned} &y_1 = r\cos\theta = R\cos(\Theta+\alpha Z) = R\cos\Theta\cos\alpha Z - R\sin\Theta\sin\alpha Z = x_1\cos\alpha x_3 - x_2\sin\alpha x_3,\\ &y_2 = r\sin\theta = R\sin(\Theta+\alpha Z) = R\sin\Theta\cos\alpha Z + R\cos\Theta\sin\alpha Z = x_2\cos\alpha x_3 + x_1\sin\alpha x_3,\\ &y_3 = x_3. \end{aligned}\right\}$$

Then calculate the components of $\mathbf{B}$ in the cylindrical polar basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ by using the basis change formula for 2-tensors.

The body is composed of a generalized neo-Hookean material characterized by its strain energy function

$$W = W(I_1), \qquad W(3) = 0, \qquad W'(I_1) > 0 \quad \text{for } I_1 \geq 3; \tag{xii}$$

the positivity of W' is a consequence of the Baker-Ericksen inequalities together with equation (i) of Problem 2.23. The corresponding constitutive relation for the Cauchy stress is

$$\mathbf{T} = -q\mathbf{I} + 2W'(I_1)\mathbf{B}. \tag{$xiii$}$$

In view of (vi), the deformation is characterized by the single kinematic parameter $\gamma = \alpha r$ and so it is natural to express the strain energy function $W(I_1)$ in terms of γ. Thus let $w(\gamma)$ be the restriction of $W(I_1)$ to a torsional deformation:

$$w(\gamma) := W(3+\gamma^2). \tag{xiv}$$

Substituting (ix) and (xiv) into $(xiii)$ gives the Cauchy stress tensor to be

$$\mathbf{T} = (2W'(I_1) - q)\mathbf{I} + \gamma w'(\gamma)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + w'(\gamma)(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta), \tag{xv}$$

and so the cylindrical polar components of the Cauchy stress are[5]

$$\begin{aligned} &T_{rr} = -q + 2W'(3+\gamma^2), \qquad T_{\theta\theta} = -q + 2W'(3+\gamma^2) + \gamma w'(\gamma),\\ &T_{zz} = -q + 2W'(3+\gamma^2), \qquad T_{r\theta} = T_{rz} = 0, \qquad T_{\theta z} = w'(\gamma). \end{aligned} \tag{xvi}$$

Observe from (xvi) that the shear stress component $T_{z\theta}$ is known, and that once q has been determined, the normal stress components will also be known.

Since the torque about the z-axis is due to the shear stress $T_{z\theta}$, and this stress component is known, we can calculate the component m_z of the moment on the end $x_3 = L$ of the cylinder to be

$$m_z = \int_0^A r\, T_{\theta z}\, 2\pi r\, dr = 2\pi \int_0^A r^2 w'(\gamma)\, dr \overset{(vii)}{=} \frac{2\pi}{\alpha^3}\int_0^{A\alpha} \gamma^2 w'(\gamma)\, d\gamma. \qquad \square$$

[5] Observe that these expressions can be simplified by absorbing the term $2W'(3+\gamma^2)$ into q.

Given the twist angle α and the material $w(\gamma)$, this equation gives the value of m_z. This is conditional of course on being able to satisfy all the field equations and boundary conditions, which requires that the stress field (xvi) satisfy the equilibrium equations and the following traction-free boundary conditions on the curved lateral boundary:

$$T_{rr} = T_{r\theta} = T_{rz} = 0 \qquad \text{at } r = A. \tag{$xvii$}$$

Note from (xvi) that the second and third boundary conditions in $(xvii)$ hold trivially and so only the first has to be enforced.

It would be natural to assume that q is independent of θ and z and depends only on r. However it is easy to show that this must necessarily be true (without having to assume it): the equilibrium equations in cylindrical polar coordinates were given in (3.95). Substituting (xvi) into $(3.95)_2$ and $(3.95)_3$ yields

$$\frac{\partial q}{\partial \theta} = 0, \qquad \frac{\partial q}{\partial z} = 0, \tag{$xviii$}$$

which tells us that q does not depend on θ and z and so is a function of r alone: $q = q(r)$. All stress components are now functions of r only. Finally, the radial equilibrium equation $(3.95)_1$ specializes to the ordinary differential equation

$$\frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0, \qquad 0 \le r < A. \tag{xix}$$

A straightforward approach in which to proceed would be to substitute (xvi) into (xix) to obtain a differential equation for $q(r)$. After this has been solved for $q(r)$, one can then calculate T_{rr} from (xvi) and finally enforce the boundary condition at $r = A$. However, since we are not particularly interested in $q(r)$ we shall proceed in a way that avoids having to find it. Since the only nontrivial boundary condition is on $T_{rr}(A)$ it is natural to use (xix) to solve for T_{rr} directly if possible. To this end, observe from (xvi) that $T_{\theta\theta} - T_{rr}$ does not involve q:

$$T_{\theta\theta} - T_{rr} = \gamma w'(\gamma). \tag{xx}$$

Substituting (xx) into (xix) gives

$$\frac{dT_{rr}}{dr} = \alpha w'(\gamma), \qquad 0 \le r < A. \tag{xxi}$$

We now integrate this from an arbitrary radius r to the outer radius $r = A$ and use the boundary condition $T_{rr}(A) = 0$:

$$\cancel{T_{rr}(A)} - T_{rr}(r) = \int_r^A \alpha w'(\gamma) dr \overset{(vii)}{=} \int_{r\alpha}^{A\alpha} w'(\gamma) d\gamma = w(A\alpha) - w(r\alpha). \tag{$xxii$}$$

Thus the equilibrium equations and boundary conditions have been satisfied and the radial stress field is given by

$$T_{rr}(r) = w(r\alpha) - w(A\alpha), \qquad 0 \le r \le A. \tag{xxiii}$$

Therefore we conclude that the assumed deformation (v) does indeed satisfy all of the requirements of the problem.

Observe from $(xvi)_1$ and $(xxiii)$ that

$$q(r) = 3W'(3+\gamma^2) - w(r\alpha) + w(A\alpha), \qquad 0 \le r \le A,$$

and note, as mentioned previously in Chapter 4.5, that $q(r)$ is a field, not a constant (in general).

Exercise: Suppose the shaft was hollow with its inner and outer boundaries being traction-free. Can you satisfy *both* boundary conditions $T_{rr}(A) = 0$ and $T_{rr}(B) = 0$ (where B is the inner radius)? If not, how would you proceed?

Next we calculate the loading on the ends (having already calculated the torque above).

Observe from (xvi) that due to the non-zero stress T_{zz} there is a normal traction on the the ends of the cylinder. Thus we now calculate the resultant force on a cross-section. This is simply the integral of T_{zz} over the cross-section. From (xvi) we see that $T_{zz} = T_{rr}$ and so the axial stress is

$$T_{zz} = w(r\alpha) - w(A\alpha), \qquad 0 \le r \le A. \tag{xxiv}$$

The axial force to be applied on a cross-section can now be calculated:

$$f_z = \int_0^A T_{zz}\, 2\pi r dr \overset{(xxi),(iii)}{=} \frac{2\pi}{\alpha^2} \int_0^{A\alpha} \gamma \big[w(\gamma) - w(A\alpha)\big]\, d\gamma. \qquad \square \tag{xxv}$$

Exercise: Use symmetry arguments to infer that the components f_x and f_y of the resultant force and the components m_x and m_y of the resultant torque vanish.

5.2.1 Discussion.

– When the preceding results are specialized to the neo-Hookean material $W(I_1) = \frac{\mu}{2}(I_1 - 3)$, one has $w(\gamma) = \frac{\mu}{2}\gamma^2$ and so

$$m_z \overset{(\square)}{=} \frac{2\pi\mu}{\alpha^3} \int_0^{A\alpha} \gamma^3\, d\gamma = \frac{\pi}{2}\mu\alpha A^4, \tag{xxvii}$$

$$f_z \overset{(xxv)}{=} \frac{\pi\mu}{\alpha^2}\int_0^{A\alpha}[\gamma^3 - A^2\alpha^2\gamma]\,d\gamma = -\frac{\pi}{4}\mu A^4\alpha^2. \qquad (xxviii)$$

– Our analysis in this section was limited to a generalized neo-Hookean material. For a general isotropic incompressible material characterized by the strain energy function $W(I_1, I_2)$, one can show that

$$f_z = -2\pi\alpha^2\int_0^A r^3\left(\frac{\partial W}{\partial I_1} + 2\frac{\partial W}{\partial I_2}\right)\bigg|_{I_1=I_2=3+\alpha^2r^2} dr, \qquad (xxix)$$

$$m_z = 4\pi\alpha\int_0^A r^3\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)\bigg|_{I_1=I_2=3+\alpha^2r^2} dr. \qquad (xxx)$$

Again, since the torsional deformation involves a single parameter γ, it is convenient to express the strain energy function $W(I_1, I_2)$ in terms of it. Thus let $w(\gamma)$ be the restriction of $W(I_1, I_2)$ to a torsional deformation:

$$w(\gamma) := W(3+\gamma^2, 3+\gamma^2), \qquad (xxxi)$$

where we have used (xi). One can show that equation (xxx) can be written in terms of w as

$$m_z = 2\pi\int_0^A r^2 w'(\alpha r)\,dr = \frac{2\pi}{\alpha^3}\int_0^{A\alpha}\gamma^2 w'(\gamma)\,d\gamma, \qquad (xxxii)$$

which coincides with what we got for the generalized neo-Hookean material except that here w is defined by $(xxxi)$.

– If a normal force was not applied on the two end plates, then the plates would displace in the x_3-direction and so the length of the cylinder would change. In this case one would consider a deformation of the form

$$r = \widehat{r}(R,\Theta,Z) = R, \qquad \theta = \widehat{\theta}(R,\Theta,Z) = \Theta + \alpha\Lambda Z, \qquad z = \widehat{z}(R,\Theta,Z) = \Lambda Z, \quad (xxxiii)$$

where the stretch Λ is to be determined (from the zero resultant axial force condition).

Exercise: In Problem 5.3 you are asked to carry out the calculations underlying the preceding remark and determine Λ.

– Recall that according to the classical linearized theory of elasticity, in order to subject a circular cylindrical shaft to a torsional deformation one need only apply a torque about its axis; an axial force is not required. It is not surprising that the finite deformation theory says that one must also apply an axial force. Recall from Section 4.6.1 that in order to maintain a finite simple shear deformation one must apply both shear and normal stresses.

Locally, at each point of the shaft, a torsional deformation is just a simple shear together with a rigid rotation. The need for an axial force here is simply a manifestation of the need for normal stresses in simple shear.

– To see another consequence of the presence of normal stresses in simple shear, consider a large thin sheet that contains a small planar crack in its interior. Far from the crack the sheet is subjected to a simple shear deformation in the plane of the sheet with the direction of shearing being parallel to the crack – a so-called Mode II loading. If the crack was not present, the sheet would undergo a simple shear deformation and in particular, there would be a normal stress acting on the plane where we intend there to be a crack. If we now introduce a crack with traction-free faces, it follows that in addition to sliding, the crack faces will either move apart and so the crack will open up, or the crack faces will press together and be in contact. Which of these occurs depends on whether the normal stress in the direction perpendicular to the crack faces is tensile or compressive in the absence of the crack. In contrast, in the linearized theory, the crack faces in Mode II simply slide parallel to each other.

– Finally we remark that normal stresses are also present in the shear flow of non-Newtonian fluids. If such a fluid is placed between two vertical coaxial circular cylindrical tubes with a closed horizontal base, and one of the tubes is rotated about its axis, the fluid will climb up along the tubes (in addition to rotating). This is because, in order to maintain a shear flow, a suitable normal stress must be applied, and such a stress was not applied at the free surface of the fluid in the aforementioned experiment. Thus the fluid moves in the vertical direction.

Exercises: Problems 5.2, 5.3, 5.4, 5.5, 5.14, 5.15 and 5.16.

5.3 Example (2): Deformation of an Incompressible Cube Under Prescribed Tensile Forces.

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4. R.S. Rivlin, Stability of pure homogeneous deformations of an elastic cube under dead loading, *Quarterly of Applied Mathematics*, **32**(1974), pp. 265-271

5. L.R.G. Treloar, Stresses and birefringence in rubber subjected to general homogeneous strain, *Proceedings of the Physical Society*, 60(1948), pp. 135-144.

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Equilibrium configurations of a cube: Consider an elastic body that occupies a unit cube in a reference configuration. It is composed of an (incompressible, isotropic) neo-Hookean material[6] characterized by the strain-energy function

$$W = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \qquad \mu > 0. \tag{i}$$

The principal Cauchy stresses are related to the principal stretches by the constitutive relation

$$\tau_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - q = \mu\lambda_i^2 - q, \quad i = 1, 2, 3, \qquad \text{(no sum on } i\text{)}, \tag{ii}$$

where q arises due to the incompressibility constraint.

Each of the six faces of the cube is subjected to a uniformly distributed normal traction whose resultant is a tensile force $F(> 0)$. This is illustrated in Figure 5.1 where the uniform distribution of normal traction is not shown, only the resultant forces are. We wish to determine the resulting pure homogeneous deformation of the body. This problem is frequently referred to as the "Rivlin cube problem".

It should be noted that in the problem we are considering it is the force F, or equivalently the Piola traction, that is prescribed and so we have dead loading on the entire boundary of the body. The associated Cauchy (true) tractions on the faces of the cube will depend on the areas of the faces in the deformed configuration. One could alternatively consider the problem in which the Cauchy tractions are prescribed on each face; that is a different problem to the one we study here.

[6] In Problem 10.3 you will generalize this to an arbitrary isotropic material.

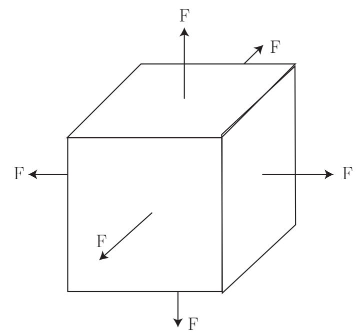

Figure 5.1: A unit cube in the reference configuration. All six of its faces are subjected to uniformly distributed normal tractions whose resultant force, on each face, has magnitude F. The figure only shows the resultant forces and not the distributed tractions.

Because of the symmetry of the body, the loading and the material, one may be inclined to assume that the deformation will also be symmetric. However, we wish to look at the possibility of not-necessarily symmetric pure homogeneous deformations, and so we shall not assume a priori that the cube deforms symmetrically. If it does, then we will find this to be the case. Thus suppose that the cube undergoes a pure homogeneous deformation

$$y_1 = \lambda_1 x_1, \qquad y_2 = \lambda_2 x_2, \qquad y_3 = \lambda_3 x_3. \tag{iii}$$

Incompressibility requires

$$\lambda_1 \lambda_2 \lambda_3 = 1. \tag{iv}$$

The deformed faces of the body have areas $\lambda_2\lambda_3$, $\lambda_3\lambda_1$ and $\lambda_1\lambda_2$ and so the prescribed boundary conditions tell us that the Cauchy stress components are

$$\tau_1 = T_{11} = \frac{F}{\lambda_2\lambda_3} \overset{(iv)}{=} F\lambda_1, \qquad \tau_2 = T_{22} = \frac{F}{\lambda_3\lambda_1} \overset{(iv)}{=} F\lambda_2, \qquad \tau_3 = T_{33} = \frac{F}{\lambda_1\lambda_2} \overset{(iv)}{=} F\lambda_3. \tag{v}$$

The problem at hand is to find the principal stretches λ_i, given F (and μ). The Piola stress tensor (field throughout the body) is

$$\mathbf{S} = \mathbf{S}(\mathbf{x}) = F\mathbf{e}_1 \otimes \mathbf{e}_1 + F\mathbf{e}_2 \otimes \mathbf{e}_2 + F\mathbf{e}_3 \otimes \mathbf{e}_3 \qquad \text{for } \mathbf{x} \in \mathcal{R}_R. \tag{vi}$$

It does not depend on the deformation. Note that

$$\mathbf{S}(\mathbf{x})\mathbf{n}_R(\mathbf{x}) = \widehat{\mathbf{s}}_R(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \partial\mathcal{R}_R. \tag{via}$$

Since the deformation is homogeneous, and assuming the reaction pressure field q to be constant, the stress field will also be homogeneous throughout the body. Therefore (ignoring body forces) the equilibrium equations are satisfied automatically. The boundary conditions have already been accounted for in (v) above. All that remains is to enforce the constitutive law (ii):

$$T_{11} = \mu\lambda_1^2 - q, \qquad T_{22} = \mu\lambda_2^2 - q, \qquad T_{33} = \mu\lambda_3^2 - q. \tag{vii}$$

Combining (vii) with (v) and using (iv) leads to

$$F\lambda_1 = \mu\lambda_1^2 - q, \qquad F\lambda_2 = \mu\lambda_2^2 - q, \qquad F\lambda_3 = \mu\lambda_3^2 - q. \tag{viii}$$

Equations $(viii)$ and (iv) provide four (nonlinear) algebraic equations involving $\lambda_1, \lambda_2, \lambda_3$ and q.

In order to solve these equations systematically it is convenient to first eliminate q. Thus, subtracting the second of $(viii)$ from the first, and similarly the third from the second leads to

$$\left.\begin{aligned} [F - \mu(\lambda_1 + \lambda_2)](\lambda_1 - \lambda_2) &= 0, \\ [F - \mu(\lambda_2 + \lambda_3)](\lambda_2 - \lambda_3) &= 0. \end{aligned}\right\} \tag{ix}$$

Equations (iv) and (ix) are to be solved for the principal stretches $\lambda_1, \lambda_2, \lambda_3$. There are three cases to consider:

Case (1): Suppose first that all of the λ's are distinct: $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. Then (ix) yields

$$F = \mu(\lambda_1 + \lambda_2), \qquad F = \mu(\lambda_2 + \lambda_3),$$

which implies that $\lambda_1 + \lambda_2 = \lambda_2 + \lambda_3$ whence

$$\lambda_1 = \lambda_3.$$

This contradicts the assumption that the $\lambda's$ are all distinct. Thus there is no solution in which the three λ's are distinct. (If such a solution had existed, it would have described an *orthorhombic configuration* of the body.)

Case (2): Suppose next that all of the λ's are equal: $\lambda_1 = \lambda_2 = \lambda_3$. This describes a *cubic configuration*. In this case equations (ix) are automatically satisfied and (iv) tells us that

$$\lambda_1 = \lambda_2 = \lambda_3 = 1. \tag{x}$$

Thus one solution of the problem, for every value of the applied force F, is given by (iii), (x). This corresponds to a configuration of the body in which, geometrically, it remains a unit cube, but one that is under stress.

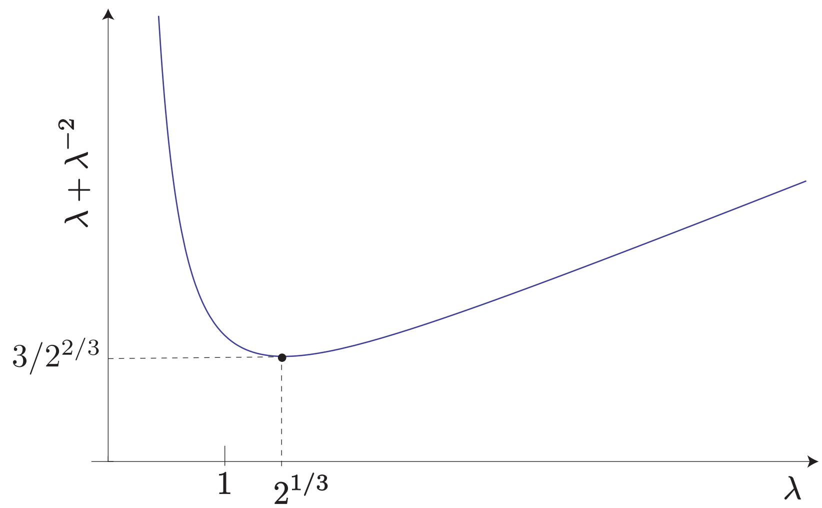

Figure 5.2: Graph of $h(\lambda) = \lambda + \lambda^{-2}$ versus λ.

Case (3): Finally, consider the remaining possibility that two λ's are equal and different to the third. This describes a *tetragonal configuration*. Suppose that

$$\lambda_1 = \lambda_2 = \lambda \text{ (say)}, \qquad \lambda_3 \neq \lambda. \tag{xi}$$

Incompressibility (iv) together with (xi) requires

$$\lambda_3 = \lambda^{-2}, \quad \lambda \neq 1, \tag{xii}$$

while the pair of equations (ix) reduce to $F = \mu(\lambda_2 + \lambda_3) = \mu(\lambda + \lambda^{-2})$, i.e.

$$\lambda + \lambda^{-2} = f \qquad \text{where we have set} \quad f := F/\mu. \tag{xiii}$$

Given f, if $(xiii)$ can be solved for one or more real roots $\lambda > 0$, then $(iii), (xi), (xiii)$ provides the corresponding solution to the problem[7]. Whether $(xiii)$ can be solved or not depends on the value of f.

In order to examine the solvability of $(xiii)$, let $h(\lambda) := \lambda + \lambda^{-2}$ for $\lambda > 0$, and observe that $h \to \infty$ as $\lambda \to 0^+$ and $h \to \infty$ as $\lambda \to \infty$. Moreover, $h'(\lambda) = 1 - 2\lambda^{-3}$ and so $h'(\lambda) < 0$ for $0 < \lambda < 2^{1/3}$, $h'(\lambda) = 0$ for $\lambda = 2^{1/3}$, and $h'(\lambda) > 0$ for $\lambda > 2^{1/3}$. Thus the graph of $h(\lambda)$ versus λ is as shown in Figure 5.2. The minimum value of h is $h(2^{1/3}) = 3/2^{2/3}$.

From Figure 5.2 and $(xiii)$ we see that

[7]There are of course additional configurations corresponding to permutations of the λ's, e.g. $\lambda_3 = \lambda_1 = \lambda$, $\lambda_2 = \lambda^{-2}$.

if $f < 3/2^{2/3}$, equation $(xiii)$ has no roots,

if $f = 3/2^{2/3}$, equation $(xiii)$ has one root $\lambda = 2^{1/3}$, and

if $f > 3/2^{2/3}$, equation $(xiii)$ has two roots.

For a solution with $\lambda > 1$ one has $\lambda_1 = \lambda_2 > 1, \lambda_3 < 1$ and so the deformed body has two relatively long equal edges and one relatively short unequal edge, i.e. the block has a flattened shape as depicted by the upper inset in Figure 5.3. On the other hand $\lambda < 1$ describes configurations in which $\lambda_1 = \lambda_2 < 1, \lambda_3 > 1$ where the deformed body has two relatively short equal edges and one relatively long unequal edge, i.e. the block has a pillar-like shape as in the lower inset.

Thus in *summary*, there are two types of configurations which the body can adopt. In one, the body remains a unit cube in the deformed configuration and this is possible for all values of the applied force F. The other is possible only if $F/\mu \geq 3/2^{2/3}$ and here the deformed body is no longer a cube. Rather, it has a tetragonal shape where two sides are equal and the third is different, and there are two possibilities of this form corresponding to the two roots of $(xiii)$.

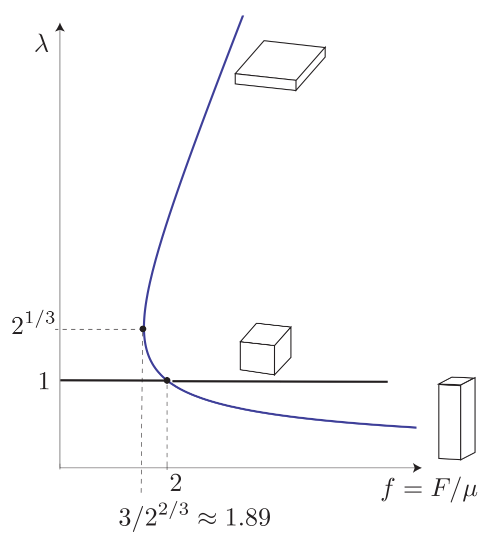

Figure 5.3: Equilibrium configurations of the cube: the symmetric (cubic) configuration corresponds to the line $\lambda = 1$. The curve corresponds to the asymmetric (tetragonal) configurations given by $(xiii)$.

Both types of solutions are depicted in Figure 5.3. The cubic solution corresponds to the horizontal line $\lambda = 1$ which extends indefinitely to the right. The tetragonal solution corresponds to the curve (which is the same curve as in Figure 5.2 but with the axes switched).

The figure shows that

> if $f < 3/2^{2/3}$ the body must be in the cubic configuration,
>
> if $f > 3/2^{2/3}$ the body can be in either a cubic or tetragonal configuration there being two configurations of the latter type.

Thus the solution to the equilibrium problem is *non-unique*.

Stability of the cube: The lack of uniqueness prompts us to examine the stability of the various equilibrium configurations.

Remarks on the stability of an equilibrium configuration: An alternative approach for studying equilibrium configurations of an elastic solid/structure is via the minimization of the potential energy. One considers *all geometrically possible deformation fields* $\mathbf{z}(\mathbf{x})$, and minimizes the potential energy Φ over this class of functions. If the potential energy has an extremum at say $\mathbf{z}(\mathbf{x}) = \mathbf{y}(\mathbf{x})$ then $\mathbf{y}(\mathbf{x})$ describes an equilibrium configuration of the body. If this extremum corresponds to a minimum of the potential energy, then we presume that this configuration is stable.

Suppose that an elastic body occupies a region $\mathcal{R}_\mathrm{R}$ in a reference configuration and that the body force $\mathbf{b}_R(\mathbf{x})$ is prescribed on $\mathcal{R}_\mathrm{R}$, the deformation $\widehat{\mathbf{y}}(\mathbf{x})$ is prescribed on a portion $\mathcal{S}_1$ of its boundary, and the Piola traction ("dead load") $\widehat{\mathbf{s}}(\mathbf{x})$ is prescribed on the remaining portion $\mathcal{S}_2$ of the boundary; here $\partial\mathcal{R}_\mathrm{R} = \mathcal{S}_1 \cup \mathcal{S}_2$. A kinematically possible deformation field ("a virtual deformation field") is any smooth enough vector field $\mathbf{z}(\mathbf{x})$ defined on $\mathcal{R}_\mathrm{R}$ that obeys all geometric constraints. One geometric requirement is that $\mathbf{z}(\mathbf{x})$ coincide with the prescribed deformation on $\mathcal{S}_1$. If there are internal kinematic constraints such as incompressibility, then these too must be enforced. The potential energy associated with a geometrically possible deformation field $\mathbf{z}(\mathbf{x})$ is

$$\Phi = \int_{\mathcal{R}_\mathrm{R}} W(\nabla\mathbf{z})\, dV_x \ - \int_{\mathcal{R}_\mathrm{R}} \mathbf{b}_R \cdot \mathbf{z}\, dV_x \ - \int_{\mathcal{S}_2} \widehat{\mathbf{s}} \cdot \mathbf{z}\, dA_x; \qquad (xiv)$$

Appendix 5.3.1 on page 479 explains how this expression arises. The first term on the right-hand side describes the elastic energy stored in the body while the next two terms correspond to the potential energy of the loading. One seeks to minimize this functional over the set of all geometrically possible deformation fields $\mathbf{z}(\mathbf{x})$.

In the case of the tri-axially loaded incompressible cube, dead-loading is prescribed on the entire boundary and so one can replace $\mathcal{S}_2$ in (xiv) by $\partial\mathcal{R}_{\rm R}$ and write

$$\Phi = \int_{\mathcal{R}_{\rm R}} W(\nabla\mathbf{z})\ dV_x \ - \int_{\partial\mathcal{R}_R} \widehat{\mathbf{s}}\cdot\mathbf{z}\ dA_x \overset{(via)}{=} \int_{\mathcal{R}_{\rm R}} W(\nabla\mathbf{z})\ dV_x \ - \int_{\partial\mathcal{R}_R} \mathbf{S}\mathbf{n}_R\cdot\mathbf{z}\ dA_x. \quad (xv)$$

We have also omitted the body force term. Since the deformation is not prescribed anywhere on $\partial\mathcal{R}_R$, the only requirement of an admissible deformation $\mathbf{z}(\mathbf{x})$ is that due to incompressibility. Thus (xv) is to be minimized over the set of all smooth enough vector fields $\mathbf{z}(\mathbf{x})$ subject to $\det\nabla\mathbf{z} = 1$.

Rather than minimizing this over the set of all geometrically possible kinematic fields suppose that we minimize over the smaller class of all geometrically possible *homogeneous* deformation fields: $\mathbf{z}(\mathbf{x}) = \mathbf{F}\mathbf{x}$ where $\mathbf{F}$ is an arbitrary constant tensor with unit determinant. (In Problem 10.4.8 you will consider all virtual deformations.) Then the potential energy specializes to

$$\Phi = \int_{\mathcal{R}_{\rm R}} W(\mathbf{F})\ dV_x \ - \int_{\partial\mathcal{R}_{\rm R}} \mathbf{S}\mathbf{n}_{\rm R}\cdot\mathbf{F}\mathbf{x}\ dA_x = \int_{\mathcal{R}_{\rm R}} \Big[W(\mathbf{F}) \ - \ \mathbf{S}\cdot\mathbf{F}\Big]\ dV_x,$$

where we have used the divergence theorem in getting to the second equality and $\mathbf{S}$ is given by (vi). Since $\mathbf{F}$ and $\mathbf{S}$ are constants, this leads to

$$\Phi = W(\mathbf{F}) - \mathbf{S}\cdot\mathbf{F}, \qquad (xvi)$$

which is to be minimized over all tensors $\mathbf{F}$ with $\det\mathbf{F} = 1$.

Finally, suppose that we further limit attention to geometrically possible deformation fields of the even more restricted form

$$z_1 = \lambda_1 x_1, \quad z_2 = \lambda_2 x_2, \quad z_3 = \lambda_3 x_3, \qquad \lambda_1\lambda_2\lambda_3 = 1; \qquad (xvii)$$

here $\lambda_1, \lambda_2, \lambda_3$ *are arbitrary* subject only to $(xvii)_4$ and $\lambda_k > 0$. The potential energy (xvi) (for the neo-Hookean material with $\mathbf{S}$ given by (vi)) now takes the explicit form

$$\frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) - F(\lambda_1 + \lambda_2 + \lambda_3). \qquad (xviii)$$

We are to minimize this over all $(\lambda_1, \lambda_2, \lambda_3)$ subject to the constraint $\lambda_1\lambda_2\lambda_3 = 1$. We can simplify this by eliminating λ_3 using the incompressibility equation. After dropping an inessential constant, dividing by μ and letting $f = F/\mu$ as before, we can write the function to be minimized as

$$\Phi(\lambda_1, \lambda_2) = \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}) - f\,(\lambda_1 + \lambda_2 + \lambda_1^{-1}\lambda_2^{-1}). \qquad (xix)$$

This function is to be minimized over all $\lambda_1 > 0, \lambda_2 > 0$.

For convenience let

$$\Phi_\alpha = \frac{\partial \Phi}{\partial \lambda_\alpha}, \qquad \Phi_{\alpha\beta} = \frac{\partial^2 \Phi}{\partial \lambda_\alpha \partial \lambda_\beta}, \qquad [P] = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12} & \Phi_{22} \end{pmatrix}, \qquad D = \det[P]. \tag{xx}$$

The equilibrium configurations correspond to the extrema of the potential energy function $\Phi(\lambda_1, \lambda_2)$. They are found by setting

$$\Phi_1 = \Phi_2 = 0.$$

An equilibrium configuration is locally stable if the corresponding extremum is a local minimum of $\Phi(\lambda_1, \lambda_2)$. To study the character of an extremum one evaluates the Hessian matrix $[P]$ at that extremum. The extremum is a local minimum if the Hessian matrix is positive definite, i.e. if both eigenvalues of $[P]$ are positive; it is a local maximum if both eigenvalues of $[P]$ are negative; and it is a saddle if one eigenvalue of $[P]$ is positive and the other is negative. Thus

$$\begin{array}{lll} \text{An extremum is a local minimum if} & D > 0, & \Phi_{11} > 0, \\ \text{An extremum is a local maximum if} & D > 0, & \Phi_{11} < 0, \\ \text{An extremum is a saddle if} & D < 0. & \end{array}$$

Differentiating (xix) yields

$$\left.\begin{array}{l} \Phi_1 = \lambda_1 - \lambda_1^{-3}\lambda_2^{-2} - f(1 - \lambda_2^{-1}) = [\lambda_1 + \lambda_1^{-1}\lambda_2^{-1} - f](\lambda_2 - \lambda_1^{-2})\lambda_2^{-1}, \\ \\ \Phi_2 = \lambda_2 - \lambda_2^{-3}\lambda_1^{-2} - f(1 - \lambda_2^{-2}\lambda_1^{-1}) = [\lambda_2 + \lambda_2^{-1}\lambda_1^{-1} - f](\lambda_1 - \lambda_2^{-2})\lambda_1^{-1}. \end{array}\right\} \tag{xxi}$$

Setting $\Phi_1 = \Phi_2 = 0$ leads to a pair of algebraic equations. There are four cases to consider since each equation involves two factors, each of which could vanish. For example the vanishing of the terms in both square brackets leads to case (b) below, while the vanishing of the terms in the first square bracket and the second parenthesis gives case (d). In this way we find the set of equilibrium configurations to be

$$(a) \quad \lambda_1 = \lambda_2 = 1, \; (\lambda_3 = 1),$$

$$(b) \quad \lambda_1 + \lambda_1^{-2} = f, \quad \lambda_1 = \lambda_2, \; (\lambda_3 = \lambda_1^{-2}),$$

$$(c) \quad \lambda_1 + \lambda_1^{-2} = f, \quad \lambda_3 = \lambda_1, \; (\lambda_2 = \lambda_1^{-2}),$$

$$(d) \quad \lambda_2 + \lambda_2^{-2} = f, \quad \lambda_2 = \lambda_3, \; (\lambda_1 = \lambda_2^{-2}).$$

By using the incompressibility condition $\lambda_1\lambda_2\lambda_3 = 1$, we see that solution (a) corresponds to the cubic configuration $\lambda_1 = \lambda_2 = \lambda_3 = 1$; solution (b) corresponds to the tetragonal configuration $\lambda_1 = \lambda_2 \neq \lambda_3$; solution (c) corresponds to the tetragonal configuration $\lambda_1 = \lambda_3 \neq \lambda_2$; and solution (d) corresponds to the tetragonal configuration $\lambda_2 = \lambda_3 \neq \lambda_1$. Thus the equilibrium configurations corresponding to solutions (a) and (b) are the same ones we found earlier, while solutions (c) and (d) are permutations of solution (b).

Since (c) and (d) are simply permutations of (b), we will shortly ignore them (as we did before). However, in order to better understand these extrema we shall plot energy contours on the λ_1, λ_2-plane, and when we do this we will be forced to confront all of the extrema. Thus for a short while longer we will continue to consider all extrema $(a) - (d)$. The λ_1, λ_2-plane with these seven extrema marked is shown in Figure 5.4.

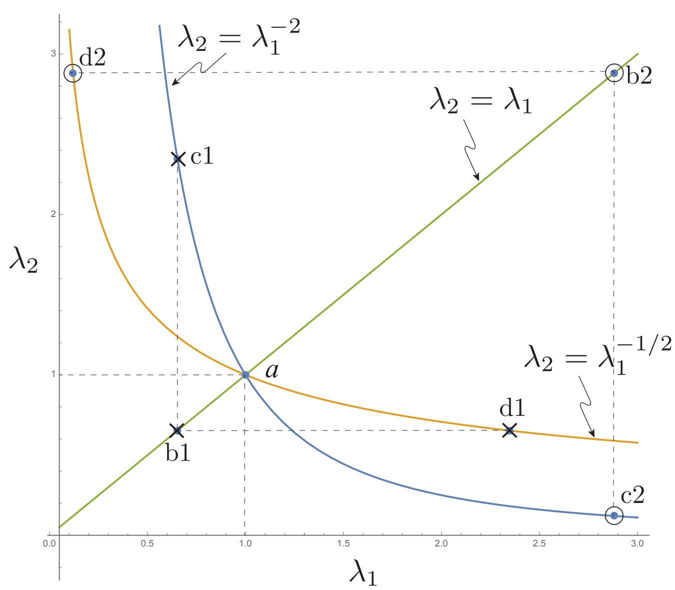

Figure 5.4: Extrema of the potential energy function $\Phi(\lambda_1, \lambda_2)$ on the λ_1, λ_2-plane. The point a corresponds to solution (a); points $b1, b2$ to solution (b); points $c1, c2$ to solution (c); and points $d1, d2$ to solution (d). The figure has been drawn for $f = 3$. Figures 5.5 and 5.6 tell us which of these are local minima, (local maxima and saddle points).

To examine the stability of these equilibrium configurations we next calculate the second

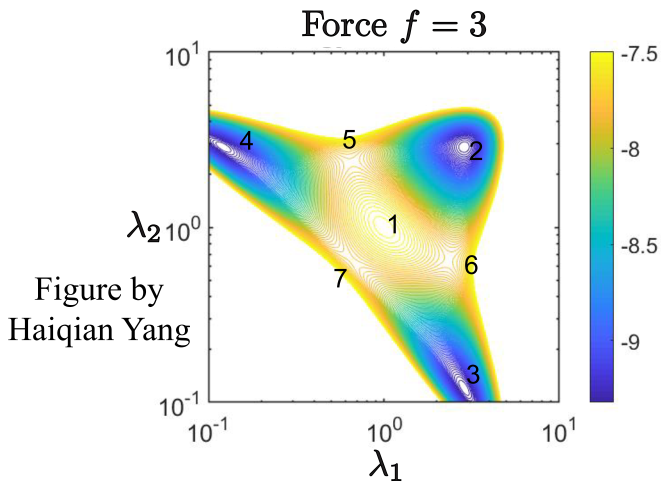

Figure 5.5: Energy contours of potential energy $\Phi(\lambda_1, \lambda_2)$ on the λ_1, λ_2-plane for $f = 3$. The energy has a local maximum at point 1 (corresponding to point a in Figure 5.4). It has saddle points at 7, 5 and 6 (corresponding to $b1, c1, d1$ in Figure 5.4). The energy has local minima at 2, 3 and 4 (corresponding to $b2, c2, d2$ in Figure 5.4). Figure provided by Haiqian Yang (student in 2.074 in 2020).

derivatives of Φ:

$$\left.\begin{aligned} \Phi_{11} &= 1 + 3\lambda_1^{-4}\lambda_2^{-2} - 2f\ \lambda_1^{-3}\lambda_2^{-1}, \\ \Phi_{22} &= 1 + 3\lambda_2^{-4}\lambda_1^{-2} - 2f\ \lambda_2^{-3}\lambda_1^{-1}, \\ \Phi_{12} &= 2\lambda_1^{-3}\lambda_2^{-3} - f\lambda_1^{-2}\lambda_2^{-2}. \end{aligned}\right\} \qquad (xxii)$$

Considering the cubic configuration, we set $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in $(xxii)$ and $(xx)_4$ which gives

$$\Phi_{11} = 2(2 - f), \qquad \Phi_{22} = 2(2 - f), \qquad \Phi_{12} = 2 - f, \qquad D = 3(2 - f)^2.$$

Therefore by the statement above (xxi), this extremum is a local minimum and *the cubic configuration is stable* if ($D > 0$, i.e. $f \neq 2$ and) $\Phi_{11} > 0$:

$$f < 2.$$

(It is a local maximum for $f > 2$.) At a tetragonal solution, say (b), we have ($\lambda_3 = \lambda^{-2}$ and)

$$\lambda_1 = \lambda_2 = \lambda \neq 1, \qquad f = \lambda + \lambda^{-2}, \qquad (xxiii)$$

and to examine its stability we evaluate $(xxii)$ and $(xx)_4$ at $(xxiii)$:

$$\Phi_{11} = (1 - \lambda^{-3})^2 > 0, \qquad \Phi_{22} = (1 - \lambda^{-3})^2, \qquad \Phi_{12} = -(1 - \lambda^{-3})\lambda^{-3},$$

$$D = \lambda^{-3}(1 - \lambda^{-3})^2\,(\lambda^3 - 2)\,.$$

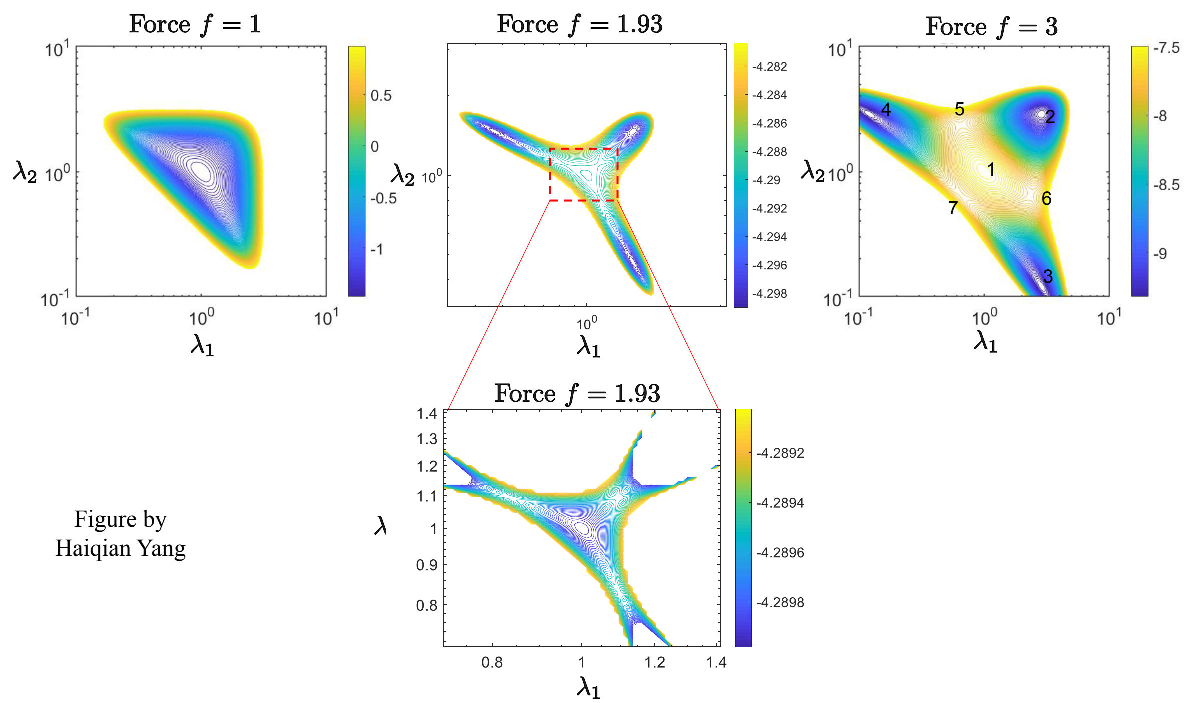

Figure 5.6: Energy contours of the potential energy at three values of force. For $f = 1$ it has a single energy well at the cubic configuration. The force $f = 1.93$ corresponds to the intermediate range in Figure 5.3. The energy has four energy wells corresponding to the cubic and three flattened configurations. When f exceeds 2, the cubic configuration is no longer a local minimum. Figure provided by Haiqian Yang (student in 2.074 in 2020).

By the statement above (xxi), this extremum is a minimum and *the tetragonal configuration is stable* if $D > 0$:

$$\lambda^3 > 2 \qquad \Rightarrow \qquad \lambda > 2^{1/3}. \tag{xxiv}$$

(It is a saddle point when it is not a minimum.)

Figure 5.5 shows the energy contours of the potential energy $\Phi(\lambda_1, \lambda_2)$. The figure has been drawn for $f = 3$. At this value of force, the energy has a local maximum at point 1 (corresponding to point a in Figure 5.4); saddle points at 7, 5 and 6 (corresponding to $b1, c1, d1$ in Figure 5.4); and energy wells (local minima) at 2, 3 and 4 (corresponding to $b2, c2, d2$ in Figure 5.4). The saddle points $b1, c1$ and $d1$ correspond to unstable pillar like configurations, while the local minima $b2, c2$ and $d2$ correspond to flattened stable configurations.

Figure 5.6 shows the energy contours at several values of f. For $f = 1$ it has a single energy well and it occurs at the cubic configuration. Since $3/2^{2/3} < 1.93 < 2$, the force

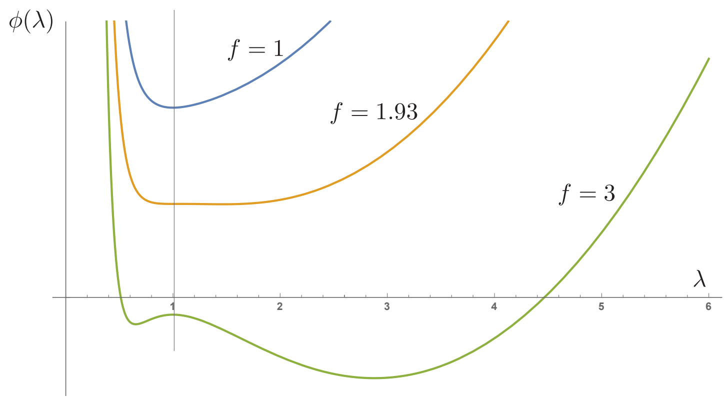

Figure 5.7: Graphs of the potential energy $\phi(\lambda) = \Phi(\lambda, \lambda)$ versus λ for three fixed values of the force f, one > 2, the second in the interval $(3/2^{2/3}, 2)$ and the third $< 3/2^{2/3}$. The leftmost local minimum of the curve corresponding to $f = 3$ is in fact a saddle point of the energy $\Phi(\lambda_1, \lambda_2)$ on the two-dimensional λ_1, λ_2-plane.

$f = 1.93$ corresponds to the intermediate range in Figure 5.3, and the energy in Figure 5.6 has four energy wells corresponding to the cubic and three flattened configurations. When f exceeds 2, the cubic configuration is no longer a local minimum.

Figure 5.7 shows a plot of the potential energy

$$\phi(\lambda) := \Phi(\lambda_1, \lambda_2)\Big|_{\lambda_1=\lambda, \lambda_2=\lambda} \overset{(xix)}{=} \frac{1}{2}(2\lambda^2 + \lambda^{-4}) - f(2\lambda + \lambda^{-2})$$

versus the stretch λ for three different values of force. This is the slice of the graph of $\Phi(\lambda_1, \lambda_2)$ on the λ_1, λ_2-plane along the straight line $\lambda_2 = \lambda_1$. In keeping with Figure 5.6, observe that the extremum at $\lambda = 1$ is an energy well (local minimum) for $f < 2$ while it is a local maximum for $f > 2$. For $3/2^{1/3} < f < 2$ the energy has two energy wells, one at $\lambda = 1$ and the other at a value of $\lambda > 2^{1/3}$, and a local maximum between them. For $f > 2$ the graph in Figure 5.7 shows two energy wells with a local maximum at $\lambda = 1$. We know from the preceding analysis that what appears to be an energy-well at the smaller stretch here is in fact a saddle point when viewed on the λ_1, λ_2-plane. It merely appears to be a minimum along the particular slice of the energy shown in Figure 5.7.

Finally we return to considering solutions (a) and (b) only and ignore the various permutations. Figure 5.8 depicts the solutions again, now with the solid line/curve corresponding to the stable solutions and the dashed line/curve the unstable ones. For $f < 3/2^{1/3}$ we have a unique stable cubic configuration. For $f > 2$ we have a unique (to within permutations) stable tetragonal configuration; see also Figure 5.4. On the intermediate range $3/2^{1/3} < f < 2$ there is one stable cubic configuration and one stable tetragonal configuration. (Strictly, there are 3 asymmetric solutions since permutations of the λ_i's should be considered.) Observe that λ is > 1 for the stable tetragonal configurations and these, as noted previously, correspond to configurations where the deformed body is flattened.

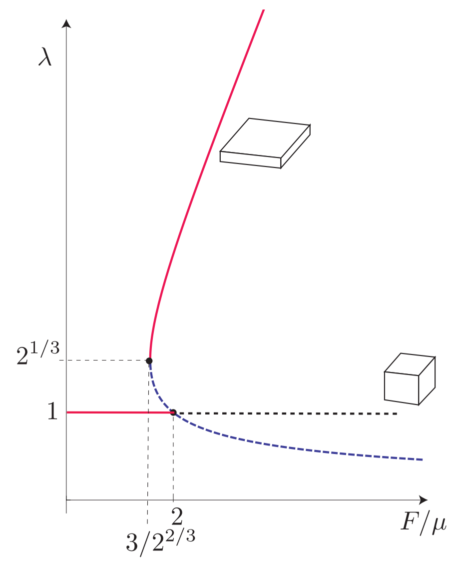

Figure 5.8: The stable and unstable solutions are depicted by the solid and dashed curves respectively.

Remark: We were supposed to minimize the potential energy (xvi) over all $\mathbf{F}$ with $\det \mathbf{F} = 1$. However we only considered admissible deformations of the form $(xvii)$ where $\mathbf{F}$ was coaxial with $\mathbf{S}$, i.e. $\mathbf{F}$ and $\mathbf{S}$ were diagonal in the same basis. When the more restricted analysis we carried out claims an equilibrium state to be a local minimizer, it may or may not be a minimizer under the wider class of all deformations. However if the restricted analysis shows an equilibrium state to be not a local minimizer, then it is not a minimizer even under the wider class of all deformations. Therefore equilibrium states that we found to be unstable are indeed unstable. Those we found to be stable may not be stable in the context of a wider class of admissible deformations.

Remark: The analogous plane stress problem was analyzed by Kearsley [2] in the context of the experiments of Treloar. In this case the stress-free body is a thin sheet and it is subjected to equal in-plane biaxial forces F, the other two faces of the sheet being traction-free. See references given at the beginning of this section (including the monograph by Ericksen [1]). The neo-Hookean material model does not exhibit asymmetric configurations in this case but a Mooney-Rivlin material does; see Problem 5.7.

Exercises: Problems 5.6 and 5.7.

5.3.1 Appendix: Potential energy of an elastic body subjected to conservative loading:

A brief video on potential energy can be found here.

Consider a motion $\mathbf{y}(\mathbf{x}, t)$. By taking the scalar product of the equation of motion $\text{Div}\,\mathbf{S} + \mathbf{b}_R = \rho_R \dot{\mathbf{v}}$ with the particle velocity $\mathbf{v} = \dot{\mathbf{y}}$, and integrating over $\mathcal{R}_R$, one can readily show for an elastic body that

$$\int_{\partial\mathcal{R}_R} \mathbf{S}\mathbf{n}_R \cdot \mathbf{v}\, dA_x + \int_{\mathcal{R}_R} \mathbf{b}_R \cdot \mathbf{v}\, dV_x = \frac{d}{dt}\int_{\mathcal{R}_R} \left(\frac{1}{2}\rho_R \mathbf{v}\cdot\mathbf{v} + W\right) dV_x, \qquad (i)$$

where ρ_R is the mass density in the reference configuration. This is a statement of the fact that the rate at which work is done on the body by the traction on $\partial\mathcal{R}_R$ and the body force on $\mathcal{R}_R$ equals the rate at which the kinetic energy and the potential energy due to deformation (the strain energy) increase. When the loading is conservative, the rate at which the loading does work can also be expressed as the rate of increase of an associated potential energy.

Suppose that the deformation is prescribed on a part $\mathcal{S}_1$ of the boundary to be $\widehat{\mathbf{y}}(\mathbf{x})$, and the Piola traction is prescribed on the complementary part $\mathcal{S}_2$ to be $\widehat{\mathbf{s}}(\mathbf{x})$. Thus for all t,

$$\mathbf{y}(\mathbf{x}, t) = \widehat{\mathbf{y}}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathcal{S}_1, \qquad \mathbf{S}(\mathbf{x}, t)\mathbf{n}_R = \widehat{\mathbf{s}}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathcal{S}_2. \qquad (ii)$$

A body force $\mathbf{b}_R(\mathbf{x})$ is applied on $\mathcal{R}_R$. Observe that the prescribed deformation $\widehat{\mathbf{y}}$, the traction $\widehat{\mathbf{s}}$ and the body force $\mathbf{b}_R$ have all been assumed to be time-independent[8]. Moreover, note that the loading $\widehat{\mathbf{s}}$ and $\mathbf{b}_R$ are independent of the deformation – they are said to be "dead loadings". Differentiating $(ii)_1$ with respect to t shows that the particle velocity $\mathbf{v}$ vanishes on $\mathcal{S}_1$. Thus we can write (i) as

$$\int_{\mathcal{S}_2} \widehat{\mathbf{s}} \cdot \mathbf{v}\, dA_x + \int_{\mathcal{R}_R} \mathbf{b}_R \cdot \mathbf{v}\, dV_x = \frac{d}{dt}\int_{\mathcal{R}_R} \left(W + \frac{1}{2}\rho_R \mathbf{v}\cdot\mathbf{v}\right) dV_x, \qquad (iii)$$

having also used $(ii)_2$. Since $\widehat{\mathbf{s}}$ and $\mathbf{b}_R$ (as well as $\mathcal{S}_2$ and $\mathcal{R}_R$) are time independent, and $\mathbf{v} = \dot{\mathbf{y}}$, this can be written as

$$\frac{d}{dt}\int_{\mathcal{S}_2} \widehat{\mathbf{s}} \cdot \mathbf{y}\, dA_x + \frac{d}{dt}\int_{\mathcal{R}_R} \mathbf{b}_R \cdot \mathbf{y}\, dV_x = \frac{d}{dt}\int_{\mathcal{R}_R} \left(W + \frac{1}{2}\rho_R \mathbf{v}\cdot\mathbf{v}\right) dV_x, \qquad (iv)$$

[8] This does not imply that the body is in equilibrium. It could, for example, be vibrating while the loading remains constant.

which yields

$$\frac{d}{dt}\left[\int_{\mathcal{R}_R}\frac{1}{2}\rho_R\mathbf{v}\cdot\mathbf{v}\,dV_x+\int_{\mathcal{R}_R}W\,dV_x-\int_{\mathcal{S}_2}\widehat{\mathbf{s}}\cdot\mathbf{y}\,dA_x-\int_{\mathcal{R}_R}\mathbf{b}_R\cdot\mathbf{y}\,dV_x\right]=0. \qquad (v)$$

This states that the sum of the kinetic and potential energies is conserved, the first term being the kinetic energy and the next three terms the potential energy Φ:

$$\Phi := \int_{\mathcal{R}_R}W\,dV_x-\int_{\mathcal{S}_2}\widehat{\mathbf{s}}\cdot\mathbf{y}\,dA_x-\int_{\mathcal{R}_R}\mathbf{b}_R\cdot\mathbf{y}\,dV_x. \qquad (vi)$$

The first term in (vi) is the elastic potential energy of the body due to deformation and the next two terms represent the potential energy of the loading.

Remark: More generally, suppose instead of being a dead load, the body force $\mathbf{b}_R$ is merely conservative in the sense that there is an associated potential $\varphi(\mathbf{x},\mathbf{y})$ such that

$$\mathbf{b}_R=-\frac{\partial\varphi}{\partial\mathbf{y}}.$$

The special case of a dead load corresponds to $\varphi(\mathbf{x},\mathbf{y})=-\mathbf{b}_R(\mathbf{x})\cdot\mathbf{y}$. The rate of working of the body force, the second term on the left-hand side of (iii), can now be written as

$$\int_{\mathcal{R}_R}\mathbf{b}_R\cdot\mathbf{v}\,dV_x=-\int_{\mathcal{R}_R}\frac{\partial\varphi}{\partial\mathbf{y}}\cdot\dot{\mathbf{y}}\,dV_x=-\int_{\mathcal{R}_R}\frac{\partial\varphi}{\partial t}\,dV_x=-\frac{d}{dt}\int_{\mathcal{R}_R}\varphi\,dV_x.$$

Similarly if the traction $\widehat{\mathbf{s}}$ is conservative with an associated potential $\psi(\mathbf{y},\mathbf{x})$,

$$\widehat{\mathbf{s}}=-\frac{\partial\psi}{\partial\mathbf{y}}, \qquad (vii)$$

the first term on the left-hand side of (iii) can be written as

$$\int_{\mathcal{S}_2}\widehat{\mathbf{s}}\cdot\mathbf{v}\,dA_x-\frac{d}{dt}\int_{\mathcal{S}_2}\psi\,dA_x.$$

The expression (vi) for the total potential energy now reads

$$\Phi=\int_{\mathcal{R}_R}(W+\varphi)\,dV_x+\int_{\mathcal{S}_2}\psi\,dA_x. \qquad (viii)$$

Remark: A potential energy of the form $\psi(\mathbf{x},\mathbf{y})$ does not cover all prescribed tractions of interest since we are sometimes concerned with loadings of the form $\widehat{\mathbf{s}}(\mathbf{x},\mathbf{y},\mathbf{F})$, e.g. the pressure loading in (5.10). See Section 5.4.2 of Ogden [2] for a discussion on how to handle such loadings.

5.4 Example (3): Growth of a Cavity.

References:

1. A.N. Gent and P.B. Lindley, Internal rupture of bonded rubber cylinders in tension, *Proceedings of the Royal Society (London)*, **A249**, (1958), 195-205.
2. J.M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, *Philosophical Transactions of the Royal Society (London)*, **A306**, (1982), 557-611.
3. H. Wang and S. Cai, Drying-induced cavitation in a constrained hydrogel, *Soft Matter*, 11(2015), pp. 1058-1061.

See also Problems 10.4.3, 10.4.6 and 10.1.

A body occupies a hollow spherical region of inner radius A and outer radius B in a reference configuration. It is composed of a generalized neo-Hookean material. A uniformly distributed radial tensile dead load (Piola traction) of magnitude σ is applied on the outer surface of the body while the inner surface remains traction-free. We wish to determine the deformation and stress fields in the body. *Our particular interest is in the radius a of the deformed cavity as a function of the applied stress σ.*

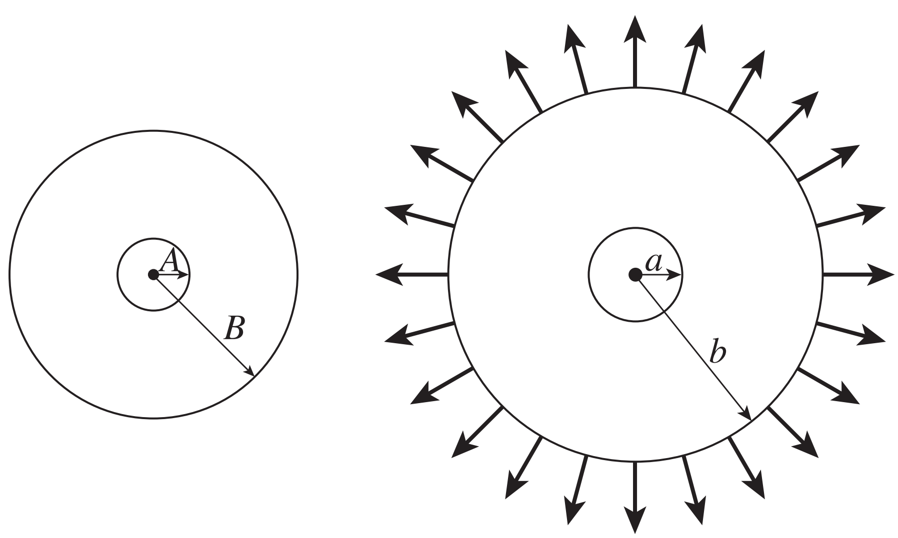

Figure 5.9: A hollow sphere in a reference configuration (left) and in the deformed configuration (right). A uniform radial dead load is applied on the outer surface of the sphere.

Let (R, Θ, Φ) and (r, θ, ϕ) be the spherical polar coordinates of a particle in the reference and deformed configurations respectively with associated basis vectors $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$. The geometric, material and loading symmetries, suggest that we assume the deformation to have the spherically symmetric form

$$r = \widehat{r}(R, \Theta, \Phi) = r(R), \qquad \theta = \widehat{\theta}(R, \Theta, \Phi) = \Theta, \qquad \phi = \widehat{\phi}(R, \Theta, \Phi) = \Phi. \tag{i}$$

Thus the displacement of a particle is in the radial direction and its magnitude depends only on the radial coordinate. Moreover, $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi\} = \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$. In Chapter 2.7.2 we derived formulae for the components of the left Cauchy-Green deformation tensor $\mathbf{B}$ in spherical polar coordinates. Substituting (i) into (2.87) yields

$$\mathbf{B} = \big[r'(R)\big]^2 \mathbf{e}_r \otimes \mathbf{e}_r + \left[\frac{r(R)}{R}\right]^2 (\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi). \tag{ii}$$

Since $\mathbf{B}$ is diagonal in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ we can read off the principal stretches to be

$$\lambda_r = r'(R), \qquad \lambda_\theta = \lambda_\phi = r(R)/R,$$

having assumed $r'(R) > 0$. Since the material is incompressible, $\lambda_r \lambda_\theta \lambda_\phi = \lambda_r \lambda_\theta^2 = 1$ and so it is convenient to work in terms of the circumferential stretch which we denote by λ. Then

$$\lambda_\theta = \lambda_\phi = \lambda := \frac{r}{R}, \qquad \lambda_r = \lambda^{-2} = r'. \tag{iii}$$

Incompressibility requires

$$\lambda_r \lambda_\theta \lambda_\phi = r' \, \frac{r^2}{R^2} = 1. \tag{iv}$$

Integrating (iv) gives $r^3 = R^3 + \mathsf{constant}$. Let a denote the as-yet-unknown radius of the cavity in the deformed configuration. Then, since $r = a$ when $R = A$ we can write this as

$$r(R) = \big[R^3 + a^3 - A^3\big]^{1/3}. \tag{v}$$

The value of the constant a is to be determined. We could have written (v) directly by equating the volume between the spherical surfaces of radii A and R in the reference configuration to the volume between the surfaces of radii a and r in the deformed configuration: $\frac{4}{3}\pi(R^3 - A^3) = \frac{4}{3}\pi(r^3 - a^3)$. The outer radius of the deformed body, $b = r(B)$, is

$$b = \big[B^3 + a^3 - A^3\big]^{1/3}. \tag{vi}$$

The deformation $r = r(R)$ is completely determined by (i) and (v) once we determine a.

Turning to the constitutive relation we first note from (iii) that

$$I_1 = \mathrm{tr}\,\mathbf{B} = \lambda_r^2 + \lambda_\theta^2 + \lambda_\phi^2 = 2\lambda^2 + \lambda^{-4}. \qquad (vii)$$

From $(ii), (iii)$ and the constitutive relation (4.67) for a generalized neo-Hookean material, we find the Cauchy stress components to be

$$T_{rr} = -q + 2\lambda^{-4}W'(I_1), \qquad T_{\theta\theta} = T_{\phi\phi} = -q + 2\lambda^2 W'(I_1), \qquad T_{r\theta} = T_{\theta z} = T_{zr} = 0, \quad (viii)$$

where q arises from the incompressibility constraint. Note that the stress components are fully determined by $(viii)$ when $q(r, \theta, \phi)$ and a are known.

We assume that the stress components are functions of the radial coordinate r alone, which, by $(viii)$, implies that q depends only on r: $q = q(r)$. The equilibrium equations (3.98) in spherical polar coordinates now reduce to the single equation

$$\frac{dT_{rr}}{dr} + \frac{2}{r}(T_{rr} - T_{\theta\theta}) = 0. \qquad (ix)$$

The direct way in which to proceed is to substitute $(viii)$ into (ix) to obtain a differential equation for $q(r)$, and after it has been solved, to then calculate the stresses from $(viii)$. However, since we are not particularly interested in q, and the boundary conditions are given on the radial stress, it is more natural to work with T_{rr}.

Observe from $(viii)$ that $T_{rr} - T_{\theta\theta}$ does not depend on q. In fact,

$$T_{rr} - T_{\theta\theta} = 2(\lambda^{-4} - \lambda^2)W'(I_1), \qquad (x)$$

and so we can write (ix) as

$$\frac{dT_{rr}}{dr} = \frac{4}{r}(\lambda^2 - \lambda^{-4})W'(I_1). \qquad (xi)$$

While we can integrate both sides with respect to r, it turns out to be preferable to integrate the right-hand side with respect to λ instead. This requires us to convert the left-hand side to $dT_{rr}/d\lambda$. We achieve this by changing variables, first from r to R using $r = r(R)$, and then from R to λ using $\lambda = r(R)/R$:

$$\frac{dT_{rr}}{dr} = \frac{1}{r'}\frac{dT_{rr}}{dR} = \frac{1}{r'}\frac{Rr' - r}{R^2}\frac{dT_{rr}}{d\lambda} = \frac{1 - \lambda^3}{R}\frac{dT_{rr}}{d\lambda}$$

having also used $\lambda = r/R$ and $r' = \lambda^{-2}$ in the last step. Therefore from the two preceding equation we obtain

$$\frac{dT_{rr}}{d\lambda} = \frac{4(\lambda - \lambda^{-5})}{1 - \lambda^3}W'(I_1) \qquad \text{where } I_1 = 2\lambda^2 + \lambda^{-4}. \qquad (xiii)$$

It is convenient to express the strain energy as a function of the stretch λ, i.e. to define a function $w(\lambda)$ by

$$w(\lambda) := W(I_1)\Big|_{I_1=2\lambda^2+\lambda^{-4}}. \tag{xiv}$$

Differentiating (xiv) with respect to λ gives

$$w'(\lambda) = 4(\lambda - \lambda^{-5})W'(I_1), \tag{xv}$$

and so we can finally write $(xiii)$ as

$$\frac{dT_{rr}}{d\lambda} = \frac{w'(\lambda)}{1-\lambda^3}. \tag{xvi}$$

On integrating (xvi) from the inner boundary to the outer boundary we get

$$T_{rr}(b) - T_{rr}(a) = \int_{\lambda_a}^{\lambda_b} \frac{w'(\lambda)}{1-\lambda^3}\, d\lambda = \int_{\lambda_b}^{\lambda_a} \frac{w'(\lambda)}{\lambda^3-1}\, d\lambda, \tag{$xvii$}$$

where λ_a and λ_b are the circumferential stretches at the corresponding boundaries:

$$\lambda_a := \frac{a}{A}, \qquad \lambda_b := \frac{b}{B} \overset{(vi)}{=} \left[1 - \frac{A^3}{B^3} + \frac{a^3}{B^3}\right]^{1/3}. \tag{$xviii$}$$

As for the boundary conditions, we are told the inner boundary is traction-free:

$$T_{rr} = 0 \qquad \text{at } r = a. \tag{xix}$$

We are also told that there is a radial Piola stress $S_{rR}(B) = \sigma$ at the outer boundary. The corresponding radial Cauchy stress $T_{rr}(b)$ can be calculated using the general tensor relation between $\mathbf{S}$ and $\mathbf{T}$. Alternatively, since the area of the outer surface in the reference configuration is $4\pi B^2$ while the corresponding area in the deformed configuration is $4\pi b^2$ we must have $4\pi B^2 S_{rR}(B) = 4\pi b^2 T_{rr}(b)$ and so the boundary condition at the outer surface is

$$T_{rr} = \sigma\frac{B^2}{b^2} \overset{(xviii)}{=} \frac{\sigma}{\lambda_b^2} \qquad \text{at } r = b. \tag{xx}$$

On substituting (xix) and (xx) into $(xvii)$,

$$\sigma = \lambda_b^2 \int_{\lambda_b}^{\lambda_a} \frac{w'(\lambda)}{\lambda^3-1}\, d\lambda. \tag{xxi}$$

By $(xviii)$, λ_a and λ_b are functions of the unknown deformed cavity radius a which is therefore the only unknown in equation (xxi). Thus given σ, equation (xxi) constitutes an algebraic equation for a.

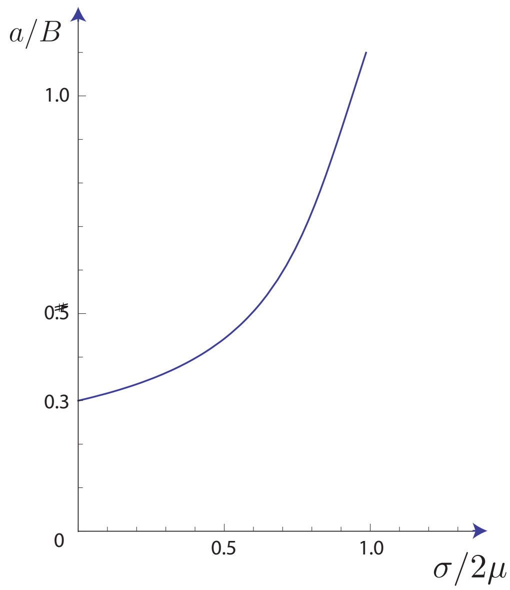

Figure 5.10: Variation of the deformed cavity radius a with applied stress σ for a neo-Hookean material. The figure has been drawn for the case $A/B = 0.3$.

To illustrate the behavior predicted by (xxi) consider a neo-Hookean material. In this case

$$W = \frac{\mu}{2}\big(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\big) \qquad \Rightarrow \qquad w(\lambda) = \frac{\mu}{2}(\lambda^{-4} + 2\lambda^2 - 3).$$

Substituting this into (xxi) and evaluating the integral leads to

$$\frac{\sigma}{2\mu} = \left[\frac{1}{\lambda_b} + \frac{1}{4}\frac{1}{\lambda_b^4} - \frac{1}{\lambda_a} - \frac{1}{4}\frac{1}{\lambda_a^4}\right]\lambda_b^2. \qquad (xxii)$$

This equation together with $(xviii)$ tells us how the cavity radius a in the deformed configuration depends on the stress σ.

Equation $(xxii)$ with $(xviii)$ is of the form $\sigma/2\mu = h(a)$. One can show that $h(a)$ increases monotonically with a, and moreover that $h(A) = 0$ and $h(a) \to \infty$ as $a \to \infty$. Thus, for each given value of the stress $\sigma > 0$, the equation $\sigma/2\mu = h(a)$ can be solved for a unique root a. The graph in Figure 5.10 shows the variation of the cavity radius a/B with the stress $\sigma/2\mu$ according to $(xxii)$; the figure has been drawn for a cavity of initial radius $A/B = 0.3$.

Thus far, this problem in the nonlinear theory, has not been qualitatively different to the corresponding problem in linear elasticity. However, if we plot graphs of a versus σ for progressively decreasing initial cavity radii A, the family of curves obtained shows an interesting trend as seen in Figure 5.11. As $A/B \to 0$, the graph of a/B versus $\sigma/2\mu$

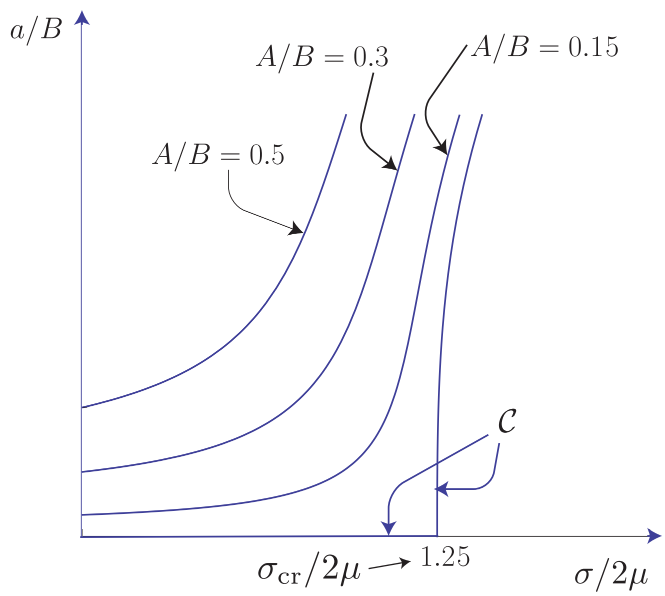

Figure 5.11: Variation of the deformed cavity radius a with applied stress σ for a neo-Hookean material. The different curves correspond to different values of the undeformed cavity radius A. Observe that as $A/B \to 0$ these curves approach the curve $\mathcal{C}$.

approaches the curve $\mathcal{C}$. Note that $\mathcal{C}$ is composed of two segments: the straight line segment $a = 0$ for $0 < \sigma/2\mu \leq 1.25$ and the curved portion for $\sigma/2\mu \geq 1.25$. The curve $\mathcal{C}$ describes the growth of a cavity whose radius is infinitesimal in the undeformed configuration. According to it, for $\sigma/2\mu \leq 1.25$ the cavity radius in the deformed configuration is also zero. However when $\sigma/2\mu > 1.25$, the cavity has a positive radius and has opened. Thus "cavitation" occurs at the critical stress $\sigma/2\mu = 1.25$.

To examine this analytically, we first determine the curved portion of $\mathcal{C}$ by taking the limit of the right-hand side of $(xxii)$ as $A/B \to 0$ at fixed $a/B > 0$. Observe from $(xviii)$ that in this limit

$$\lambda_a = \frac{a}{A} = \frac{a}{B}\frac{B}{A} \to \infty, \qquad \lambda_b \to \left[1 + \frac{a^3}{B^3}\right]^{1/3}. \tag{$xxiii$}$$

Substituting these limiting values into $(xxii)$ gives

$$\frac{\sigma}{2\mu} = \lambda_b + \frac{1}{4}\frac{1}{\lambda_b^2} = \frac{5/4 + a^3/B^3}{(1 + a^3/B^3)^{2/3}},$$

which is the equation of the curved portion of $\mathcal{C}$. To find $\sigma_{\rm cr}$, i.e. the point at which $\mathcal{C}$

departs from the horizontal axis, we let $a/B \to 0$ in this equation which yields

$$\frac{\sigma_{\rm cr}}{2\mu} = 5/4. \qquad (xxiv)$$

This is indicated in Figure 5.11.

Remark: Observe from $(xxiii)$ that $\lambda_b \to 1$ in the limit $a/B \to 0$ and therefore from (xx) that

$$T_{cr} = \sigma_{cr}.$$

In summary, we have shown for a neo-Hookean material that a cavity that is infinitesimally small in the undeformed configuration remains infinitesimally small as the stress σ increases until it reaches the critical value σ_{cr}. When σ exceeds σ_{cr}, the cavity opens and grows (i.e. $a > 0$) in the manner described by the curved portion of $\mathcal{C}$. This describes the phenomenon of *cavitation*.

We now return to the generalized neo-Hookean material (from the neo-Hookean material) and concern ourselves with (xxi). On taking the limit $A/B \to 0$ at fixed $a/B > 0$ and keeping $(xxiii)$ in mind, (xxi) yields

$$\sigma = \lambda_b^2 \int_{\lambda_b}^{\infty} \frac{w'(\lambda)}{\lambda^3 - 1}\, d\lambda \qquad \text{where} \quad \lambda_b = \left[1 + \frac{a^3}{B^3}\right]^{1/3}. \qquad (xxv)$$

This relates the deformed radius a of a cavity that was infinitesimal in the reference configuration to the stress σ. To find the critical stress σ_{cr} for cavitation we let $a/B \to 0$ in (xxv) to obtain the formal expression

$$\sigma_{cr} = \int_1^{\infty} \frac{w'(\lambda)}{\lambda^3 - 1} d\lambda. \qquad (xxvi)$$

As noted in the exercise below, this expression for the cavitation stress continues to hold for an arbitrary isotropic incompressible material (with $w(\lambda)$ defined by $(xxix)$ below). Observe that the integrand in $(xxvi)$ has a potential singularity at $\lambda = 1$ unless $w'(1)$ behaves suitably. Moreover, since the range of this integral is infinite, its convergence depends on the behavior of $w(\lambda)$ as $\lambda \to \infty$. If $w(\lambda) \sim \lambda^m$ for large λ, the integral will converge if $m < 3$ and not otherwise. This is essentially Ball's condition (4.125). Thus for example for the one-term Ogden material (4.143), cavitation will not occur, i.e. the critical stress at cavitation will be infinite, if the constitutive parameter $n \geq 3$. In summary, for certain elastic materials, i.e. certain functions W, the integral in $(xxvi)$ will not converge and so an infinitesimally small void will remain infinitesimally small for all values of applied stress. For other materials (this integral will converge and) an infinitesimally small cavity will begin to grow when σ exceeds the critical value given by $(xxvi)$.

Remark: Ball [2] studied this problem using energy minimization which therefore addresses the issue of stability as well. Moreover, he studied this as a bifurcation problem for an (initially) solid sphere.

Exercise: *Arbitrary incompressible isotropic material:* The preceding analysis (for a generalized neo-Hookean material) can be carried over quite easily to an arbitrary incompressible isotropic material. Rather than working with $W(I_1, I_2)$ work with the form $W(\lambda_1, \lambda_2, \lambda_3)$ and use the constitutive relation (4.67) to show that

$$T_{rr} = \lambda_1 \frac{\partial W}{\partial \lambda_1} - q = \lambda^{-2} \frac{\partial W}{\partial \lambda_1}\bigg|_{\lambda_1=\lambda^{-2}, \lambda_2=\lambda_3=\lambda} - q, \qquad (xxvii)$$

$$T_{\theta\theta} = T_{\phi\phi} = \lambda_2 \frac{\partial W}{\partial \lambda_2} - q = \lambda \frac{\partial W}{\partial \lambda_2}\bigg|_{\lambda_1=\lambda^{-2}, \lambda_2=\lambda_3=\lambda} - q. \qquad (xxviii)$$

Let $w(\lambda)$ be the restriction of the strain energy function to deformations of the sort at hand:

$$w(\lambda) := W(\lambda^{-2}, \lambda, \lambda). \qquad (xxix)$$

Show that

$$T_{rr} - T_{\theta\theta} = -\frac{1}{2}\lambda w'(\lambda), \qquad (xxx)$$

and that equation $(xxvi)$ continues to hold except that w is now defined by $(xxix)$.

Exercises: Problems 5.10, 5.11, 5.14, 5.17.

5.5 Example (4): Limit point instability of a thin-walled hollow sphere.

The limit point instability of a thin-walled hollow sphere (or cylinder) refers to the loss of monotonicity of the function $p(\lambda)$ that gives the pressure p as a function of the mean circumferential stretch λ. Since we studied the deformation of a hollow spherical body in Section 5.4, it is convenient to make use of those results here, specializing them to the thin-walled case[9]. The two differences, (at least to start with), between the problem in Section 5.4 and that considered here is that one, the loading here is an internal pressure p per unit deformed area applied on the inner boundary with the outer boundary being traction-free, and two, the body is composed of an arbitrary isotropic incompressible elastic material. (We

[9] Alternatively one can derive the results pertinent to this case directly as in the next problem in Section 5.6.

will want to specialize the results to a Mooney-Rivlin material, and so it is not sufficient to limit attention to a generalized neo-Hookean material.)

We start be rewriting the relevant equations from Section 5.4 (making use of the results in the Exercise at the end of that section): the deformation is given by either of the equivalent forms

$$r^3(R) = R^3 + a^3 - A^3 = R^3 + b^3 - B^3, \tag{i}$$

where a and b are the inner and outer radii in the deformed configuration; and the circumferential stretch $\lambda(R)$ is

$$\lambda(R) = \frac{r(R)}{R}. \tag{ii}$$

From the constitutive relation we obtain

$$T_{\theta\theta} - T_{rr} = \frac{1}{2}\lambda w'(\lambda), \tag{iii}$$

and the equilibrium equations reduce to

$$\frac{dT_{rr}}{d\lambda} = -\frac{w'(\lambda)}{\lambda^3 - 1}, \tag{iv}$$

where

$$w(\lambda) := W(\lambda^{-2}, \lambda, \lambda). \tag{v}$$

Here $W(\lambda_1, \lambda_2, \lambda_3)$ is the strain energy function characterizing the incompressible isotropic material. The boundary conditions in the problem considered here are

$$T_{rr}(a) = -p, \qquad T_{rr}(b) = 0. \tag{vi}$$

Integrating (iv) from $r = a$ to $r = b$ and using the boundary conditions (vi) leads to

$$p = \int_{\lambda_b}^{\lambda_a} \frac{w'(\lambda)}{\lambda^3 - 1}\, d\lambda, \tag{vii}$$

where λ_a and λ_b are the circumferential stretches at the inner and outer surfaces respectively:

$$\lambda_a = a/A, \qquad \lambda_b = b/B. \tag{$viii$}$$

Equation (vii) together with $(viii)$ and $b^3 - B^3 = a^3 - A^3$ provides a relation between the pressure p and the deformed inner radius a.

We now specialize the preceding results to the case where the body is *thin-walled*, i.e. when the wall thickness $T = B - A$ is small compared to the mean radius $\overline{R} = (A + B)/2$:

$$\epsilon := T/\overline{R} \ll 1. \tag{ix}$$

The inner and outer undeformed radii can be expressed in terms of $\overline{R}$ and ε as

$$A = \overline{R} - T/2 = \overline{R}(1 - \epsilon/2), \qquad B = \overline{R} + T/2 = \overline{R}(1 + \epsilon/2). \tag{x}$$

If $\overline{\lambda} = \lambda(\overline{R})$ is the mean circumferential stretch we can approximate (vii) as

$$p \approx \frac{w'(\overline{\lambda})}{\overline{\lambda}^3 - 1}\,(\lambda_a - \lambda_b).$$

Thus our immediate task is to find an approximate expression for $\lambda_a - \lambda_b$ for small ε. We shall do this by deriving expressions of the form $\lambda_a = \overline{\lambda} + \varepsilon\alpha + \mathrm{O}(\varepsilon^2)$ and $\lambda_b = \overline{\lambda} + \varepsilon\beta + \mathrm{O}(\varepsilon^2)$.

From (i), together with (ii) and $(viii)$, we have

$$\lambda_a^3 = \frac{a^3}{A^3} = \frac{R^3}{A^3}(\lambda^3 - 1) + 1, \qquad \lambda_b^3 = \frac{b^3}{B^3} = \frac{R^3}{B^3}(\lambda^3 - 1) + 1, \tag{xi}$$

where $\lambda = \lambda(R)$ is the stretch at the radius R. Now evaluate $(xi)_1$ at the mean radius $\overline{R}$ where the stretch is $\overline{\lambda}$, use $(x)_1$ and drop terms smaller than $\mathrm{O}(\epsilon)$:

$$\begin{aligned}
\lambda_a \quad &\overset{(xi)_1}{=} \left[\frac{\overline{R}^3}{A^3}(\overline{\lambda}^3 - 1) + 1\right]^{1/3} \overset{(x)_1}{=} \left[\left(1 - \frac{\epsilon}{2}\right)^{-3}(\overline{\lambda}^3 - 1) + 1\right]^{1/3} = \\
&= \left[\left(1 + \frac{3\epsilon}{2}\right)(\overline{\lambda}^3 - 1) + 1\right]^{1/3} + \mathrm{O}(\epsilon^2) = \overline{\lambda}\left[1 + \frac{3\epsilon}{2}\frac{\overline{\lambda}^3 - 1}{\overline{\lambda}^3}\right]^{1/3} + \mathrm{O}(\epsilon^2) = \\
&= \overline{\lambda} + \frac{\overline{\lambda}^3 - 1}{2\overline{\lambda}^2}\epsilon + \mathrm{O}(\epsilon^2).
\end{aligned} \tag{xii}$$

Likewise from $(xi)_2$ and $(x)_2$ one obtains

$$\lambda_b = \overline{\lambda} - \frac{\overline{\lambda}^3 - 1}{2\overline{\lambda}^2}\epsilon + \mathrm{O}(\epsilon^2). \tag{$xiii$}$$

We can now approximate (vii) as

$$p = \frac{w'(\overline{\lambda})}{\overline{\lambda}^3 - 1}\,(\lambda_a - \lambda_b) + \mathrm{O}(\epsilon^2) \overset{(xii),(xiii)}{=} \epsilon\frac{w'(\overline{\lambda})}{\overline{\lambda}^2} + \mathrm{O}(\epsilon^2). \tag{xiv}$$

Therefore to leading order, the pressure p is related to the mean circumferential stretch $\overline{\lambda}$ by

$$p = p(\overline{\lambda}) \;:=\; \epsilon\,\frac{w'(\overline{\lambda})}{\overline{\lambda}^2} = \frac{T}{\overline{R}}\,\frac{w'(\overline{\lambda})}{\overline{\lambda}^2}. \tag{xv}$$

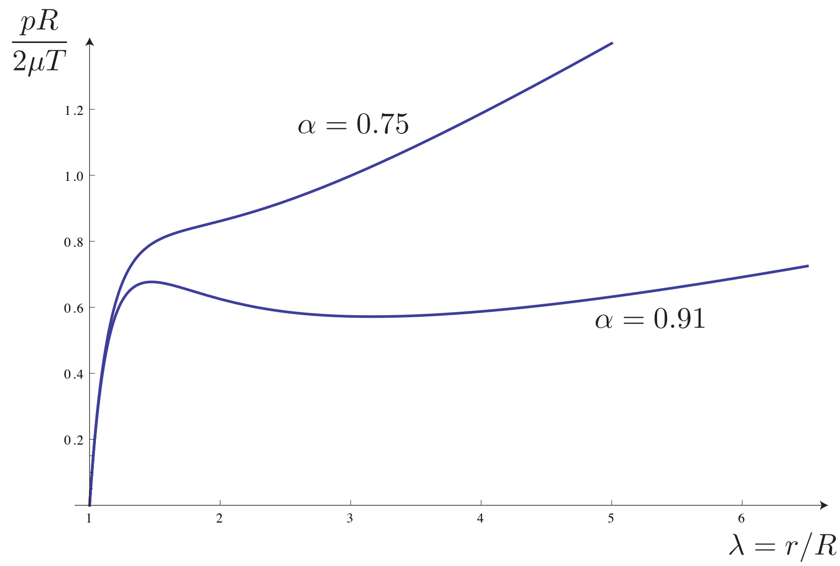

Figure 5.12: Pressure p versus the mean circumferential stretch $\overline{\lambda}$ for a thin-walled spherical shell of a Mooney-Rivlin material. The figure has been drawn for $\alpha = 0.75$ and 0.91

To illustrate the response of the thin-walled shell, consider a Mooney-Rivlin material:

$$W = \frac{\mu}{2}\Big[\alpha(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + (1-\alpha)(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3)\Big], \qquad \mu > 0, \quad 0 < \alpha < 1.$$

The associated energy function $w(\lambda)$ defined by (v) is

$$w(\lambda) = \frac{\mu}{2}\Big[\alpha(\lambda^{-4} + 2\lambda^2 - 3) + (1-\alpha)(\lambda^4 + 2\lambda^{-2} - 3)\Big],$$

and so from (xv) we find that

$$\frac{pR}{T} = \frac{w'(\overline{\lambda})}{\overline{\lambda}^2} = 2\mu(\overline{\lambda} - \overline{\lambda}^{-5})(1 - \alpha + \alpha\overline{\lambda}^{-2}). \qquad (xvi)$$

Figure 5.12 shows a plot of the pressure p versus the circumferential stretch $\overline{\lambda}$ based on (xvi) for two values of α. For $\alpha = 0.75$, the pressure increases monotonically with increasing stretch but the curvature is seen to change. The curve for $\alpha = 0.91$ is quite different (once $\overline{\lambda}$ exceeds about 1.25). In particular, the relation between p and r/R loses monotonicity – the so-called limit point instability. The curve now has two rising branches connected by a declining branch. The multiple branches of the curve indicate the existence of multiple equilibrium configurations if the pressure is in a suitable range. The configuration associated with the left-most branch has a relatively small deformed radius while the configuration associated with the right-most branch has a significantly larger radius. Transition between these configurations occurs at the local maximum and minimum.

Remark 1: Before leaving this problem we shall calculate the radial and circumferential stress components T_{rr} and $T_{\theta\theta}$. For convenience we shall drop the overline on the mean stretch. First, considering the thick-walled shell, the radial stress is found by integrating (iv) from an arbitrary radius to the outer boundary leading to

$$T_{rr}(r) = -\int_{\lambda_b}^{r/R} \frac{w'(\lambda)}{\lambda^3 - 1}\, d\lambda, \tag{xvii}$$

and then the circumferential stress is given by (iii) and $(xvii)$ to be

$$T_{\theta\theta}(r) = \frac{1}{2}\lambda w'(\lambda) - \int_{\lambda_b}^{r/R} \frac{w'(\lambda)}{\lambda^3 - 1}\, d\lambda. \tag{xviii}$$

Approximating the expression $(xvii)$ for the radial stress gives

$$T_{rr} = -\frac{w'(\lambda)}{\lambda^3 - 1}\,(\lambda - \lambda_b) + \mathrm{O}(\epsilon^2) \overset{(xiii)}{=} -\frac{1}{2}\epsilon\frac{w'(\lambda)}{\lambda^2} + \mathrm{O}(\epsilon^2) \overset{(xiv)}{=} -\frac{p}{2} + \mathrm{O}(\epsilon^2). \tag{xix}$$

Observe from (xvi) that

$$w'(\lambda) = \frac{p\lambda^2}{\epsilon} + \mathrm{O}(\epsilon). \tag{xx}$$

Finally we turn to the circumferential stress $T_{\theta\theta}$. Substituting (xix) and (xx) into (iii) yields

$$T_{\theta\theta} = T_{rr} + \frac{1}{2}\lambda w'(\lambda) = -\frac{p}{2} + \frac{p\lambda^3}{2\epsilon} + \mathrm{O}(\epsilon) = \frac{p\lambda^3}{2\epsilon} + \mathrm{O}(1). \tag{xxi}$$

The familiar expression for $T_{\theta\theta}$ in a thin-walled spherical shell involves the mean radius r and wall thickness t in the *deformed* configuration. Setting $\lambda = r/R$ and $\epsilon = T/R$ in (xxi) gives

$$T_{\theta\theta} \approx \frac{pr^3}{2TR^2}. \tag{xxii}$$

However by incompressibility, the volumes $4\pi R^2 T$ and $4\pi r^2 t$ of the undeformed and deformed shell must be equal whence

$$R^2 T = r^2 t. \tag{xxiii}$$

Observe from this and $\lambda = r/R$ that

$$t = T/\lambda^2. \tag{xxiv}$$

Using $(xxiii)$ in $(xxii)$ gives the circumferential stress in the familiar form

$$T_{\theta\theta} \approx \frac{pr}{2t}, \tag{xxv}$$

where r and t are the mean radius and wall thickness in the deformed configuration.

In summary, given the geometric parameters R and T, the constitutive description of the material $w(\lambda)$ and the applied loading p, one solves (xv) for the stretch λ, determines the deformed radius and wall-thickness from (ii) and $(xxiv)$, and calculates the circumferential stress $T_{\theta\theta}$ from (xxv).

Remark 2: Volume is the work-conjugate kinematic variable corresponding to pressure. Accordingly if v denotes the volume contained within the deformed spherical shell, it is related to the stretch by

$$v = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3 \lambda^3.$$

Let $\widehat{w}(v) = w(\lambda)$. Since w denotes the energy per unit reference volume, the total elastic energy in the shell is

$$\widehat{E}(v) = 4\pi R^2 T\, \widehat{w}(v).$$

It is readily seen by differentiating the two preceding equations that

$$\widehat{E}\,'(v) = \frac{dE}{dv} = 4\pi R^2 T\, \frac{d\widehat{w}(v)}{dv} = 4\pi R^2 T\, \frac{dw(\lambda)}{d\lambda}\frac{d\lambda}{dv} = 4\pi R^2 T\, w'(\lambda)\, \frac{1}{4\pi R^3 \lambda^2} = \frac{T}{R}\, \frac{w'(\lambda)}{\lambda^2} \overset{(xv)}{=} p.$$

Thus we have

$$p = \widehat{p}(v) = \widehat{E}\,'(v). \qquad (xxvi)$$

Remark 3: In the thin-walled limit, where the pressurized sphere is like an inflated membrane, it is natural to absorb the wall-thickness into the other variables. Accordingly let $\mathcal{W}(\lambda)$ denote the elastic energy per unit deformed *area* so that then, since w is the energy per unit reference volume,

$$(4\pi r^2)\mathcal{W} = (4\pi R^2 T) w \qquad \Rightarrow \qquad \mathcal{W} = \frac{T w(\lambda)}{\lambda^2}. \qquad (xxvii)$$

Also, let τ be the circumferential (hoop) force per unit deformed *length* – the "surface tension". Since $T_{\theta\theta}$ is the circumferential force per unit deformed area, they are related by

$$(2\pi r)\tau = (2\pi r t) T_{\theta\theta} \qquad \Rightarrow \qquad \tau = t T_{\theta\theta}. \qquad (xxviii)$$

Therefore we can rewrite $T_{\theta\theta} = pr/(2t)$ as

$$p = \frac{2\tau}{r}. \qquad (xxix)$$

This is the well-known Young-Laplace equation for a spherical bubble relating the pressure p, radius r and surface tension τ. Moreover the equation $p = (T/R)w'(\lambda)/\lambda^2$ can be rewritten as $\tau = (\lambda^2 \mathcal{W})'/(2\lambda)$, i.e.

$$\tau = \mathcal{W} + \frac{\lambda \mathcal{W}'}{2}. \qquad (xxx)$$

In the special case where the energy per unit deformed area $\mathcal{W}$ is a constant (corresponding to $w(\lambda) = \mathsf{constant}\ \lambda^2$), then $\tau = \mathcal{W}$. This is a second commonly used relation in the study of bubbles: the surface tension equals the surface energy per unit area. The results ($xxix$) and (xxx) are derived directly in Problem 5.1.

5.6 Example (5): Two-Phase Configurations of a Thin-Walled Tube.

<u>References:</u>

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3. S. Kyriakides and Y-C. Chang, On the initiation and propagation of a localized instability in an inflated elastic tube, *International Journal of Solids and Structures*, **27**(1991), 1085-1111.
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This example provides a toy model for a *phase transition* where a body can exist in one of several phases, and transform between them when the (mechanical or thermal) loading is varied. This is because different phases are "preferred" (stable) at different load-levels. Often, for some intermediate range of loading, multiple phases will[10] *co-exist*. An introduction to this topic can be found in Chapter 7.

Consider a long *thin-walled* circular cylindrical tube that has length L, mean radius R and wall thickness $T\,(\ll R)$ in a stress-free reference configuration. The tube is composed of an incompressible isotropic elastic material and is subjected to an internal pressure p (per unit deformed area). In the deformed configuration the tube has (an unknown) mean radius r and wall thickness t. We assume a state of plane strain so that particles do not undergo any displacement in the axial direction; in particular, this implies that the length of the tube in the deformed configuration is also L.

Let $\lambda = r/R$ denote the mean circumferential stretch. Then the principal stretches are

$$\lambda_R = \lambda^{-1}, \qquad \lambda_\Theta = \lambda, \qquad \lambda_Z = 1 \qquad \text{where} \quad \lambda = \frac{r}{R}, \tag{i}$$

having used the plane strain assumption to write λ_Z, and incompressibility to write λ_R. Let $w(\lambda)$ be the restriction of the strain energy function to deformations of the present form, i.e. let

$$w(\lambda) \coloneqq W^*(\lambda^{-1}, \lambda, 1). \tag{ii}$$

[10]One could say that the Rivlin cube problem in Section 5.3 involves cubic and tetragonal phases, though one does not encounter configurations in which both of these phases co-exist.

The pressure-stretch relation $p = p(\lambda)$ can be shown to be

$$p = p(\lambda) = \frac{T}{R}\,\frac{w'(\lambda)}{\lambda}. \tag{iii}$$

This can be derived by specializing the solution to the thick-walled tube to the case $T/R \ll 1$ (as we did for the thin-walled sphere in Section 5.5, the resulting equation there being (xv) on page 491), or directly, as in the appendix at the end of this section on page 505.

The total elastic energy per unit length of the tube is $E = 2\pi RTw$ since w is the strain energy per unit reference volume and the reference volume of the tube per unit length is $2\pi RT$. Since volume is work-conjugate to pressure, it is natural to convert the kinematic variable of choice from $\lambda \to v$ where

$$v := \pi r^2 \overset{(i)_4}{=} \pi R^2 \lambda^2 \tag{iv}$$

is the *volume* enclosed by *a unit length* of the tube in the deformed configuration. We now express the energy E and the pressure p as functions of v:

$$\widehat{E}(v) \;:=\; 2\pi RT\, w(\lambda)\Big|_{\lambda=(v/\pi R^2)^{1/2}}, \qquad\qquad \widehat{p}(v) \;:=\; p(\lambda)\Big|_{\lambda=(v/\pi R^2)^{1/2}}. \tag{v}$$

Differentiating $(v)_1$ with respect to v and using (iii) and $(v)_2$ leads to the following pressure-volume relation (please carry out this calculation):

$$p = \widehat{p}(v) = \widehat{E}\,'(v). \tag{vi}$$

Observe that the pressure is the gradient of the energy with respect to volume reflecting the work-conjugacy of p and v.

Given a specific strain energy function, one can work out the details above and obtain an explicit expression for the pressure-volume relation $p = \widehat{p}(v)$. For example for the neo-Hookean strain energy function,

$$W^*(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu}{2}\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3),$$

we find

$$w(\lambda) \overset{(ii)}{=} \frac{\mu}{2}(\lambda^{-2} + \lambda^2 - 2), \qquad \widehat{E}(v) \overset{(iv),(v)}{=} \mu\pi RT\left(\frac{\pi R^2}{v} + \frac{v}{\pi R^2} - 2\right).$$

Then (vi) takes the explicit form

$$p = \widehat{p}(v) = \mu\,\frac{T}{R}\left\{1 - \left(\frac{\pi R^2}{v}\right)^2\right\}.$$

For certain strain energy functions such as the neo-Hookean and Gent models, the pressure is found to be a monotonically increasing function of volume. Consequently given the pressure p, there is a unique corresponding value of volume v.

However for certain other strain energy functions this relation is non-monotonic, an example of which is the following 3-term Ogden strain energy function that models the particular latex rubber material studied by Kyriakides and Chang [3]:

$$W^* = \sum_{n=1}^{3} \frac{\mu_n}{\alpha_n}(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3),$$

where

$$\mu_1 = 617\,\text{kPa}, \quad \mu_2 = 1.86\,\text{kPa}, \quad \mu_3 = -9.79\,\text{kPa}, \quad \alpha_1 = 1.30, \quad \alpha_2 = 5.08, \quad \alpha_3 = -2.00.$$

For this material, as the volume v increases, the pressure p first increases until it reaches a

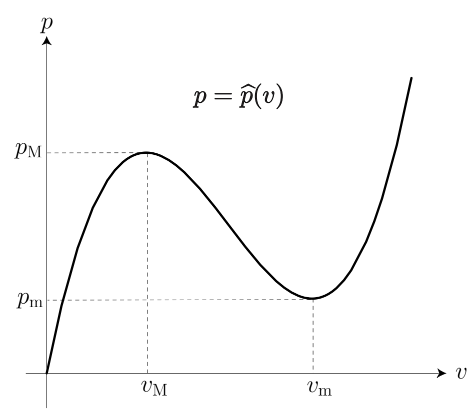

Figure 5.13: Schematic graph of $p = \widehat{p}(v) = \widehat{E}'(v)$ versus v for a certain class of strain energy functions. The pressure reaches a (local) maximum value $p_{\rm M}$ at $v = v_{\rm M}$ and a (local) minimum value p_m at $v = v_m$.

value $p_{\rm M}$, it then decreases until it reaches a value p_m, and finally increases again. Figure 5.13 depicts this *schematically* where the (local) maximum value of pressure $p = p_{\rm M}$ is attained at $v = v_{\rm M}$, and the (local) minimum value of pressure $p = p_m$ is attained at $v = v_m$.

We shall now discuss the consequences of having such a rising-falling-rising pressure-volume curve.

Loading by a "soft device": pressure controlled loading. If the prescribed value of pressure is $< p_m$ or $> p_{\rm M}$ we see from Figure 5.13 that there is a unique corresponding value of v. However if the pressure lies in the intermediate range $p_m < p < p_M$, there are

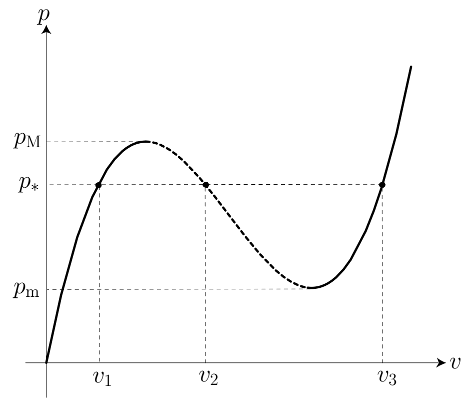

Figure 5.14: Three values of volume, v_1, v_2 and v_3, correspond to the given pressure p_*. The tube has a relatively small radius in the configuration associated with v_1 and a large radius in the configuration associated with v_3. The configuration associated with v_2 is unstable.

three values of v, say v_1, v_2 and v_3, corresponding to the three branches of the pressure-volume curve as depicted in Figure 5.14. Thus the solution to the equilibrium problem is non-unique. Additional considerations must be taken into account in order to resolve this non-uniqueness.

Equilibrium solutions that are observed in the laboratory must be stable. Thus it is natural to look at the stability of these multiple equilibrium states. In order to examine this, one must describe more carefully the manner in which the loading is controlled. Suppose that the pressure is controlled – often called loading by a "soft device". This can be achieved, for example, by inflating the tube with an incompressible fluid using a piston carrying a weight. Changing the magnitude of the weight changes the pressure. The potential energy of the elastic tube and a soft loading device is

$$\Phi(v; p) = \widehat{E}(v) - pv, \qquad v > 0. \tag{vii}$$

(In Section 5.3 we discussed the potential energy of an elastic body in a general setting.)

At each value of p, equation (vii) defines the potential energy Φ for all positive v. The particular values of v corresponding to equilibrium configurations are given by the extrema of $\Phi(\cdot\,; p)$; moreover, we say that such an equilibrium configuration is stable against small disturbances (*locally stable*) if the extremum is a local minimum. To determine the stable equilibria we first calculate the first and second variations, $\delta\Phi$ and $\delta^2\Phi$, of Φ:

$$\delta\Phi = \widehat{E}\,'(v)\,\delta v - p\,\delta v = \big[\widehat{E}\,'(v) - p\big]\,\delta v, \qquad \delta^2\Phi = \widehat{E}\,''(v)\big(\delta v\big)^2. \tag{viii}$$

Equilibria are found by requiring the first variation $\delta\Phi$ to vanish for all variations δv:

$$\delta\Phi = \big[\widehat{E}\,'(v) - p\big]\,\delta v = 0 \qquad \Rightarrow \qquad p = \widehat{E}'(v) \overset{(vi)}{=} \widehat{p}(v). \tag{ixa}$$

Given p, this tells us how to determine v. An equilibrium configuration is locally stable if the second variation (evaluated at that configuration) is positive for all (nontrivial) variations δv:

$$\delta^2\Phi = \widehat{E}\,''(v)\big(\delta v\big)^2 > 0 \qquad \Rightarrow \qquad \widehat{E}\,''(v) > 0 \qquad \Rightarrow \qquad \widehat{p}\,'(v) > 0. \tag{ixb}$$

Thus according to (ixb), the p, v-curve must be rising at a stable equilibrium configuration. This implies that the configuration associated with v_2 in Figure 5.14 is unstable, but those associated with v_1 and v_3 are both stable.

Figure 5.15 illustrates this in terms of the potential energy function: $\Phi(\cdot\,; p)$ has a single energy-well (local minimum) for $p < p_m$ corresponding to the first branch of the p, v-curve; a single energy-well for $p > p_{\rm M}$ corresponding to the third branch of the p, v-curve; and for $p_m < p < p_{\rm M}$, it has two energy-wells, one, v_1, associated with the first branch and the other, v_3, the third branch. The local maximum between them, at v_2, is associated with the second branch.

Observe that there are two (locally) stable configurations corresponding to each value of pressure in the range $p_m < p < p_{\rm M}$ and so the solution to the equilibrium problem (continues to be) nonunique[11].

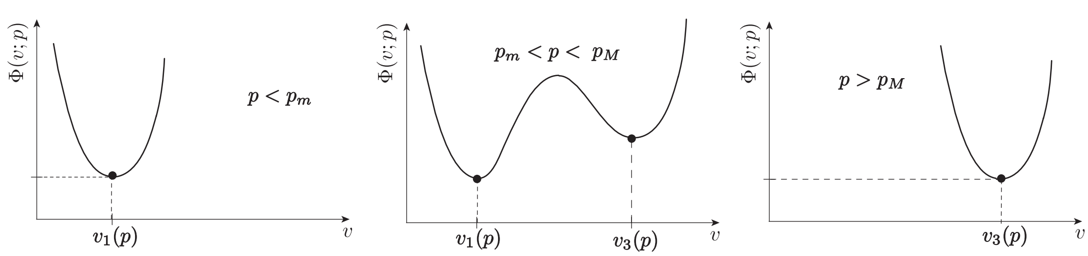

Figure 5.15: Potential energy $\Phi(v;p)$ versus volume v at three values of the pressure. Though the middle figure shows the energy-well on the left being lower than that on the right, this need not be the case; it depends on the value of $p \in (p_m, p_{\rm M})$; see Figure 5.19.

One approach to understanding this non-uniqueness is to consider the *process* by which the tube is pressurized instead of considering a strictly equilibrium problem. Say the pressure

[11]As can be seen from Figure 5.8, this also occurs in the Rivlin cube problem studied in Section 5.3 for applied force values in the range $3/2^{2/3} < F/\mu < 2$.

in the tube is p_* as shown in Figure 5.16. The observed value of the corresponding volume v may depend on the process by which the pressure p_* is reached: one might reasonably expect based on Figure 5.16 that if the pressure had increased monotonically from 0 to p_* the associated volume would be v_1; but that if instead the pressure had decreased monotonically from some large value to p_*, the associated volume would be v_3.

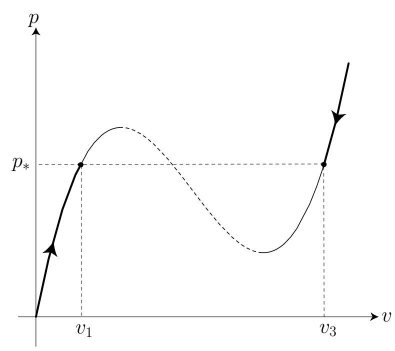

Figure 5.16: A process during which the pressure increases from 0 to p_*; and a second process during which the pressure decreases from some large value to p_*. The first process necessarily starts on the first branch of the p, v-curve, while the second process necessarily starts on the third branch of the p, v-curve.

An alternative approach would be to require an equilibrium configuration to be a *global* minimizer of the potential energy. Suppose there are two local minimizers v_1 and v_3 as shown in Figure 5.17:

$$p = \widehat{p}(v_1) = \widehat{p}(v_3), \qquad \widehat{p}\,'(v_1) > 0, \quad \widehat{p}\,'(v_3) > 0. \tag{x}$$

In the current problem, the global minimizer would be the solution with the smaller value of potential energy. Thus we must compare $\Phi(v_3; p) = \widehat{E}(v_3) - pv_3$ with $\Phi(v_1; p) = \widehat{E}(v_1) - pv_1$:

$$\Phi(v_3; p) - \Phi(v_1; p) = \Big[\widehat{E}(v_3) - pv_3\Big] - \Big[\widehat{E}(v_1) - pv_1\Big] = \int_{v_1}^{v_3} \widehat{p}(v)\, dv - p(v_3 - v_1), \tag{xi}$$

where in getting to the second equality we used $\widehat{p}(v) = \widehat{E}'(v)$. The first term represents the area below the p, v-curve between v_1 and v_3 (see Figure 5.17) while the second is the area of the rectangle with base $v_3 - v_1$ and height p. Therefore in terms of the areas shown in Figure 5.17,

$$\Phi(v_3; p) - \Phi(v_1; p) = \mathsf{Area}\ A\ - \mathsf{Area}\ B. \tag{xii}$$

Thus $\Phi(v_1; p) < \Phi(v_3; p)$ when $\mathsf{Area}\ A > \mathsf{Area}\ B$ in which case the configuration associated with v_1 is the global minimizer. On the other hand $\Phi(v_3; p) < \Phi(v_1; p)$ when $\mathsf{Area}\ A < \mathsf{Area}\ B$

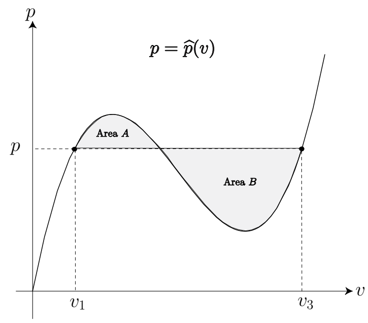

Figure 5.17: Areas A and B of two lobes cut off by the $p =$ constant line.

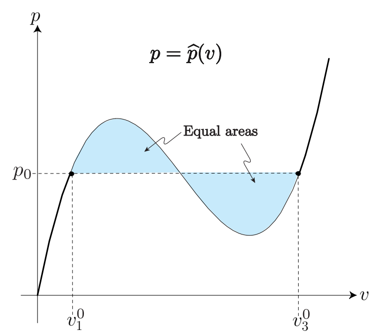

Figure 5.18: The Maxwell pressure p_0 cuts off lobes of equal area.

and so the configuration associated with v_3 is the global minimizer in this case. It is readily shown that there is a unique value of pressure, say p_0, at which these areas are equal; see Figure 5.18. It is called the *Maxwell pressure* and is given by

$$\int_{\overline{v}_1(p_0)}^{\overline{v}_3(p_0)} \widehat{p}(v)\, dv = p_0 \Big[\overline{v}_3(p_0) - \overline{v}_1(p_0) \Big]. \qquad (xiii)$$

Here the functions $\overline{v}_1(p)$ and $\overline{v}_3(p)$ are the inverses of $\widehat{p}(v)$ when it is restricted to the first and third branch respectively. Thus we conclude that the solutions $v = v_1$ and $v = v_3$ are the respective global minimizers for $p < p_0$ and $p > p_0$. This is illustrated by the bold curves in Figure 5.18.

Figure 5.19 shows plots of the potential energy $\Phi(v;p) = \widehat{E}(v) - pv$ versus v at three different values of $p \in (p_m, p_{\rm M})$; see Figure 5.15 for plots corresponding to $0 < p < p_m$ and $p > p_{\rm M}$. The energy-well associated with the first branch of the p, v-curve is lower than that associated with the third branch for $p_m < p < p_0$, and the reverse is true for $p_0 < p < p_{\rm M}$.

Loading by a "hard device": volume controlled loading. In this case the total

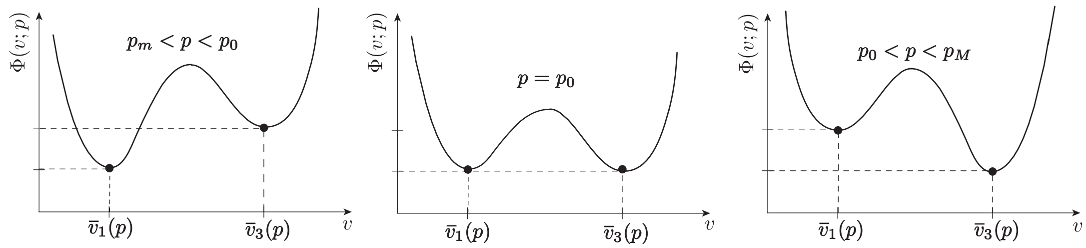

Figure 5.19: Potential energy $\Phi(v;p) = \widehat{E}(v) - pv$ versus volume v at three values of the pressure $p \in (p_m, p_{\rm M})$; see Figure 5.15 for plots corresponding to $0 < p < p_m$ and $p > p_{\rm M}$

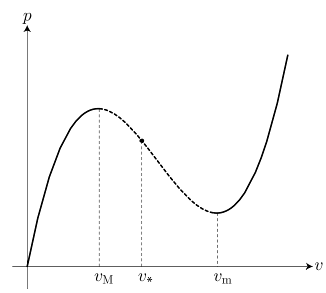

Figure 5.20: Volume controlled loading: There is only one homogeneous equilibrium configuration of the tube corresponding to any value v_* of the prescribed volume. If $v_M < v_* < v_m$ as in the figure, this configuration is associated with the falling branch of the pressure-volume curve and it is unstable.

volume within the tube is controlled – often called loading by a "hard device". This can be achieved, for example, by inflating the tube with an incompressible fluid using a screw: moving the screw in or out would increase or decrease the prescribed volume.

As can be seen from Figure 5.20, there is a unique value of pressure corresponding to *any* value v_* of the prescribed volume. The solutions corresponding to $v_* < v_M$ and $v_* > v_m$ are associated with the two rising branches of the pressure-volume curve, and they are stable against small disturbances. The solution associated with the intermediate range $v_M < v_* < v_m$ is associated with the falling branch of the pressure-volume curve, and it is unstable; see Ericksen [1] for a proof of this. Thus if the prescribed volume lies in the range $v_M < v_* < v_m$ there is no stable solution to the problem within the class of homogeneous solutions we have considered (*non-existence*).

However, since we control the volume in this experiment, we are free to prescribe a value such as v_* shown in Figure 5.20. What configuration does the tube take, given that the homogeneous configuration is unstable?

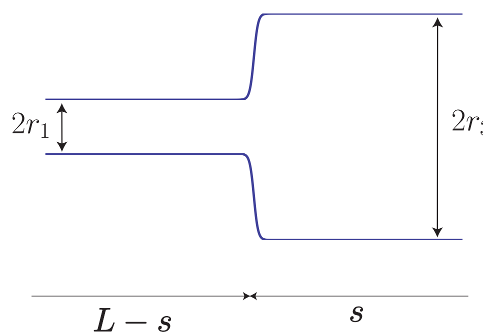

Figure 5.21: A configuration of the tube in which a length $L-s$ of the tube has a radius r_1 (where $v_1 = \pi r_1^2$ is associated with the first branch of the p, v-curve); and the remaining length s of the tube has a radius r_3 (where $v_3 = \pi r_3^2$ is associated with the third branch of the p, v-curve);

Based on the experiments of Kyriakides and Chang [3], it turns out that when $v_* \in (v_M, v_m)$ the tube adopts a *piecewise homogeneous* deformed configuration as depicted in Figure 5.21: a configuration that involves two segments, each homogeneous, but different to the other (*co-existence*). One homogeneous segment is associated with the first branch of the pressure-volume curve and has some volume per unit length $v_1(< v_*)$; and the other homogeneous segment is associated with the third branch of the pressure-volume curve with some volume per unit length $v_3(> v_*)$. See Figure 5.22, and also Figure 5.23. These two values of volume, v_1 and v_3, average out to give the value v_*. Thus the equilibrium configuration of the tube involves lengths of two different radii (with a transition zone joining them) as depicted in Figure 5.21. In our one-dimensional model we treat the transition zone as a sharp (jump) discontinuity.

When v_* is close to $v_{\rm M}$ (see Figure 5.20), most of the tube will be associated with the first branch; and when v_* is close to v_m, most of it will be associated with the third branch. As the value of v_* increases from $v_{\rm M}$ to v_m, more and more of the tube transforms from the first to the third branch and the segment associated with the first branch gets monotonically shorter. See Kyriakides and Chang [3] and Kyriakides and Lee [4] for experiments that exhibit this behavior.

To make this quantitative, now consider *piecewise homogeneous* configurations of the tube, as depicted in Figure 5.21, in which a length s of the tube is associated with the third branch of the pressure-volume curve and has volume (per unit length) v_3. The rest of the tube of length $L - s$ is associated with the first branch of the pressure-volume curve and has volume (per unit length) v_1. Let the (prescribed) total volume in the tube be $V_* = v_* L$.

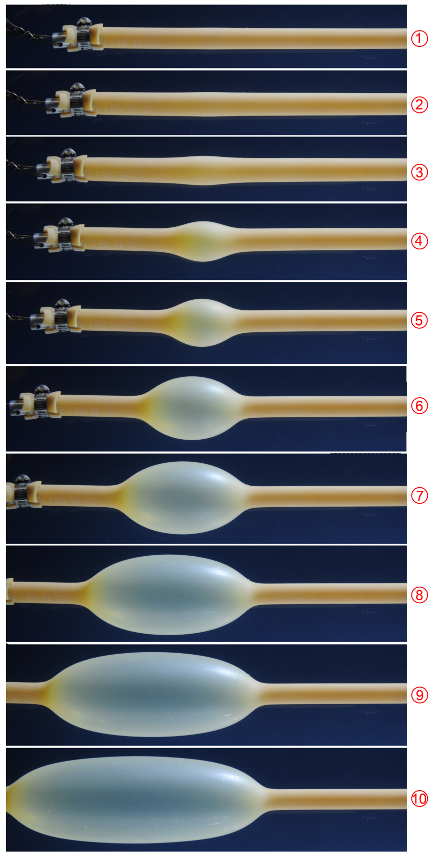

Figure 5.22: Sequence of photographs provided by Stelios Kyriakides (see also Kyriakides and Chang [3]) showing a "two-phase" equilibrium configuration of an inflated latex rubber tube with the larger radius phase growing at the expense of the other.

Then

$$(L - s)v_1 + sv_3 = V_* = v_* L. \qquad (xiv)$$

(We can add a constant to account for any volume inside the loading device if we wish.) Since

$$\frac{s}{L} = \frac{v_* - v_1}{v_3 - v_1}$$

and $v_3 > v_1$, it follows that s will increase as v_* increases (assuming v_1 and v_3 remain constant). In fact, when v_* increases from v_1 to v_3, the length s will increase from 0 to L. Thus the length of the segment of the tube associated with branch-3 increases as stated in the preceding paragraph.

The potential energy of the system is[12]

$$\Phi(v_1, v_3, s) = \widehat{E}(v_1)(L - s) + \widehat{E}(v_3)s, \qquad (xv)$$

there being no potential energy associated with the hard loading device. We are to minimize (xv) subject to the constraint (xiv). The constraint can be accounted for in the usual way through a Lagrange multiplier q and so we consider the modified function:

$$\Psi(v_1, v_3, s) = \widehat{E}(v_1)(L - s) + \widehat{E}(v_3)s - q[(L - s)v_1 + sv_3 - V_*]. \qquad (xvi)$$

On calculating the first variation $\delta\Psi$ we obtain

$$\begin{aligned} \delta\Psi &= \widehat{E}'(v_1)(L - s)\,\delta v_1 - \widehat{E}(v_1)\,\delta s + \widehat{E}'(v_3)s\,\delta v_3 + \widehat{E}(v_3)\,\delta s - \\ &\quad -q(L - s)\,\delta v_1 + qv_1\,\delta s - qs\,\delta v_3 - qv_3\delta s \;= \\ &= \Big[\widehat{E}'(v_1) - q\Big](L - s)\,\delta v_1 + \Big[\widehat{E}'(v_3) - q\Big]s\,\delta v_3 + \Big[\widehat{E}(v_3) - \widehat{E}(v_1) - q(v_3 - v_1)\Big]\delta s. \end{aligned} \qquad (xvii)$$

Thus on setting $\delta\Psi = 0$ for all variations $\delta v_1, \delta v_3$ and δs we find

$$\widehat{E}'(v_1) - q = 0, \qquad \widehat{E}'(v_3) - q = 0, \qquad \widehat{E}(v_3) - \widehat{E}(v_1) - q(v_3 - v_1) = 0. \qquad (xviii)$$

This gives

$$q = \widehat{E}'(v_1) = \widehat{E}'(v_3), \qquad \Leftrightarrow \qquad q = \widehat{p}(v_1) = \widehat{p}(v_3), \qquad (xix)$$

and

$$\widehat{E}(v_3) - \widehat{E}(v_1) - q(v_3 - v_1) = 0 \qquad \Leftrightarrow \qquad \int_{v_1}^{v_3} \widehat{p}(v)\,dv = q(v_3 - v_1). \qquad (xx)$$

Equation (xix) tells us that the pressures $\widehat{p}(v_1)$ and $\widehat{p}(v_3)$ in the two parts of the tube are equal, and that the value of the Lagrange multiplier q is in fact this pressure. From (xx) we conclude that $q = p_0$ is the ("equal-area") Maxwell pressure and therefore that v_1 and v_3 have the values

$$v_1 = v_1^0 = \overline{v}_1(p_0), \qquad v_3 = v_3^0 = \overline{v}_3(p_0), \qquad (xxi)$$

shown in Figure 5.18. The length s of the tube associated with the third branch is given by (xiv) with $v_1 = v_1^0$ and $v_3 = v_3^0$:

$$s = \frac{v_* - v_1^0}{v_3^0 - v_1^0}\,L. \qquad (xxii)$$

[12]Since the configuration of interest is not homogeneous we have to calculate the potential energy of the entire length of the tube. In contrast previously, we only needed to consider the potential energy per unit length.

These solutions are relevant for $v_1^0 < v_* < v_3^0$. As the prescribed volume v_* increases from v_1^0 to v_3^0 equation $(xxii)$ tells us that s increases from 0 to L, and so the length of the segment associated with the third branch gradually increases as more and more of the tube transforms from the first to the third branch.

To examine the stability of these configurations we calculate the second variation $\delta^2\Psi$ using $(xvii)$:

$$\begin{aligned}\delta^2\Psi = \ & \widehat{E}''(v_1)(L-s)\,(\delta v_1)^2 - \big[\widehat{E}'(v_1)-q\big]\delta s\,\delta v_1 \\ & +\widehat{E}''(v_3)s\,(\delta v_3)^2 + \big[\widehat{E}'(v_3)-q\big]\delta s\,\delta v_3 + \Big[\big(\widehat{E}'(v_3)-q\big)\delta v_3 - \big(\widehat{E}'(v_1)-q\big)\delta v_1\Big]\delta s.\end{aligned} \tag{$xxiii$}$$

Evaluating the second variation at the extrema given by $(xviii)$

$$\begin{aligned}\delta^2\Psi\Big|_{\text{at extremizer}} & \overset{(xxiii),(xviii)}{=} \quad \widehat{E}''(v_1)(L-s)\,(\delta v_1)^2 + \widehat{E}''(v_3)s\,(\delta v_3)^2 = \\ & = \quad \widehat{p}'(v_1)(L-s)\,(\delta v_1)^2 + \widehat{p}'(v_3)s\,(\delta v_3)^2.\end{aligned}$$

It follows that $\delta^2\Psi \geq 0$ since $\widehat{p}'(v_1) > 0, \widehat{p}'(v_3) > 0$ and therefore the equilibrium configurations $(xviii)$ are locally stable.

In summary, we see from Figure 5.18 that for $0 < v_* < v_1^0$ there is only the homogeneous solution involving the first branch; for $v_* > v_3^0$ there is only the homogeneous solution involving the third branch; and for $v_M < v_* < v_m$ there is only the piecewise homogeneous solution involving the first and third branches. However for values of volume in the intermediate range $v_1^0 < v_* < v_{\rm M}$ we have a homogeneous solution associated with the first branch *and* a piecewise homogeneous solution. Likewise for values of volume in $v_m < v_* < v_3^0$ we have a homogeneous solution associated with the third branch and a piecewise homogeneous solution.

The experiments of Kyriakides and Chang [3] and Kyriakides and Lee [4] show that, as the prescribed volume is increased, the pressure rises along the first branch of the pressure-volume curve all the way until the pressure reaches the value $p_{\rm M}$. Two-phase configurations then emerge and the pressure drops to the Maxwell pressure p_0. As the volume continues to increase, the pressure remains constant at the value p_0. This can be seen in the videos here provided to us by Kyriakides taken from [4].

Appendix: Derivation of equation (iii)**:** Our approach will be to *directly* construct an *approximate solution*, exploiting the fact that the tube is thin-walled. The results can be justified by first solving the

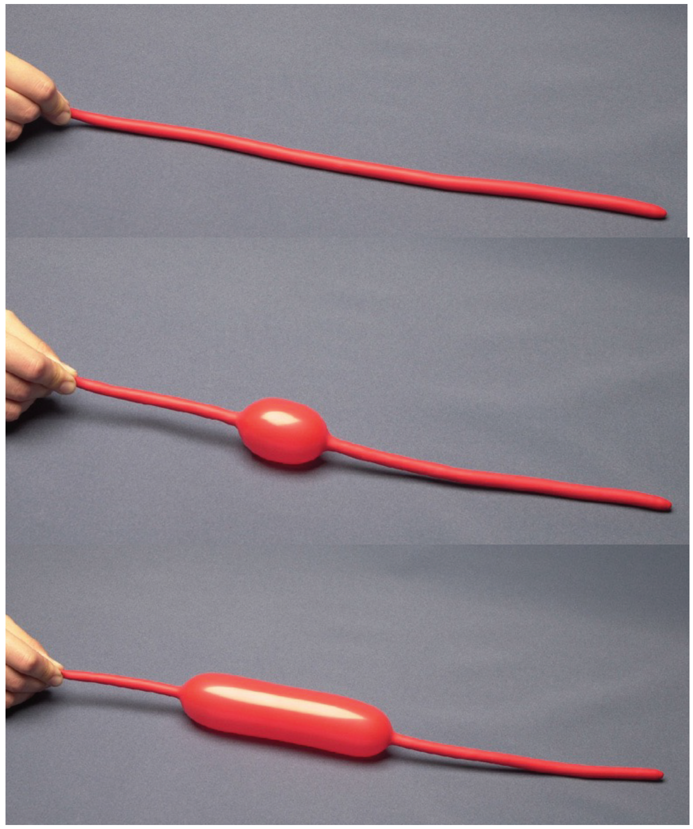

Figure 5.23: Top: Entire balloon in the small radius phase. Middle: Balloon in a 2-phase configuration, with part in the small radius phase and the rest in the large radius phase. Bottom: The extent of the large radius phase has increases, at the expense of the extent of the small radius phase. https://www.doitpoms.ac.uk/tlplib/bioelasticity/index.php. Department of Materials Science & Metallurgy, University of Cambridge, 2008.

corresponding problem for a thick-walled tube, and then taking the limit $T/R \to 0$ of those results. This was the approach we followed when studying a thin-walled spherical shell in Section 5.5.

The volume of material (per unit axial length of the tube) is $2\pi RT$ in the undeformed configuration and $2\pi rt$ in the deformed configuration. Incompressibility tells us they must be equal: $2\pi RT = 2\pi rt$. Thus $t/T = R/r$ and so we can write the deformed radius r and the deformed wall-thickness t in terms of the stretch λ as

$$r = \lambda R, \qquad t = T/\lambda. \tag{a}$$

By symmetry, the principal Cauchy stresses are $T_{rr}, T_{\theta\theta}$ and T_{zz}. The radial stress T_{rr} varies from the value $-p$ at the inner wall to the value zero at the outer wall over a small distance t. Thus we approximate T_{rr} to be

$$T_{rr} \approx -\frac{p}{2}. \tag{b}$$

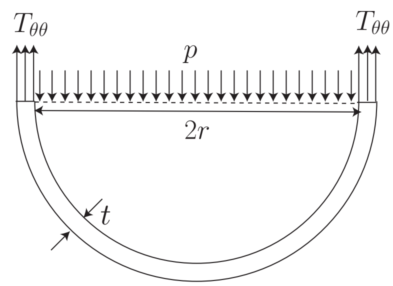

Figure 5.24: Free body diagram of a longitudinal section of the tube including the fluid it contains (in the deformed configuration). The tube has thickness t and the hoop stress is $T_{\theta\theta}$. The relevant portion of fluid has length $2r$ and pressure p. The force $2 \times (T_{\theta\theta}\, t)$ must balance the force $p(2r)$.

Next, consider the equilibrium of the longitudinal section of the tube shown in Figure 5.24. Observe that the figure shows the tube in the *deformed* configuration. Note also that the figure depicts a free body diagram of the lower half of the tube *and* the fluid inside it. Equilibrium requires the resultant force on this free body diagram to vanish, i.e. we must have $p \times 2r = 2(T_{\theta\theta} \times t)$ where $T_{\theta\theta}$ is the mean Cauchy hoop stress. Thus

$$T_{\theta\theta} \approx \frac{pr}{t} \overset{(xvi)}{=} \frac{pR}{T}\,\lambda^2. \tag{c}$$

Observe that $T_{\theta\theta} = \mathrm{O}\big(\frac{R}{T}\big)$ while $T_{rr} = O(1)$ as $T/R \to 0$ and so $T_{\theta\theta} \gg T_{rr}$.

The principal Cauchy stress τ_i is related to the principal stretches by the constitutive relation $\tau_i = \lambda_i \partial W/\partial\lambda_i - q$ (no sum on i). By taking the 1- and 2-directions to refer to the radial and circumferential directions respectively we have, upon using $\lambda_r = \lambda^{-1}, \lambda_\theta = \lambda$ from (i),

$$T_{rr} = \lambda_r \frac{\partial W}{\partial \lambda_1} - q \overset{(i)}{=} \lambda^{-1}\frac{\partial W}{\partial \lambda_1} - q, \qquad T_{\theta\theta} = \lambda_\theta \frac{\partial W}{\partial \lambda_2} - q \overset{(i)}{=} \lambda \frac{\partial W}{\partial \lambda_2} - q.$$

The reaction pressure q can be eliminated by subtracting the first equation from the second leading to

$$T_{\theta\theta} - T_{rr} = \lambda\frac{\partial W}{\partial \lambda_2} - \lambda^{-1}\frac{\partial W}{\partial \lambda_1}.$$

Since $T_{\theta\theta} \gg T_{rr}$ we drop T_{rr} and write

$$T_{\theta\theta} \approx \lambda\frac{\partial W}{\partial \lambda_2} - \lambda^{-1}\frac{\partial W}{\partial \lambda_1}. \tag{d}$$

Finally turning to the constitutive relation, we differentiate $w(\lambda) = W^*(\lambda^{-1}, \lambda, 1)$ with respect to λ to find

$$w'(\lambda) = -\lambda^{-2}\frac{\partial W}{\partial \lambda_1} + \frac{\partial W}{\partial \lambda_2},$$

and so we can write (d) as

$$T_{\theta\theta} = \lambda w'(\lambda). \tag{e}$$

On combining (c) and (e) we get

$$p = \frac{T}{R}\,\frac{w'(\lambda)}{\lambda} \tag{xxi}$$

which is equation (iii).

5.7 Example(6): Surface instability of a neo-Hookean half-space.

References:

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In this section we study a homogeneously deformed body that is in equilibrium under a certain loading, and inquire as to the conditions under which there may exist a second equilibrium configuration "close" to the homogeneous one, satisfying *the same loading*. If such a configuration exists, and if it is energetically preferred, this would indicate the instability of the first.

In analyzing the question at hand there are three configurations to consider as shown schematically in Figure 5.25: a stress-free reference configuration, a homogeneously deformed configuration, and an inhomogeneously deformed configuration. We shall let $\mathbf{z}, \mathbf{x}$ and $\mathbf{y}$ denote the respective positions of a particle in these three configurations. Since we will have

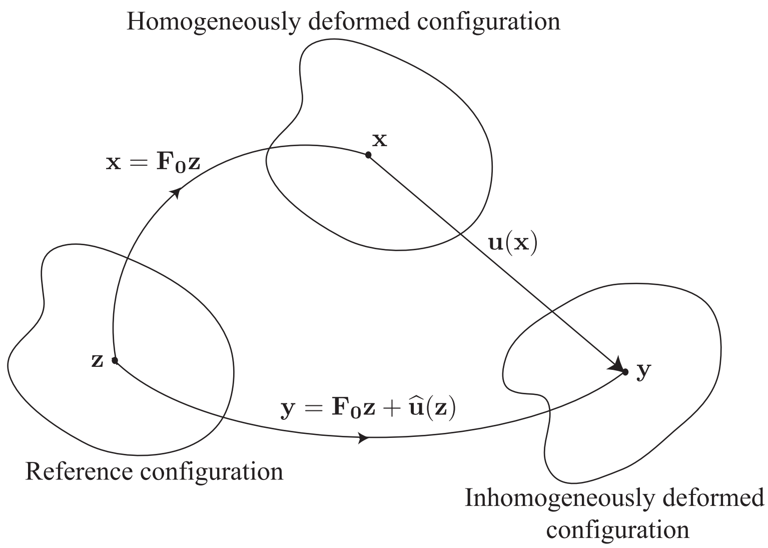

Figure 5.25: A homogeneous deformation takes $\mathbf{z} \to \mathbf{x}$. An inhomogeneous deformation takes $\mathbf{z} \to \mathbf{y}$. The displacement of a particle from the homogeneously deformed configuration to the inhomogeneously deformed configuration is $\mathbf{u}(\mathbf{x})$. In the problem of interest to us here, the two deformed configurations are "close" to each other.

to calculate the gradients of various fields with respect to different configurations, we shall append a subscript to refer to the configuration. Thus for example $\text{grad}_z\mathbf{f}, \text{grad}_x\mathbf{f}$ and $\text{grad}_y\mathbf{f}$ will denote the gradients of $\mathbf{f}(\mathbf{z}), \mathbf{f}(\mathbf{x})$ and $\mathbf{f}(\mathbf{y})$ with respect to the reference configuration, the homogeneously deformed configuration and the inhomogeneously deformed configuration respectively; they have cartesian components $\partial f_i/\partial z_j, \partial f_i/\partial x_j$ and $\partial f_i/\partial y_j$.

We start in Section 5.7.1 by carrying out all calculations explicitly in the context of a specific boundary-value problem. This analysis will be generalized in Section 5.7.2 where we consider an arbitrary small deformation superimposed on an arbitrary homogeneous finite deformation. Problem 5.21 is concerned with an arbitrary small deformation superposed on an arbitrary (not necessarily homogeneous) finite deformation.

5.7.1 Example: Surface instability of a neo-Hookean half-space.

The wrinkling and creasing of surfaces under compression are of interest in various applications as described in the papers listed above and the reference is them. The particular problem we consider is the following: in a stress-free reference configuration the body occupies the half-space $z_2 > 0$ as depicted in Figure 5.26. The rectangular cartesian coordinates

$$\left.\begin{array}{lcl} S_{11} & \to & -S \\ S_{22}, S_{12}, S_{21} & \to & 0 \end{array}\right\} \quad \text{as } z_2 \to \infty$$

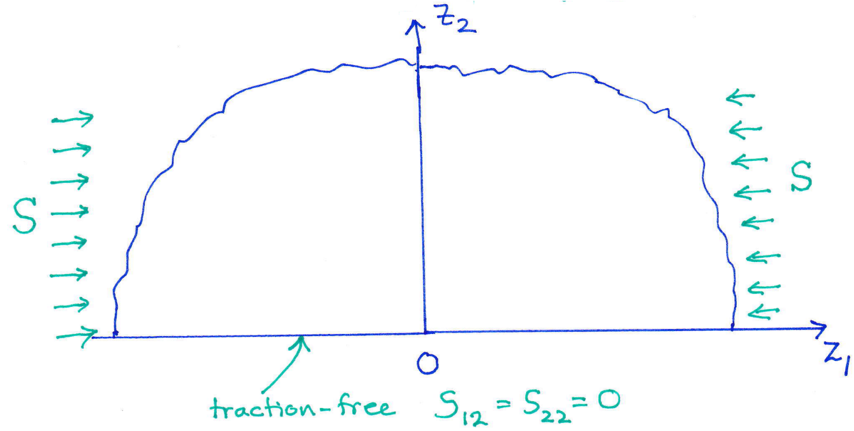

Figure 5.26: Semi-infinite neo-Hookean body with traction-free surface subjected to a uniaxial compression.

of a generic particle in this configuration are denoted by (z_1, z_2, z_3). All components are taken with respect to a fixed basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The surface $z_2 = 0$ is traction-free which implies that $\mathbf{S}\mathbf{e}_2 = \mathbf{0}$:

$$S_{12} = S_{22} = S_{32} = 0 \qquad \text{for } z_2 = 0. \tag{i}$$

A uniform compressive Piola normal stress of magnitude S parallel to the z_1-axis is applied remotely as depicted in Figure 5.26. We model this by requiring

$$S_{11} \to -\mathrm{S}, \quad \text{all other } \ S_{ij} \text{ (except } S_{33}) \ \to 0 \qquad \text{as } |\mathbf{z}| \to \infty; \tag{ii}$$

the stress component S_{33} tends to some finite value as will be discussed below.

We **first** consider a homogeneous deformation that is consistent with (the field equations and) the preceding boundary conditions. Let (x_1, x_2, x_3) be the rectangular cartesian coordinates of a particle in the homogeneously deformed configuration, the deformation that takes $(z_1, z_2, z_3) \to (x_1, x_2, x_3)$ being

$$x_1 = \lambda_1 z_1, \quad x_2 = \lambda_2 z_2, \quad x_3 = \lambda_3 z_3. \tag{iii}$$

The constant stretches λ_1, λ_2 and λ_3 are to be determined. The deformation gradient tensor associated with (iii) is

$$\mathbf{F}_0 = \mathrm{Grad}_z \mathbf{x}(\mathbf{z}) = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{iv}$$

For a tensor $\mathbf{A}$ associated with the homogeneous configuration we will interchangeably use the notation $\mathbf{A}^0$ and $\mathbf{A}_0$. (The former is more convenient when, for example, we want to show its components A^0_{ij}, whereas the latter is preferred when, say, we want to write $\mathbf{A}_0^{-1}$). Incompressibility requires

$$\det \mathbf{F}_0 = \lambda_1\lambda_2\lambda_3 = 1. \tag{v}$$

Assume the material to be neo-Hookean. The Piola stress tensor $\mathbf{S}_0$ is then related to the deformation gradient tensor $\mathbf{F}_0$ by the constitutive relation

$$\mathbf{S}_0 = \mu\mathbf{F}_0 - q_0\mathbf{F}_0^{-T}, \tag{vi}$$

where the constant q_0 is the reactive pressure associated with the incompressibility constraint. From (iv) and (vi), the Piola stress tensor $\mathbf{S}_0$ is

$$\mathbf{S}_0 = (\mu\lambda_1 - q_0\lambda_1^{-1})\,\mathbf{e}_1\otimes\mathbf{e}_1 + (\mu\lambda_2 - q_0\lambda_2^{-1})\,\mathbf{e}_2\otimes\mathbf{e}_2 + (\mu\lambda_3 - q_0\lambda_3^{-1})\,\mathbf{e}_3\otimes\mathbf{e}_3. \tag{vii}$$

Since this stress field is uniform, the equilibrium equation $\text{Div}_z\,\mathbf{S}_0(\mathbf{z}) = \mathbf{o}$ holds automatically. In order to conform with the boundary conditions (i) we must have $S^0_{22} = 0$ and so from (vii),

$$q_0 = \mu\lambda_2^2. \tag{$viii$}$$

The remote prescribed loading condition (ii) requires $S^0_{11} = -S$, which by (vii) and $(viii)$ leads to the stress- stretch relation

$$S = \mu\left(\lambda_2^2\lambda_1^{-1} - \lambda_1\right). \tag{ix}$$

Thus we have

$$\mathbf{S}_0 = -S\,\mathbf{e}_1\otimes\mathbf{e}_1 + \mu(\lambda_3 - \lambda_2^2\lambda_3^{-1})\,\mathbf{e}_3\otimes\mathbf{e}_3.$$

The stretches are to be determined from $(v), (ix)$ and one more condition pertaining to either λ_3 or S^0_{33}. By leaving this condition unspecified we are able to describe several sub-cases. For example, if the homogeneous deformation is one of *plane strain* in the z_1, z_2-plane, one has $\lambda_3 = 1$ and therefore it follows from (v) and (vii) that

$$\lambda_2 = \lambda_1^{-1}, \qquad \lambda_3 = 1, \qquad S^0_{33} = \mu(1 - \lambda_1^{-2}). \tag{x}$$

On the other hand if the body is in a state of uniaxial stress in the z_1-direction, one similarly finds using $S^0_{33} = 0$ that

$$\lambda_2 = \lambda_1^{-1/2}, \qquad \lambda_3 = \lambda_1^{-1/2}, \qquad S^0_{33} = 0. \tag{xi}$$

If instead the body is in a state of equi-biaxial stretch $\lambda_1 = \lambda_3$ one finds

$$\lambda_2 = \lambda_1^{-2}, \qquad \lambda_3 = \lambda_1, \qquad S_{33}^0 = \mu(\lambda_1 - \lambda_1^{-5}). \tag{xii}$$

The Cauchy stress tensor corresponding to (vii) is given by $\mathbf{T}_0 = \mathbf{S}_o\mathbf{F}_0^T$:

$$\mathbf{T}_0 = T\,\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_3 S_{33}^0\,\mathbf{e}_3 \otimes \mathbf{e}_3, \qquad \text{where} \quad T := -S\lambda_1 = \mu(\lambda_1^2 - \lambda_2^2). \tag{xiii}$$

Given $S > 0$, we seek $\lambda_1, \lambda_2, \lambda_3$ from $(v), (ix)$ and either $(x)_1, (xi)_1$ or $(xii)_1$.

Since the deformation has the form (iii), we see that the region occupied by the body in the homogeneously deformed configuration is

$$x_2 > 0, \qquad -\infty < x_1 < \infty; \tag{xiv}$$

see the middle figure in Figure 5.27.

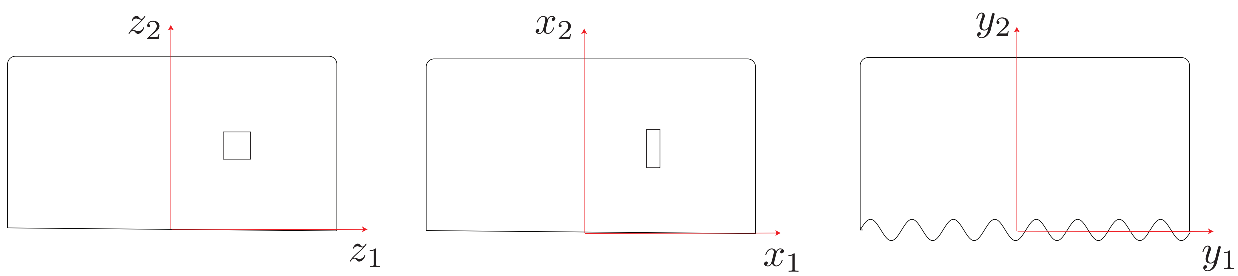

Figure 5.27: Coordinate systems: Left: reference configuration. Middle: Homogeneously deformed configuration. Right: Inhomogeneously deformed configuration.

We now seek a **second** deformation corresponding to the same loading (i.e. boundary conditions). This deformation, which is necessarily inhomogeneous, takes the particle located at $\mathbf{z}$ in the reference configuration to the location $\mathbf{y}$ in the deformed configuration:

$$\mathbf{y} = \mathbf{F}_0\mathbf{z} + \widehat{\mathbf{u}}(\mathbf{z}). \tag{xv}$$

We continue to use the preceding stress-free configuration as the reference configuration. Then the deformation gradient tensor, equilibrium equation and constitutive relation are

$$\mathbf{F} = \mathrm{Grad}_z\mathbf{y}, \qquad \mathrm{Div}_z\mathbf{S} = \mathbf{o}, \qquad \mathbf{S} = \mu\mathbf{F} - q\mathbf{F}^{-T}. \tag{xvi}$$

One could of course use the homogeneously deformed configuration as the reference configuration in which case the constitutive relation has to be modified in order to take into account the stress in this reference configuration.

Observe from $(iii), (iv)$ and (xv) that $\mathbf{y} = \mathbf{x} + \widehat{\mathbf{u}}(\mathbf{z})$ and so $\widehat{\mathbf{u}}$ is the displacement *from the homogeneously deformed* configuration to the inhomogeneously deformed configuration, see Figure 5.25. As one might expect, it is convenient to change variables and express this displacement field as a function of $\mathbf{x}$ rather than $\mathbf{z}$ by introducing the function

$$\mathbf{u}(\mathbf{x}) = \widehat{\mathbf{u}}(\mathbf{z}) \quad \text{where} \quad \mathbf{z} = \mathbf{F}_0^{-1}\mathbf{x}. \tag{xvii}$$

The deformation (xv) can then be expressed as

$$\mathbf{y} = \mathbf{x} + \mathbf{u}(\mathbf{x}). \tag{xviii}$$

We shall limit attention to superposed displacement fields $\mathbf{u}$ of the plane strain form

$$\mathbf{u}(\mathbf{x}) = u_1(x_1, x_2)\mathbf{e}_1 + u_2(x_1, x_2)\mathbf{e}_2. \tag{xix}$$

The inhomogeneous deformation can therefore be written out as

$$\left.\begin{aligned} y_1 &= \lambda_1 z_1 + u_1(x_1, x_2), \\ y_2 &= \lambda_2 z_2 + u_2(x_1, x_2), \\ y_3 &= \lambda_3 z_3, \end{aligned}\right\} \tag{xx}$$

where the x_i's are related to the z_i's by (iii). In the analysis going forward, we assume that the inhomogeneous deformation is close to the homogeneous deformation in the sense that

$$\epsilon := |\mathrm{grad}_x \mathbf{u}| \ll 1. \tag{xxi}$$

Accordingly we shall consistently drop terms that are of $\mathrm{O}(\epsilon^2)$. Observe that in previous chapters we linearized the equations of finite elasticity about the reference configuration whereas here we will be linearizing about the homogeneously deformed configuration; the former therefore corresponds to a special case of the latter. In order to minimize the cumbersomeness of various expressions to follow, it will convenient to use a comma followed by a subscript to indicate partial differentiation with respect to the corresponding x-coordinate, for example to write

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}. \tag{xxii}$$

We shall use this convention from hereon in the rest of this section.

The components of the deformation gradient tensor with respect to the reference configuration are given by $F_{ij} = \partial y_i/\partial z_j$. From this, (xx) and (iii) one finds[13]

$$[F] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} + \begin{pmatrix} u_{1,1}\lambda_1 & u_{1,2}\lambda_2 & 0 \\ u_{2,1}\lambda_1 & u_{2,2}\lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (xxiii)$$

Incompressibility requires[14]

$$1 = \det \mathbf{F} \overset{(xxiii)}{=} 1 + u_{1,1} + u_{2,2} + \mathrm{O}(\epsilon^2), \qquad (xxiv)$$

where we have used $\det \mathbf{F}_0 = 1$. Thus to leading order, incompressibility requires

$$u_{1,1} + u_{2,2} = 0, \qquad \text{for } -\infty < x_1 < \infty,\ x_2 \geq 0. \qquad (xxv)$$

The components of $\mathbf{F}^{-1}$ can be readily calculated from $(xxiii)$ to be[15]

$$[F]^{-1} = \begin{pmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{pmatrix} + \begin{pmatrix} -\lambda_1^{-1}u_{1,1} & -\lambda_1^{-1}u_{1,2} & 0 \\ -\lambda_2^{-1}u_{2,1} & -\lambda_2^{-1}u_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathrm{O}(\epsilon^2)\ . \qquad (xxvi)$$

Using $(xxiii)$ and $(xxvi)$ in the constitutive law $\mathbf{S} = \mu\mathbf{F} - q\mathbf{F}^{-T}$ and linearizing leads to

$$\left.\begin{aligned} S_{11} &= -S + \mu(\lambda_1^2 + \lambda_2^2)\lambda_1^{-1}u_{1,1} - \lambda_1^{-1}\tilde{q}, \\ S_{12} &= \mu\lambda_2\left(u_{1,2} + u_{2,1}\right), \\ S_{21} &= \mu(\lambda_1^2 u_{2,1} + \lambda_2^2 u_{1,2})\lambda_1^{-1}, \\ S_{22} &= 2\mu\lambda_2 u_{2,2} - \tilde{q}\lambda_2^{-1}, \\ S_{33} &= S_{33}^0 - \widetilde{q}\lambda_3^{-1}, \qquad S_{13} = S_{23} = S_{31} = S_{32} = 0. \end{aligned}\right\} \qquad (xxvii)$$

In arriving at $(xxvii)$ we have dropped terms of $\mathrm{O}(\epsilon^2)$ and approximated the reactive pressure as

$$q(\mathbf{x}) = q_0 + \widetilde{q}(\mathbf{x}), \qquad \widetilde{q} = \mathrm{O}(\epsilon). \qquad (xxviii)$$

[13] Note that $F_{ij} = \partial y_i/\partial z_j = F^0_{ij} + \partial u_i/\partial z_j = F^0_{ij} + (\partial u_i/\partial x_k)(\partial x_k/\partial z_j) = F^0_{ij} + (\partial u_i/\partial x_k)F^0_{kj}$ and so $\mathbf{F} = \mathbf{F}_0 + \mathbf{H}\mathbf{F}_0$ where $\mathbf{H} = \mathrm{grad}_x\mathbf{u}$.

[14] Note that $\det \mathbf{F} = \det(\mathbf{I}+\mathbf{H})\mathbf{F}_0 = \det \mathbf{F}_0 \det(\mathbf{I}+\mathbf{H}) = \det(\mathbf{I}+\mathbf{H}) = 1 + \mathrm{tr}\mathbf{H} + \mathrm{O}(\epsilon^2)$.

[15] Note that $\mathbf{F}^{-1} = [(\mathbf{I}+\mathbf{H})\mathbf{F}_0]^{-1} = \mathbf{F}_0^{-1}(\mathbf{I}+\mathbf{H})^{-1} = \mathbf{F}_0^{-1}(\mathbf{I}-\mathbf{H}) + \mathrm{O}(\epsilon^2)$.

Upon using the chain rule we can write the equilibrium equation $\text{Div}_z\,\mathbf{S} = \mathbf{o}$ as $\text{Div}_x\,(\mathbf{S}\mathbf{F}_0^T) = \mathbf{o}$. Substituting $(xxvii)$ into this leads to

$$\left.\begin{aligned} \widetilde{q}_{,1} &= \mu(\lambda_1^2 + \lambda_2^2)u_{1,11} + \mu\lambda_2^2\,(u_{1,22} + u_{2,12})\,, \\ \widetilde{q}_{,2} &= \mu\lambda_1^2 u_{2,11} + \mu\lambda_2^2\,(u_{1,12} + 2u_{2,22})\,. \end{aligned}\right\} \qquad (xxix)$$

The third equilibrium equation $\partial S_{31}/\partial z_1 + \partial S_{32}/\partial z_2 + \partial S_{33}/\partial z_3 = 0$ yields $\partial\widetilde{q}/\partial z_3 = 0$ which tells us that

$$\widetilde{q}(\mathbf{x}) = \widetilde{q}(x_1, x_2). \qquad (xxx)$$

In the far-field the Piola stress tensor $\mathbf{S} \to \mathbf{S}_0$ and so we must have

$$u_{\alpha,\beta} \to 0 \ \text{ for } \ \alpha, \beta = 1, 2, \qquad \widetilde{q} \to 0 \qquad \text{as} \quad |\mathbf{x}| \to \infty. \qquad (xxxi)$$

The traction-free boundary condition requires $\mathbf{S}\mathbf{e}_2 = \mathbf{0}$ on $z_2 = 0$ which by $(xxvii)$ leads to

$$\left.\begin{aligned} u_{1,2} + u_{2,1} &= 0, \\ 2\mu\lambda_2^2 u_{2,2} - \tilde{q} &= 0, \end{aligned}\right\} \qquad \text{for} \quad x_2 = 0. \qquad (xxxii)$$

In summary, the unknown fields $u_1(x_1, x_2), u_2(x_1, x_2), \widetilde{q}(x_1, x_2)$ must obey the incompressibility equation (xxv), the equilibrium equations $(xxix)_1$ and $(xxix)_2$, the boundary conditions $(xxxii)$ at $x_2 = 0$ and the decay condition $(xxxi)$ in the far-field. Observe that $u_1(x_1, x_2) = 0, u_2(x_1, x_2) = 0, \widetilde{q}(x_1, x_2) = 0$ is one solution of this problem (corresponding to the homogeneous deformation). If any non-trivial solutions exist we expect them to do so at particular values of the applied stretch λ_1 in which case this would be an eigenvalue problem.

Simplification: Before solving the boundary value problem just formulated, it is possible to simplify it in two ways.

First we eliminate the pressure field $\widetilde{q}$ from the problem as follows. Differentiating $(xxix)_1$ with respect to x_2 and $(xxix)_2$ with respect to x_1 and equating the resulting expressions eliminates $\widetilde{q}$ from the field equations and leads to the differential equation

$$\lambda_1^2 u_{1,112} + \lambda_2^2 u_{1,222} - \lambda_1^2 u_{2,111} - \lambda_2^2 u_{2,122} = 0. \qquad (xxxiii)$$

Similarly $\widetilde{q}$ can be eliminated from the boundary conditions as follows: since $(xxxii)$ holds along the boundary, i.e. for all x_1, it may be differentiated with respect to x_1. Thereafter $(xxix)_1$ can be used to eliminate $\partial\widetilde{q}/\partial x_1$ from the result. This leads to

$$(\lambda_1^2 + \lambda_2^2)u_{1,11} + \lambda_2^2 u_{1,22} - \lambda_2^2 u_{2,12} = 0 \qquad \text{for} \quad x_2 = 0. \qquad (xxxiv)$$

Thus in summary the displacement fields $u_1(x_1,x_2), u_2(x_1,x_2)$ must obey the field equations (xxv) and $(xxxiii)$, i.e.

$$\left.\begin{aligned} u_{1,1}+u_{2,2} &= 0, \\ \lambda_1^2 u_{1,112}+\lambda_2^2 u_{1,222}-\lambda_1^2 u_{2,111}-\lambda_2^2 u_{2,122} &= 0 \end{aligned}\right\} \quad \text{for} \quad -\infty < x_1 < \infty,\ x_2 \geq 0, \qquad (xxxv)$$

and the boundary conditions $(xxxii)_1$ and $(xxxiv)$, i.e.

$$\left.\begin{aligned} u_{1,2}+u_{2,1} &= 0, \\ (\lambda_1^2+\lambda_2^2)u_{1,11}+\lambda_2^2 u_{1,22}-\lambda_2^2 u_{2,12} &= 0, \end{aligned}\right\} \quad \text{for} \quad x_2 = 0. \qquad (xxxvi)$$

In addition, in the far field

$$u_{i,j} \to 0 \qquad \text{as} \quad x_2 \to \infty \text{ at each fixed } x_1. \qquad (xxxvii)$$

The problem can be simplified even further since the general solution of the incompressibility equation $(xxxv)_1$ can be written down in terms of an arbitrary scalar potential $\phi(x_1,x_2)$ as

$$u_1 = \phi_{,2}, \quad u_2 = -\phi_{,1}. \qquad (xxxviii)$$

Substituting this into the differential equation $(xxxiii)$ leads to

$$\lambda_1^2\ \phi_{,1111} + (\lambda_1^2+\lambda_2^2)\ \phi_{,1122} + \lambda_2^2\ \phi_{,2222} = 0 \qquad \text{for } x_2>0,\ -\infty<x_1<\infty, \qquad (xxxix)$$

while the boundary conditions $(xxxvi)$ yield

$$(\lambda_1^2+2\lambda_2^2)\phi_{,112}+\lambda_2^2\phi_{,222} = 0 \qquad \text{for } x_2 = 0, \qquad (XL)$$

$$\phi_{,22}-\phi_{,11} = 0 \qquad \text{for } x_2 = 0. \qquad (XLI)$$

The boundary value problem $(xxxix), (XL), (XLI)$ is in fact an eigenvalue problem. We are interested in the values of λ_1/λ_2 for which it has a nontrivial solution $\phi(x_1,x_2)$.

Solution: We look for solutions of the differential equation $(xxxix)$ that are (a) periodic in the x_1-direction (and therefore of the form e^{ikx_1}) and (b) exponentially decaying away from the free-surface (and therefore of the form e^{skx_2} where $ks<0$.) Thus we seek solutions of the form

$$\phi = \mathrm{e}^{skx_2+ikx_1}, \qquad (XLII)$$

where s and k are unknown constants. Without loss of generality we can assume $k > 0$. Substituting $(XLII)$ into $(xxxix)$ leads to the following quartic equation for determining s:

$$\lambda_2^2 s^4 - (\lambda_1^2 + \lambda_2^2)s^2 + \lambda_1^2 = 0, \tag{XLIII}$$

the roots of which are

$$s = \pm 1, \qquad \pm\lambda_1/\lambda_2. \tag{XLIV}$$

Since we want the displacement field to decay as $x_2 \to +\infty$, and since the displacement arising from $(xxxviii), (XLII)$ involves the term $\mathrm{e}^{skx_2},\ k > 0$, we discard the two positive roots $s = +1, +\lambda_1/\lambda_2$.

Thus we have two linearly independent solutions, $\mathrm{e}^{-kx_2+ikx_1}$ and $\mathrm{e}^{-(\lambda_1/\lambda_2)kx_2+ikx_1}$, of the differential equation $(xxxix)$. This leads us to seek a solution of the complete boundary-value problem of the form

$$= \left[A\mathrm{e}^{-kx_2} + B\mathrm{e}^{-(\lambda_1/\lambda_2)kx_2}\right]\mathrm{e}^{ikx_1}. \tag{XLV}$$

Equation (XLV) satisfies the equilibrium equation for any choice of the constants A and B. Substituting (XLV) into the boundary conditions (XL) and (XLI) yields a pair of algebraic equations which we write in matrix form as

$$\begin{pmatrix} \lambda_1^2 + \lambda_2^2 & 2\lambda_1\lambda_2 \\ 2 & 1 + \lambda_1^2/\lambda_2^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{XLVI}$$

If $(XLVI)$ is to have a nontrivial solution for A and B, the determinant of the 2×2 matrix must vanish and this requires

$$\left(\frac{\lambda_1}{\lambda_2}\right)^4 + 2\left(\frac{\lambda_1}{\lambda_2}\right)^2 - 4\left(\frac{\lambda_1}{\lambda_2}\right) + 1 = 0. \tag{XLVII}$$

One root of this equation is $\lambda_1/\lambda_2 = 1$. However by (ix), this corresponds to $S = 0$ and so we discard this root. We therefore cancel out the factor $\lambda_1/\lambda_2 - 1$ and obtain

$$\left(\frac{\lambda_1}{\lambda_2}\right)^3 + \left(\frac{\lambda_1}{\lambda_2}\right)^2 + 3\left(\frac{\lambda_1}{\lambda_2}\right) - 1 = 0. \tag{XLVIII}$$

We are interested in the real positive roots λ_1/λ_2 of (L). Consider the function $f(\xi) = \xi^3 + \xi^2 + 3\xi - 1$ for $\xi \geq 0$ and note that $f'(\xi) > 0$ for $\xi > 0$. Therefore f increases monotonically with ξ. Since $f(0) = -1 < 0$ and $f(1) = 4 > 0$ it follows that $f(\xi)$ has a unique positive zero in the interval $(0,1)$. This proves that equation $(XLVIII)$ has exactly one real positive root. Numerical solution gives this root to be

$$\frac{\lambda_1}{\lambda_2} \approx 0.295598 \qquad \text{and so by (v)}, \quad \lambda_1 \approx \frac{0.543689}{\sqrt{\lambda_3}}. \tag{$XLIX$}$$

As noted previously in the context of $(x), (xi)$ and (xii), our analysis covers several sub-cases. Consider for example *the case of plane strain* where by (x),

$$\lambda_2 = \lambda_1^{-1}, \qquad \lambda_3 = 1. \tag{L}$$

Then $(XLIX)$ gives

$$\lambda_1 = \lambda_{\text{cr}} \approx 0.543689. \tag{LI}$$

Thus we conclude that an inhomogeneous deformation of the form (xv) is possible if the stretch λ_1 has the value λ_{cr}, the corresponding value of stress S_{cr} being given by (ix).

To determine the complete deformation we obtain the ratio A/B from $(XLVI)$ and then the displacement field from $(xxxviii)$ and (XLV). Observe that the constant k (which represents the reciprocal of the wave length of the oscillations in the x_1-direction) remains arbitrary. This is because our problem statement for the semi-infinite body involves no length scale.

The original solution of this problem is due to Biot. For a discussion of stability, see Chen, Yang and Wheeler, and for an analysis not limited to neo-Hookean materials, see Dowaikh and Ogden. For other modes of surface instability (such as creasing), see Cao and Hutchinson. See M.K. Kang and R. Huang, Soft Matter, (2010), DOI: 10.1039/c0sm00335b, for a treatment of wrinkling in a hydrogel.

5.7.2 An arbitrary small deformation superimposed on an arbitrary homogeneous finite deformation.

The preceding analysis can be carried out rather generally (and then specialized to the specific problem of interest, whether it be the problem studied in the previous section or some other problem). We now illustrate this by considering an arbitrary small deformation superimposed on an arbitrary homogeneous deformation. Many of the results below hold even if the deformation about which we are linearizing is not homogenous, see Problem 5.21.

We know that the natural stress measures to use when working on the reference configuration and the current configuration are the Piola stress tensor field $\mathbf{S}(\mathbf{z})$ and the Cauchy stress tensor field $\mathbf{T}(\mathbf{y})$ respectively. As we shall see below, it will be convenient to work with a stress measure $\mathbf{\Sigma}(\mathbf{x})$ when working on the intermediate configuration.

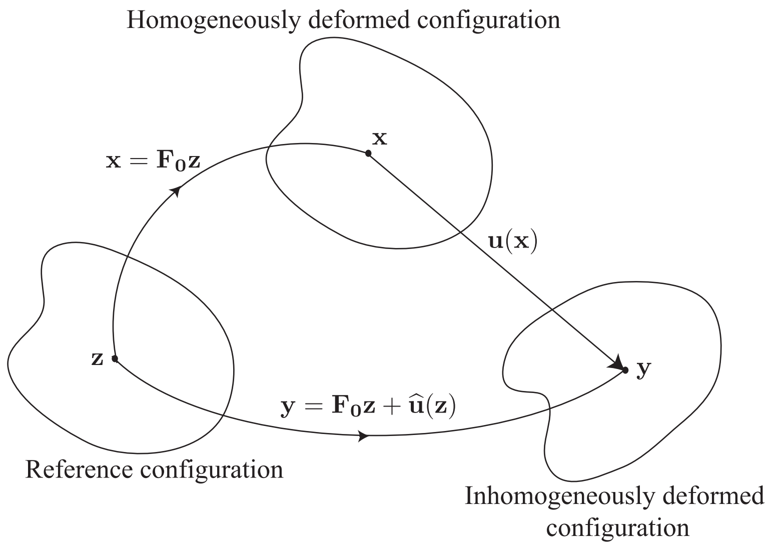

Figure 5.28: A homogeneous deformation takes $\mathbf{z} \to \mathbf{x}$. An inhomogeneous deformation takes $\mathbf{z} \to \mathbf{y}$. The displacement of a particle from the homogeneously deformed configuration to the inhomogeneously deformed configuration is $\mathbf{u}(\mathbf{x})$. The two deformed configurations are "close" to each other.

Consider a homogeneous deformation of the body

$$\mathbf{x} = \mathbf{F}_0\mathbf{z} \tag{5.11}$$

that takes a particle located at $\mathbf{z}$ in the reference configuration to the location $\mathbf{x}$ in the deformed configuration, the deformation gradient tensor $\mathbf{F}_0$ being constant. The material is incompressible and so

$$\det \mathbf{F}_0 = 1. \tag{5.12}$$

Assuming the material to be an arbitrary homogeneous, incompressible elastic material, the Piola stress tensor $\mathbf{S}_0$ is related to the deformation gradient tensor $\mathbf{F}_0$ by

$$\mathbf{S}_0 = \left.\frac{\partial W}{\partial \mathbf{F}}\right|_{\mathbf{F}=\mathbf{F}_0} - q_0\mathbf{F}_0^{-T}, \tag{5.13}$$

where the constant q_0 is the reactive pressure associated with the incompressibility constraint. Since the stress field is uniform, the equilibrium equations hold automatically.

Now consider an inhomogeneous deformation

$$\mathbf{y} = \mathbf{F}_0\mathbf{z} + \widehat{\mathbf{u}}(\mathbf{z}), \tag{5.14}$$

in which the particle located at $\mathbf{z}$ in the reference configuration is carried to the location $\mathbf{y}$ in the deformed configuration. Observe from (5.11) and (5.14) that $\mathbf{y} = \mathbf{x} + \widehat{\mathbf{u}}(\mathbf{z})$ and so $\widehat{\mathbf{u}}$ is the displacement *from the homogeneously deformed configuration* to the inhomogeneously deformed configuration, see Figure 5.28. As one might expect, it is convenient to change variables and express this displacement field as a function of $\mathbf{x}$ rather than $\mathbf{z}$ by introducing the function

$$\mathbf{u}(\mathbf{x}) = \widehat{\mathbf{u}}(\mathbf{z}) \quad \text{with} \quad \mathbf{x} = \mathbf{F}_0\mathbf{z}. \tag{5.15}$$

The deformation (5.14) can now be written

$$\mathbf{y} = \mathbf{x} + \mathbf{u}(\mathbf{x}). \tag{5.16}$$

When the context makes clear as to whether we are working with $\mathbf{u}(\mathbf{x})$ or $\widehat{\mathbf{u}}(\mathbf{z})$, we will omit the hat.

Let $\nabla_z\mathbf{u}$ and $\nabla_x\mathbf{u}$ denote the displacement gradient tensors whose cartesian components are

$$(\nabla_z\mathbf{u})_{ij} = \frac{\partial u_i}{\partial z_j}, \qquad (\nabla_x\mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{5.17}$$

It will be convenient to let

$$\mathbf{H} = \nabla_x\mathbf{u}. \tag{5.18}$$

By the chain rule,

$$\nabla_z\mathbf{u} = \nabla_x\mathbf{u}\,\mathbf{F}_0 = \mathbf{H}\mathbf{F}_0. \tag{5.19}$$

The deformation gradient tensor associated with (5.14) (with respect to the reference configuration) is

$$\mathbf{F} = \nabla_z\mathbf{y} = \mathbf{F}_0 + \nabla_z\mathbf{u} = \mathbf{F}_0 + \mathbf{H}\,\mathbf{F}_0 \;\;=\;\; (\mathbf{I}+\mathbf{H})\mathbf{F}_0, \tag{5.20}$$

and the Jacobian determinant is

$$\det\mathbf{F} = \det\Big[(\mathbf{I}+\mathbf{H})\mathbf{F}_0\Big] \;=\; \det(\mathbf{I}+\mathbf{H}), \tag{5.21}$$

having used (5.12).

Going forward, we shall assume that

$$\epsilon := |\nabla_x\mathbf{u}| = |\mathbf{H}| \ll 1 \tag{5.22}$$

and approximate all equations based on this, neglecting terms that are quadratic or smaller in ϵ. Thus in particular the incompressibility requirement $\det\mathbf{F} = \det(\mathbf{I}+\mathbf{H}) = 1$ gives, to leading order,

$$\operatorname{tr}\mathbf{H} + \mathrm{O}(\epsilon^2) = 0 \qquad \Rightarrow \quad \operatorname{div}_x\mathbf{u} + \mathrm{O}(\epsilon^2) = 0. \tag{5.23}$$

Likewise from (5.20)

$$\mathbf{F}^{-1} = \mathbf{F}_0^{-1}(\mathbf{I}+\mathbf{H})^{-1} = \mathbf{F}_0^{-1}(\mathbf{I}-\mathbf{H}) + \mathrm{O}(\epsilon^2), \qquad \mathbf{F}^{-T} = (\mathbf{I}-\mathbf{H}^T)\mathbf{F}_0^{-T} + \mathrm{O}(\epsilon^2). \tag{5.24}$$

Turning next to the constitutive relation, we first let $\mathbb{A}(\mathbf{F})$ and $\mathbb{B}(\mathbf{F})$ be the 4-tensor functions of $\mathbf{F}$ defined as the tensors with components

$$\mathbb{A}_{ijk\ell}(\mathbf{F}) := \frac{\partial^2 W(\mathbf{F})}{\partial F_{ij}\partial F_{k\ell}}, \qquad \mathbb{B}_{ipkq}(\mathbf{F}) := \mathbb{A}_{ijk\ell}(\mathbf{F})F_{pj}F_{q\ell}, \tag{5.25}$$

Observe that when evaluated at $\mathbf{F} = (\mathbf{I}+\mathbf{H})\mathbf{F}_0$ and linearized,

$$\frac{\partial W}{\partial F_{ij}}(\mathbf{F})\bigg|_{\mathbf{F}=(\mathbf{I}+\mathbf{H})\mathbf{F_0}} = \frac{\partial W}{\partial F_{ij}}(\mathbf{F}_0) + \mathbb{A}_{ijk\ell}(\mathbf{F}_0)\ \overset{o}{F}_{q\ell}\ H_{kq} + \mathrm{O}(\epsilon^2).$$

In what follows we shall omit the argument $\mathbf{F}_0$ from the partial derivatives of W (including $\mathbb{A}$). Setting $\mathbf{F} = (\mathbf{I}+\mathbf{H})\mathbf{F}_0$ and

$$q = q_0 + \widetilde{q}, \tag{5.26}$$

where $\widetilde{q}$ is assumed to be $\mathrm{O}(\epsilon)$, in the constitutive equation

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) - q\mathbf{F}^{-T},$$

and linearizing gives

$$\begin{aligned} S_{ij} &= \frac{\partial W}{\partial F_{ij}} + \mathbb{A}_{ijk\ell}\ \overset{o}{F}_{q\ell}\ H_{kq} - (q_0+\widetilde{q})(\overset{o}{F}{}^{-1}_{ji} - H_{pi}\ \overset{o}{F}{}^{-1}_{jp}) = \\ &= S^o_{ij} + \mathbb{A}_{ijk\ell}\ \overset{o}{F}_{q\ell}\ H_{kq} - \widetilde{q}\ \overset{o}{F}{}^{-1}_{ji} + q_0 H_{pi}\ \overset{o}{F}{}^{-1}_{jp} = \\ &= S^o_{ij} + \left[\mathbb{A}_{isk\ell}\ \overset{o}{F}_{q\ell}\overset{o}{F}_{ps}\ H_{kq} - \widetilde{q}\delta_{pi} + q_0 H_{pi}\right]\ \overset{o}{F}{}^{-1}_{jp} = \\ &= S^o_{ij} + [\mathbb{B}_{ipkq}H_{kq} + q_0 H_{pi} - \widetilde{q}\delta_{pi}]\ \overset{o}{F}{}^{-1}_{jp} = \\ &= S^o_{ij} + \Sigma_{ip}\ \overset{o}{F}{}^{-1}_{jp}. \end{aligned} \tag{5.27}$$

Thus

$$\mathbf{S} = \mathbf{S}^o + \mathbf{\Sigma}\mathbf{F}_0^{-T},$$

where

$$\mathbf{\Sigma} = \mathbb{B}\mathbf{H} + q_0\mathbf{H}^T - \widetilde{q}\mathbf{I}; \tag{5.28}$$

$\mathbf{\Sigma}\mathbf{F}_0^{-T}$ is the perturbation of the Piola stress. Equation (5.28) is effectively the constitutive relation for $\mathbf{\Sigma}$.

It is easy to show to $O(\epsilon^2)$ that $\mathrm{div}_z\mathbf{S} = \mathrm{div}_z\mathbf{S}_0 + \mathrm{div}_z(\boldsymbol{\Sigma}\mathbf{F}_0^{-T}) = \mathrm{div}_z(\boldsymbol{\Sigma}\mathbf{F}_0^{-T}) = \mathrm{div}_x(\boldsymbol{\Sigma})$, noting that in the last step the divergence is taken with respect to the homogeneously deformed configuration. Thus the equilibrium equation $\mathrm{div}_z\mathbf{S} = \mathbf{0}$ to leading order can be written as [16]

$$\mathrm{div}_x\boldsymbol{\Sigma} = 0, \tag{5.29}$$

for the stress tensor field $\boldsymbol{\Sigma}(\mathbf{x})$.

Thus in summary, the perturbed problem involves the fields $\mathbf{u}(\mathbf{x}), \mathbf{H}(\mathbf{x})$ and $\boldsymbol{\Sigma}(\mathbf{x})$ and they obey the incompressibility equation (5.23), the equilibrium equation (5.29) and the constitutive equation (5.28).

In order to recover what we had before, we now specialize (5.28) for the neo-Hookean material. It is readily found from $(5.25)_1$ and

$$W = \frac{\mu}{2}(\mathbf{F}\cdot\mathbf{F} - 1)$$

that

$$\mathbb{A}_{ijk\ell} = \mu\delta_{j\ell}\delta_{ik},$$

and therefore from $(5.25)_2$ that

$$\mathbb{B}_{ijpq} = \mu B_{jq}\delta_{ip},$$

whence

$$\mathbb{B}_{ijpq}H_{pq} = \mu B^0_{jq}H_{iq} = \mu(\mathbf{H}\mathbf{B}_0)_{ij}.$$

Equation (5.28) for the stress $\boldsymbol{\Sigma}$ now specializes to

$$\boldsymbol{\Sigma} = \mu\mathbf{H}\mathbf{B}_0 + q_0\mathbf{H}^T - \widetilde{q}\mathbf{I} = \mathbf{H}(\mu\mathbf{B}_0 - q_0\mathbf{I}) + q_0(\mathbf{H}+\mathbf{H}^T) - \widetilde{q}\mathbf{I} = \mathbf{H}\mathbf{T}_0 + q_0(\mathbf{H}+\mathbf{H}^T) - \widetilde{q}\mathbf{I}$$

and so we recover (xix).

When these equations are specialized to the plane strain perturbation (ixb), the traction-free boundary condition on $z_2 = 0$ and $\mathbf{S} \to -S\mathbf{e}_1\otimes\mathbf{e}_1$ as $|\mathbf{z}| \to \infty$ one recovers the equations we had above.

In terms of components in an arbitrary fixed cartesian basis, (5.28) reads

$$\Sigma_{ij} = \mathbb{B}_{ijk\ell}\frac{\partial u_k}{\partial x_\ell} + q_0\frac{\partial u_j}{\partial x_i} - \widetilde{q}\,\delta_{ij}. \tag{5.30}$$

Note that since the deformation (5.11) is homogeneous, the elastic moduli $\mathbb{B}_{ijk\ell}$ and the scalar q_0 are constants. Substituting (5.30) into the equilibrium equations $\partial\Sigma_{ij}/\partial x_j = 0$ and

[16] As an exercise show that $\mathrm{div}_z\widetilde{\mathbf{S}} = \mathrm{div}_x\boldsymbol{\Sigma}$ where $\widetilde{\mathbf{S}} = \boldsymbol{\Sigma}\mathbf{F}_0^{-T}$ even if $\mathbf{F}_0$ is not a constant, i.e. if it is a field $\mathbf{F}_0(\mathbf{z})$.

using the incompressibility equation $\partial u_i/\partial x_i = 0$ leads to

$$\mathbb{B}_{ijk\ell} \frac{\partial^2 u_k}{\partial x_j \partial x_\ell} - \frac{\partial \widetilde{q}}{\partial x_i} = 0. \tag{5.31}$$

This, together with the incompressibility equation $\partial u_i/\partial x_i = 0$, are to be solved for $u_i(\mathbf{x})$ and $\widetilde{q}(\mathbf{x})$.

Exercises: Problems 5.18, 5.19, 5.20 and 5.21.

5.8 Exercises.

Problem 5.1. Consider a spherical elastic shell that undergoes a large (spherically symmetric) deformation when subjected to an internal pressure p.

(I) First, model the shell as a two-dimensional entity (a membrane). Let r and R denote the radii of the membrane in the deformed and reference configurations respectively; let σ be the circumferential *force* in the membrane per unit deformed (circumferential) *length*; and let $\mathcal{W}$ be the elastic energy in the membrane per unit deformed *area*. The constitutive relation for the energy is $\mathcal{W} = \mathcal{W}(\lambda)$ where $\lambda = r/R$ is the circumferential stretch of the membrane.

(a) Use force balance to show that $p = 2\sigma/r$.

(b) Balance the rate of external working with the rate of increase of stored energy in a quasi-static motion and show that

$$p = \frac{2}{r}\mathcal{W} + \frac{1}{R}\mathcal{W}', \qquad \sigma = \frac{1}{2}\lambda\mathcal{W}' + \mathcal{W}. \tag{i}$$

Remark: You may have seen these results in the context of a soap bubble with surface tension σ and the surface energy $\mathcal{W} = \mathsf{constant}$.

(II) Now model the shell as a hollow spherical solid composed of an incompressible, isotropic material, whose wall-thickness is small but positive. The quantities r and R introduced above now represent the *mean* radii in the deformed and undeformed configurations, and let $T\,(\ll R)$ denote the wall-thickness in the undeformed configuration. Note by symmetry and incompressibility that $\lambda_\theta = \lambda_\phi = \lambda, \lambda_r = \lambda^{-2}$ where $\lambda = r/R$ is the mean circumferential stretch. Let $w(\lambda)$ denote the elastic energy in the shell per unit *reference volume*. Show that $w(\lambda) = \lambda^2\mathcal{W}(\lambda)/T$. Substituting this into $(i)_1$ gives

$$p = \frac{T}{R}\,\frac{w'(\lambda)}{\lambda^2}. \tag{ii}$$

which coincides with equation (xv) on page 491 as it should.

Problem 5.2. *Pressurized hollow circular cylinder.* A thick-walled hollow circular tube has inner and outer radii A and B respectively in the undeformed configuration. It is composed of an isotropic incompressible elastic material. Determine the radii a and b of the tube in the deformed configuration when it is subjected to an internal pressure p on the inner curved surface. Assume that particles do not displace in the axial direction. Calculate the forces that must be applied on the two end faces of the tube in order to prevent axial deformation.

Problem 5.3. *Torsion.* Consider the torsional deformation of an isotropic incompressible solid circular cylindrical body as in Section 5.2 but now assume that no *resultant* axial force is applied on the two rigid plates at its ends. (The axial stress T_{zz} need not be zero everywhere, only its resultant on a cross-section must vanish.) Under these conditions, in addition to twisting about the z-axis, the cylinder will change in length (and therefore so will its radius). Thus now consider the deformation

$$r = \widehat{r}(R, \Theta, \Phi) = r(R), \qquad \theta = \widehat{\theta}(R, \Theta, \Phi) = \Theta + \alpha\lambda Z, \qquad z = \widehat{z}(R, \Theta, \Phi) = \lambda Z.$$

Assume the material is neo-Hookean and that in the undeformed configuration the body has radius A. Derive an algebraic equation in which λ is the only unknown. (The parameters μ, α and A are given.)

Problem 5.4. *Combined axial and azimuthal shear of a tube.* Consider a long hollow circular tube that in a reference configuration has length L and inner and outer radii A and B respectively. The outer surface $R = B$ of the tube is held fixed. A rigid solid cylinder of radius A is inserted into the cavity and firmly bonded to the hollow tube on their common interface $R = A$. An axial force $F\,\mathbf{e}_z$ and a torque $T\,\mathbf{e}_z$ are applied on the rigid cylinder where $\mathbf{e}_z$ is a unit vector parallel to the axis of the cylinder.

Suppose the hollow tube is elastic and is composed of a neo-Hookean material.

Use cylindrical polar coordinates (R, Θ, Z) and (r, θ, z) in the reference and deformed configurations respectively. Assume that the resulting deformation is of the form

$$r = R, \qquad \theta = \Theta + \phi(R), \qquad z = Z + w(R). \tag{i}$$

If only the axial force was applied, we would take the rotation $\phi = 0$; and likewise if only the torque was applied, we would take the axial displacement $w = 0$. The deformation described by $\phi(R)$ is referred to as an azimuthal shear, and that associated with $w(R)$, an axial (or telescopic) shear.

(a) Determine the rotation and axial displacement of the rigid solid cylinder.

(b) Calculate the radial stress field $T_{rr}(r)$ to within an unknown constant. Explain how you might find the constant but do not carry out any calculations to find it.

Solution:

Kinematics: From (i) and the formula for the deformation gradient tensor in cylindrical polar coordinates (see Chapter 2.7),

$$\mathbf{F} = \mathbf{e}_r \otimes \mathbf{e}_R + R\phi'\mathbf{e}_\theta \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + w'\mathbf{e}_z \otimes \mathbf{e}_R + \mathbf{e}_z \otimes \mathbf{e}_Z.$$

According to (i), the axial displacement $w(R)$ translates each cylindrical surface $R =$ constant in the axial direction leading to a simple shear with shearing direction $\mathbf{e}_z$ and glide plane normal $\mathbf{e}_r$; the (nondimensional) amount of shear is $w'(R)$. Moreover the circumferential rotation $\phi(R)$ rotates each cylindrical surface $R =$ constant about the z-axis leading to a simple shear with shearing direction $\mathbf{e}_\theta$ and glide plane normal $\mathbf{e}_r$; the (nondimensional) amount of shear is $R\phi'(R)$. Denote these amounts of shear by

$$k_1 = R\phi'(R), \qquad k_2 = w'(R), \tag{ii}$$

so that we can write $\mathbf{F}$ as

$$\mathbf{F} = \mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_z \otimes \mathbf{e}_Z + k_1\mathbf{e}_\theta \otimes \mathbf{e}_R + k_2\mathbf{e}_z \otimes \mathbf{e}_R. \tag{iii}$$

Remark: One can factor (iii) and write it in the illuminating form

$$\mathbf{F} = \underbrace{(\mathbf{I} + k_2\mathbf{e}_z \otimes \mathbf{e}_r)}_{\mathbf{F_2}}\underbrace{(\mathbf{I} + k_1\mathbf{e}_\theta \otimes \mathbf{e}_r)}_{\mathbf{F_1}}\underbrace{(\mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_z \otimes \mathbf{e}_Z)}_{\mathbf{Q}}$$

where $\mathbf{F}_1$ describes a simple shear of amount $k_1 = R\phi'(R)$, shearing direction $\mathbf{e}_\theta$ and glide plane normal $\mathbf{e}_r$; $\mathbf{F}_2$ describes a simple shear of amount $k_2 = w'(R)$, shearing direction $\mathbf{e}_z$ and glide plane normal $\mathbf{e}_r$; and $\mathbf{Q}$ is the rotation tensor that takes the basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ into the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. Therefore the deformation can be decomposed into a rotation $\mathbf{Q}$ followed by the simple shear $\mathbf{F}_1$ followed by the simple shear $\mathbf{F}_2$.

Since the (neo-Hookean) material is incompressible we must have $\det \mathbf{F} = 1$. Calculating the determinant of $\mathbf{F}$ gives

$$\det \mathbf{F} = \det[F] = \det \begin{pmatrix} 1 & 0 & 0 \\ k_1 & 1 & 0 \\ k_2 & 0 & 1 \end{pmatrix} = 1,$$

and so $\det \mathbf{F} = 1$ automatically. Therefore incompressibility imposes no restrictions on the functions $\phi(R)$ and $w(R)$.

The associated left Cauchy Green tensor is

$$\begin{aligned} \mathbf{B} = \mathbf{F}\mathbf{F}^T \quad &= \quad \mathbf{I} + k_1(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + k_2(\mathbf{e}_z \otimes \mathbf{e}_r + \mathbf{e}_r \otimes \mathbf{e}_z) + \\ &\qquad + k_1k_2(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta) + k_1^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta + k_2^2\mathbf{e}_z \otimes \mathbf{e}_z, \end{aligned}$$

and so the matrix of components of $\mathbf{B}$ in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is

$$[B] = \begin{pmatrix} 1 & k_1 & k_2 \\ k_1 & 1 + k_1^2 & k_1k_2 \\ k_2 & k_1k_2 & 1 + k_2^2 \end{pmatrix}. \qquad (iv)$$

Stress and equilibrium. The components of $\mathbf{B}$ depend only on the r-coordinate (and not θ and z). For simplicity let's assume that the reactive stress q in the constitutive relation also depends only on r. Then, it follows from the constitutive relation that the Cauchy stress components in cylindrical polar coordinates also do not depend on θ and z. *See the remark below on page 527 for an easier and direct way in which to arrive at (xii) and $(xiii)$ making use of the simplicity of the present problem.* The equilibrium equations in Chapter 3.10 (in the absence of body force) thus specialize to

$$\frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0, \qquad \frac{dT_{r\theta}}{dr} + 2\frac{T_{r\theta}}{r} = 0, \qquad \frac{dT_{rz}}{dr} + \frac{T_{rz}}{r} = 0. \qquad (v)$$

The second and third of these equations can be written as

$$\frac{d}{dr}(r^2T_{r\theta}) = 0, \qquad \frac{d}{dr}(rT_{rz}) = 0,$$

which can be integrated to obtain

$$T_{r\theta}(r) = \frac{c_1}{r^2}, \qquad T_{rz}(r) = \frac{c_2}{r}, \qquad A \le r \le B, \qquad (vi)$$

where c_1 and c_2 are constants of integration (to be found using the boundary conditions).

To find the boundary conditions at $r = A$ we now consider the equilibrium *of the rigid cylinder.* Force balance in the $\mathbf{e}_z$ direction requires

$$F + \int_{\mathcal{S}} \mathbf{t} \cdot \mathbf{e}_z \, dA_y = 0, \qquad (vii)$$

where $\mathcal{S}$ is the interface $r = A$ between the cylinders and $\mathbf{t}$ is the traction *on the rigid cylinder* at this surface. It can be calculated using $\mathbf{t} = \mathbf{T}\mathbf{n}$, $\mathbf{n} = \mathbf{e}_r$ and $r = A$ (note that $\mathbf{n}$ points out of the rigid cylinder):

$$\mathbf{t} = T_{rr}(A)\mathbf{e}_r + T_{r\theta}(A)\mathbf{e}_\theta + T_{rz}(A)\mathbf{e}_z. \qquad (viii)$$

Substituting $(viii)$ into (vii) and using $dA_y = LAd\theta$ leads to

$$F + LAT_{rz}(A) \int_0^{2\pi} d\theta = 0 \qquad \Rightarrow \qquad F + 2\pi LAT_{rz}(A) = 0,$$

where L is the length of $\mathcal{S}$ in the z-direction.This gives the boundary condition

$$T_{rz}(A) = -\frac{F}{2\pi AL}. \qquad (ix)$$

We next consider moment balance of the rigid cylinder about its axis. This requires

$$T + \int_{\mathcal{S}} (\mathbf{y} \times \mathbf{t}) \cdot \mathbf{e}_z \, dA_y = 0. \qquad (x)$$

Since $\mathbf{y} = r\mathbf{e}_r + z\mathbf{e}_z = A\mathbf{e}_r + z\mathbf{e}_z$ at a point on $\mathcal{S}$, we have

$$\begin{aligned} \mathbf{y} \times \mathbf{t} \quad &= (A\mathbf{e}_r + z\mathbf{e}_z) \times \Big[T_{rr}(A)\mathbf{e}_r + T_{r\theta}(A)\mathbf{e}_\theta + T_{rz}(A)\mathbf{e}_z\Big] = \\ &= AT_{r\theta}(A)\mathbf{e}_z - [AT_{rz}(A) - zT_{rr}(A)]\mathbf{e}_\theta - zT_{r\theta}(A)\mathbf{e}_r \quad \text{on } \mathcal{S}. \end{aligned}$$

Substituting this into (x), using $dA_y = LAd\theta$ and simplifying the integrals as above leads to

$$T + LA^2 T_{r\theta}(A) \int_0^{2\pi} d\theta = 0$$

from which we obtain the boundary condition

$$T_{r\theta}(A) = -\frac{T}{2\pi A^2 L}. \qquad (xi)$$

On using the boundary condition (ix) in the stress field $(vi)_2$ we get $c_2 = -F/(2\pi L)$ and so the shear stress field $T_{rz}(r)$ in the elastic body is

$$T_{rz}(r) = -\frac{F}{2\pi rL}, \qquad A \le r \le B. \qquad (xii)$$

Similarly from (xi) and $(vi)_1$ we find

$$T_{r\theta}(r) = -\frac{T}{2\pi r^2 L}, \qquad A \le r \le B. \qquad (xiii)$$

Remark: A direct way in which to arrive at (xii) and $(xiii)$ is as follows: Consider as a free body diagram the solid rigid cylinder together with the portion of the elastic cylinder between the radii A and r. The area of the outer surface is $2\pi rL$ and the axial force due to the stress T_{zr} on it is $2\pi rLT_{zr}$. Thus force balance in the axial direction gives

$$2\pi rLT_{zr} + F = 0 \qquad \Rightarrow \quad T_{zr}(r) = -\frac{F}{2\pi rL}, \qquad A \le r \le B. \tag{xii}$$

Similarly, the moment arm to this surface is r and therefore the moment about the z-axis due to the stress $T_{\theta r}$ is $(2\pi rL)T_{\theta r}\, r$. Thus moment balance about the z-axis is

$$2\pi r^2 LT_{\theta r} + T = 0 \qquad \Rightarrow \quad T_{\theta r}(r) = -\frac{T}{2\pi r^2 L}, \qquad A \le r \le B. \tag{xiii}$$

Constitutive relation. Since the constitutive relation for the Cauchy stress for a neo-Hookean material is

$$\mathbf{T} = \mu\mathbf{B} - q\mathbf{I},$$

we conclude that the matrix of components of $\mathbf{T}$ in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is

$$[T] = \begin{pmatrix} T_{rr} & T_{r\theta} & T_{rz} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta z} \\ T_{zr} & T_{z\theta} & T_{zz} \end{pmatrix} = \begin{pmatrix} \mu - q & \mu k_1 & \mu k_2 \\ \mu k_1 & \mu(1+k_1^2) - q & \mu k_1 k_2 \\ \mu k_2 & \mu k_1 k_2 & \mu(1+k_2^2) - q \end{pmatrix}. \tag{xiv}$$

Therefore, in particular,

$$T_{rr} = \mu - q, \qquad T_{\theta\theta} = \mu(1+k_1^2) - q, \qquad T_{r\theta} = \mu k_1, \qquad T_{rz} = \mu k_2. \tag{xv}$$

Substituting $(ii), (xii)$ and $(xiii)$ into $(xv)_3$ and $(xv)_4$ gives

$$\mu r\phi'(r) = -\frac{T}{2\pi r^2 L}, \qquad \mu w'(r) = -\frac{F}{2\pi rL}. \tag{xvi}$$

Since the outer wall of the hollow cylinder in fixed, we have the kinematic boundary conditions $\phi(B) = 0, w(B) = 0$. Integrating (xvi) and using these boundary conditions gives

$$\phi(r) = \frac{T}{4\pi\mu L}\left(r^{-2} - B^{-2}\right), \qquad w(r) = -\frac{F}{2\pi\mu L}\ln(r/B), \qquad A \le r \le B. \tag{xvii}$$

Therefore the rotation of the rigid shaft is

$$\phi(A) = \frac{T}{4\pi\mu L}\left(A^{-2} - B^{-2}\right), \qquad \square$$

and its displacement in the axial direction is

$$w(A) = \frac{F}{2\pi\mu L}\ln(B/A). \qquad \square$$

(b) The only equilibrium equation remaining to be satisfied is

$$\frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0. \tag{xviii}$$

From $(xv), (ii)$ and $(xvii)$,

$$T_{rr} - T_{\theta\theta} = -\mu k_1^2 = -\mu\big(r\phi'(r)\big)^2 = -\mu\left(\frac{T}{2\pi\mu r^2 L}\right)^2$$

and so the equilibrium equation $(xviii)$ gives

$$\frac{dT_{rr}}{dr} = \mu \left(\frac{T}{2\pi\mu L}\right)^2 \frac{1}{r^5} \qquad \Rightarrow \qquad T_{rr} = -\mu \left(\frac{T}{2\pi\mu L}\right)^2 \frac{1}{4r^4} + \underbrace{c}_{\text{constant}}. \qquad \square$$

We need another boundary condition in order to determine the constant of integration. Since all (field equations and) boundary conditions at $r = A$ and $r = B$ have been used, this condition will have to come from information about the loading on, say, the flat end $Z = 0$ of the hollow cylinder. Since it is only a constant scalar parameter that remains to be determined, the additional condition will be something like the vanishing of some resultant force component on the hollow elastic cylinder at $Z = 0$ (rather than the vanishing of the traction at each point on the boundary $Z = 0$). To calculate the resultant force one would integrate S_{ZZ} on the flat end $Z = 0, A \leq R \leq B$ (or the true traction on the deformed image of that cross-section which is *not* a flat plane). Since from (xiv), $T_{rr} = \mu - q$ and $T_{zz} = \mu(1 + k_2^2) - q$ we can use $T_{zz} = \mu(1 + k_2^2) - \mu + T_{rr}$ to calculate T_{zz} (and it will involve the same unknown constant c as above).

Problem 5.5. *Internal pressure and axial loading of a circular tube.* In a stress-free reference configuration, a hollow circular cylindrical tube has inner and outer radii A and B respectively. It is composed of an isotropic incompressible material. The tube is subjected to an internal pressure p and an axial force N and undergoes a deformation of the axi-symmetric form

$$r = r(R), \qquad \theta = \Theta, \qquad z = \Lambda Z, \tag{i}$$

where Λ is the (unknown) axial stretch of the tube. The pressure p and force N are given. Denote the (unknown) inner and outer radii of the tube in the deformed configuration by a and b.

Derive three equations in which a, b and Λ are the only unknowns. You will find it convenient to let $\lambda = r/R$ and introduce the function

$$w(\lambda, \Lambda) = W^*(\lambda^{-1}\Lambda^{-1}, \lambda, \Lambda), \tag{ii}$$

because the principal stretches are $\lambda^{-1}\Lambda^{-1}, \lambda, \Lambda$.

Problem 5.6. *Instability of a cube.* Reconsider the "Rivlin cube problem" that we considered in Section 5.3 where a unit cube of homogeneous incompressible isotropic elastic material was subjected to normal dead-load forces of magnitude F. Show for a Mooney-Rivlin material with stored-energy function

$$W(\lambda_1, \lambda_2, \lambda_3) = \alpha(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \beta(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3), \tag{i}$$

where $\alpha > 0$ and $\beta > 0$ are constants, that there can be solutions of the form

$$y_1 = \lambda_1 x_1, \quad y_2 = \lambda_2 x_2, \quad y_3 = \lambda_3 x_3, \tag{ii}$$

with *all three* λ_i*'s different*, provided α and β are suitably restricted. What is the restriction?

Problem 5.7. *Instability of a thin sheet.* [Experiments of this nature have been carried out by Treloar.] Consider a body that in an unstressed reference configuration is a square sheet $a \times a \times t$, $(a > t)$. The long edges of the sheet are parallel to the x_1- and x_2-axes. Tensile normal forces F (per unit reference area) act on the four edges $x_1 = \pm a/2$ and $x_2 = \pm a/2$ of the sheet (through the application of uniform normal traction distributions). The two faces $x_3 = \pm t/2$ are traction-free. The sheet is made of a Mooney-Rivlin material

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu}{2}\Big[\alpha\big(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\big) + (1-\alpha)\big(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3\big)\Big], \qquad \mu > 0, \quad 0 < \alpha \le 1. \qquad (i)$$

Assume the deformation to be a pure homogeneous stretch

$$\mathbf{y} = \mathbf{F}\mathbf{x}, \qquad \mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 \ + \ \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 \ + \ \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3. \qquad (ii)$$

(a) Show that for all values of the force F there is a symmetric equilibrium configuration with $\lambda_1 = \lambda_2$.

(b) For a neo-Hookean material ($\alpha = 1$) show that there are no asymmetric equilibrium configurations with $\lambda_1 \neq \lambda_2$.

(c) When $\alpha \neq 1$ show that there is an asymmetric equilibrium configuration with $\lambda_1 \neq \lambda_2$. At what value of force does the asymmetric solution bifurcate from the symmetric solution?

(d) For what range of values of F is the symmetric solution stable?

Solution:

See (1) E. A. Kearsley, Asymmetric Stretching of a Symmetrically Loaded Elastic Sheet, *International Journal of Solids and Structures*, 22(1986), issue 2, pp. 111-119, and (2) J. L. Ericksen, Introduction to the Thermodynamics of Solids, Chapman Hall, 1991, Chapter 6.

The deformation is given by (ii) together with the incompressibility requirement

$$\lambda_1 \lambda_2 \lambda_3 = 1. \qquad (iii)$$

Since the edges of the sheet in the deformed configuration have areas $\lambda_2 a \lambda_3 t$ and $\lambda_1 a \lambda_3 t$, and the magnitudes of the applied forces are Fat, the boundary conditions give

$$\mathbf{T} = \frac{F}{\lambda_2\lambda_3}\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{F}{\lambda_1\lambda_3}\mathbf{e}_2 \otimes \mathbf{e}_2 \stackrel{(iii)}{=} F\lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + F\lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2. \qquad (iv)$$

The constitutive equation

$$T_{ii} = \lambda_i \frac{\partial W}{\partial \lambda_i} - q \qquad \text{(no sum on } i\text{)} \qquad (v)$$

together with (i) and (iv) give

$$\left.\begin{aligned} F\lambda_1 &= -q + \mu\alpha\lambda_1^2 - \mu(1-\alpha)\lambda_1^{-2}, \\ F\lambda_2 &= -q + \mu\alpha\lambda_2^2 - \mu(1-\alpha)\lambda_2^{-2}, \\ 0 &= -q + \mu\alpha\lambda_3^2 - \mu(1-\alpha)\lambda_3^{-2}. \end{aligned}\right\} \qquad (vi)$$

On eliminating q from (vi) we obtain

$$\left.\begin{aligned} F\lambda_1 &= \mu\alpha(\lambda_1^2-\lambda_3^2)+\mu(1-\alpha)(\lambda_1^2-\lambda_3^2)\lambda_2^2,\\ F\lambda_2 &= \mu\alpha(\lambda_2^2-\lambda_3^2)+\mu(1-\alpha)(\lambda_2^2-\lambda_3^2)\lambda_1^2, \end{aligned}\right\} \qquad (vii)$$

which can be written as

$$\frac{F\lambda_1}{\mu\alpha}=\big[1+\beta\lambda_2^2\big](\lambda_1^2-\lambda_3^2), \qquad \frac{F\lambda_2}{\mu\alpha}=\big[1+\beta\lambda_1^2\big](\lambda_2^2-\lambda_3^2), \qquad (viii)$$

having set

$$\beta=\frac{1-\alpha}{\alpha}\ \geq 0.$$

The three equations (iii), $(viii)_1$ and $(viii)_2$ are to be solved for λ_1, λ_2 and λ_3.

(a) When $\lambda_1 = \lambda_2$ the two equations in $(viii)$ coincide, and after eliminating λ_3 using (iii) we get the following equation relating the force F to the stretch λ_1:

$$\frac{F}{\mu\alpha}=h(\lambda_1):=\lambda_1^{-5}(1+\beta\lambda_1^2)(\lambda_1^6-1). \qquad (viiia)$$

Since $h(\lambda_1)\to-\infty$ as $\lambda_1\to 0$, $h(\lambda_1)\to\infty$ as $\lambda_1\to\infty$ and h is continuous on $(0,\infty)$, it follows that the range of the function $h(\lambda_1)$ is $(-\infty,\infty)$ and therefore the equation $h(\lambda_1)=F/(\mu\alpha)$ has at least one root $\lambda_1(>0)$ corresponding to each value of F. Thus symmetric solutions exist for all values of F.

(b) We next examine the solutions on the λ_1,λ_2-plane. Eliminating F from $(viii)$ and simplifying leads to

$$(\lambda_1-\lambda_2)\Big[1+\lambda_1^3\lambda_2^3+\beta(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2-\lambda_1^4\lambda_2^4)\Big]=0. \qquad (ix)$$

Clearly, a *symmetric configuration*

$$\mathcal{C}_{\rm symm}: \qquad \lambda_1=\lambda_2, \qquad (x)$$

satisfies (ix). For a neo-Hookean material one has $\beta=0$ (i.e. $\alpha=1$) in which case we see from (ix) that the symmetric configuration is the only possible solution and so no asymmetric solutions are possible.

(c) On the other hand for $\beta>0$ there is an *asymmetric configuration*, $\lambda_1\neq\lambda_2$, that according to (ix) is described by

$$\mathcal{C}_{\rm asymm}: 1+\lambda_1^3\lambda_2^3+\beta(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2-\lambda_1^4\lambda_2^4)=0. \qquad (xi)$$

The curves $\mathcal{C}_{\rm symm}$ and $\mathcal{C}_{\rm asymm}$ on the λ_1,λ_2-plane are shown in Figure 5.29. The figure has been drawn for $\beta=1/8$ which corresponds to the value $\alpha=8/9$ in Treloar's experiments. The curves intersect at the critical stretch $\lambda_{\rm cr}$ found by setting $\lambda_1=\lambda_2=\lambda_{\rm cr}$ in (xi):

$$1+\lambda_{\rm cr}^6+\beta(3\lambda_{\rm cr}^2-\lambda_{\rm cr}^8)=0. \qquad (xii)$$

The corresponding critical value of force is found by setting $\lambda_1=\lambda_{cr}$ in $(viiia)$:

$$\frac{F_{\rm cr}}{\mu\alpha}=\lambda_{\rm cr}^{-5}\big[1+\beta\lambda_{\rm cr}^2\big](\lambda_{\rm cr}^6-1). \qquad (xiv)$$

When $\beta=1/8$ equations (xii) and (xiv) give $\lambda_{\rm cr}\approx 2.84$ and $F_{\rm cr}/\mu\alpha\approx 5.69$.

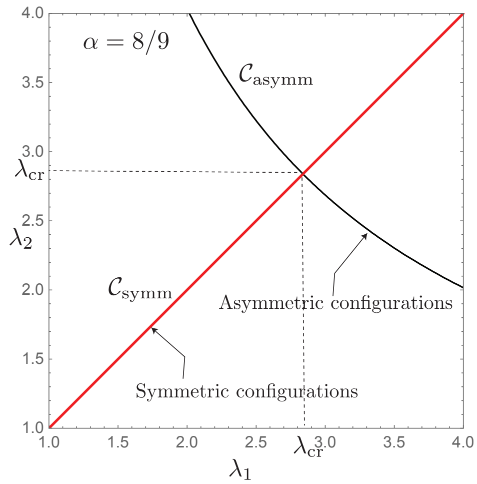

Figure 5.29: The curves $\mathcal{C}_{\text{symm}}$ and $\mathcal{C}_{\text{asymm}}$ corresponding to the symmetric and asymmetric configurations of the sheet. The figure has been drawn for $\alpha = 8/9$ in which case $\lambda_{\text{cr}} \doteq 2.84$.

We now derive relations between the force F and the through-thickness stretch λ_3 for the two types of solutions by returning to the three equations (iii), $(viii)_1$ and $(viii)_2$ and eliminating λ_1 and λ_2. (This calculation is not needed to answer the questions asked in this problem). In the case of the symmetric solution, we set $\lambda_1 = \lambda_2 = \lambda_3^{-1/2}$ in $(viii)_1$ and find

$$\frac{F}{\mu\alpha} = \lambda_3^{-3/2}(\beta + \lambda_3)(1 - \lambda_3^3). \qquad (xiii)$$

To derive the corresponding relation in the asymmetric case $\lambda_1 \neq \lambda_2$ we subtract $(viii)_2$ from $(viii)_1$ and simplify to obtain

$$\lambda_1 + \lambda_2 = \frac{F/\mu}{\alpha + (1-\alpha)\lambda_3^2}.$$

Similarly, adding $(viii)_1$ and $(viii)_2$ and simplifying gives

$$\frac{F}{\mu}(\lambda_1 + \lambda_2) = \Big[\alpha - (1-\alpha)\lambda_3^2\Big](\lambda_1 + \lambda_2)^2 - 2\lambda_3^{-2}(1 + \lambda_3^3)[\alpha\lambda_3 - (1-\alpha)].$$

We now eliminate $\lambda_1 + \lambda_2$ from the two preceding equations to obtain the desired relation

$$\left(\frac{F}{\mu\alpha}\right)^2 = \frac{(1 + \lambda_3^3)(\beta - \lambda_3)(1 + \beta\lambda_3^2)^2}{\beta\lambda_3^4}. \qquad (xv)$$

(d) *Stability:* The potential energy of the system can be written as

$$\Phi(\lambda_1, \lambda_2) = \frac{1}{2}\Big[\big(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3\big) + \beta\big(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2\lambda_2^2 - 3\big)\Big] - f\lambda_1 - f\lambda_2, \qquad (xvi)$$

where we have set $f = F/(\mu\alpha)$ and the elastic potential energy was obtained from (i) with $\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}$. Thus

$$\Phi_1 = \frac{\partial\Phi}{\partial\lambda_1} = \big(\lambda_1 - \lambda_1^{-3}\lambda_2^{-2}\big) + \beta\big(-\lambda_1^{-3} + \lambda_1\lambda_2^2\big) - f,$$

$$\Phi_2 = \frac{\partial \Phi}{\partial \lambda_2} = \big(\lambda_2 - \lambda_2^{-3}\lambda_1^{-2}\big) + \beta\big(-\lambda_2^{-3} + \lambda_2\lambda_1^2\big) - f,$$

$$\Phi_{11} = \frac{\partial^2 \Phi}{\partial \lambda_1^2} = \big(1 + 3\lambda_1^{-4}\lambda_2^{-2}\big) + \beta\big(3\lambda_1^{-4} + \lambda_2^2\big),$$

$$\Phi_{22} = \frac{\partial^2 \Phi}{\partial \lambda_2^2} = \big(1 + 3\lambda_2^{-4}\lambda_1^{-2}\big) + \beta\big(3\lambda_2^{-4} + \lambda_1^2\big),$$

$$\Phi_{12} = \frac{\partial^2 \Phi}{\partial \lambda_1 \partial \lambda_2} = 2\lambda_1^{-3}\lambda_2^{-3} + 2\beta\lambda_1\lambda_2.$$

Stability requires

$$\Phi_{11} > 0, \qquad \Phi_{11}\Phi_{22} - \Phi_{12}^2 > 0.$$

At the *symmetric solution* $\lambda_1 = \lambda_2$ this specializes to

$$\Phi_{11} = \big(1 + 3\lambda_1^{-6}\big) + \beta\big(3\lambda_1^{-4} + \lambda_1^2\big), \qquad \Phi_{22} = \big(1 + 3\lambda_1^{-6}\big) + \beta\big(3\lambda_1^{-4} + \lambda_1^2\big), \qquad \Phi_{12} = 2\lambda_1^{-6} + 2\beta\lambda_1^2.$$

Clearly $\Phi_{11} > 0$. On the other hand

$$\Phi_{11}\Phi_{22} - \Phi_{12}^2 = \lambda_1^{-6}\big[1 + \lambda_1^6 + \beta(3\lambda_1^2 - \lambda_1^8)\big]\big[1 + 5\lambda_1^{-6} + 3\beta(\lambda_1^{-4} + \lambda_1^2)\big].$$

Thus $\Phi_{11}\Phi_{22} - \Phi_{12}^2 > 0$ corresponds to

$$1 + \lambda_1^6 + \beta(3\lambda_1^2 - \lambda_1^8) > 0;$$

cf. (*xii*). One can show that the function $g(\lambda) = 1 + \lambda^6 + \beta(3\lambda^2 - \lambda^8)$ is positive for $\lambda < \lambda_{\rm cr}$ and negative for $\lambda > \lambda_{\rm cr}$ (and vanishes for $\lambda = \lambda_{\rm cr}$ of course). Therefore we conclude that the symmetric configuration is stable for $\lambda < \lambda_{\rm cr}$ and unstable for $\lambda > \lambda_{\rm cr}$.

Problem 5.8. *Stability of the "Rivlin Cube" with respect to arbitrary perturbations.* Reconsider the stability of the "Rivlin cube" studied in Section 5.3. There, we first determined the various pure homogeneous deformations the body could undergo, and second, investigated whether these deformations minimized the potential energy. In this latter calculation, we limited attention to virtual deformations that were homogeneous and coaxial with the pure homogeneous deformations we were studying. In the present problem, you are asked to consider all virtual deformations.

In the "Rivlin cube" problem the unit cube is subjected to the dead loading $\mathbf{s} = \overline{\mathbf{S}}\mathbf{n}_{\rm R}$ on $\partial\mathcal{R}_{\rm R}$ where $\overline{\mathbf{S}}$ is a given constant tensor. The associated deformation whose stability we want to study is

$$\mathbf{y}(\mathbf{x}) = \mathbf{F}\mathbf{x} \qquad \text{for} \quad \mathbf{x} \in \mathcal{R}_{\rm R}, \tag{i}$$

where the constant tensor $\mathbf{F}$ has $\det \mathbf{F} = 1$ and

$$\overline{\mathbf{S}} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) - q\,\mathbf{F}^{-T}. \tag{ii}$$

In order to study the stability of a deformation (*i*), consider virtual deformations of the form

$$\mathbf{z}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) + \epsilon\boldsymbol{\eta}(\mathbf{x}) = \mathbf{F}\mathbf{x} + \epsilon\boldsymbol{\eta}(\mathbf{x}) \qquad \text{for} \quad \mathbf{x} \in \mathcal{R}_{\rm R}. \tag{iii}$$

Here $\mathbf{z}(\mathbf{x})$ is the virtual deformation, $\mathbf{y}(\mathbf{x})$ is the deformation whose stability we wish the study, and $\epsilon\boldsymbol{\eta}(\mathbf{x})$ is the virtual displacement. The associated virtual deformation gradient tensor is

$$\mathbf{G} = \nabla_x \mathbf{z} = \mathbf{F} + \epsilon \nabla_x \boldsymbol{\eta}. \qquad (iv)$$

Here and in what follows, a subscript, e.g. x, on ∇ indicates that the gradient is being taken with respect to the position, e.g. $\mathbf{x}$. In (iii), ϵ is a scalar parameter and $\boldsymbol{\eta}(\mathbf{x})$ is an arbitrary smooth function subject only to the incompressibility requirement

$$\det \mathbf{G} = 1. \qquad (v)$$

The potential energy associated with a virtual deformation $\mathbf{z}(\mathbf{x})$ is

$$\Phi = \int_{\mathcal{R}_{\mathrm{R}}} W(\nabla_x \mathbf{z}) dV_x - \int_{\partial\mathcal{R}_{\mathrm{R}}} \overline{\mathbf{S}}\mathbf{n}_{\mathrm{R}} \cdot \mathbf{z}\, dA_x.$$

It is convenient to incorporate the kinematic constraint (v) into the potential energy through a Lagrange multiplier q, and to therefore consider

$$\Phi = \int_{\mathcal{R}_{\mathrm{R}}} \left(W(\nabla_x \mathbf{z}) - q \det(\nabla_x \mathbf{z})\right) dV_x - \int_{\partial\mathcal{R}_{\mathrm{R}}} \overline{\mathbf{S}}\mathbf{n}_{\mathrm{R}} \cdot \mathbf{z}\, dA_x. \qquad (vi)$$

On substituting the virtual deformation (iii) into the potential energy (vi), and keeping $\boldsymbol{\eta}(\mathbf{x})$ fixed for the moment, we can view the potential energy as a function of the scalar parameter ϵ:

$$\Phi = \Phi(\epsilon). \qquad (vii)$$

Since $\mathbf{z}(\mathbf{x}) = \mathbf{y}(\mathbf{x})$ when $\epsilon = 0$, see (iii), it follows that if $\mathbf{y}(\mathbf{x})$ is a minimizer of the potential energy then $\epsilon = 0$ is a minimizer of $\Phi(\epsilon)$. This requires

$$\left.\frac{\mathrm{d}\Phi}{d\epsilon}\right|_{\epsilon=0} = 0, \qquad \left.\frac{\mathrm{d}^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} \geq 0. \qquad (viii).$$

It will be convenient in what follows to let

$$\mathbf{H} \coloneqq \nabla_y \boldsymbol{\eta} = \nabla_x \boldsymbol{\eta}\, \mathbf{F}^{-1}. \qquad (ix)$$

(a) Show that

$$\det \mathbf{G} = 1 + \operatorname{tr} \mathbf{H} + \mathrm{O}(\epsilon^2) \qquad \text{as} \quad \epsilon \to 0, \qquad (x)$$

so that the incompressibility requirement (v) tells us that $\operatorname{tr} \mathbf{H} = 0 + \mathrm{O}(\epsilon)$ as $\epsilon \to 0$.

(b) Evaluate $d\Phi/d\epsilon$ and show that, in view of (ii), the first requirement $(viii)_1$ holds automatically.

(c) Show that

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_{\mathrm{R}}} \left[\frac{\partial^2 W}{\partial F_{ij}\partial F_{k\ell}}(\mathbf{F})\eta_{i,j}\eta_{k,\ell} - q(H_{ii}H_{jj} - H_{ij}H_{ji})\right] dV_x. \qquad (xi)$$

(d) Next consider a neo-Hookean material:

$$W = \frac{\mu}{2}(\mathbf{F}\cdot\mathbf{F} - 1), \qquad (xii)$$

and show that (xi) now specializes to

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_\mathrm{R}} [\mu B_{kj} H_{ij} H_{ik} + q H_{ij} H_{ji}]\, dV_x, \tag{$xiii$}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$.

(e) Now consider the stability of the cubic solution $\mathbf{F} = \mathbf{I}$. Recall that the loading is in fact an equi-triaxial dead loading, i.e. $\overline{\mathbf{S}} = S\,\mathbf{I}$. In this case $(ii), (xii)$ gives $q = \mu - S$. Show that

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_\mathrm{R}} \Big[(2\mu - S)\varepsilon_{i,j}\varepsilon_{i,j} + S\omega_{i,j}\omega_{i,j})\Big] dV_x, \tag{xiv}$$

where we have set $\varepsilon_{ij} := \frac{1}{2}(\eta_{i,j} + \eta_{j,i})$ and $\omega_{ij} := \frac{1}{2}(\eta_{i,j} - \eta_{j,i})$.

Thus far we kept $\boldsymbol{\eta}(\mathbf{x})$ fixed. But in fact it is arbitrary, subject only to the requirement stemming from incompressibility. Thus, for stability, it is necessary that the expression in the previous equation be non-negative for all such $\eta_{i,j}$. Show from this that the cubic deformation is stable for $0 < S < 2\mu$. And unstable for $S > 2\mu$ and $S < 0$. What is the nature of a virtual deformation that makes the cubic configuration unstable in the case $S < 0$?

References:

– R, Hill, On uniqueness and stability in the theory of finite elastic strain, *Journal of the Mechanics and Physics of Solids*, 5 (1957), pp. 229–241.
– R.S. Rivlin, Stability of pure homogeneous deformations of an elastic cube under dead loading, *Quarterly Journal of Applied Mathematics*, 1974, pp. 265-271.

Problem 5.9. *Stability of the "Rivlin cube" for an arbitrary isotropic material.* In Section 5.3 we examined the stability of a neo-Hookean cube subjected to an equi-triaxial dead loading (Piola traction). Generalize that analysis to a cube composed of an arbitrary isotropic (unconstrained) elastic material by extremizing

$$\Phi(\mathbf{F}) = W(\mathbf{F}) - \mathbf{S} \cdot \mathbf{F}, \tag{i}$$

over all geometrically admissible homogeneous deformations. Assume, in keeping with the equi-triaxial dead loading, that

$$\mathbf{S} = \sum_{i=1}^{3} S_i \mathbf{e}_i \otimes \mathbf{e}_i, \tag{ii}$$

and consider only deformations of the form $\mathbf{y} = \mathbf{F}\mathbf{x}$ where

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i, \tag{iii}$$

i.e. where $\mathbf{F}$ and $\mathbf{S}$ are coaxial.

How would your analysis change if the material is incompressible?

Problem 5.10. *Cavitation.* Derive a formula for the cavitation stress for a general incompressible isotropic elastic material in plane strain by considering the growth of the cylindrical cavity in a tube of undeformed inner and outer radii A and B. Under what conditions on $w(\lambda) := W(\lambda^{-1}, \lambda, 1)$ is the critical stress for cavitation finite?

Problem 5.11. *Pressurized spherical shell with radial inextensibility constraint.* A hollow spherical shell has inner and outer radii A and B respectively in the undeformed configuration. It is composed of an elastic material whose strain energy function is of the form[17] $W(\lambda_1, \lambda_2, \lambda_3)$. Very stiff fibers, that you can model as inextensible, oriented in the radial direction have been embedded throughout the body. The shell is subjected to an internal pressure p on the inner curved surface, the outer surface being traction-free. Derive two algebraic equations in which the only unknowns are the radii a and b in the deformed configuration (which one could in principle solve for a and b.)

Discuss the case where, in addition to the inextensibility constraint, the material is incompressible.

Problem 5.12. *Steadily rotating cylinder.* An incompressible solid circular cylinder has radius A and length L in a reference configuration. Its mass density is ρ. The cylinder undergoes a steady rotation about its axis of symmetry at the constant angular speed ω. Assume the motion to be described by

$$r = r(R), \qquad \theta = \Theta + \omega t, \qquad z = \lambda Z. \tag{i}$$

Calculate the acceleration of a particle by differentiating $\mathbf{y}(\mathbf{x}, t)$ with respect to time t at a fixed particle $\mathbf{x}$, i.e. by differentiating $\mathbf{y}(R, \Theta, Z, t) = r\,\mathbf{e}_r(\theta) + z\mathbf{e}_z$ at fixed R, Θ, Z.

Use incompressibility to determine $r(R)$.

Suppose that the curved boundary of the cylinder is traction-free, and the resultant force on its two ends are zero. Moreover, suppose the cylinder is composed of a *generalized* neo-Hookean material. Derive an algebraic equation relating the axial stretch λ to the angular speed ω. Specialize your answer to a neo-Hookean material. *Note*: Since inertial effects are being taken into account, you must use the equations of motion (i.e. the equilibrium equations (3.95) with the term $\rho\mathbf{a}$ added to the right-hand side where $\mathbf{a} = \ddot{\mathbf{y}}$ is the acceleration). Neglect the body force due to gravity.

Solution: The position vector of a particle $\mathbf{x}$ at time t is

$$\mathbf{y}(\mathbf{x}, t) = r\,\mathbf{e}_r(\theta) + z\,\mathbf{e}_z, \qquad r = r(R), \quad \theta = \Theta + \omega t, \quad z = \lambda Z, \tag{ii}$$

[17]Recall that the inextensibility of a fiber in direction $\mathbf{m}_R$ is characterized by the constraint $\phi(\mathbf{C}) = 0$ where $\phi(\mathbf{C}) = \mathbf{C}\mathbf{m}_R \cdot \mathbf{m}_R - 1$. This function $\phi(\mathbf{C})$ not an isotropic function. Thus, even if the strain energy has the form $W(\lambda_1, \lambda_2, \lambda_3)$, the stress response will not be isotropic. We will see in Chapter 6 that $\mathbf{C}\mathbf{m}_R \cdot \mathbf{m}_R$ is in fact the invariant I_4 for an anisotropic material with one preferred direction.

where $\mathbf{x} = R\mathbf{e}_R + Z\mathbf{e}_Z$ and

$$\mathbf{e}_r = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2. \tag{iii}$$

Differentiating $\mathbf{y}$ with respect to t at fixed $\mathbf{x}$, (i.e. at fixed R, Θ, Z) gives the particle velocity:

$$\dot{\mathbf{y}}(\mathbf{x},t) \overset{(ii)}{=} r\,\dot{\mathbf{e}}_r = r\frac{d\mathbf{e}_r}{d\theta}(\theta)\,\dot\theta \overset{(iii)}{=} r\,(-\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2)\,\omega. \tag{iv}$$

Differentiating (iv) with respect to t gives the acceleration:

$$\ddot{\mathbf{y}}(\mathbf{x},t) = r\,(-\cos\theta\,\mathbf{e}_1 - \sin\theta\,\mathbf{e}_2)\,\omega^2 \overset{(iii)}{=} -r\omega^2\,\mathbf{e}_r. \qquad \square \tag{v}$$

From (i) and (2.79) the left Cauchy-Green deformation tensor is

$$\mathbf{B} = \big(r'(R)\big)^2\,\mathbf{e}_r\otimes\mathbf{e}_r + \frac{r^2}{R^2}\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \lambda^2\mathbf{e}_z\otimes\mathbf{e}_z. \tag{vi}$$

Since the material is incompressible, it is necessary that $\det\mathbf{B} = 1$ whence (assuming $r'(R) > 0$)

$$\lambda r(R)r'(R) = R \qquad \Rightarrow \qquad \lambda r^2(R) = R^2 + \mathsf{constant}. \tag{vii}$$

Since $r = 0$ at $R = 0$ the constant of integration in (vii) vanishes and so

$$r(R) = \lambda^{-1/2}R. \qquad \square \tag{viii}$$

Remark: Alternatively and more easily, consider a part of the cylinder that has unit length and radius R in the reference configuration. In the deformed configuration it has length λ and radius r. Equating the volumes of this part in the two configurations gives $\pi R^2 = \pi r^2\lambda$ from which $(viii)$ follows. Equation (vi) can now be written as

$$\mathbf{B} = \lambda^{-1}\mathbf{e}_r\otimes\mathbf{e}_r + \lambda^{-1}\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \lambda^2\mathbf{e}_z\otimes\mathbf{e}_z. \tag{ix}$$

For a generalized neo-Hookean material $W = W(I_1)$, the constitutive relation (4.63) specializes to

$$\mathbf{T} = T_{rr}\mathbf{e}_r\otimes\mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta\otimes\mathbf{e}_\theta + T_{zz}\mathbf{e}_z\otimes\mathbf{e}_z, \tag{x}$$

where

$$T_{rr} = T_{\theta\theta} = 2\lambda^{-1}W'(I_1) - q, \qquad T_{zz} = 2\lambda^2 W'(I_1) - q. \tag{xi}$$

Assume that $q = q(r)$. The equations of motion (i.e. equations (3.95) with the term $\rho\mathbf{a}$ added to the right-hand side where $\mathbf{a} = \ddot{\mathbf{y}}$ is the acceleration) reduce to

$$\frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} = \rho a_r \qquad \overset{(v),(xi)_1}{\Rightarrow} \qquad \frac{dT_{rr}}{dr} = -\rho r\omega^2. \tag{xii}$$

Integrating this from r to a and using the boundary condition $T_{rr}(a) = 0$ gives

$$\cancel{T_{rr}(a)} - T_{rr}(r) = -\frac{1}{2}\rho\omega^2(a^2 - r^2) \qquad \Rightarrow \qquad T_{rr}(r) = \frac{1}{2}\rho\omega^2(a^2 - r^2). \tag{xiii}$$

Here

$$a = \lambda^{-1/2}A, \tag{xiv}$$

is the radius of the cylinder in the rotating configuration; see $(viii)$. The remaining boundary conditions $T_{r\theta} = T_{rz} = 0$ on the lateral surface hold automatically. From (xi),

$$T_{zz} - T_{rr} = 2(\lambda^2 - \lambda^{-1})W'(I_1) = \lambda w'(\lambda) \qquad \text{where} \quad w(\lambda) := W(\lambda^2 + 2\lambda^{-1}), \tag{xv}$$

and therefore after using $(xiii)$,

$$T_{zz}(r) = \lambda w'(\lambda) + \frac{1}{2}\rho\omega^2(a^2 - r^2). \tag{xvi}$$

Since the resultant axial force on a cross-section vanishes,

$$\int_{\mathcal{A}} T_{zz}\, dA_y = \int_0^a 2\pi r T_{zz} dr = 0 \quad \overset{(xvi)}{\Rightarrow} \quad \lambda w'(\lambda) + \frac{1}{4}\rho\omega^2 a^2 = 0, \tag{xvii}$$

which after using (xiv) yields

$$\lambda^2 w'(\lambda) + \frac{1}{4}\rho\omega^2 A^2 = 0. \qquad \square \tag{xviii}$$

For a neo-Hookean material $w(\lambda) = \frac{\mu}{2}(\lambda^2 + 2\lambda^{-1} - 3)$ and so $(xviii)$ specializes to

$$\lambda = \left(1 - \frac{\rho\omega^2 A^2}{4\mu}\right)^{1/3}. \qquad \square$$

Observe that as ω increases, the length λL of the cylinder decreases and its radius $\lambda^{-1/2}A$ increases. According to this model, the angular speed cannot exceed $\sqrt{4\mu/(\rho A^2)}$.

Problem 5.13. *Oscillation of a tube.* A hollow circular tube of infinite length has inner radius A and outer radius B in a stress-free reference configuration. It is composed of a homogeneous, isotropic, incompressible elastic material characterized by a strain energy function $W^*(\lambda_1, \lambda_2, \lambda_3)$. The tube undergoes a cylindrically symmetric plane strain motion

$$r = r(R,t), \quad \theta = \Theta, \quad z = Z \qquad \text{for} \qquad A \le R \le B, \quad t \ge 0, \tag{i}$$

where (R, Θ, Z) and (r, θ, z) are cylindrical polar coordinates in the reference and deformed configuration respectively. The inner and outer curved boundaries of the tube are traction-free. Let

$$a(t) = r(A,t) \qquad \text{and} \qquad b(t) = r(B,t), \tag{ii}$$

be the inner and outer radii of the tube at time t.

- (a) Derive an ordinary differential equation in which $a(t)$ is the only unknown.
- (b) Specialize it to a neo-Hookean material.
- (c) If at the initial instant $a(0) = a_0 \ne A$ and $\dot{a}(0) = 0$, is the resulting motion periodic? If so, determine the period of oscillation.

Solution:

J.K. Knowles, Large amplitude oscillations of a tube of incompressible elastic material, *Quarterly of Applied Mathematics*, **18** (1960), pp. 71-77. For a related problem see J.K. Knowles, On a class of oscillations in the finite deformation theory of elasticity, *ASME Journal of Applied Mechanics*, **29** (1962), pp. 283-286.

(a) Incompressibility requires $\pi R^2 - \pi A^2 = \pi r^2 - \pi a^2$ whence

$$r(R,t) = \Big[R^2 + a^2(t) - A^2\Big]^{1/2}, \tag{iii}$$

and so in particular

$$b(t) = \Big[B^2 + a^2(t) - A^2\Big]^{1/2}. \tag{iv}$$

Let

$$\lambda(r,t) = r(R,t)/R \tag{v}$$

be the circumferential stretch. The principal stretches are then

$$\lambda_r = \frac{\partial r}{\partial R} = \lambda^{-1}, \qquad \lambda_\theta = \lambda, \qquad \lambda_z = 1, \tag{vi}$$

and the circumferential stretches at the inner and outer tube wall are

$$\lambda_a(t) = a(t)/A, \qquad \lambda_b = b(t)/B. \tag{vii}$$

Equation (iii) can alternatively be obtained by integrating $\lambda_r\lambda_\theta\lambda_z = (r/R)\partial r/\partial R = 1$. Differentiating (iii) with respect to t at fixed R gives the particle velocity

$$\dot r = \frac{1}{2}\Big[R^2 + a^2(t) - A^2\Big]^{-1/2} 2a\dot a = \frac{a\dot a}{r}, \tag{viii}$$

which when differentiated again (with respect to t at fixed R) gives the particle acceleration

$$\ddot r = \frac{a\ddot a}{r} + \left(1 - \frac{a^2}{r^2}\right)\frac{\dot a^2}{r}. \tag{ix}$$

The constitutive law for an isotropic incompressible material tells us that $T_{ii} = \lambda_i \partial W^*/\partial\lambda_i - q$ (no sum on i). Thus the radial and circumferential Cauchy stress components are

$$T_{rr} = \lambda^{-1}\frac{\partial W^*}{\partial \lambda_1} - q, \qquad T_{\theta\theta} = \lambda\frac{\partial W^*}{\partial\lambda_2} - q. \tag{x}$$

Let $w(\lambda)$ be the restriction of the strain energy function W^* to the class of deformations under consideration:

$$w(\lambda) := W^*(\lambda^{-1}, \lambda, 1), \qquad \lambda > 0. \tag{xi}$$

Differentiating (xi) with respect to λ and using (x) then gives

$$T_{rr} - T_{\theta\theta} = -\lambda w'(\lambda). \tag{xii}$$

The radial equation of motion is

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = \rho\ddot r. \tag{xiii}$$

Integrating $(xiii)$ with respect to r from $r = a$ to $r = b$ and using the traction-free boundary conditions yields

$$\cancel{T_{rr}(b,t)} - \cancel{T_{rr}(a,t)} - \int_a^b \frac{\lambda w'(\lambda)}{r} dr = \int_a^b \rho \ddot r \, dr, \tag{xiv}$$

where we have used (xii). The right-hand side of (xiv) can be evaluated using (ix):

$$\int_a^b \rho\ddot r\, dr = \int_a^b \rho\left[\frac{\dot a^2}{r} + \frac{a\ddot a}{r} - \frac{a^2\dot a^2}{r^3}\right] dr = \rho(\dot a^2 + a\ddot a)\ln(b/a) + \frac{1}{2}\rho a^2\dot a^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right). \tag{xv}$$

We now aim to change dr to $d\lambda$ in the left-hand side of (xiv). To this end we first note from (v) that

$$\frac{d\lambda}{dR} = \frac{Rdr/dR - r}{R^2} \overset{(v),(vi)}{=} \frac{\lambda^{-1} - \lambda}{R}. \tag{xvi}$$

Thus the left-hand side of (xiv) can be written as

$$\int_a^b \frac{\lambda w'(\lambda)}{r} dr \overset{(vi)_1}{=} \int_A^B \frac{\lambda w'(\lambda)}{r} \lambda^{-1} dR \overset{(xvi)}{=} \int_{\lambda_a}^{\lambda_b} \frac{w'(\lambda)}{r} \frac{R}{\lambda^{-1} - \lambda} d\lambda \overset{(v)}{=} \int_{\lambda_a}^{\lambda_b} \frac{w'(\lambda)}{1 - \lambda^2} d\lambda. \tag{$xvii$}$$

Finally substituting (xv) and $(xvii)$ into (xiv) yields

$$\rho a \ddot{a} \ln\left(\frac{b^2}{a^2}\right) + \rho\left[\ln\left(\frac{b^2}{a^2}\right) + a^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\right]\dot{a}^2 = \int_{\lambda_a}^{\lambda_b} \frac{2w'(\lambda)}{\lambda^2 - 1} d\lambda. \qquad \square \tag{$xviii$}$$

In view of (iv) and (vii), this can be viewed as a differential equation for $a(t)$.

In order to study $(xviii)$ it is convenient to let

$$x(t) := a(t)/A = \lambda_a(t), \tag{xix}$$

and to introduce the parameter

$$\xi := B^2/A^2 - 1 \ > 0. \tag{xx}$$

From (iv) and $(vii)_2$,

$$\lambda_b^2 = \frac{b^2}{B^2} = \frac{x^2 + \xi}{1 + \xi}, \qquad \frac{b^2}{a^2} = 1 + \xi/x^2, \qquad \frac{a^2}{b^2} - 1 = -\frac{\xi}{\xi + x^2}. \tag{xxi}$$

On using (xix) and (xxi) we can write $(xviii)$ as

$$x \ln\left(1 + \frac{\xi}{x^2}\right)\ddot{x} + \left[\ln\left(1 + \frac{\xi}{x^2}\right) - \frac{\xi}{\xi + x^2}\right]\dot{x}^2 = \frac{2}{\rho A^2}\int_x^{\lambda_b} \frac{w'(\lambda)}{\lambda^2 - 1} d\lambda,$$

which in turn can be written as

$$x \ln\left(1 + \frac{\xi}{x^2}\right)\dot{x}\frac{d\dot{x}}{dx} + \left[\ln\left(1 + \frac{\xi}{x^2}\right) - \frac{\xi}{\xi + x^2}\right]\dot{x}^2 = \frac{2}{\rho A^2}\int_x^{\lambda_b} \frac{w'(\lambda)}{\lambda^2 - 1} d\lambda,$$

and thus

$$\frac{1}{x}\frac{d}{dx}\left[\frac{1}{2}\dot{x}^2 x^2 \ln\left(1 + \frac{\xi}{x^2}\right)\right] = \frac{2}{\rho A^2}\int_x^{\lambda_b} \frac{w'(\lambda)}{\lambda^2 - 1} d\lambda. \tag{$xxii$}$$

(b) We now specialize $(xxii)$ to a neo-Hookean material

$$W = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3).$$

Since

$$w(\lambda) = \frac{\mu}{2}(\lambda^2 + \lambda^{-2} - 2),$$

we have

$$\int_x^{\lambda_b} \frac{w'(\lambda)}{\lambda^2 - 1} d\lambda = \frac{\mu}{2}\left[\ln\left(\frac{\lambda_b^2}{x^2}\right) + \frac{1}{x^2} - \frac{1}{\lambda_b^2}\right].$$

Substituting this into $(xxii)$ yields

$$\frac{d}{dx}\left[\frac{1}{2}\dot{x}^2x^2\ln\left(1+\frac{\xi}{x^2}\right)\right]=\frac{\mu}{\rho A^2}\left[\frac{1}{x}-\frac{x}{\lambda_b^2}+x\ln\left(\frac{\lambda_b^2}{x^2}\right)\right]$$

where λ_b is given in terms of x by $(xxi)_1$.

Integrating both sides of the preceding equation with respect to x yields

$$\frac{1}{2}\dot{x}^2x^2\ln\left(1+\frac{\xi}{x^2}\right)=\frac{\mu}{\rho A^2}\left[\frac{x^2-1}{2}\,\ln\left(\frac{x^2+\xi}{(1+\xi)x^2}\right)-\frac{\xi}{2}\right]+\mathsf{constant}$$

which we can write as

$$\frac{1}{2}\dot{x}^2x^2\ln\left(1+\xi/x^2\right)+F(x)=c$$

where

$$F(x):=\frac{1}{2}\,\frac{\mu}{\rho A^2}\,(1-x^2)\,\ln\left(\frac{1+\xi/x^2}{1+\xi}\right)\qquad\text{for}\quad x>0,$$

and c is a constant. Observe that F is positive and decreases monotonically for $0<x<1$; vanishes at $x=1$; and is again positive and increases monotonically for $x>1$. Using the initial conditions

$$x_0:=x(0)=a_0/A,\qquad \dot{x}(0)=\dot{a}(0)/A=0,$$

we get

$$c=F(x_0)=\frac{1}{2}\,\frac{\mu}{\rho A^2}\,(1-x_0^2)\,\ln\left(\frac{1+\xi/x_0^2}{1+\xi}\right)$$

and so

$$\frac{1}{2}\dot{x}^2x^2\ln\left(1+\xi/x^2\right)+F(x)=F(x_0). \qquad (xxiii)$$

(c) We can write $(xxiii)$ as

$$\dot{x}^2=g(x) \qquad (xxiv)$$

where

$$g(x)=\frac{2F(x_0)-2F(x)}{x^2\ln\left(1+\xi/x^2\right)}.$$

Since $\dot{x}^2\geq 0$ it follows that we must have $F(x(t))\leq F(x_0)$ which then ensures that $(xxiv)$ will give two values, $\dot{x}=\pm\sqrt{g}$, for each value of x. To show that equation $(xxiv)$ describes a periodic motion, we must show that it describes a closed curve (trajectory) on the $x,\dot{x}$-phase plane. It is only necessary for this that there be exactly two distinct positive values of x for which $F(x)=F(x_0)$, i.e. $\dot{x}=0$. Since $F(x)$ decreases monotonically for $0<x<1$ with $F(x)\to\infty$ as $x\to 0^+$ and $F(1)=0$, it follows that the equation $F(x)=F(x_0)$ has exactly one positive root <1 for every value of $F(x_0)>0$. Similarly since $F(x)$ increases monotonically for $x>1$ with $F(1)=0$ and $F(x)\to\infty$ as $x\to\infty$, it follows that the equation $F(x)=F(x_0)$ has exactly one root >1 for every value of $F(x_0)>0$. (When an analogous analysis is carried out on equation $(xxii)$ for a general material $w(\lambda)$, one finds that periodic oscillatory motions are possible if and only if $w(\lambda)$ obeys certain growth conditions on the strain energy function for $\lambda\to 0^+$ and $\to\infty$; see the paper Knowles (1960) referenced above.) The two roots $x=x_{min}<1$ and $x=x_{max}>1$ are the minimum and maximum amplitudes of the oscillation. The period of the motion is

$$T=2\int_{x_{min}}^{x_{max}}\frac{dx}{\dot{x}}=2\int_{x_{min}}^{x_{max}}\sqrt{\frac{x^2\ln(1+\xi/x^2)}{2F(x_0)-2F(x)}}\,dx. \qquad \square$$

Problem 5.14. *Eversion of a hollow sphere.* In a stress-free reference configuration a body occupies a spherical shell of inner radius $R_1 > 0$ and outer radius $R_2 > R_1$. It is composed of an incompressible isotropic elastic material with strain energy function $W(I_1, I_2)$. The body is everted – turned inside out – (say by cutting the body in half, everting each part, and then gluing the two halves together). In the deformed configuration it occupies a spherical shell of inner radius $r_2 > 0$ and outer radius $r_1 > r_2$. *The outer surface $R = R_2$ of the undeformed shell maps into the inner surface $r = r_2$ of the deformed shell and the inner surface $R = R_1$ of the undeformed shell maps into the outer surface $r = r_1$ of the deformed shell.* Here R and r are the radial spherical polar coordinates in the undeformed and deformed configurations. The inner and outer surfaces of the deformed sphere, $r = r_2$ and $r = r_1$ are traction-free. Determine r_1 and r_2.

J. L. Ericksen, Inversion of a perfectly elastic spherical shell, *Zeitschrift für Angewandte Mathematik und Mechanik*, 35(1955), issue 9-10, pp. 382-385.

Problem 5.15. *Eversion of a cylinder.* In the reference configuration, the hollow circular cylindrical body occupies the region $A \leq R \leq B$, $0 \leq \Theta \leq 2\pi$, $-L/2 \leq Z \leq L/2$. We are concerned with a deformation that turns the body inside out (everts it) as sketched in Figure 5.30. (Imagine turning a sock inside out.)

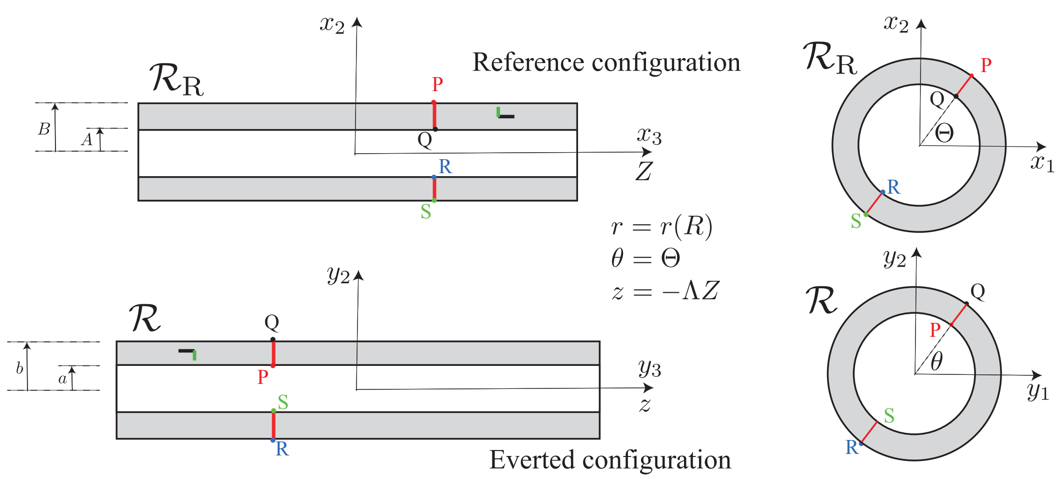

Figure 5.30: Hollow circular cylinder in an unstressed reference configuration (top) and an everted configuration (bottom). Note the locations of the particles P, Q, R and S in the two configurations. The deformation has the form $r = r(R), \theta = \Theta, z = -\Lambda Z$. Observe that $r(A) = b$, $r(B) = a$ and $r(A) > r(B)$; also that as R increases from A to B (point Q to point P in upper figure), r decreases from b to a (point Q to point P in lower figure), indicating that $r'(R) < 0$. Corresponding material fibers in the two configurations are shown in green and black; observe that, in addition to stretch, they have rotated by 180^o about $\mathbf{e}_\theta$.

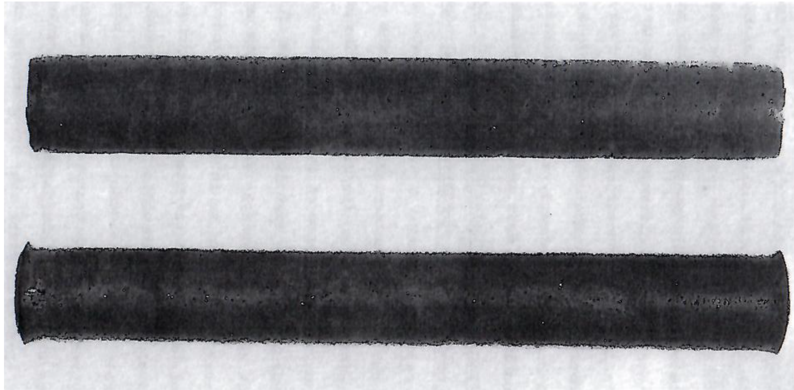

Figure 5.31: A segment of a rubber tube in unstressed (top) and everted (bottom) configurations; figure taken from Truesdell [3]. The entire boundary of the tube is traction-free. Observe that the everted tube is slightly longer, has slightly smaller diameter, and has flared ends.

In the deformed configuration it occupies the region $a \le r \le b$, $0 \le \theta \le 2\pi$, $-\ell/2 \le z \le \ell/2$. Using cylindrical polar coordinates, the position of a particle in the reference and deformed configurations are

$$\mathbf{x} = R\,\mathbf{e}_R \,+\, Z\,\mathbf{e}_Z, \qquad \mathbf{x} = r\,\mathbf{e}_r \,+\, z\,\mathbf{e}_z.$$

Assume the deformation describing the eversion has the form

$$r = \widehat{r}(R,\Theta,Z) = r(R), \qquad \theta = \widehat{\theta}(R,\Theta,Z) = \Theta, \qquad z = -\Lambda Z, \tag{i}$$

(where $r(R)$ *and* $\Lambda(>0)$ are unknown); see Figure 5.30. According to this deformation the everted body is also a hollow circular cylinder of some as yet unknown inner radius a, outer radius b and length $\ell = \Lambda L$. The deformation maps the undeformed cross-sectional plane $Z = L/2$ into $z = -\ell/2$ (and $Z = -L/2$ into $z = \ell/2$). Since the inner surface $R = A$ in the reference configuration is mapped into the outer surface $r = b$ in the deformed configuration, and likewise since the outer surface $R = B$ goes into the inner surface $r = a$, the deformation must be such that

$$r(A) = b, \qquad r(B) = a, \tag{ii}$$

where $A < B$ and $r(A) > r(B)$. Observe also that as R increases monotonically from the value A to B (point Q to point P in the upper figure), r decreases monotonically from the value b to a (point Q to point P in lower figure), indicating that

$$r'(R) < 0 \qquad \text{for} \quad A \le R \le B. \tag{iii}$$

The two curved surfaces are given to be traction-free:

$$T_{rr}(a) = T_{rr}(b) = 0. \tag{iv}$$

The assumed form (i) of the deformation requires the two flat ends of the tube to remain flat. This is only possible if some suitable traction distribution acts on the two ends[18] (and so T_{zz} will not be identically zero

[18]It the ends of the tube are traction-free, the everted tube will not be perfectly cylindrical in the deformed configuration; its ends will be flared as shown in Figure 5.31.

on $z = \pm\ell/2$). The resultant axial force however is taken to vanish (imagine that the two ends of the tube are attached to two rigid end-plates after eversion):

$$2\pi \int_a^b rT_{zz}(r,z)\,dr = 0 \qquad \text{for } z = \pm\ell/2. \tag{v}$$

The material is incompressible and isotropic.

(a) Determine $r(R)$ and Λ.

(b) Calculate the rotation tensor $\mathbf{R}$ (in the polar decomposition $\mathbf{F} = \mathbf{RU}$).

References:

1. R.S. Rivlin, Large elastic deformations of isotropic materials. VI Further results in the theory of torsion, shear and flexure, *Philosophical Transactions of the Royal Society of London*, Series A. Mathematical and Physical Sciences. Volume 242 (1949), pp. 173-195.

2. P. Chadwick, The existence and uniqueness of solutions to two problems in the Mooney-Rivlin theory for rubber, *Journal of Elasticity*, Volume 2, Issue 2, 1972, pp. 123 - 128.

3. C. Truesdell, Some challenges offered to analysis by rational thermomechanics: Lecture 1, in Contemporary Developments in Continuum Mechanics and Partial Differential Equations, Proceedings of the International Symposium on Continuum Mechanics and Partial Differential Equations, Rio de Janeiro, 1977. Edited by G.M. de La Penha and L.A. Medeiros, North Holland, 1978, pp. 497-540.

Solution: Upon using (2.77), the deformation gradient tensor associated with the deformation (i) is

$$\mathbf{F} = r'(R)\,\mathbf{e}_r \otimes \mathbf{e}_r + \frac{r(R)}{R}\,\mathbf{e}_\theta \otimes \mathbf{e}_\theta - \Lambda \mathbf{e}_z \otimes \mathbf{e}_z.$$

Keeping (iii) in mind, the principal stretches, all positive, are

$$\lambda_r = -r', \qquad \lambda_\theta = r/R, \qquad \lambda_z = \Lambda. \tag{vi}$$

(If you wish you can jump ahead to part (b) below and return here thereafter.) By incompressibility

$$\lambda_r\lambda_\theta\lambda_z = -\Lambda r'(R)\,\frac{r(R)}{R} = 1 \qquad \Rightarrow \qquad r(R) = \Big[a^2 + \frac{B^2 - R^2}{\Lambda}\Big]^{1/2}, \tag{vii}$$

having enforced $r(B) = a$. From $r(A) = b$ we get

$$b = \Big[a^2 + \frac{B^2 - A^2}{\Lambda}\Big]^{1/2}, \tag{vii}$$

and so the deformation (i), $(vii)_2$ is fully determined once a and Λ have been found. Denote the hoop stretch by

$$\lambda(R) := \frac{r(R)}{R}, \tag{$viii$}$$

so that in particular,

$$\lambda(B) \overset{(ii)}{=} \frac{a}{B} = \frac{\alpha}{N} \quad \text{and} \quad \lambda(A) \overset{(ii)}{=} \frac{b}{A} \overset{(vii)}{=} \left[\frac{N^2 - 1 + \alpha^2\Lambda}{\Lambda}\right]^{1/2} \quad \text{where} \quad \alpha := \frac{a}{A},\ N := \frac{B}{A}. \tag{ix}$$

From $(viii), (vi)$ and incompressibility we have

$$\lambda_r = \lambda^{-1}\Lambda^{-1}, \qquad \lambda_\theta = \lambda, \qquad \lambda_z = \Lambda. \tag{x}$$

From the constitutive relation $T_{ii} = \lambda_i \, \partial W^*/\partial \lambda_i - q$ (no sum on i) we have

$$T_{rr} = \lambda^{-1}\Lambda^{-1} W_1^* - q, \qquad T_{\theta\theta} = \lambda W_2^* - q, \qquad T_{zz} = \Lambda W_3^* - q, \tag{xi}$$

where $q(r)$ is the reaction stress due to incompressibility, and we are using the short-hand $W_i^* = \partial W^*/\partial\lambda_i$.

Finally, since all shear stress components vanish and normal stress components depend on r only and not θ and z, the equilibrium equations (3.95) reduce to

$$\frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0. \tag{xii}$$

The problem can now be solved in principle as follows. Substituting (xi) into (xii) gives a first-order ordinary differential equation for $q(r)$ that, together with the boundary condition $(iv)_1$ can be solved for $q(r)$. The remaining unknown parameters a and Λ can then be determined from the two algebraic equations resulting from substituting (xi) into the boundary conditions $(iv)_2$ and (v).

We can avoid determining $q(r)$ by proceeding as follows, noting first from (xi) that

$$T_{rr} - T_{\theta\theta} = \lambda^{-1}\Lambda^{-1} W_1^* - \lambda W_2^*, \qquad T_{zz} - T_{rr} = \Lambda W_3^* - \lambda^{-1}\Lambda^{-1} W_1^*, \tag{$xiii$}$$

where the right-hand sides of $(xiii)$ are viewed as functions of r: $W_i^*(\Lambda^{-1}/\lambda(r), \lambda(r), \Lambda)$. We now substitute $(xiii)_1$ into (xii), integrate from $r = a$ to $r = b$ and use the boundary conditions (iv) to get

$$\int_a^b \frac{\lambda W_2^* - \lambda^{-1}\Lambda^{-1} W_1^*}{r} \, dr \;=\; 0. \tag{xiv}$$

Likewise, substituting $(xiii)_1$ into (xii) and integrating from a to r and using the boundary condition $(iv)_1$ gives

$$T_{rr} = \int_a^r \frac{\lambda W_2^* - \lambda^{-1}\Lambda^{-1} W_1^*}{r} \, dr, \tag{xv}$$

(where r in the integrand here and below is a dummy variable of integration). Substituting this into $(xiii)_2$ leads to

$$T_{zz} = \Lambda W_3^* - \lambda^{-1}\Lambda^{-1} W_1^* + \int_a^r \frac{\lambda W_2^* - \lambda^{-1}\Lambda^{-1} W_1^*}{r} \, dr. \tag{xvi}$$

Finally we use (xvi) in the boundary condition (v) to get

$$\int_a^b r \left[\Lambda W_3^* - \lambda^{-1}\Lambda^{-1} W_1^* + \int_a^r \frac{\lambda W_2^* - \lambda^{-1}\Lambda^{-1} W_1^*}{r} \, dr \right] dr = 0. \tag{$xvii$}$$

The only unknowns in (xiv) and $(xvii)$ are the parameters a and Λ and so they can be found (provided these two equations have positive roots a and Λ).

The equations (xiv) and $(xviii)$ can be simplified as follows: recall that the principal stretches are $\lambda^{-1}\Lambda^{-1}, \lambda$ and Λ and so the restriction of W^* to the class of deformations at hand is

$$w(\lambda, \Lambda) := W^*(\lambda^{-1}\Lambda^{-1}, \lambda, \Lambda), \qquad \lambda > 0, \Lambda > 0. \tag{$xviii$}$$

Thus

$$\frac{\partial w}{\partial \lambda} = -\lambda^{-2}\Lambda^{-1}W_1^* + W_2^*, \qquad \frac{\partial w}{\partial \Lambda} = -\lambda^{-1}\Lambda^{-2}W_1^* + W_3^*. \tag{xix}$$

Moreover

$$\frac{d\lambda}{dr} \overset{(viii)}{=} \frac{d}{dr}\left(\frac{r}{R(r)}\right) = \frac{R - r\,dR/dr}{R^2} \overset{(vi)_1,(viii)}{=} \frac{1+\lambda/\lambda_r}{R} \overset{(x)_1}{=} \frac{1+\lambda^2\Lambda}{R}. \tag{xx}$$

From $(xix)_1$ and (xx) we see that the boundary condition (xiv) can be written as

$$\int_{a/B}^{b/A} \frac{1}{1+\lambda^2\Lambda}\frac{\partial w}{\partial \lambda}\,d\lambda = 0. \tag{xxi}$$

Likewise from $(xix)_2$ and (xx) we can write (xvi) as

$$T_{zz} = \Lambda\frac{\partial w}{\partial \Lambda} + T_{rr} \overset{(xvii)}{=} \Lambda\frac{\partial w}{\partial \Lambda} + \int_{a/B}^{r/R} \frac{1}{1+\lambda^2\Lambda}\frac{\partial w}{\partial \lambda}\,d\lambda, \tag{xxii}$$

and so the boundary condition $(xvii)$ can be expressed as

$$\int_a^b \left[r\Lambda\frac{\partial w}{\partial \Lambda}(\lambda(r),\Lambda) + r\int_{a/B}^{r/R} \frac{1}{1+\lambda^2\Lambda}\frac{\partial w}{\partial \lambda}\,d\lambda\right] dr = 0. \tag{xxiii}$$

This is a second algebraic equation involving a and Λ. One now seeks to determine a and Λ from $(xxii)$ and $(xxiii)$, after which the deformation is known from (i) and $(vii)_2$.

For a neo-Hookean material we have

$$w(\lambda,\Lambda) = \frac{\mu}{2}\Big[\lambda^{-2}\Lambda^{-2} + \lambda^2 + \Lambda^2 - 3\Big],$$

in which case the formulae (xxi) and $(xxiii)$ take the explicit forms

$$\frac{1}{N^2 - 1 + \Lambda\alpha^2} - \frac{N^2}{\Lambda\alpha^2} + \ln\left[\frac{N^2(N^2-1+\Lambda\alpha^2)}{\Lambda\alpha^2}\right] = 0, \tag{xxiv}$$

$$\Lambda^3 - \frac{(N^2+\Lambda\alpha^2)^2}{2\Lambda\alpha^2(N^2-1+\Lambda\alpha^2)} + 1 = 0, \tag{xxv}$$

where α and N were introduced in (ix). Chadwick [2] has shown that there exists precisely one pair of positive numbers (α, Λ) that satisfy $(xxiv), (xxv)$.

Motivated by the experiments described by Truesdell [3], suppose we take $B = 0.5''$, $B - A = 1/12''$ (and $L = 8''$) which gives $N = B/A = 1.2$. Following Chadwick [2] , we first solve $(xxiv)$ for $\Lambda\alpha^2$, next determine Λ from (xxv), and finally circle back to determine $\alpha = a/A$. This leads to $\Lambda = 1.007''$ and $a = 0.42''$.

(b) The deformation gradient tensor above can be written as

$$\mathbf{F} = -\lambda_r\,\mathbf{e}_r\otimes\mathbf{e}_r + \lambda_\theta\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta - \lambda_z\mathbf{e}_z\otimes\mathbf{e}_z,$$

where the principal stretches $\lambda_r = -r', \lambda_\theta = r/R, \lambda_z = \Lambda$ are all positive (as is necessary). Note that $\det\mathbf{F} > 0$ and so $\mathbf{F}$ is nonsingular. However $\mathbf{F}$ is not positive definite and so is not the stretch tensor. The right stretch tensor is the symmetric positive definite tensor

$$\mathbf{U} = \lambda_r\,\mathbf{e}_r\otimes\mathbf{e}_r + \lambda_\theta\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \lambda_z\mathbf{e}_z\otimes\mathbf{e}_z,$$

and so the rotation tensor $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ is the proper orthogonal tensor

$$\mathbf{R} = -\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta - \mathbf{e}_z \otimes \mathbf{e}_z.$$

Since $\mathbf{R}\mathbf{e}_\theta = \mathbf{e}_\theta$ this represents a rotation about $\mathbf{e}_\theta$. (Through what angle?)

Problem 5.16. *Harmonic material.* (Goriely et al.)

(a) Consider the cylindrically symmetric deformation of a hollow circular tube that has inner and outer radii A and B in a stress-free reference configuration. It is composed of an arbitrary unconstrained isotropic elastic material. The deformation is described by $r = r(R), \theta = \Theta, z = Z$ where (R, Θ, Z) and (r, θ, z) are the cylindrical polar coordinates of a particle in the reference and deformed configurations respectively. Show that $r(R)$ satisfies the ordinary differential equation

$$\frac{d}{dR}\left(R\frac{\partial W}{\partial \lambda_1}\right) - \frac{\partial W}{\partial \lambda_2} = 0 \qquad \text{for} \quad A \le R \le B, \tag{i}$$

where $W(\lambda_1, \lambda_2) \coloneqq W^*(\lambda_1, \lambda_2, 1)$, $\lambda_1 = r'(R)$ and $\lambda_2 = r(R)/R$.

(b) The so-called Harmonic strain energy function is a model for a homogeneous, unconstrained, isotropic elastic material. It is given by

$$W^*(\lambda_1, \lambda_2, \lambda_3) = F(j_1) + \xi(j_2 - 3) + \eta(j_3 - 1), \tag{ii}$$

where ξ and η are material constants and $F(j_1)$ is a constitutive function. Here

$$j_1 = \lambda_1 + \lambda_2 + \lambda_3, \qquad j_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \qquad j_3 = \lambda_1\lambda_2\lambda_3. \tag{iii}$$

Determine the restrictions on F, ξ and η needed to ensure that the strain energy and stress vanish in the reference configuration. Determine also the restrictions imposed by the Baker-Ericksen inequality.

(c) Show that in plane strain, this model can be reduced to

$$W(\lambda_1, \lambda_2) = f(i_1) - \alpha(i_2 - 1), \tag{iv}$$

where α is a material constant, $f(i_1)$ is a constitutive function, and

$$i_1 = \lambda_1 + \lambda_2, \qquad i_2 = \lambda_1\lambda_2. \tag{v}$$

Determine (by specializing your answers to part (b)) the restrictions on f and α needed to ensure that the strain energy and stress vanish in the reference configuration and that the Baker-Ericksen inequalities hold.

(d) Solve equation (i) for the Harmonic material.

(e) Suppose that the outer radius B is infinite and that $T_{rr}(R) \to 0$ as $R \to \infty$. Moreover, let $T_{rr}(A) = -p < 0$. Determine $T_{\theta\theta}(R)$ for $R \ge A$. At which point R in the body, and at what values of pressure p, does $T_{\theta\theta}(R)$ become unbounded.

Problem 5.17. *Limit point instability.* (Goriely et al.) The strain energy function for the simplified isotropic Fung model for soft tissue was given in (4.147) where $\mu > 0$ and $\beta > 0$ are material constants.

Consider a thin-walled hollow sphere composed of this material. The "limit-point instability" refers to the loss of monotonicity of the function $p(\lambda)$ where p is the internal pressure and λ the mean circumferential stretch.

Determine the critical value of the parameter β, say β_{cr}, above which the limit point instability disappears. Plot four curves of p versus λ corresponding to the four choices (a) $\beta = 0$, (b) $\beta = \beta_{cr}/2$, (c) $\beta = \beta_{cr}$ and (d) $\beta = 1.5\beta_{cr}$.

Determine a realistic value of β for soft tissue from the literature, and reach a conclusion about the existence of this instability in a Fung material (proposed as a model for aneurysm rupture).

For analyses of the limit point instability for other constitutive models, including less simplified Fung models, see Chapter 8 of *Cardiovascular Solid Mechanics* by Jay Humphrey, Springer, 2002.

Solution: From the analysis in Section 5.5 (or see Problem 5.1) we know that the pressure-stretch relation is

$$p = \frac{T}{R}\,\frac{w'(\lambda)}{\lambda^2} \qquad \text{where} \qquad w(\lambda) := W^*(\lambda^{-2}, \lambda, \lambda), \tag{i}$$

(and R and T are the mean radius and wall thickness in the reference configuration). On using the strain energy function (4.147) for the isotropic Fung material this specializes to

$$\frac{pR}{2\mu T} = p(\lambda) = \frac{\lambda^6 - 1}{\lambda^7}\,\exp\left[\beta(2\lambda^2 + \lambda^{-4} - 3)\right]. \tag{ii}$$

To find a local extremum of $p(\lambda)$ we set $dp/d\lambda = 0$ which leads to

$$\beta = h(\lambda) \tag{iii}$$

where

$$h(\lambda) := \frac{\lambda^4(\lambda^6 - 7)}{4(\lambda^6 - 1)^2}, \qquad \lambda > 0. \tag{iv}$$

If there is a $\lambda\,(> 1)$ at which $dp/d\lambda = 0$, then, given β, we must be able to solve the equation $\beta = h(\lambda)$ for a root $\lambda > 1$. One can verify that as λ increases, the function $h(\lambda)$ increases monotonically from $-\infty$ at $\lambda = 1^+$ until it reaches the value $h(\lambda_{\max})$ at $\lambda = \lambda_{\max}$ and then decreases monotonically to zero. Thus $\beta = h(\lambda)$ is solvable provided $\beta \le h(\lambda_{\max})$. By setting $dh/d\lambda = 0$ we find that $\lambda_{\max}$ is a root of

$$\lambda_{\max}^{12} - 23\lambda_{\max}^6 - 14 = 0 \qquad \Rightarrow \qquad \lambda_{\max} = \left[\frac{23 + \sqrt{585}}{2}\right]^{1/6} \doteq 1.693.$$

This gives $h(\lambda_{\max}) \doteq 0.0668$ and so the limit point instability exists provided

$$0 < \beta \le \beta_{cr} \qquad \text{where} \qquad \beta_{cr} \doteq 0.0668.$$

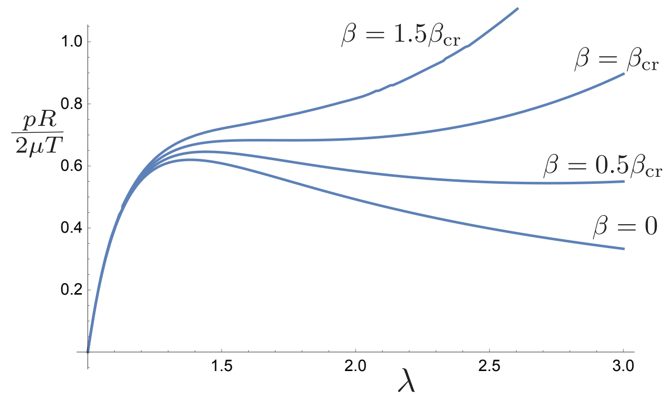

Figure 5.32: Pressure p versus mean circumferential stretch λ for the isotropic Fung model for different values of the constitutive parameter β. The limit point instability exists for $\beta \leq \beta_{cr}$.

Figure 5.32 shows plots of p versus λ according to (iii) for different values of β. According to Table 11.1 of Goriely (see reference on page 387) values of β in the range $3 < \beta < 20$ provides a reasonable model for soft biological tissues. Thus the isotropic Fung model in this range does not exhibit the limit point instability.

Problem 5.18. *Surface instability.* Reconsider the surface instability of a half-space as in Section 5.7 but now consider an arbitrary isotropic incompressible material. Determine conditions for the onset of a surface instability. Specialize your results to a Gent material.

Problem 5.19. *Surface instability.* Reconsider the surface instability of a half-space as in Section 5.7. Using the same basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as there, now consider the following cases:

(a) Suppose the homogeneous configuration involves an *equibiaxial stretch*, i.e.

$$\mathbf{F}_0 = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3, \qquad \text{with } \lambda_1 = \lambda_3.$$

(b) Suppose the homogeneous configuration involves a uniaxial stress, i.e.

$$\mathbf{F}_0 = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3, \qquad \mathbf{T}_0 = T \mathbf{e}_1 \otimes \mathbf{e}_1,$$

where $T_{22} = T_{33} = 0$.

Assume the material to be neo-Hookean. Determine the conditions (if any) under which a surface instability occurs. Consider only plane strain perturbations where the displacement from the homogeneously deformed configuration has the form $\mathbf{u}(\mathbf{x}) = u_1(x_1, x_2)\mathbf{e}_1 + u_2(x_1, x_2)\mathbf{e}_2$.

Problem 5.20. *Surface instability with surface tension.* Recall that the "Biot Problem" – the instability of a half-space under compression – did not involve a length-scale and so the critical value of stretch at instability was independent of the wave number of the sinusoidal surface undulation. Thus the analysis did not pick a particular wave number of the undulation at instability. In this problem you are to endow the free-surface with (constant) surface tension γ (force per unit length). The ratio $\ell_0 := \gamma/\mu$ then has the dimension of length. You are to determine the critical stretch λ_1 at instability. (Limit attention plane strain deformations and a neo-Hookean solid.)

In the simplest model of *surface tension*, the traction-free boundary condition $\mathbf{Tn} = \mathbf{o}$ at a free surface is replaced by

$$\mathbf{Tn} = -\gamma\kappa\mathbf{n}, \tag{i}$$

where κ is the (mean) curvature and $\mathbf{n}$ the unit outward normal vector, both associated with the deformed surface.

Problem 5.21. *Small deformation superposed on an arbitrary finite deformation.* A body is composed of an arbitrary unconstrained elastic material. It occupies a region $\mathcal{R}_{\mathrm{R}}$ in a homogeneous reference configuration and a region $\mathcal{R}$ in deformed configuration-1. A particle $\mathbf{z} \in \mathcal{R}_{\mathrm{R}}$ is taken to $\mathbf{x}_0(\mathbf{z}) \in \mathcal{R}$ by the equilibrium deformation-1:

$$\mathbf{x} = \mathbf{x}_0(\mathbf{z}).$$

The associated deformation gradient tensor is

$$\mathbf{F}_0 = \nabla_z \mathbf{x}_0(\mathbf{z}).$$

The associated Cauchy stress tensor field $\mathbf{T}_0(\mathbf{x})$ obeys the equilibrium equation

$$\operatorname{div}_x \mathbf{T}_0 = \mathbf{o}.$$

Deformation-2 takes $\mathbf{z} \to \widehat{\mathbf{y}}(\mathbf{z})$ where

$$\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{z}) = \mathbf{x}_0(\mathbf{z}) + \widehat{\mathbf{u}}(\mathbf{z}),$$

with associated deformation gradient tensor

$$\mathbf{F} = \nabla_z \widehat{\mathbf{y}}(\mathbf{z}) = \mathbf{F}_0 + \nabla_z \widehat{\mathbf{u}}(\mathbf{z}).$$

Introduce the following representation of the displacement field (from configuration-1 to configuration-2)

$$\mathbf{u}(\mathbf{x}) := \widehat{\mathbf{u}}(\mathbf{z}) \qquad \text{where} \qquad \mathbf{x} = \mathbf{x}_0(\mathbf{z}),$$

with associated gradient

$$\mathbf{H} = \nabla_x \mathbf{u}(\mathbf{x}).$$

Let deformation-2 be close to deformation-1 in the sense that

$$\varepsilon := |\mathbf{H}| \ll 1.$$

Let $\mathbf{T}$ be the Cauchy stress tensor associated with deformation-2 and it obeys the equilibrium equation

$$\mathrm{div}_y \mathbf{T} = \mathbf{o}.$$

Show that

(a) $\mathbf{T} = \mathbf{T}_0 + \widetilde{\mathbf{T}} + \mathrm{O}(\varepsilon^2)$ where

$$\widetilde{\mathbf{T}} = -\mathrm{tr}\,\mathbf{H}\ \mathbf{T}_0 + \mathbf{H}\mathbf{T}_0 + \mathbf{T}_0\mathbf{H}^T + \mathbb{C}\mathbf{E},$$

$$\mathbb{C} = \frac{4}{J_0}\mathbb{F}\mathbb{A}\mathbb{F}^T, \qquad \mathbb{F} := \mathbf{F}_0 \boxtimes \mathbf{F}_0, \qquad \mathbb{A}_{ijk\ell} := \frac{\partial^2 W(\mathbf{C}_0)}{\partial C_{ij}\partial C_{k\ell}}.$$

See Problem 1.64 for the definition and properties of the tensor product of two 2-tensors.

(b) $\mathrm{div}_x \widetilde{\mathbf{T}}(\mathbf{x}) = \mathbf{o}$.

(c) the 4-tensor $\mathbb{C}$ has the first and second minor symmetries and the major symmetry. See Problem 1.64 for the definition and some results concerning the minor and major symmetries of a 4-tensor.

Problem 5.22. *Universal deformation.* A deformation that can be maintained in equilibrium by the application of surface tractions only, and no body forces, *for arbitrary W* is said to be a *universal deformation.* Ericksen showed that for an unconstrained material the most general universal deformation is a homogenous one; see Problem 4.17. For incompressible materials however there are certain additional universal deformations (essentially because of the presence of the reaction pressure field $q(\mathbf{x})$). This problem is concerned with one of them.

When studying the inflation, extension and twisting of a tube of an incompressible *isotropic* material we encountered the following deformation:

$$r(R) = \sqrt{c + R^2/\Lambda}, \qquad \theta = \Theta + \alpha\Lambda Z, \qquad z = \Lambda Z,$$

where c, α and Λ are constants. Here (R, Θ, Z) and (r, θ, z) are cylindrical polar coordinates in the reference and deformed configurations respectively. Show that *any* incompressible elastic body in equilibrium can sustain this deformation purely by applying suitable tractions on its boundaries (with no body forces)[19] . This, therefore, is a universal deformation for an incompressible elastic material. In view of this, this is also a possible deformation field for an incompressible *anisotropic* material.

References:

1. A. Goriely, A. Erlich and C. Goodbrake, C5.1 Solid Mechanics: Online problem sheets, https://courses.maths.ox.ac.uk/node/36846/materials, Oxford University, 2018.

2. R.W. Ogden, Chapter 5 of *Non-Linear Elastic Deformations*, Chapter 3, Dover, 1997.

3. D. J. Steigmann, Chapters 7 and 8 of Finite Elasticity Theory, Oxford, 2017.

[19] Of course the tractions that have to be applied will depend on the material.

Chapter 6

Anisotropic Elastic Solids.

Our treatment of anisotropy in this chapter is limited to that arising from the presence of either one or two preferred directions, the former corresponding to transverse isotropy. For a more complete treatment of anisotropy, see, for example, Spencer [7, 8].

6.1 One family of fibers. Transversely isotropic material.

Consider a material with one preferred direction $\mathbf{m}_{\mathrm{R}}$ in the reference configuration. For example this may be due to the presence of a family of fibers in that direction. For simplicity, we will sometimes refer to a preferred direction as a "fiber direction" (even if there are no fibers). In general, the fibers will *not* be inextensible.

The constitutive response functions for the Piola stress and strain energy function for such a material will depend on both the deformation gradient tensor $\mathbf{F}$ and the fiber direction $\mathbf{m}_{\mathrm{R}}$:

$$\mathbf{S} = \widehat{\mathbf{S}}(\mathbf{F}, \mathbf{m}_{\mathrm{R}}), \qquad W = \widehat{W}(\mathbf{F}, \mathbf{m}_{\mathrm{R}}).$$

The elastic power identity $\mathbf{S} \cdot \dot{\mathbf{F}} = \dot{W}$ implies, as before, that

$$\widehat{\mathbf{S}}(\mathbf{F}, \mathbf{m}_{\mathrm{R}}) = \frac{\partial \widehat{W}}{\partial \mathbf{F}}(\mathbf{F}, \mathbf{m}_{\mathrm{R}}).$$

Material frame indifference requires that

$$\widehat{W}(\mathbf{F}, \mathbf{m}_{\mathrm{R}}) = \widehat{W}(\mathbf{QF}, \mathbf{m}_{\mathrm{R}}) \qquad \text{for all orthogonal } \mathbf{Q},$$

keeping in mind that $\mathbf{m}_{\rm R}$ is a direction in the reference configuration and so $\mathbf{Q}$ does not transform it. By the same argument as in Chapter 4, this holds if and only if the strain energy function depends on the deformation through the right Cauchy-Green tensor. Thus, there is a function $\overline{W}$ such that

$$\widehat{W}(\mathbf{F}, \mathbf{m}_{\rm R}) = \overline{W}(\mathbf{C}, \mathbf{m}_{\rm R}), \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F},$$

and the Cauchy stress is related to the deformation through

$$\mathbf{T} = \frac{2}{J}\mathbf{F}\frac{\partial \overline{W}}{\partial \mathbf{C}}\mathbf{F}^T. \tag{6.1}$$

Up to this point the anisotropy of the material has not had any (substantive) effect.

Now consider material symmetry. In the particular reference configuration at hand, the material with the single preferred direction $\mathbf{m}_{\rm R}$ is *transversely isotropic* with respect to that direction in the sense that the strain energy function is invariant under all rotations (of the reference configuration) about $\mathbf{m}_{\rm R}$ and under reflection in the plane perpendicular to $\mathbf{m}_{\rm R}$. (The latter implies it is invariant to replacing $\mathbf{m}_{\rm R}$ by $-\mathbf{m}_{\rm R}$). Thus the material symmetry group for such a material is

$$\mathcal{G} = \{\mathbf{Q} \ : \ \mathbf{Q}\mathbf{Q}^T = \mathbf{I}, \quad \mathbf{Q}\mathbf{m}_{\rm R} = \pm\mathbf{m}_{\rm R}\}, \tag{vi}$$

see Problem 1.11(b), and material symmetry tells us that

$$\overline{W}(\mathbf{C}, \mathbf{m}_{\rm R}) = \overline{W}(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \mathbf{Q}\mathbf{m}_{\rm R}) \qquad \text{for all } \mathbf{Q} \in \mathcal{G}, \tag{vii}$$

and all symmetric positive definite tensors $\mathbf{C}$. Note that in this step, $\mathbf{Q}$ acts on the reference configuration and so it does transform $\mathbf{m}_{\rm R}$. However in view of (vi) we can write this equivalently as

$$\overline{W}(\mathbf{C}, \mathbf{m}_{\rm R}) = \overline{W}(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \pm\mathbf{m}_{\rm R}) \qquad \text{for all } \mathbf{Q} \in \mathcal{G}. \tag{$viii$}$$

It is shown in Problem 6.1 that the following group of orthogonal tensors,

$$\mathcal{G}' = \{\mathbf{Q} \ : \ \mathbf{Q}\mathbf{Q}^T = \mathbf{I}, \quad \mathbf{Q}\mathbf{M}\mathbf{Q}^T = \mathbf{M}\} \qquad \text{where} \quad \mathbf{M} := \mathbf{m}_{\rm R} \otimes \mathbf{m}_{\rm R}, \tag{6.2}$$

is identical[1] to the group $\mathcal{G}$ defined by (vi). Thus we can replace $\mathcal{G}$ by $\mathcal{G}'$ in ($viii$). It is then shown in Problem 6.2 that the strain energy function $\overline{W}(\mathbf{C}, \mathbf{m}_{\rm R})$ obeys the invariance ($viii$) over the set $\mathcal{G}'$ if and only if the function $\check{W}(\mathbf{C}, \mathbf{M})$ obeys the invariance

$$\check{W}(\mathbf{C}, \mathbf{M}) = \check{W}(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \mathbf{Q}\mathbf{M}\mathbf{Q}^T) \qquad \text{for all orthogonal tensors } \mathbf{Q}, \tag{6.3}$$

[1]When $\mathbf{m}_R \to \mathbf{Q}\mathbf{m}_R$ note that $\mathbf{M} \to (\mathbf{Q}\mathbf{m}_R) \otimes (\mathbf{Q}\mathbf{m}_R) = \mathbf{Q}(\mathbf{m}_R \otimes \mathbf{m}_R)\mathbf{Q}^T = \mathbf{Q}\mathbf{M}\mathbf{Q}^T$.

where

$$\overline{W}(\mathbf{C}, \mathbf{m}_{\mathrm{R}}) = \breve{W}(\mathbf{C}, \mathbf{M}), \qquad \mathbf{M} \coloneqq \mathbf{m}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}}.$$

Note that (6.3) holds for *all* orthogonal $\mathbf{Q}$ not just those in $\mathcal{G}'$. Therefore the function $\breve{W}$ is jointly isotropic in both arguments.

The tensor $\mathbf{M} \coloneqq \mathbf{m}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}}$ is referred to as the **structural tensor** (for transverse isotropy). It characterizes the "internal structure" of the material in the reference configuration.

Finally, it is claimed in Problem 6.3 that a strain energy function that obeys (6.3) can be expressed as

$$\overline{W}(\mathbf{C}, \mathbf{M}) = \widetilde{W}(I_1, I_2, I_3, I_4, I_5), \tag{6.4}$$

where

$$\begin{gathered} I_1(\mathbf{C}) = \operatorname{tr} \mathbf{C}, \qquad I_2(\mathbf{C}) = \tfrac{1}{2}\Big[(\operatorname{tr} \mathbf{C})^2 - \operatorname{tr} \mathbf{C}^2\Big], \qquad I_3(\mathbf{C}) = \det \mathbf{C}, \\ I_4(\mathbf{C}, \mathbf{M}) = \mathbf{C} \cdot \mathbf{M}, \qquad I_5(\mathbf{C}, \mathbf{M}) = \mathbf{C}^2 \cdot \mathbf{M}; \end{gathered} \tag{6.5}$$

see Ericksen and Rivlin [1]. We would expect this list to also include $I_1(\mathbf{M}), I_2(\mathbf{M})$ and $I_3(\mathbf{M})$, but recall from Problem 1.19 that $I_1(\mathbf{M}) = 1$ and $I_2(\mathbf{M}) = I_3(\mathbf{M}) = 0$. Moreover it does not include $\mathbf{C} \cdot \mathbf{M}^2$ since it is readily seen that $\mathbf{C} \cdot \mathbf{M}^2 = \mathbf{C} \cdot \mathbf{M}$; this follows because $\mathbf{M}^n = \mathbf{M}$ for any positive integer n. It is sometimes convenient to express the invariants in terms of $\mathbf{m}_{\mathrm{R}}$ by substituting $\mathbf{M} = \mathbf{m}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}}$ into (6.5). This yields

$$\begin{gathered} I_1 = \operatorname{tr} \mathbf{C}, \qquad I_2 = \tfrac{1}{2}\Big[(\operatorname{tr} \mathbf{C})^2 - \operatorname{tr} \mathbf{C}^2\Big], \qquad I_3 = \det \mathbf{C}, \\ I_4 = \mathbf{C}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}}, \qquad I_5 = \mathbf{C}^2\mathbf{m}_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}}. \end{gathered} \tag{6.6}$$

Strictly, the scalar-valued functions I_4 and I_5 are not invariants in the usual sense of invariance (i.e. invariance over the set of *all* orthogonal tensors) though that term is often used. Other authors refer to them as *pseudo-invariants*. Observe that

$$I_4 = \mathbf{C}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}} = \mathbf{F}^T\mathbf{F}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}} = \mathbf{F}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{F}\mathbf{m}_{\mathrm{R}} = |\mathbf{F}\mathbf{m}_{\mathrm{R}}|^2, \tag{6.7}$$

and so I_4 denotes the (square of the) stretch in the fiber direction $\mathbf{m}_{\mathrm{R}}$. The fiber direction in the deformed configuration is

$$\mathbf{m} = \frac{\mathbf{F}\mathbf{m}_{\mathrm{R}}}{|\mathbf{F}\mathbf{m}_{\mathrm{R}}|} \overset{(6.7)}{=} \frac{\mathbf{F}\mathbf{m}_{\mathrm{R}}}{\sqrt{I_4}}. \tag{6.8}$$

Observe that:

$$I_5 = \mathbf{C}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{C}\mathbf{m}_{\mathrm{R}} = \mathbf{F}^T\mathbf{F}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{F}^T\mathbf{F}\mathbf{m}_{\mathrm{R}} \overset{(6.8)}{=} I_4\, \mathbf{F}^T\mathbf{m} \cdot \mathbf{F}^T\mathbf{m} = I_4\, \mathbf{F}\mathbf{F}^T\mathbf{m} \cdot \mathbf{m} = I_4\, \mathbf{B}\mathbf{m} \cdot \mathbf{m}$$

where $\mathbf{m}$ is the fiber direction in the deformed configuration.

From (6.1), (6.4) and the chain rule, the constitutive relation for the Cauchy stress tensor can be written as

$$\mathbf{T} = \frac{2}{J}\mathbf{F}\left[\sum_{i=1}^{5} \frac{\partial \widetilde{W}}{\partial I_i}\frac{\partial I_i}{\partial \mathbf{C}}\right]\mathbf{F}^T. \tag{6.9}$$

The terms $\partial I_i/\partial \mathbf{C}$ for $i = 1, 2, 3$ were calculated previously when we considered isotropic materials, while it is readily shown from $(6.5)_{4,5}$ and $(6.6)_{4,5}$ that

$$\frac{\partial I_4}{\partial \mathbf{C}} = \mathbf{M} = \mathbf{m}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}}, \qquad \frac{\partial I_5}{\partial \mathbf{C}} = \mathbf{CM} + \mathbf{MC} = \mathbf{C}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}} + \mathbf{m}_{\mathrm{R}} \otimes \mathbf{C}\mathbf{m}_{\mathrm{R}}. \tag{6.10}$$

Observe that the two tensors $\partial I_4/\partial \mathbf{C}$ and $\partial I_5/\partial \mathbf{C}$ are symmetric (as they must be since $I_4(\cdot, \mathbf{M})$ and $I_5(\cdot, \mathbf{M})$ are defined on the set of symmetric tensors). On substituting the expressions for $\partial I_i/\partial \mathbf{C}$ into (6.9) we get the following explicit form for the constitutive relation for $\mathbf{T}$:

$$\begin{aligned}\mathbf{T} \ &= 2JW_3\,\mathbf{I} \ + \ \frac{2}{J}\left[W_1 + I_1 W_2\right]\mathbf{B} \ - \ \frac{2}{J}W_2\mathbf{B}^2 + \\ &+\frac{2}{J}W_4(\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{F}\mathbf{m}_{\mathrm{R}}) \ + \ \frac{2}{J}W_5\Big[(\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{B}\mathbf{F}\mathbf{m}_{\mathrm{R}}) + (\mathbf{B}\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{F}\mathbf{m}_{\mathrm{R}})\Big]\ ,\end{aligned} \tag{6.11}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and we have used the notation

$$W_i = \frac{\partial \widetilde{W}}{\partial I_i}, \qquad i = 1, 2, \ldots, 5. \tag{6.12}$$

The Cauchy stress tensor given by (6.11) is automatically symmetric (as it must be since it stemmed from (6.9) that yields a symmetric stress tensor). Observe also that the principal directions of $\mathbf{T}$ and $\mathbf{B}$ *no longer coincide* in general. One can use (6.8) to express the constitutive relation in terms of the direction $\mathbf{m}$ of the (stretched) fiber in the deformed configuration.

If the material is *incompressible*, then

$$I_3 = J^2 = \det \mathbf{F} = 1$$

and the strain energy function has the form $W = \widetilde{W}(I_1, I_2, \cancel{I_3}, I_4, I_5)$. We must drop the term involving W_3 from the constitutive relation (6.11) and replace it with the reaction stress associated with the incompressibility constraint (i.e. a pressure $-q\,\mathbf{I}$). This leads to

$$\begin{aligned}\mathbf{T} \ &= -q\mathbf{I} + 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2) + 2W_4(\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{F}\mathbf{m}_{\mathrm{R}}) + \\ &+2W_5\Big[(\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{B}\mathbf{F}\mathbf{m}_{\mathrm{R}}) + (\mathbf{B}\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{F}\mathbf{m}_{\mathrm{R}})\Big]\ ,\end{aligned} \tag{6.13}$$

having set $J = 1$ throughout.

If the fibers are *inextensible*, then

$$I_4 = |\mathbf{F}\mathbf{m}_{\mathrm{R}}|^2 = 1$$

and the strain energy function has the form $W = \widetilde{W}(I_1, I_2, I_3, \cancel{I_4}, I_5)$. Now we must drop the term involving W_4 from the constitutive relation (6.11) and replace it with the reaction stress associated with the inextensibility constraint, i.e. a uniaxial stress $q\,\mathbf{m} \otimes \mathbf{m}$ in the direction $\mathbf{m}$ – the fiber direction in the deformed configuration (see Problem 4.22 (*a*)). This leads to

$$\begin{aligned} \mathbf{T} &= 2JW_3\,\mathbf{I} \;+\; \frac{2}{J}\left[W_1 + I_1 W_2\right]\mathbf{B} \;-\; \frac{2}{J}W_2\mathbf{B}^2 + \\ &+q\,\mathbf{m}\otimes\mathbf{m} \;+\; \frac{2}{J}W_5\left[(\mathbf{F}\mathbf{m}_{\mathrm{R}}\otimes\mathbf{B}\mathbf{F}\mathbf{m}_{\mathrm{R}}) + (\mathbf{B}\mathbf{F}\mathbf{m}_{\mathrm{R}}\otimes\mathbf{F}\mathbf{m}_{\mathrm{R}})\right], \end{aligned} \tag{6.14}$$

where $\mathbf{m} = \mathbf{F}\mathbf{m}_{\mathrm{R}}$.

6.1.1 Example: pure homogeneous stretch of a cube.

Consider a rectangular block with its edges parallel to the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The material is incompressible and transversely isotropic with respect to the direction

$$\mathbf{m}_{\mathrm{R}} = \cos\Theta\,\mathbf{e}_1 + \sin\Theta\,\mathbf{e}_2, \qquad 0 \le \Theta \le \pi/2. \tag{6.15}$$

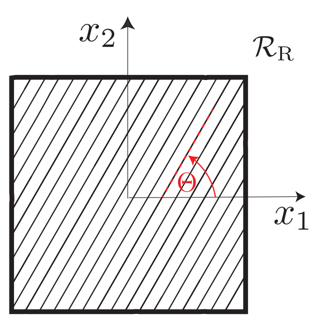

Figure 6.1: Rectangular block in reference configuration occupied by a material involving one family of fibers in the x_1, x_2-plane.

The body is subjected to the pure homogeneous deformation

$$\mathbf{y} = \mathbf{F}\mathbf{x} \qquad \text{where} \qquad \mathbf{F} = \lambda_1\mathbf{e}_1\otimes\mathbf{e}_1 + \lambda_2\mathbf{e}_2\otimes\mathbf{e}_2 + \lambda_3\mathbf{e}_3\otimes\mathbf{e}_3. \tag{6.16}$$

Since the material is incompressible

$$\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}. \tag{6.17}$$

The image of the fiber $\mathbf{m}_{\mathrm{R}}$ in the deformed configuration is

$$\mathbf{F}\mathbf{m}_{\mathrm{R}} \overset{(6.15),(6.16)}{=} \lambda_1 \cos\Theta\,\mathbf{e}_1 + \lambda_2 \sin\Theta\,\mathbf{e}_2. \tag{6.18}$$

Let

$$\mathbf{m} = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2, \tag{6.19}$$

denote the direction of a fiber in the deformed configuration. Then, since $\mathbf{F}\mathbf{m}_{\mathrm{R}}$ is parallel to $\mathbf{m}$, it follows from the two preceding equations that

$$\frac{\lambda_1\cos\Theta}{\cos\theta} = \frac{\lambda_2\sin\Theta}{\sin\theta} \qquad \Rightarrow \qquad \tan\theta = \frac{\lambda_2}{\lambda_1}\tan\Theta. \tag{6.20}$$

This gives the fiber orientation θ in the deformed configuration. In particular it tells us how θ varies with the deformation (unless, $\Theta = 0$ or $\pi/2$ in which case the angle θ in the deformed configuration does not depend on the deformation and remains at $\theta = \Theta$).

We now calculate the terms in the constitutive relation (6.13). From $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and (6.18) we find

$$\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{F}\mathbf{m}_{\mathrm{R}} = \lambda_1^2\cos^2\Theta\mathbf{e}_1\otimes\mathbf{e}_1 + \lambda_1\lambda_2\cos\Theta\sin\Theta(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1) + \lambda_2^2\sin^2\Theta\mathbf{e}_2\otimes\mathbf{e}_2,$$

$$\begin{aligned}\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{B}\mathbf{F}\mathbf{m}_{\mathrm{R}} + \mathbf{B}\mathbf{F}\mathbf{m}_{\mathrm{R}} \otimes \mathbf{F}\mathbf{m}_{\mathrm{R}} = {} & 2\lambda_1^4\cos^2\Theta\ \mathbf{e}_1\otimes\mathbf{e}_1 + 2\lambda_2^4\sin^2\Theta\ \mathbf{e}_2\otimes\mathbf{e}_2 + \\ & + \lambda_1\lambda_2(\lambda_1^2+\lambda_2^2)\cos\Theta\sin\Theta\ (\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1).\end{aligned}$$

The invariants (6.6) specialize for the deformation (6.16), (6.17) and fiber direction (6.15) to

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}, \qquad I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2\lambda_2^2, \tag{6.21}$$

$$I_4 = \lambda_1^2\cos^2\Theta + \lambda_2^2\sin^2\Theta, \qquad I_5 = \lambda_1^4\cos^2\Theta + \lambda_2^4\sin^2\Theta. \tag{6.22}$$

Note that I_4 and I_5 are *not* symmetric in λ_1, λ_2 in general. Consequently, in contrast to the isotropic case, if we replace the I's in the strain energy function W with the λ's using (6.21), (6.22), the resulting expression will *not* be invariant to a change $\lambda_1 \leftrightarrow \lambda_2$.

The constitutive relation (6.13) now gives

$$\left.\begin{aligned}
T_{11} &= -q + 2W_1\lambda_1^2 + 2W_2(I_1\lambda_1^2 - \lambda_1^4) + 2W_4\lambda_1^2\cos^2\Theta + 4W_5\lambda_1^4\cos^2\Theta,\\
T_{22} &= -q + 2W_1\lambda_2^2 + 2W_2(I_1\lambda_2^2 - \lambda_2^4) + 2W_4\lambda_2^2\sin^2\Theta + 4W_5\lambda_2^4\sin^2\Theta,\\
T_{33} &= -q + 2W_1\lambda_3^2 + 2W_2(I_1\lambda_3^2 - \lambda_3^4),\\
T_{12} &= 2[W_4 + W_5(\lambda_1^2+\lambda_2^2)]\lambda_1\lambda_2\sin\Theta\cos\Theta, \qquad T_{23} = T_{31} = 0.
\end{aligned}\right\} \tag{6.23}$$

Observe that in contrast to the isotropic case, the shear stress[2] $T_{12} \neq 0$. This shear stress is required in order to maintain the deformation (6.16). Thus the deformation (6.16) cannot be sustained (for example) by a state of uniaxial stress $\mathbf{T} = T_{11}\mathbf{e}_1 \otimes \mathbf{e}_1$. Note also that the directions $\mathbf{e}_1, \mathbf{e}_2$ are *not* principal directions for $\mathbf{T}$ though they are principal directions for $\mathbf{B}$. This too is a consequence of anisotropy.

As we have done repeatedly in the isotropic case, it is natural to introduce the restriction of the strain energy function W to the setting at hand by introducing the function

$$w(\lambda_1, \lambda_2, \Theta) := W(I_1, I_2, I_4, I_5), \tag{6.24}$$

where the invariants have the expressions in (6.21) and (6.22). As noted previously, $w(\lambda_1, \lambda_2, \Theta)$ $\neq w(\lambda_2, \lambda_1, \Theta)$ due to the anisotropy. Differentiating w with respect to its arguments and keeping (6.24), (6.21), (6.22) and (6.23) in mind shows that (Ogden [5])

$$T_{11} - T_{33} = \lambda_1 \frac{\partial w}{\partial \lambda_1}, \qquad T_{22} - T_{33} = \lambda_2 \frac{\partial w}{\partial \lambda_2}, \qquad T_{12} = \frac{\lambda_1 \lambda_2}{\lambda_2^2 - \lambda_1^2} \frac{\partial w}{\partial \Theta}, \tag{6.25}$$

where it should be kept in mind that T_{11} and T_{22} are not principal stresses.

To illustrate the response described by (6.23) consider the particular material

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{\mu\beta}{2}(I_4 - 1)^2, \qquad \mu > 0,\ \beta > 0, \tag{i}$$

and the special case of plane strain with vanishing normal stress $T_{22} = 0$:

$$\lambda_3 = 1, \qquad T_{22} = 0. \tag{ii}$$

Note that $T_{33} \neq 0$ and therefore this is not a state of uniaxial stress. Incompressibility gives $\lambda_2 = \lambda^{-1}$ where we have set $\lambda_1 = \lambda$, and (6.21) and (6.22) specialize to

$$I_1 = \lambda^2 + \lambda^{-2} + 1, \qquad I_4 = \lambda^2 \cos^2 \Theta + \lambda^{-2} \sin^2 \Theta. \tag{iii}$$

Setting $T_{22} = 0$ in $(6.23)_2$ allows one to solve for q and use the result to eliminate q from $(6.23)_1$. This leads to

$$\begin{aligned}
T_{11}/\mu \quad &= \lambda^2 - \lambda^{-2} + 2\beta\big[\lambda^4 \cos^4 \Theta - \lambda^{-4} \sin^4 \Theta - \lambda^2 \cos^2 \Theta + \lambda^{-2} \sin^2 \Theta\big], \\
T_{12}/(2\mu\beta) \quad &= \big[\lambda^2 \cos^2 \Theta + \lambda^{-2} \sin^2 \Theta - 1\big] \sin\Theta \cos\Theta = \lambda^{-2}(\lambda^2 - 1)(\lambda^2 - \tan^2 \Theta) \sin\Theta \cos^3 \Theta.
\end{aligned} \tag{iv}$$

[2] It does vanish in the special cases $\Theta = 0$ or $\Theta = \pi/2$, i.e. when the fibers are oriented in one of the principal directions for $\mathbf{B}$.

The stress component T_{33} needed to maintain the plane strain deformation is given by $(6.23)_3$. Figure 6.2(a) shows plots of the normal stress T_{11}/μ versus λ for three different values of the anisotropy parameter β; while Figure 6.2(b) shows plots of the shear stress $T_{12}/(2\mu\beta)$ versus λ for three different values of the fiber angle Θ.

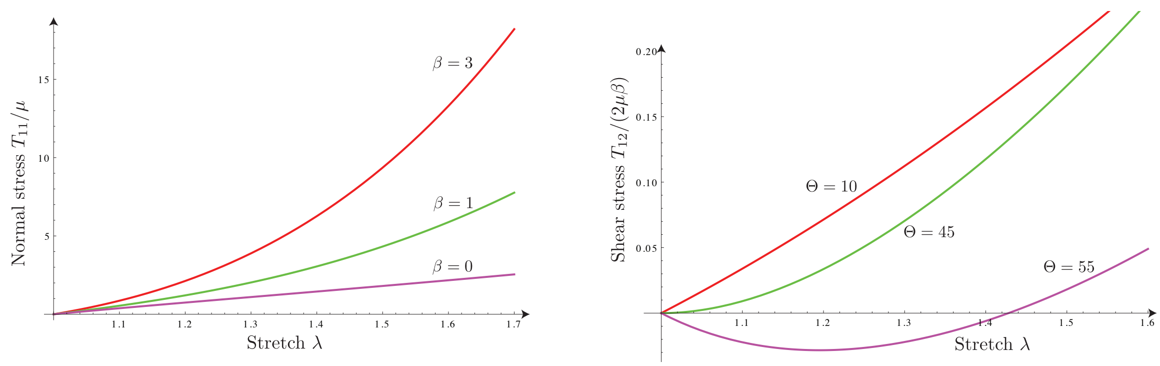

Figure 6.2: Variation of the normal and shear stress components T_{11} and T_{12} with the stretch $\lambda_1 = \lambda$ for the material described by (i) under conditions where $\lambda_3 = 1$ and $T_{22} = 0$.

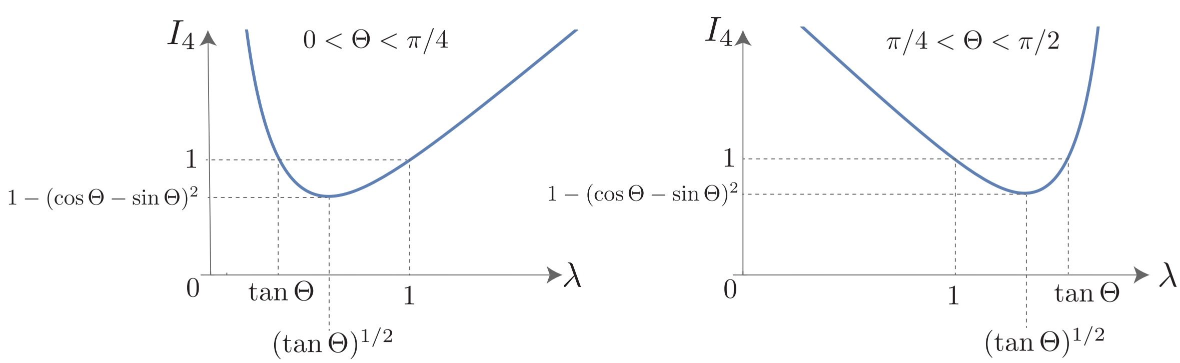

Figure 6.3: Schematic plots of the square of the fiber stretch (I_4) versus the imposed stretch (λ). The left and right figures correspond to the cases $0 < \Theta < \pi/4$ and $\pi/4 < \Theta < \pi/2$ respectively.

Equation $(iii)_2$ can be rewritten equivalently as

$$I_4 = 1 + (1 - \lambda^{-2})(\lambda^2 - \tan^2\Theta)\cos^2\Theta. \qquad (v)$$

Figure 6.3 shows plots of I_4 – the square of the fiber stretch – versus the imposed stretch $\lambda\,(= \lambda_1)$. Consider the left-hand figure corresponding to the case $0 < \Theta < \pi/4$ (in which event $\tan\Theta < 1$). When λ increases monotonically from unity, I_4 starts from the value 1 and increases monotonically, eventually tending to infinity as $\lambda \to \infty$. On the other hand when λ decreases monotonically from the value 1, the fiber stretch starts from the value 1 and initially decreases until it reaches a minimum at $\lambda = \sqrt{\tan\Theta}$. It increases monotonically thereafter, passing through the value 1 (again) when $\lambda = \tan\Theta$ and tending to infinity as $\lambda \to 0^+$; the fact that the fibers initially contract in this case (before subsequently elongating) can be intuitively seen by visualizing fibers that are almost horizontal initially. The right-hand figure corresponds to the case $\pi/4 < \Theta < \pi/2$ ($\tan\Theta > 1$) where a similar sort of behavior is seen, with $\lambda > 1$ and $\lambda < 1$ reversed. When $\Theta = \pi/4$, the fiber elongates for all values of λ.

The minimum value of I_4 in Figure 6.3 is $I_4 = 1 - (\cos\Theta - \sin\Theta)^2$. Note from (6.20) that the fiber angle in the deformed configuration is $\theta = \pi/2 - \Theta$ when $\lambda = \tan\Theta$.

Observe from $(iv)_2$ that the shear stress T_{12} vanishes at the two values of stretch $\lambda = 1$ and $\lambda = \tan\Theta$ at which the fiber stretch is unity.

6.2 Two families of fibers.

There are many examples of materials involving two families of fibers, biological tissue with collagen fibers being one. Let the fiber directions (in the reference configuration) be $\mathbf{m}_\mathrm{R}$ and $\mathbf{m}'_\mathrm{R}$. Then the strain energy function will depend on, see Spencer [7, 8], the invariants I_1, I_2, I_3, the invariants I_4 and I_5 associated with the first family of fibers, the analogous invariants I_6 and I_7 for the second family of fibers,

$$I_6 = \mathbf{C}\mathbf{m}'_\mathrm{R} \cdot \mathbf{m}'_\mathrm{R} \qquad I_7 = \mathbf{C}^2\mathbf{m}'_\mathrm{R} \cdot \mathbf{m}'_\mathrm{R}, \tag{6.26}$$

and the invariants I_8 and I_9 *that couple* $\mathbf{m}_\mathrm{R}$ and $\mathbf{m}'_\mathrm{R}$:

$$I_8 = \mathbf{C}\mathbf{m}'_\mathrm{R} \cdot \mathbf{m}_\mathrm{R}, \qquad I_9 = (\mathbf{m}_\mathrm{R} \cdot \mathbf{m}'_\mathrm{R})^2. \tag{6.27}$$

Thus the strain energy function has the form

$$W = \widetilde{W}(I_1, I_2, \ldots, I_8), \tag{6.28}$$

where we have omitted I_9 since it does not involve the deformation gradient[3]. We remark that the form in which I_8 has been written is *not* invariant[4] to the replacement of $\mathbf{m}_\mathrm{R}$ by $-\mathbf{m}_\mathrm{R}$

[3] In view of the forms of I_4 to I_8, one might expect the quantity $\mathbf{C}^2\mathbf{m}_\mathrm{R} \cdot \mathbf{m}'_\mathrm{R}$ to also appear in this list of invariants. In Problem 6.4 you are asked to show that this quantity can be expressed in terms of the other invariants.

[4] Or said differently, it has not been written as a function of $\mathbf{C}, \mathbf{M}$ and $\mathbf{M}'$.

(while keeping $\mathbf{m}'_{\mathrm{R}}$ fixed). This can be addressed by working with, for example, $(\mathbf{C}\mathbf{m}'_{\mathrm{R}}\cdot\mathbf{m}_{\mathrm{R}})^2$ or $(\mathbf{C}\mathbf{m}'_{\mathrm{R}}\cdot\mathbf{m}_{\mathrm{R}})(\mathbf{m}_{\mathrm{R}}\cdot\mathbf{m}'_{\mathrm{R}})$. Note also that, in contrast to I_4, I_5, I_6 and I_7, the invariant $I_8 \neq 1$ in the reference configuration.

From (6.1), (6.28) and the chain rule, together with (6.6), (6.26) and (6.27), the constitutive equation for $\mathbf{T}$ (assuming the material to be incompressible) reads

$$
\begin{aligned}
\mathbf{T} \; &= -q\mathbf{I} + 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2) + \\
&+2W_4\mathbf{F}\mathbf{m}_{\mathrm{R}}\otimes\mathbf{F}\mathbf{m}_{\mathrm{R}} + 2W_6\mathbf{F}\mathbf{m}'_{\mathrm{R}}\otimes\mathbf{F}\mathbf{m}'_{\mathrm{R}} + \\
&+2W_5(\mathbf{F}\mathbf{m}_{\mathrm{R}}\otimes\mathbf{B}\mathbf{F}\mathbf{m}_{\mathrm{R}} + \mathbf{B}\mathbf{F}\mathbf{m}_{\mathrm{R}}\otimes\mathbf{F}\mathbf{m}_{\mathrm{R}}) + 2W_7(\mathbf{F}\mathbf{m}'_{\mathrm{R}}\otimes\mathbf{B}\mathbf{F}\mathbf{m}'_{\mathrm{R}} + \mathbf{B}\mathbf{F}\mathbf{m}'_{\mathrm{R}}\otimes\mathbf{F}\mathbf{m}'_{\mathrm{R}}) + \\
&+W_8(\mathbf{F}\mathbf{m}_{\mathrm{R}}\otimes\mathbf{F}\mathbf{m}'_{\mathrm{R}} + \mathbf{F}\mathbf{m}'_{\mathrm{R}}\otimes\mathbf{F}\mathbf{m}_{\mathrm{R}})
\end{aligned}
\tag{6.29}
$$

where we have set $W_i = \partial\widetilde{W}/\partial I_i,\ i = 1,\dots 8$.

Keep in mind that $I_4 = \mathbf{C}\mathbf{m}_{\mathrm{R}}\cdot\mathbf{m}_{\mathrm{R}}$ and $I_6 = \mathbf{C}\mathbf{m}'_{\mathrm{R}}\cdot\mathbf{m}'_{\mathrm{R}}$ are corresponding quantities for the two families of fibers. Thus if the two families are *mechanically equivalent* then the energy should be unaffected by an exchange of I_4 and I_6. The same goes for I_5 and I_7. Thus for two mechanically equivalent families of fibers the strain energy function must have the property

$$
\widetilde{W}(I_1, I_2, I_4, I_5, I_6, I_7, I_8) = \widetilde{W}(I_1, I_2, I_6, I_5, I_4, I_7, I_8) = \widetilde{W}(I_1, I_2, I_4, I_7, I_6, I_5, I_8). \tag{6.30}
$$

There are many strain energy functions that have been proposed in the literature for modeling soft biological tissues. One example is

$$
\widetilde{W}(I_1, I_4, I_6) = \frac{\mu_1}{2}(I_1 - 3) + \frac{1}{2}\frac{\mu_4}{k_4}\Big[\exp[k_4(I_4-1)^2] - 1\Big] + \frac{1}{2}\frac{\mu_6}{k_6}\Big[\exp[k_6(I_6-1)^2] - 1\Big], \tag{6.31}
$$

where $\mu_1, \mu_4, \mu_6, k_1, k_4, k_6$ are material constants, see Holzapfel et. al. [4]. If the two families of fibers are mechanically equivalent, one would take $\mu_4 = \mu_6$ and $k_4 = k_6$. Observe that (6.31) has the neo-Hookean form for the I_1 term while the terms involving I_4, I_6 are of the Fung form (Section 4.7). If $I_4 - 1$ and $I_6 - 1$ are small, this can be replaced by

$$
\widetilde{W} = \frac{\mu_1}{2}(I_1 - 3) + \frac{\mu_4}{2}(I_4 - 1)^2 + \frac{\mu_6}{2}(I_6 - 1)^2; \tag{6.32}
$$

the special case of this described by (6.51) below is known as the "standard fiber reinforcing model".

6.2.1 Example: pure homogeneous stretch of a cube.

Consider a rectangular block with its edges parallel to the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The material is incompressible and involves two families of fibers that, in the reference configuration, are

$$\mathbf{m}_{\mathrm{R}} = \cos\Theta\,\mathbf{e}_1 + \sin\Theta\,\mathbf{e}_2, \qquad \mathbf{m}'_{\mathrm{R}} = \cos\Theta\,\mathbf{e}_1 - \sin\Theta\,\mathbf{e}_2; \tag{6.33}$$

see Figure 6.4. This material is *orthotropic* with the symmetry planes coinciding with the three coordinate planes.

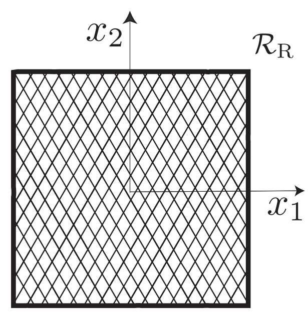

Figure 6.4: Region occupied (in a reference configuration) by an incompressible rectangular block with two families of fibers in the x_1, x_2-plane.

The body is subjected to a homogeneous deformation

$$\mathbf{y} = \mathbf{F}\mathbf{x} \qquad \text{where} \qquad \mathbf{F} = \lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3\mathbf{e}_3 \otimes \mathbf{e}_3. \tag{6.34}$$

Since the material is incompressible,

$$\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}. \tag{6.35}$$

Let the fiber directions in the deformed configuration be denoted by

$$\mathbf{m} = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2, \qquad \mathbf{m}' = \cos\theta\,\mathbf{e}_1 - \sin\theta\,\mathbf{e}_2. \tag{6.36}$$

Since $\mathbf{F}\mathbf{m}_{\mathrm{R}} = \lambda_1 \cos\Theta\,\mathbf{e}_1 + \lambda_2 \sin\Theta\,\mathbf{e}_2$ and $\mathbf{F}\mathbf{m}'_{\mathrm{R}} = \lambda_1 \cos\Theta\,\mathbf{e}_1 - \lambda_2 \sin\Theta\,\mathbf{e}_2$ it is readily shown that the fiber angle in the deformed configuration is given by

$$\tan\theta = \frac{\lambda_2}{\lambda_1}\,\tan\Theta; \tag{6.37}$$

see (6.20).

The invariants specialize, for the deformation (6.34), (6.35) and fiber directions (6.33), to

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}, \qquad I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2\lambda_2^2, \tag{6.38}$$

$$I_4 = I_6 = \lambda_1^2 \cos^2\Theta + \lambda_2^2 \sin^2\Theta, \tag{6.39}$$

$$I_5 = I_7 = \lambda_1^4 \cos^2\Theta + \lambda_2^4 \sin^2\Theta, \tag{6.40}$$

$$I_8 = \lambda_1^2 \cos^2\Theta - \lambda_2^2 \sin^2\Theta. \tag{6.41}$$

The constitutive relation (6.29) now gives

$$T_{11} = -q + 2W_1\lambda_1^2 + 2W_2(I_1\lambda_1^2 - \lambda_1^4) + 2(W_4 + W_6 + W_8)\lambda_1^2\cos^2\Theta + 4(W_5 + W_7)\lambda_1^4\cos^2\Theta, \tag{6.42}$$

$$T_{22} = -q + 2W_1\lambda_2^2 + 2W_2(I_1\lambda_2^2 - \lambda_2^4) + 2(W_4 + W_6 - W_8)\lambda_2^2\sin^2\Theta + 4(W_5 + W_7)\lambda_2^4\sin^2\Theta, \tag{6.43}$$

$$T_{33} = -q + 2W_1\lambda_3^2 + 2W_2(I_1\lambda_3^2 - \lambda_3^4). \tag{6.44}$$

$$T_{12} = 2[W_4 - W_6 + (W_5 - W_7)(\lambda_1^2 + \lambda_2^2)]\lambda_1\lambda_2 \sin\Theta\cos\Theta, \qquad T_{23} = T_{31} = 0, \tag{6.45}$$

Observe from (6.45) that the shear stress $T_{12} \neq 0$ in general and so the principal directions of $\mathbf{T}$ do not coincide with those of $\mathbf{B}$. However note from (6.39) and (6.40) that $I_4 = I_6$ and $I_5 = I_7$. Therefore if the two fiber families are mechanically equivalent, i.e. if (6.30) holds, then $W_4 = W_6$ and $W_5 = W_7$. In this case (6.45) gives $T_{12} = 0$ (as one would expect).

Again, it is natural to introduce the restriction of the strain energy function W to the setting at hand by introducing the function

$$w(\lambda_1, \lambda_2, \Theta) = W(I_1, I_2, I_4, I_5, I_6, I_7, I_8), \tag{6.46}$$

where the invariants are expressed in terms of $\lambda_1, \lambda_2, \Theta$ by (6.38), (6.39), (6.40) and (6.41). Note that $w(\lambda_1, \lambda_2, \Theta) \neq w(\lambda_2, \lambda_1, \Theta)$ due to anisotropy. Differentiating w with respect to λ_1 and λ_2 shows that (Ogden [5])

$$T_{11} - T_{33} = \lambda_1 \frac{\partial w}{\partial \lambda_1}, \qquad T_{22} - T_{33} = \lambda_2 \frac{\partial w}{\partial \lambda_2}, \tag{6.47}$$

keeping in mind that T_{11} and T_{22} are not principal stresses. The direct analog of $(6.25)_3$ does not hold.

Remark: In contrast to Problem 2.3, the fibers here are not constrained to being inextensible. In Section 6.2.2 we shall consider the inextensible case.

Remark: In certain materials one may wish to allow fibers to elongate but not to shorten. Since the fiber stretch is $|\mathbf{Fm}_{\rm R}| = |\mathbf{Fm}'_{\rm R}| = [\lambda_1^2\cos^2\Theta + \lambda_2^2\sin^2\Theta]^{1/2}$, in such a situation one must constrain the stretches to obey

$$\lambda_1^2\cos^2\Theta + \lambda_2^2\sin^2\Theta \geq 1. \tag{6.48}$$

Suppose now that the fiber families are mechanically equivalent and therefore that the shear stress T_{12} vanishes automatically. Suppose further that the boundary conditions on the block lead to a state of *uniaxial stress*[5]: $\mathbf{T} = T_{11}\,\mathbf{e}_1 \otimes \mathbf{e}_1$. Setting $T_{22} = T_{33} = 0$ in (6.47) gives

$$T_{11} = \lambda_1 \frac{\partial w}{\partial \lambda_1}, \qquad \frac{\partial w}{\partial \lambda_2} = 0. \tag{6.49}$$

The second of these is an equation involving the two stretches λ_1 and λ_2 (and Θ). If it can be solved (in principle) for λ_2, we would have a relation $\lambda_2 = \lambda_2(\lambda_1, \Theta)$ between the axial stretch λ_1 and the transverse stretch λ_2. (The third principal stretch is $\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}$.) Substituting this back into the first equation in (6.49) gives the following stress-stretch relation between T_{11} and λ_1

$$T_{11} = T_{11}(\lambda_1) = \lambda_1 \frac{\partial w}{\partial \lambda_1}(\lambda_1, \lambda_2, \Theta)\bigg|_{\lambda_2 = \lambda_2(\lambda_1, \Theta)}. \tag{6.50}$$

To illustrate the response in uniaxial stress[6] consider the so-called "standard fiber reinforcing model",

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{\mu\beta}{2}\Big[(I_4 - 1)^2 + (I_6 - 1)^2\Big], \qquad \mu > 0, \beta > 0. \tag{6.51}$$

This is the special case of (6.32) with

$$\mu_1 = \mu, \qquad \mu_4 = \mu_6 = \beta\mu. \tag{6.52}$$

Since this strain energy function obeys (6.30) the two fiber families are mechanically equivalent. Observe that the parameter $\beta > 0$ effectively characterizes the stiffness of the fibers (due to both the actual fiber stiffness and the concentration of fibers). Large β corresponds to stiff fibers.

The relation between the transverse stretch λ_2 and the longitudinal stretch λ_1 as given by $(6.49)_2$ specializes for the material (6.51) to the following cubic equation for λ_2^2:

$$4\beta\sin^4\Theta\,\lambda_2^6 + \Big[1 + 4\beta(\lambda_1^2\cos^2\Theta - 1)\sin^2\Theta\Big]\,\lambda_2^4 - \lambda_1^{-2} = 0. \tag{6.53}$$

[5]Note that the deformation (6.34) would not be compatible with a state of *uniaxial stress* in the $\mathbf{e}_1$-direction if $T_{12} \neq 0$.

[6]Based on Chapter 11 of Goriely [2] .

In principle, one solves this for λ_2 to get

$$\lambda_2 = \lambda_2(\lambda_1, \Theta). \tag{6.54}$$

Suppose we want to calculate the Poisson's ratio at infinitesimal deformations that measures the contraction in the x_2-direction with respect to the stretch in the x_1-direction. This is given by $-d\lambda_2/d\lambda_1$ evaluated at the undeformed configuration $\lambda_2 = 1$. (Why?) Differentiating (6.53) with respect to λ_1 gives

$$24\beta \sin^4\Theta\, \lambda_2^5 \frac{d\lambda_2}{d\lambda_1} + \Big[8\beta\lambda_1 \cos^2\Theta \sin^2\Theta\Big]\, \lambda_2^4 + \Big[1 + 4\beta(\lambda_1^2 \cos^2\Theta - 1)\sin^2\Theta\Big]\, 4\lambda_2^3 \frac{d\lambda_2}{d\lambda_1} + 2\lambda_1^{-3} = 0.$$

Solving this for $d\lambda_2/d\lambda_1$ and evaluating the result at $\lambda_1 = \lambda_2 = 1$ leads to

$$\left.\frac{d\lambda_2}{d\lambda_1}\right|_{\lambda_1=1} = -\frac{1 + 4\beta\cos^2\Theta\sin^2\Theta}{2 + 4\beta\sin^4\Theta}. \tag{6.55}$$

The (negative) of this gives the particular Poisson's ratio we sought as a function of the fiber angle Θ. Observe that if $\beta = 0$, corresponding to a neo-Hookean material, this reduces to the classical value 1/2. At the other extreme, in the limit $\beta \to \infty$, this reduces to the value found previously in Problem 2.3 for rigid fibers. Observe that for *all values* of the anisotropy parameter β, this Poisson ratio has the value 1/2 at the particular fiber angle Θ_* given by[7]

$$\tan\Theta_* = \sqrt{2} \qquad \Theta_* \approx 54.74^o. \tag{6.56}$$

Since the material is anisotropic, (6.55) is *not* the Poisson's ratio that measures the contraction in the x_3-direction with respect to the x_1-direction. To determine that, we differentiate $\lambda_3 = \lambda_1^{-1}\,\lambda_2^{-1}(\lambda_1)$ with respect to λ_1 and evaluate the result at $\lambda_1 = 1$. This leads to

$$\left.\frac{d\lambda_3}{d\lambda_1}\right|_{\lambda_1=1} = -\frac{1 - 4\beta\cos^2\Theta\sin^2\Theta + 4\beta\sin^4\Theta}{2 + 4\beta\sin^4\Theta}. \tag{6.57}$$

Again, observe that if $\beta = 0$, corresponding to a neo-Hookean material, this reduces to the classical value 1/2. At the other extreme, in the limit $\beta \to \infty$, it reduces to the value we found in Problem 2.3 for rigid fibers. Again, for *all values* of the anisotropy parameter β, this Poisson ratio also has the value 1/2 at the particular fiber angle Θ_* given by (6.56).

The relation between the stress T_{11} and stretch λ_1 is now found from (6.50), (6.32), (6.46) and (6.52) :

$$T_{11} = T_{11}(\lambda_1) = \mu\Big[\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2} + 4\beta(\lambda_1^2\cos^2\Theta + \lambda_2^2\sin^2\Theta - 1)\lambda_1^2\cos^2\Theta\Big] \tag{6.58}$$

[7] This angle appears in various other contexts and is referred to as the "magic angle".

with $\lambda_2 = \lambda_2(\lambda_1, \Theta)$ given by (6.54). The graph of $\widehat{T}_{11}(\lambda_1)$ versus λ_1 describes the stress-stretch behavior in uniaxial stress.

To find the *effective Young's modulus* at infinitesimal deformations (for stress in the particular direction under consideration) we differentiate (6.58) with respect to λ_1 and evaluate the result at $\lambda_1 = \lambda_2 = 1$. This yields

$$E_{\text{eff}}(\Theta) = \left.\frac{dT_{11}}{d\lambda_1}\right|_{\lambda_1=1} = 4\mu + 8\beta\mu\cos^4\Theta + 2\mu(1 + 4\beta\sin^2\Theta\cos^2\Theta)\left.\frac{d\lambda_2}{d\lambda_1}\right|_{\lambda_1=1},$$

which upon using (6.55) simplifies to

$$E_{\text{eff}}(\Theta) = \left.\frac{dT_{11}}{\partial\lambda_1}\right|_{\lambda_1=1} = \mu\,\frac{4[3 + 5\beta + 3\beta\cos 4\Theta]}{4 + 3\beta - 4\beta\cos 2\Theta + \beta\cos 4\Theta}. \tag{6.59}$$

This gives the effective Young's modulus of the material (in the particular direction under consideration) as a function of the fiber angle Θ. If $\beta = 0$, corresponding to a neo-Hookean material, (6.59) yields

$$E_{\text{eff}}(\Theta) = 3\mu, \tag{6.60}$$

which coincides with the value of Young's modulus we found previously; see discussion below in (4.92). At the other extreme when $\beta \to \infty$, corresponding to rigid fibers, the limiting value of $E_{\text{eff}}(\Theta)$ agrees with what we will find in Section 6.2.2. One can readily verify that the effective Young's modulus (6.59) is *independent of the anisotropy parameter* β at the particular fiber angle given by (6.56) and has the value 3μ.

It is interesting to examine the variation of the effective Young's modulus with the fiber angle. It is readily found that the maximum value of $E_{\text{eff}}(\Theta)$ (as a function of the fiber angle Θ) occurs at $\Theta = 0$ corresponding to the case when the fibers are parallel to the stressing direction. Its value is

$$\left.E_{\text{eff}}\right|_{\max} = E_{\text{eff}}(0) = \mu(3 + 8\beta). \tag{6.61}$$

When the fibers are perpendicular to the stressing direction, $\Theta = \pi/2$, one finds

$$E_{\text{eff}}(\pi/2) = \mu\,\frac{3 + 8\beta}{1 + 2\beta}. \tag{6.62}$$

This, however, turns out not to be the minimum value of $E_{\text{eff}}(\Theta)$. It is not difficult to show that the minimum value occurs when the fiber angle has the value Θ_* given by (6.56) and that this value is

$$\left.E_{\text{eff}}\right|_{\min} = E_{\text{eff}}(\Theta_*) = 3\mu. \tag{6.63}$$

Figure (6.5) shows the graph of $E_{\text{eff}}(\Theta)$ versus Θ where the figure has been drawn for $\beta = 1$.

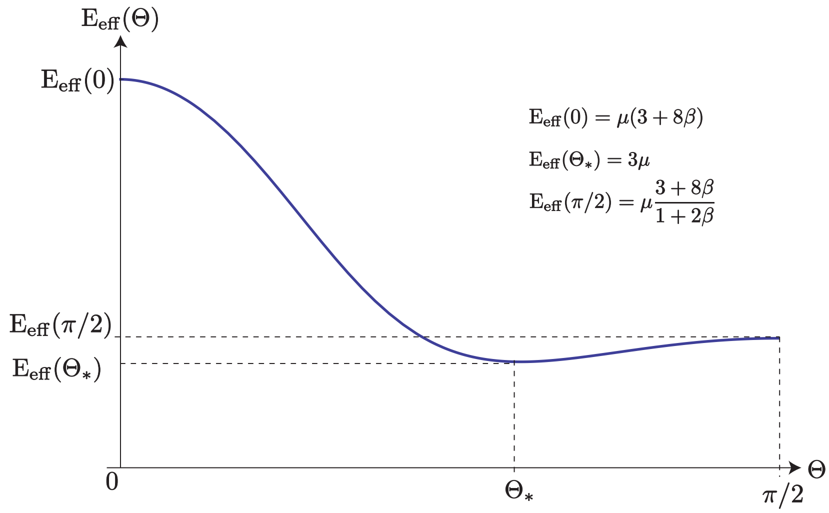

Figure 6.5: Effective Young's modulus as a function of fiber angle. Figure drawn for $\beta = 1$.

6.2.2 Inextensible fibers.

Consider again the problem studied in Section 6.2.1 but now assume the fibers to be inextensible. The kinematics of this problem was analyzed previously in Problem 2.3.

The constitutive relation for a material with two families of fibers is given by (6.29). If the fibers are inextensible, then $I_4 = |\mathbf{Fm}_{\rm R}|^2 = 1$ and $I_6 = |\mathbf{Fm}'_{\rm R}|^2 = 1$ and so the strain energy function has the form $W = \widetilde{W}(I_1, I_2, I_5, I_7, I_8)$ having also assumed the material to be incompressible. The terms involving W_3, W_4 and W_6 in the constitutive relation must be omitted, and the reaction stresses arising due to the constraints must be included. We know from Problem 4.22(a) that the reaction stress associated with inextensibility is a uniaxial stress in the direction of the deformed fibers (and a hydrostatic stress due to incompressibility). Therefore (6.29) is replace by

$$\begin{aligned}\mathbf{T} &= -q\mathbf{I} + q_4\mathbf{Fm}_{\rm R}\otimes\mathbf{Fm}_{\rm R} + q_6\mathbf{Fm}'_{\rm R}\otimes\mathbf{Fm}'_{\rm R} + 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2) + \\ &\quad + 2W_5(\mathbf{Fm}_{\rm R}\otimes\mathbf{BFm}_{\rm R} + \mathbf{BFm}_{\rm R}\otimes\mathbf{Fm}_{\rm R}) + 2W_7(\mathbf{Fm}'_{\rm R}\otimes\mathbf{BFm}'_{\rm R} + \mathbf{BFm}'_{\rm R}\otimes\mathbf{Fm}'_{\rm R}) + \\ &\quad + W_8(\mathbf{Fm}_{\rm R}\otimes\mathbf{Fm}'_{\rm R} + \mathbf{Fm}'_{\rm R}\otimes\mathbf{Fm}_{\rm R})\end{aligned} \tag{6.64}$$

where q is due to incompressibility, q_4 is due to inextensibility of the $\mathbf{m}_{\rm R}$-fibers and q_6 is due to inextensibility of the $\mathbf{m}'_{\rm R}$-fibers.

For illustrative purposes consider the strain energy function (6.31), but now omit the terms involving I_4 and I_6:

$$W = \frac{\mu}{2}(I_1 - 3), \qquad \mu > 0. \tag{6.65}$$

The constitutive relation (6.64) for the Cauchy stress now simplifies to

$$\mathbf{T} = -q\mathbf{I} + \mu\mathbf{B} + q_4\mathbf{Fm}_{\mathrm{R}} \otimes \mathbf{Fm}_{\mathrm{R}} + q_6\mathbf{Fm}'_{\mathrm{R}} \otimes \mathbf{Fm}'_{\mathrm{R}}. \tag{6.66}$$

As in Section 6.2.1, consider a rectangular block of the material with its edges parallel to the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The material is incompressible and involves two families of inextensible fibers that, in the reference configuration are oriented as in (6.33); see Figure 6.4.

The body is subjected to a homogeneous deformation $\mathbf{y} = \mathbf{Fx}$ where $\mathbf{F}$ is given by (6.34). It follows from (6.33) and (6.34) that

$$\mathbf{Fm}_{\mathrm{R}} = \lambda_1 \cos\Theta\, \mathbf{e}_1 + \lambda_2 \sin\Theta\, \mathbf{e}_2, \qquad \mathbf{Fm}'_{\mathrm{R}} = \lambda_1 \cos\Theta\, \mathbf{e}_1 - \lambda_2 \sin\Theta\, \mathbf{e}_2, \tag{6.67}$$

and so inextensibility requires

$$I_4 = I_6 = |\mathbf{Fm}_{\mathrm{R}}| = |\mathbf{Fm}'_{\mathrm{R}}| = \lambda_1^2 \cos^2\Theta + \lambda_2^2 \sin^2\Theta = 1. \tag{6.68}$$

Since the material is incompressible,

$$\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}. \tag{6.69}$$

Thus we have *two* constraints on the principal stretches. The kinematics of this problem was analyzed in detail in Problem 2.3. In particular, we found that the stretch λ_1 is restricted to the range $0 < \lambda_1 < 1/\cos\Theta$ and solving (6.68) and (6.69) for λ_2 and λ_3 in terms of λ_1 led to

$$\lambda_2 = \frac{(1 - \lambda_1^2\cos^2\Theta)^{1/2}}{\sin\Theta}, \qquad \lambda_3 = \frac{\sin\Theta}{\lambda_1(1 - \lambda_1^2\cos^2\Theta)^{1/2}}, \qquad 0 < \lambda_1 < 1/\cos\Theta. \tag{6.70}$$

Graphs of λ_2 and λ_3 versus λ_1 for $1 < \lambda_1 < 1/\cos\Theta$ are shown in Figure 6.6. Observe that λ_3 is not a monotonic function of λ_1. The slopes of these curves at $\lambda_1 = 1$ are the negatives of the Poisson's ratios (which, for the extensible case, we found previously).

We next turn to the relation between T_{11} and λ_1. It follows from (6.34) and (6.67) that

$$\mathbf{B} = \mathbf{FF}^T = \lambda_1^2\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2^2\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3^2\mathbf{e}_3 \otimes \mathbf{e}_3, \tag{6.71}$$

$$\mathbf{Fm}_{\mathrm{R}} \otimes \mathbf{Fm}_{\mathrm{R}} = \lambda_1^2\cos^2\Theta\, \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_1\lambda_2 \sin\Theta\cos\Theta(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \lambda_2^2 \sin^2\Theta\, \mathbf{e}_2 \otimes \mathbf{e}_2, \tag{6.72}$$

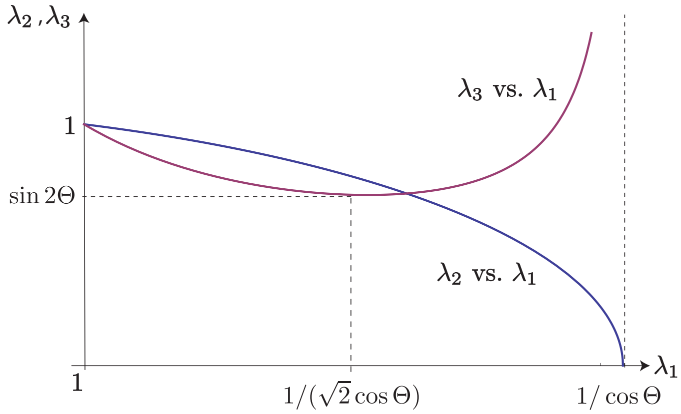

Figure 6.6: Transverse stretches λ_2 and λ_3 versus longitudinal stretch λ_1 in uniaxial stress according to (6.70) for the block shown in Figure 6.4. Figure has been drawn for $\Theta = 3\pi/8$.

$$\mathbf{Fm}'_{\mathrm{R}}\otimes\mathbf{Fm}'_{\mathrm{R}} = \lambda_1^2\cos^2\Theta\,\mathbf{e}_1\otimes\mathbf{e}_1 - \lambda_1\lambda_2\sin\Theta\cos\Theta(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1) + \lambda_2^2\sin^2\Theta\,\mathbf{e}_2\otimes\mathbf{e}_2, \quad (6.73)$$

which when substituted into the constitutive relation (6.66) gives

$$T_{11} = -q + \mu\lambda_1^2 + (q_4+q_6)\lambda_1^2\cos^2\Theta, \tag{6.74}$$

$$T_{22} = -q + \mu\lambda_2^2 + (q_4+q_6)\lambda_2^2\sin^2\Theta, \tag{6.75}$$

$$T_{33} = -q + \mu\lambda_3^2, \tag{6.76}$$

$$T_{12} = [q_4 - q_6]\lambda_1\lambda_2\sin\Theta\cos\Theta, \qquad T_{23} = T_{31} = 0. \tag{6.77}$$

Suppose now that the boundary conditions on the body lead to a state of uniaxial stress:

$$\mathbf{T} = T\,\mathbf{e}_1\otimes\mathbf{e}_1, \tag{6.78}$$

where T is the component of Cauchy stress in the loading direction. On setting $T_{22} = 0, T_{33} = 0$ and $T_{12} = 0$ in (6.75), (6.76), (6.77) we find

$$q_4 = q_6 = \frac{\mu(\lambda_3^2 - \lambda_2^2)}{2\lambda_2^2\sin^2\Theta}, \qquad q = \mu\lambda_3^2. \tag{6.79}$$

Substituting (6.79) into (6.74) gives the stress T:

$$T/\mu = (\lambda_1^2 - \lambda_3^2) + \left[\frac{\lambda_3^2}{\lambda_2^2} - 1\right]\frac{\lambda_1^2\cos^2\Theta}{\sin^2\Theta} \tag{6.80}$$

Finally we substitute for λ_2 and λ_3 from (6.70) to get

$$T/\mu = \frac{\sin^2\Theta - \cos^2\Theta}{\sin^2\Theta}\lambda_1^2 \; + \; \frac{\sin^2\Theta(2\lambda_1^2\cos^2\Theta - 1)}{\lambda_1^2(1-\lambda_1^2\cos^2\Theta)^2}. \tag{6.81}$$

Figure 6.7 shows a plot of T versus λ_1. The figure has been drawn for $\Theta = \pi/3$.

Differentiating (6.81) with respect to λ_1 and evaluating the result at $\lambda_1 = 1$ gives the effective Young's modulus (in the direction under consideration):

$$E_{\text{eff}}(\Theta) = 4\mu \ \frac{5 + 3\cos 4\Theta}{3 - 4\cos 2\Theta + \cos 4\Theta}. \tag{6.82}$$

This agrees with the limit $\beta \to \infty$ of the result (6.59) we got in the case of the extensible fibers. Observe that $E_{\text{eff}}(\Theta) \to \infty$ as $\Theta \to 0$ corresponding to the rigid fibers being in the x_1–direction. When the fibers are perpendicular to the x_1-direction we get $E_{\text{eff}}(\pi/2) = 4\mu$.

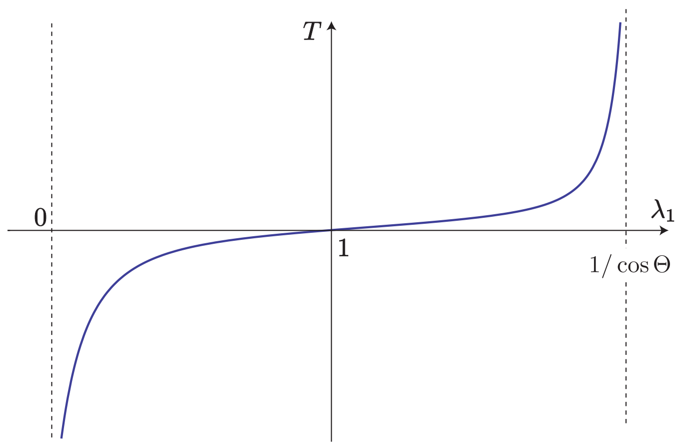

Figure 6.7: Stress T_{11}-stretch λ_1 curve in uniaxial stress according to (6.81) for the block shown in Figure 6.4.

6.2.3 Inflation, extension and twisting of a *thin*-walled tube.

Consider a *thin-walled* tube of mean radius R, wall thickness T and length L in the reference configuration[8]. Its two ends are closed. We are told that $T \ll R$. The tube wall involves two in-plane families of fibers. They are inclined, in the reference configuration, at angles Φ and $-\Psi$ from the *circumferential direction* as shown in Figure 6.11:

$$\mathbf{m}_{\text{R}} = \cos\Phi\, \mathbf{e}_\Theta + \sin\Phi\, \mathbf{e}_Z, \qquad \mathbf{m}'_{\text{R}} = -\cos\Psi\, \mathbf{e}_\Theta + \sin\Psi\, \mathbf{e}_Z. \tag{6.83}$$

The fibers are not necessarily mechanically equivalent and they are not necessarily oriented symmetrically with respect to the tube, i.e. $\Phi \neq \Psi$. The tube is subjected to an internal

[8] The corresponding problem for a thick-walled cylinder is considered in Problem 6.7.

pressure p, twisting moment (torque) M and axial force F. This loading expands the tube to a mean radius r, elongates it to a length ℓ and rotates one end of the tube with respect to the other by an angle $\alpha\ell$. Let t be the wall thickness in the deformed configuration.

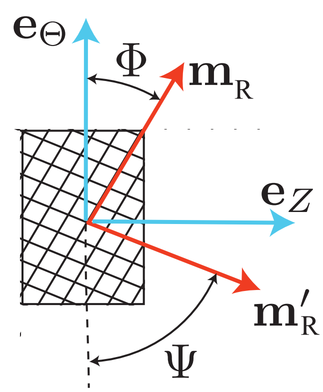

Figure 6.8: In the reference configuration the fiber directions $\mathbf{m}_\mathrm{R}$ and $\mathbf{m}'_\mathrm{R}$, locally, at each point in the tube, lie in the Θ, Z-plane as shown.

Let $\lambda = r/R$ and $\Lambda = \ell/L$ be the mean circumferential and axial stretches of the tube. The volumes of the circular part of the tube before and after deformation are $2\pi RTL$ and $2\pi rt\ell$ respectively and so, since the material is incompressible, $2\pi RTL = 2\pi rt\ell$. Thus

$$r = \lambda R, \qquad \ell = \Lambda L, \qquad t = \lambda^{-1}\Lambda^{-1}T. \tag{6.84}$$

Assume that the deformation that takes $(R, \Theta, Z) \to (r, \theta, z)$ has the form

$$r = r(R), \qquad \theta = \Theta + \alpha\Lambda Z, \qquad z = \Lambda Z. \tag{6.85}$$

The associated deformation gradient tensor is

$$\mathbf{F} = \lambda^{-1}\Lambda^{-1}\mathbf{e}_r \otimes \mathbf{e}_R + \lambda\, \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \alpha\Lambda\, r\mathbf{e}_\theta \otimes \mathbf{e}_Z + \Lambda \mathbf{e}_z \otimes \mathbf{e}_Z, \tag{6.86}$$

where incompressibility has been used in writing the first term. The left Cauchy Green tensor is

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \lambda^{-2}\Lambda^{-2}\mathbf{e}_r \otimes \mathbf{e}_r + (\lambda^2 + \alpha^2\Lambda^2 r^2)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \Lambda^2 \mathbf{e}_z \otimes \mathbf{e}_z + \alpha\Lambda^2\, r(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta), \tag{6.87}$$

The vectors $\mathbf{F}\mathbf{m}_\mathrm{R}, \mathbf{F}\mathbf{m}'_\mathrm{R}$, tensors $\mathbf{F}\mathbf{m}_\mathrm{R} \otimes \mathbf{F}\mathbf{m}_\mathrm{R}, \mathbf{F}\mathbf{m}'_\mathrm{R} \otimes \mathbf{F}\mathbf{m}'_\mathrm{R}$ and invariants I_1, I_4 and I_6 can now be readily calculated:

$$\mathbf{F}\mathbf{m}_\mathrm{R} = (\lambda\cos\Phi + r\Lambda\alpha\sin\Phi)\mathbf{e}_\theta + \Lambda\sin\Phi\mathbf{e}_z, \tag{6.88}$$

$$\mathbf{Fm}'_{\mathrm{R}} = (-\lambda\cos\Psi + r\Lambda\alpha\sin\Psi)\mathbf{e}_\theta + \Lambda\sin\Psi\mathbf{e}_z, \tag{6.89}$$

$$\begin{aligned}\mathbf{Fm}_{\mathrm{R}}\otimes\mathbf{Fm}_{\mathrm{R}} &= (\lambda\cos\Phi + r\Lambda\alpha\sin\Phi)^2\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \Lambda^2\sin^2\Phi\mathbf{e}_z\otimes\mathbf{e}_z+\\ &\quad +(\lambda\cos\Phi + r\Lambda\alpha\sin\Phi)\Lambda\sin\Phi(\mathbf{e}_z\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_z).\end{aligned} \tag{6.90}$$

$$\begin{aligned}\mathbf{Fm}'_{\mathrm{R}}\otimes\mathbf{Fm}'_{\mathrm{R}} &= (-\lambda\cos\Psi + r\Lambda\alpha\sin\Psi)^2\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \Lambda^2\sin^2\Psi\mathbf{e}_z\otimes\mathbf{e}_z+\\ &\quad +(-\lambda\cos\Psi + r\Lambda\alpha\sin\Psi)\Lambda\sin\Psi(\mathbf{e}_z\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_z).\end{aligned} \tag{6.91}$$

$$\left.\begin{aligned}I_1 &= \mathrm{tr}\,\mathbf{B} = \lambda^{-2}\Lambda^{-2} + \lambda^2 + \alpha^2\lambda^2\Lambda^2R^2 + \Lambda^2,\\ I_4 &= |\mathbf{Fm}_{\mathrm{R}}|^2 = \lambda^2(\cos\Phi + \alpha\Lambda R\sin\Phi)^2 + \Lambda^2\sin^2\Phi,\\ I_6 &= |\mathbf{Fm}'_{\mathrm{R}}|^2 = \lambda^2(-\cos\Psi + \alpha\Lambda R\sin\Psi)^2 + \Lambda^2\sin^2\Psi.\end{aligned}\right\} \tag{6.92}$$

The constitutive relation

$$\mathbf{T} = -q\mathbf{I} + 2W_1\mathbf{B} + 2W_4\mathbf{Fm}_{\mathrm{R}}\otimes\mathbf{Fm}_{\mathrm{R}} + 2W_6\mathbf{Fm}'_{\mathrm{R}}\otimes\mathbf{Fm}'_{\mathrm{R}} \tag{6.93}$$

together with (6.87), (6.90), (6.91) and (6.92) can be used to calculate the stress components $T_{rr}, T_{\theta\theta}, T_{zz}$ and $T_{z\theta}$ (the remaining two stress components $T_{r\theta}$ and T_{rz} vanish). This leads to the following expression where we have eliminated the reactive pressure q by subtracting the normal stress T_{rr} from the other two normal stresses $T_{\theta\theta}$ and T_{zz}:

$$\begin{aligned}T_{\theta\theta} - T_{rr} &= 2(\lambda^2 + \alpha^2\lambda^2\Lambda^2R^2 - \lambda^{-2}\Lambda^{-2})W_1 + 2\lambda^2(\cos\Phi + R\Lambda\alpha\sin\Phi)^2W_4+\\ &\quad +2\lambda^2(-\cos\Psi + R\Lambda\alpha\sin\Psi)^2W_6,\\ T_{zz} - T_{rr} &= 2(\Lambda^2 - \lambda^{-2}\Lambda^{-2})W_1 + 2\Lambda^2\sin^2\Phi W_4 + 2\Lambda^2\sin^2\Psi W_6\\ T_{\theta z} &= 2\alpha R\lambda\Lambda^2\left[W_1 + \sin^2\Phi W_4 + \sin^2\Psi W_6\right] + \lambda\Lambda\left[\sin 2\Phi W_4 - \sin 2\Psi W_6\right].\end{aligned} \tag{6.94}$$

By exploiting the symmetry of the problem and the fact that the tube is thin-walled, we can use the equilibrium equations to derive approximate expressions for the stress components, exactly as we did in the isotropic case. This leads to

$$T_{\theta\theta} \approx \frac{pr}{t}, \qquad T_{zz} \approx \frac{F + \pi r^2 p}{2\pi r t} \qquad T_{z\theta} \approx \frac{M}{2\pi r^2 t}, \qquad T_{rr} \approx -\frac{p}{2}, \tag{6.95}$$

where the term $\pi r^2 p$ in T_{zz} arises because the two ends of the tube are closed. Note that, due to the factor $1/t$ in $T_{\theta\theta}$ and T_{zz} these two normal stress components are significantly larger than T_{rr}.

Finally, combining (6.94) with (6.95) and dropping T_{rr} leads to

$$\begin{aligned}
\frac{pr}{t} &= 2(\lambda^2 + \alpha^2\lambda^2\Lambda^2R^2 - \lambda^{-2}\Lambda^{-2})W_1 + 2\lambda^2(\cos\Phi + \alpha\Lambda R\sin\Phi)^2 W_4 + \\
&\qquad + 2\lambda^2(-\cos\Psi + \alpha\Lambda R\sin\Psi)^2 W_6, \\
\frac{F + \pi r^2 p}{2\pi r t} &= 2(\Lambda^2 - \lambda^{-2}\Lambda^{-2})W_1 + 2\Lambda^2\sin^2\Phi W_4 + 2\Lambda^2\sin^2\Psi W_6, \\
\frac{M}{2\pi r^2 t} &= 2\alpha\lambda\Lambda^2 R\left[W_1 + \sin^2\Phi W_4 + \sin^2\Psi W_6\right] + \lambda\Lambda\left[\sin 2\Phi W_4 - \sin 2\Psi W_6\right].
\end{aligned} \tag{6.96}$$

Given p, F and M, the three equations (6.96) are to be solved for λ, Λ and α. Various special cases can be examined. For example, suppose we do not apply a twisting moment M. The third equation in (6.96) must still hold with zero on its left hand side; the rotation α of the tube will not be zero in general.

6.3 Worked Examples and Exercises.

Problem 6.1. Let $\mathbf{M}$ denote the tensor

$$\mathbf{M} = \mathbf{m}_{\mathrm{R}} \otimes \mathbf{m}_{\mathrm{R}}, \tag{6.97}$$

where $\mathbf{m}_{\mathrm{R}}$ is a unit vector. Show that

$$\mathbf{Q}\mathbf{m}_{\mathrm{R}} = \pm \mathbf{m}_{\mathrm{R}}, \tag{6.98}$$

for an orthogonal tensor $\mathbf{Q}$ if and only if

$$\mathbf{Q}\mathbf{M}\mathbf{Q}^T = \mathbf{M}. \tag{6.99}$$

An important consequence of this is that the groups

$$\mathcal{G} = \{\mathbf{Q} \ : \ \mathbf{Q}\mathbf{Q}^T = \mathbf{I}, \ \mathbf{Q}\mathbf{m}_{\mathrm{R}} = \pm\mathbf{m}_{\mathrm{R}} \ \} \quad \text{and} \quad \mathcal{G}' = \{\mathbf{Q} \ : \ \mathbf{Q}\mathbf{Q}^T = \mathbf{I}, \ \mathbf{Q}\mathbf{M}\mathbf{Q}^T = \mathbf{M} \ \} \tag{6.100}$$

are identical.

Problem 6.2. Keeping in mind that the strain energy function is invariant to replacing $\mathbf{m}_R$ by $-\mathbf{m}_R$, let

$$\overline{W}(\mathbf{C}, \mathbf{m}_R) = \breve{W}(\mathbf{C}, \mathbf{m}_R \otimes \mathbf{m}_R). \tag{i}$$

Show that

$$\overline{W}(\mathbf{C}, \mathbf{m}_R) = \overline{W}(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \mathbf{Q}\mathbf{m}_R) \qquad \text{for all } \mathbf{Q} \in \mathcal{G}, \tag{ii}$$

if and only if

$$\breve{W}(\mathbf{C}, \mathbf{m}_R \otimes \mathbf{m}_R) = \breve{W}\big(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \mathbf{Q}(\mathbf{m}_R \otimes \mathbf{m}_R)\mathbf{Q}^T\big) \qquad \text{for all orthogonal } \mathbf{Q}. \tag{iii}$$

Here $\mathcal{G}$ is the material symmetry group for transverse isotropy given in (6.100). Note that the second statement holds for *all* orthogonal $\mathbf{Q}$ not just those in $\mathcal{G}$. Therefore the function $\breve{W}$ is jointly isotropic in both arguments.

Problem 6.3. Show that $\breve{W}(\mathbf{C}, \mathbf{M}) = \breve{W}(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \mathbf{Q}\mathbf{M}\mathbf{Q}^T)$ for all orthogonal $\mathbf{Q}$ if and only if there is a function $\widetilde{W}$ such that

$$\breve{W}(\mathbf{C}, \mathbf{M}) = \widetilde{W}(I_1, I_2, I_3, I_4, I_5), \tag{6.101}$$

where

$$\begin{gathered} I_1(\mathbf{C}) = \operatorname{tr}\mathbf{C}, \qquad I_2(\mathbf{C}) = \tfrac{1}{2}\Big[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2\Big], \qquad I_3(\mathbf{C}) = \det\,\mathbf{C}, \\ I_4(\mathbf{C}, \mathbf{M}) = \mathbf{C}\cdot\mathbf{M}, \qquad I_5(\mathbf{C}, \mathbf{M}) = \mathbf{C}^2\cdot\mathbf{M}. \end{gathered} \tag{6.102}$$

Remark 1: Observe that one can equivalently write

$$I_4 = \mathbf{C}\cdot\mathbf{M} = \mathbf{C}\mathbf{m}_{\mathrm{R}}\cdot\mathbf{m}_{\mathrm{R}} \qquad I_5 = \mathbf{C}^2\cdot\mathbf{M} = \mathbf{C}^2\mathbf{m}_{\mathrm{R}}\cdot\mathbf{m}_{\mathrm{R}}. \tag{6.103}$$

For a proof, see Chapter 5 of Steigmann [9].

Problem 6.4. In the case of a material involving two families of fibers, it was claimed in Section 6.2 that the strain energy function depended on (in addition to I_1, I_2, I_3) the invariants

$$I_4 = \mathbf{C}\mathbf{m}_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}}, \quad I_5 = \mathbf{C}^2\mathbf{m}_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}}, \quad I_6 = \mathbf{C}\mathbf{m}'_{\mathrm{R}} \cdot \mathbf{m}'_{\mathrm{R}}, \quad I_7 = \mathbf{C}^2\mathbf{m}'_{\mathrm{R}} \cdot \mathbf{m}'_{\mathrm{R}},$$

and the coupling term

$$I_8 = \mathbf{C}\mathbf{m}'_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}}.$$

In view of the forms of I_4, I_5, I_6, I_7 and I_8, one might expect the quantity $\mathbf{C}^2\mathbf{m}'_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}}$ to also appear in this list of invariants. Show that $\mathbf{C}^2\mathbf{m}'_{\mathrm{R}} \cdot \mathbf{m}_{\mathrm{R}}$ can be written in terms of the other invariants.

Problem 6.5. An incompressible body occupies a unit cube in the reference configuration with its edges parallel to the coordinate axes. The body contains one family of fibers in planes parallel to the x_1, x_2-plane, oriented at an angle Θ from the x_1-axis. The body is subjected to a uniform stress $\mathbf{T} = T\mathbf{e}_1 \otimes \mathbf{e}_1 + T_{33}\mathbf{e}_3 \otimes \mathbf{e}_3$. The value of T is given while that of T_{33} is such that the deformation is a plane strain in the x_1, x_2-plane, i.e. $\lambda_3 = 1$. Note that in contrast to the problem considered in Section 6.1.1, here we have $T_{12} = 0$. Because of anisotropy, the deformation will involve a (to-be-determined) amount of shear k and so assume that the deformation has the form

$$\mathbf{y} = \mathbf{F}\mathbf{x} \qquad \text{where} \quad \mathbf{F} = \lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 + k\mathbf{e}_1 \otimes \mathbf{e}_2. \tag{i}$$

The material is characterized by the strain energy function

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{\mu\beta}{2}(I_4 - 1)^2, \qquad \mu > 0,\ \beta > 0. \tag{ii}$$

Derive two algebraic equations involving λ_1 and k as the only unknowns.

Now suppose the deformation is infinitesimal. Linearize the pair of equations you found and thus calculate the amount of shear k as a function of T (and μ, β and Θ).

Explore various limiting cases, e.g. $\beta \to 0, \beta \to \infty, \Theta \to 0$, etc. Assuming $T > 0$, when is $k > 0$ and when is it < 0?

Solution

In view of incompressibility we require $\det \mathbf{F} = \lambda_1\lambda_2 = 1$ and so we set $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^{-1}$. The deformation gradient tensor can then be written as

$$\mathbf{F} = \lambda\mathbf{e}_1 \otimes \mathbf{e}_1 + k\mathbf{e}_1 \otimes \mathbf{e}_2 + \lambda^{-1}\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{iii}$$

Thus

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\lambda^2 + k^2)\mathbf{e}_1 \otimes \mathbf{e}_1 + k\lambda^{-1}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \lambda^{-2}\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{iv}$$

$$\mathbf{F}\mathbf{m}_{\mathrm{R}} = (\lambda\cos\Theta + k\sin\Theta)\mathbf{e}_1 + \lambda^{-1}\sin\Theta\mathbf{e}_2, \tag{v}$$

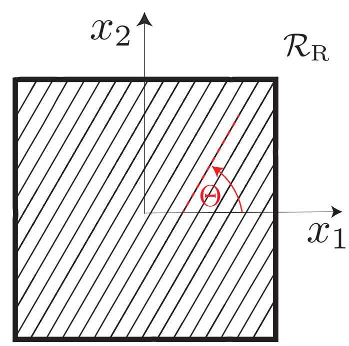

Figure 6.9: Rectangular block in reference configuration occupied by a material involving one family of fibers in the x_1, x_2-plane.

$$I_4 = |\mathbf{Fm}_{\mathrm{R}}|^2 = \lambda^2\cos^2\Theta + \lambda^{-2}\sin^2\Theta + 2k\lambda\sin\Theta\cos\Theta + k^2\sin^2\Theta, \tag{vi}$$

$$\begin{aligned}\mathbf{Fm}_{\mathrm{R}} \otimes \mathbf{Fm}_{\mathrm{R}} \quad &= (\lambda\cos\Theta + k\sin\Theta)^2\mathbf{e}_1\otimes\mathbf{e}_1 + +\lambda^{-2}\sin^2\Theta\mathbf{e}_2\otimes\mathbf{e}_2 + \\ &+(\sin\Theta\cos\Theta + k\lambda^{-1}\sin^2\Theta)(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)\end{aligned} \tag{vii}$$

The constitutive equation is

$$\mathbf{T} = -q\mathbf{I} + 2W_1\mathbf{B} + 2W_4(\mathbf{Fm}_{\mathrm{R}}\otimes\mathbf{Fm}_{\mathrm{R}}),$$

so that on using (iv) and (vii),

$$T_{11} = -q + 2W_1(\lambda^2 + k^2) + 2W_4(\lambda\cos\Theta + k\sin\Theta)^2 = T, \tag{viii}$$

$$T_{22} = -q + 2W_1\lambda^{-2} + 2W_4\lambda^{-2}\sin^2\Theta, \tag{ix}$$

$$T_{12} = 2W_1k\lambda^{-1} + 2W_4(\sin\Theta\cos\Theta + k\lambda^{-1}\sin^2\Theta). \tag{x}$$

The shear stress components T_{23} and T_{31} vanish automatically. We are told that $T_{22} = 0$. Thus we can use (ix) to eliminate q from $(viii)$. We are also told that $T_{12} = 0$. Thus we are led to the pair of equations

$$\left.\begin{aligned}(\lambda^2 + k^2 - \lambda^{-2}) + 2\beta(I_4 - 1)\left[(\lambda\cos\Theta + k\sin\Theta)^2 - \lambda^{-2}\sin^2\Theta\right] \quad &= T/\mu, \\ k\lambda^{-1} + 2\beta(I_4 - 1)(\sin\Theta\cos\Theta + k\lambda^{-1}\sin^2\Theta) \quad &= 0,\end{aligned}\right\} \qquad \square \tag{xi}$$

having used (ii) and recalling (vi).

Now suppose the deformation is infinitesimal. In this case $\lambda - 1$ and k are small. Letting $\lambda = 1 + \varepsilon$ we approximate the various terms in (xi) to leading order:

$$\lambda^2 - \lambda^{-2} + k^2 = (1+\varepsilon)^2 - (1+\varepsilon)^{-2} + \ldots = 4\varepsilon + \ldots$$

$$I_4 = (1+\varepsilon)^2\cos^2\Theta + (1+\varepsilon)^{-2}\sin^2\Theta + 2k\sin\Theta\cos\Theta + \ldots = 1 + 2\varepsilon\cos 2\Theta + k\sin 2\Theta + \ldots$$

$$(\lambda\cos\Theta + k\sin\Theta)^2 - \lambda^{-2}\sin^2\Theta = \cos 2\Theta + \ldots$$

$$k\lambda^{-1} = k + \ldots$$

$$(\sin\Theta\cos\Theta + k\lambda^{-1}\sin^2\Theta) = \sin\Theta\cos\Theta + \ldots = \frac{1}{2}\sin 2\Theta + \ldots$$

Substituting these expressions into (xi) yields the pair of linear algebraic equations for ε and k:

$$\left.\begin{aligned} 2\Big[2+\beta+\beta\cos 4\Theta\Big]\varepsilon + \beta k\sin 4\Theta = T/\mu, \\ (2+\beta-\beta\cos 4\Theta)k + 2\beta\varepsilon\sin 4\Theta = 0. \end{aligned}\right\}$$

Solving for k gives

$$k = -\frac{1}{4}\,\frac{\beta\sin 4\Theta}{1+\beta}\,\frac{T}{\mu}. \qquad \square$$

Discussion: Assuming $T > 0$, we see that

$$k \quad \begin{cases} = 0 & \text{for} \quad \Theta = 0, \\ < 0 & \text{for} \quad 0 < \Theta < \pi/4, \\ = 0 & \text{for} \quad \Theta = \pi/4, \\ > 0 & \text{for} \quad \pi/4 < \Theta < \pi/2, \\ = 0 & \text{for} \quad \Theta = \pi/2. \end{cases}$$

Also, we have $k \to 0$ when $\beta \to 0$ and

$$k \to -\frac{1}{4}\sin 4\Theta\,\frac{T}{\mu} \qquad \text{when} \quad \beta \to \infty.$$

In view of (ii), we can view the respective cases of large and small β as corresponding to strong and weak degrees of anisotropy.

Problem 6.6. In the reference configuration a *thin-walled* tube has mean radius R, wall thickness $T \ll R$ and length L. Its two ends are closed. The tube wall involves a single in-plane family of fibers. At each point of the tube wall, they lie in the Θ, Z-plane, oriented at an angle Φ with respect to the circumferential Θ-direction as shown in Figure 6.10:

$$\mathbf{m}_{\mathrm{R}} = \cos\Phi\,\mathbf{e}_\Theta + \sin\Phi\,\mathbf{e}_Z.$$

The tube is subjected to an internal pressure p and an axial force F. This loading will, in general, expand the tube to a mean radius r, elongate it to a length ℓ and rotate one end of the tube with respect to the other by an angle $\alpha\ell$. Assume the material to be incompressible and characterized by the strain energy function

$$W(I_1, I_4) = \frac{\mu}{2}(I_1 - 3) + \frac{\mu\beta}{2}(I_4 - 1)^2, \qquad \mu > 0,\ \beta > 0.$$

With $F = 0$, plot a graph of the twist angle α versus p. Does α vary monotonically with p (i.e. does the tube reverse its twist direction at some p)? Is there a value of F for which $\alpha = 0$?

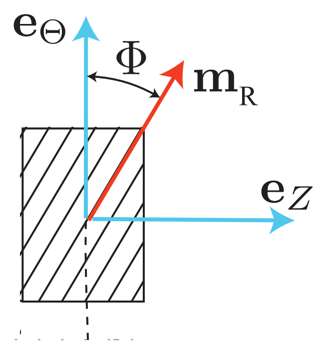

Figure 6.10: In the reference configuration the fiber direction $\mathbf{m}_\mathrm{R}$, locally, at each point in the tube wall, lies in the Θ, Z-plane as shown.

Problem 6.7. Consider a hollow (*thick-walled*) circular cylindrical tube with closed ends. It has inner radius A, outer radius B and length L in a reference configuration and is subjected to an internal pressure p, axial force F and twisting moment (torque) M. The tube is made of an incompressible material involving two families of fibers that lie, locally at each point, in the Θ, Z-plane, oriented at *different* angles Φ and Ψ with respect to the circumferential Θ-direction as shown in Figure 6.11. Formulate the problem and derive the equations to be solved to determine the resulting radial expansion, axial elongation and twist angle of the tube.

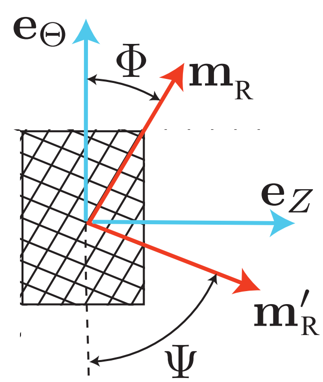

Figure 6.11: In the reference configuration the fiber directions $\mathbf{m}_\mathrm{R}$ and $\mathbf{m}'_\mathrm{R}$, locally, at each point in the tube, lie in the Θ, Z-plane as shown.

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Chapter 7

A Two-Phase Elastic Material: An Example.

7.1 A material with cubic and tetragonal phases.

We shall use the terms “energy-well” and “local minimum” interchangeably in this chapter and so we will say that the strain energy function $W(\mathbf{C})$ has an **energy-well** at $\mathbf{C} = \mathbf{C}_*$ if $W(\mathbf{C})$ has a local minimum at $\mathbf{C} = \mathbf{C}_*$:

$$\left.\frac{\partial W}{\partial C_{ij}}\right|_{\mathbf{C}=\mathbf{C}_*} = 0, \qquad \left.\frac{\partial^2 W}{\partial C_{ij}\partial C_{k\ell}}\right|_{\mathbf{C}=\mathbf{C}_*} H_{ij}H_{k\ell} > 0, \tag{7.1}$$

for all symmetric tensors $\mathbf{H} \neq \mathbf{0}$. Note from the constitutive relation for stress and $(7.1)_1$ that the body is necessarily stress-free at an energy-well. The particular strain energy functions considered previously had energy-wells at $\mathbf{C} = \mathbf{I}$ corresponding to the reference configuration.

In this section we construct a strain energy function with *multiple* energy-wells, i.e. multiple local minima. Each energy-well describes a “phase” of the material and multi-well strain energy functions describe materials that can exist in more than one phase. It should be noted that the multiple phases here are all solid phases. Figure 7.1 depicts a particular crystalline solid that has a cubic lattice in one phase and a tetragonal lattice in another phase as is the case for certain alloys of In-Tl, Mn-Ni and Mn-Cu.

Temperature plays an important role in the study of such materials. In our present discussion we hold the temperature fixed at the so-called *transformation temperature*. At this temperature, the heights of the different energy-wells are the same, i.e. the values of W

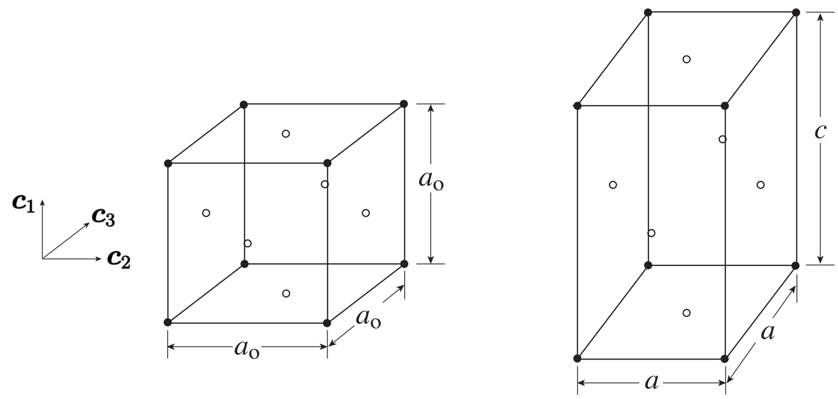

Figure 7.1: One phase has a face-centered cubic lattice with lattice parameters $a_o \times a_o \times a_o$, the other a face-centered tetragonal lattice with lattice parameters $a \times a \times c$. The unit vectors $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ are associated with the cubic directions. Solely for purposes of clarity, the atoms at the vertices are depicted by filled dots while those at the centers of the faces are shown by open dots.

are the same at all local minima. Thus we might say that there is no preferred phase at this temperature. Above the transformation temperature, one phase will have lower energy than the other, while this will switch below the transformation temperature. This is illustrated in Figure 7.5 where, at high temperatures the energy-well at $\varepsilon = 0$ has lower energy than the other two energy-wells, while the reverse is true at low temperatures. The temperature-dependent version of the material in this chapter can be found in Abeyaratne et al. [1].

Consider a material that can exist in two phases, a cubic phase and a tetragonal phase. Let $a_0 \times a_0 \times a_0$ and $a \times a \times c$ be the respective lattice parameters of the cubic and tetragonal lattices, and let $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ denote fixed unit vectors in the cubic directions. The deformation from the cubic phase to the tetragonal phase is achieved by stretching the cubic lattice equally in two of the cubic directions and unequally in the third direction, see Figure 7.1, the associated stretches being α, α and β:

$$\alpha = \frac{a}{a_0} > 1, \qquad \beta = \frac{c}{a_0}.$$

Since we can impose the unequal stretch β on any one of the $\mathbf{c}_k$-directions, $k = 1, 2, 3$, (and the stretch α on the remaining two cubic directions), there are three ways in which to stretch the cubic lattice into the tetragonal lattice as depicted in Figure 7.2. We say there are three *variants* of the tetragonal phase. With the cubic phase taken to be the reference configuration, the three variants are characterized by the stretch tensors $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3$ whose components in the basis $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ are

$$[U_1] = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \qquad [U_2] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \qquad [U_3] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}. \tag{7.2}$$

It will be useful for subsequent calculations to note that

$$\mathbf{U}_1 \;=\; \beta\, \mathbf{c}_1 \otimes \mathbf{c}_1 + \alpha\, \mathbf{c}_2 \otimes \mathbf{c}_2 + \alpha\, \mathbf{c}_3 \otimes \mathbf{c}_3. \tag{7.3}$$

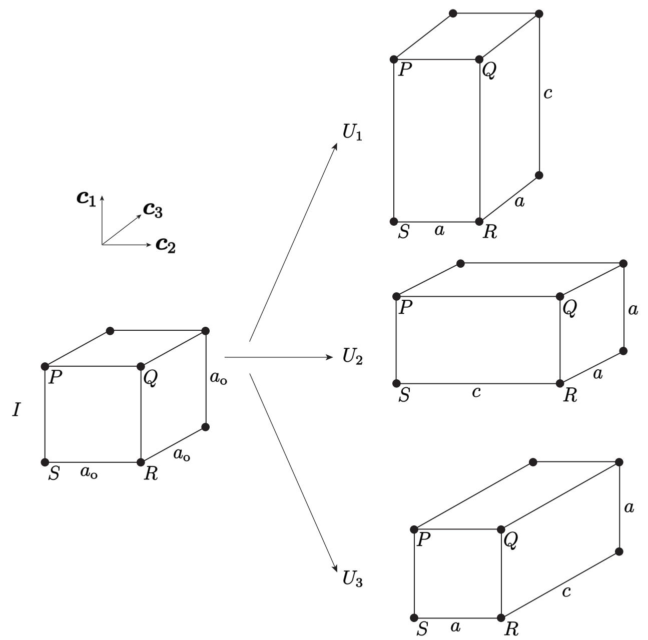

Figure 7.2: Cubic phase and three variants of the tetragonal phase. Observe that one *cannot* rigidly rotate one tetragonal variant in such a way as to make the "atoms" P,Q,R,S coincide with the locations of these same atoms in a different tetragonal variant.

We now construct an explicit strain-energy function $W(\mathbf{C})$ that can be used to describe such a material. It therefore has local minima at $\mathbf{C} = \mathbf{I}, \mathbf{U}_1^2, \mathbf{U}_2^2$ and $\mathbf{U}_3^2$. According to the claim in the exercise below, a function $W(\mathbf{C})$ that has an energy-well at $\mathbf{C} = \mathbf{U}_1^2$, will automatically have energy-wells at $\mathbf{C} = \mathbf{U}_2^2$ and $\mathbf{U}_3^2$ in view of material symmetry.

Exercise: Show using material symmetry, that if $W(\mathbf{C})$ has a local minimum at any one of the tensors $\mathbf{C}_k$, then W will automatically have energy-wells at the remaining two $\mathbf{C}_k$'s where $\mathbf{C}_k = \mathbf{U}_k^2$, $k = 1, 2, 3$.

Following Ericksen [2, 3, 4] we write the strain-energy function as a function of the Green Saint-Venant strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$. It must have energy-wells at $\mathbf{E} = \mathbf{0}, \mathbf{E}_1, \mathbf{E}_2$ and

$\mathbf{E}_3$ where $\mathbf{E}_k = \frac{1}{2}(\mathbf{C}_k - \mathbf{I}) = \frac{1}{2}(\mathbf{U}_k^2 - \mathbf{I})$. Since we have taken the reference configuration to coincide with the cubic phase, the strain-energy function $W(\mathbf{E})$ (with respect to that configuration), must possess cubic symmetry. It is known (see, for example, Smith and Rivlin [5] and Green and Adkins [6]) that, to have cubic symmetry, W must be a function of the "cubic invariants" $i_k(\mathbf{E})$:

$$\begin{aligned} i_1 &= E_{11} + E_{22} + E_{33}, & i_2 &= E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11}, \\ i_3 &= E_{11}E_{22}E_{33}, & i_4 &= E_{12}E_{23}E_{31}, \\ i_5 &= E_{12}^2 + E_{23}^2 + E_{31}^2, & &\ldots \text{etc.}, \end{aligned} \tag{7.4}$$

where E_{ij} refers to the i,j-component of $\mathbf{E}$ in the cubic basis $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$. The number of invariants in this list depends on the particular cubic sub-class under consideration (see, e.g., Section 1.11 of Green and Adkins [6]) but the analysis below is valid for all of them.

For temperatures close to the transformation temperature, Ericksen [2, 3, 4] has argued based on experimental observations that, as a first approximation, (a) all of the shear strain components (in the cubic basis) vanish and (b) the sum of the normal strains also vanishes. Accordingly he suggested a kinematically constrained theory based on the two constraints

$$i_1 = E_{11} + E_{22} + E_{33} = 0, \qquad i_5 = E_{12}^2 + E_{23}^2 + E_{31}^2 = 0. \tag{7.5}$$

In this case, the only two nontrivial strain invariants among those in the preceding list are i_2 and i_3 and so we take $W = W(i_2, i_3)$. The constraint $i_1 = 0$ can be written as $E_{33} = -E_{11} - E_{22}$ and so we can eliminate E_{33} from i_2 and i_3 to obtain

$$i_2 = -E_{11}^2 - E_{11}E_{22} - E_{22}^2, \qquad i_3 = -E_{11}^2E_{22} - E_{11}E_{22}^2. \tag{7.6}$$

Note that the constraint $i_1 = 0$ implies $\operatorname{tr} \mathbf{E} = 0$ and so in particular $\operatorname{tr} \mathbf{E}_1 = 0$. Therefore the lattice stretches α and β must be related by $2\alpha^2 + \beta^2 = 3$.

Let us start by considering a one-phase material involving only the cubic phase. Then the only local minimum of $W(\mathbf{E})$ is at $\mathbf{E} = \mathbf{0}$. The simplest form of W to consider is a polynomial in the components of $\mathbf{E}$, and in order to have one local minimum its degree must be (at least) quadratic. Thus consider

$$W(\mathbf{E}) = c_0 + c_2 i_2, \tag{7.7}$$

where c_0 and c_2 are constants and the invariant i_2 is given by $(7.6)_1$. The first derivatives of $W(E_{11}, E_{22})$ are

$$\frac{\partial W}{\partial E_{11}} = -c_2(2E_{11} + E_{22}), \qquad \frac{\partial W}{\partial E_{22}} = -c_2(2E_{22} + E_{11}), \tag{7.8}$$

and they vanish at $\mathbf{E} = \mathbf{0}$. Thus W has an extremum at $\mathbf{E} = \mathbf{0}$. To ensure that this is a local minimum, we calculate the second derivatives of $W(E_{11}, E_{22})$:

$$\frac{\partial^2 W}{\partial E_{11}^2} = -2c_2, \qquad \frac{\partial^2 W}{\partial E_{22}^2} = -2c_2, \qquad \frac{\partial^2 W}{\partial E_{11} \partial E_{22}} = -c_2. \tag{7.9}$$

The extremum is a local minimum if the Hessian matrix

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{pmatrix} \qquad \text{where } W_{ij} = \frac{\partial^2 W}{\partial E_{ii} \partial E_{jj}} \text{ (no sum on } i \text{ and } j), \tag{7.10}$$

evaluated at $\mathbf{E} = \mathbf{0}$ is positive definite. It is easily seen that this requires $c_2 < 0$. Thus the strain energy function (7.7), $(7.6)_1$ with $c_2 < 0$ has a local minimum at $\mathbf{E} = \mathbf{0}$ (and this local minimum is unique). A contour plot of this energy is shown in Figure 7.3. Note the energy-well at $\mathbf{E} = \mathbf{0}$.

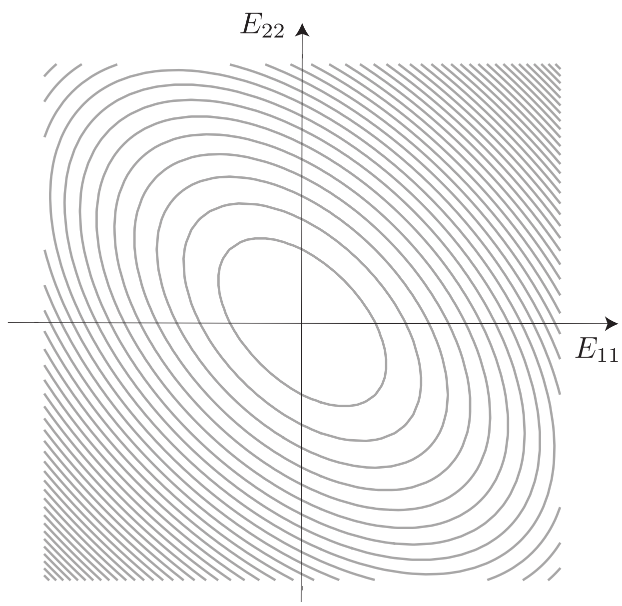

Figure 7.3: Constant energy contours on the E_{11}, E_{22}-plane for the one-phase strain energy function (7.7), $(7.6)_1$ with $c_2 < 0$. Observe the single energy-well at $\mathbf{E} = \mathbf{0}$.

We now return to the two-phase material of interest. This energy function must have one energy well at $\mathbf{E} = \mathbf{0}$ (the cubic phase) and another at $\mathbf{E} = \mathbf{E}_1$ (the tetragonal phase). Here

$$\mathbf{E}_1 = \frac{1}{2}(\mathbf{U}_1^2 - \mathbf{I}) \stackrel{(7.3)}{=} \frac{1}{2}(\beta^2 - 1)\, \mathbf{c}_1 \otimes \mathbf{c}_1 + \frac{1}{2}(\alpha^2 - 1)\, \mathbf{c}_2 \otimes \mathbf{c}_2 + \frac{1}{2}(\alpha^2 - 1)\, \mathbf{c}_3 \otimes \mathbf{c}_3. \tag{7.11}$$

The simplest form of W to consider is again a polynomial in the components of $\mathbf{E}$, but in order to endow it with two distinct local minima, its degree must be at least quartic. It follows from (7.4) and (7.5) that the most general quartic polynomial of the form $W = W(i_2, i_3)$ is

$$W = c_0 + c_2 i_2 + c_3 i_3 + c_{22} i_2^2, \tag{7.12}$$

where the c_i's are constants[1]. From (7.6) and (7.12), the first and second partial derivatives of $W(E_{11}, E_{22})$ with respect to the strain components are

$$\begin{aligned}
\frac{\partial W}{\partial E_{11}} &= -c_2(2E_{11} + E_{22}) - c_3(2E_{11}E_{22} + E_{22}^2) + 2c_{22}(E_{11}^2 + E_{11}E_{22} + E_{22}^2)(2E_{11} + E_{22}),\\
\frac{\partial W}{\partial E_{22}} &= -c_2(2E_{22} + E_{11}) - c_3(2E_{11}E_{22} + E_{11}^2) + 2c_{22}(E_{11}^2 + E_{11}E_{22} + E_{22}^2)(2E_{22} + E_{11}),
\end{aligned} \tag{7.13}$$

$$\begin{aligned}
\frac{\partial^2 W}{\partial E_{11}^2} &= -2c_2 - 2c_3E_{22} + 6c_{22}(2E_{11}^2 + 2E_{11}E_{22} + E_{22}^2),\\
\frac{\partial^2 W}{\partial E_{22}^2} &= -2c_2 - 2c_3E_{11} + 6c_{22}(2E_{22}^2 + 2E_{11}E_{22} + E_{11}^2),\\
\frac{\partial^2 W}{\partial E_{11}\partial E_{22}} &= -c_2 - 2c_3(E_{11} + E_{22}) + 6c_{22}(E_{11}^2 + 2E_{11}E_{22} + E_{22}^2).
\end{aligned} \tag{7.14}$$

Of the multiple local minima of W, one must be at $\mathbf{E} = \mathbf{0}$ corresponding to the cubic phase. It is seen immediately from (7.13) that the first derivatives of $W(E_{11}, E_{22})$ vanish automatically at $\mathbf{E} = \mathbf{0}$. Evaluating the Hessian matrix at $\mathbf{E} = \mathbf{0}$ shows that it is positive definite provided

$$c_2 < 0. \tag{7.15}$$

A second local minimum of $W(\mathbf{E})$ is at the strain $\mathbf{E}_1$ given by (7.11). A straightforward calculation based on substituting (7.11) into (7.13) shows that the first derivatives of $W(E_{11}, E_{22})$ vanish at $\mathbf{E}_1$ provided

$$c_2 = -pc_3 + 6p^2 c_{22}, \tag{7.16}$$

where we have set

$$p := \frac{1}{2}(\alpha^2 - 1) > 0. \tag{7.17}$$

In order to ensure that this extremum is a local minimum, we evaluate the second derivatives of W at $\mathbf{E} = \mathbf{E}_1$ using (7.14). The positive definiteness of the associated Hessian matrix is found to require

$$12pc_{22} > c_3 > 0. \tag{7.18}$$

[1] When thermal effects are taken into account the c_i's will be functions of temperature.

Finally, since we are working at the transformation temperature where the heights of the energy-wells are the same, we need $W(\mathbf{0}) = W(\mathbf{E}_1)$. Calculating these two values of the energy from (7.12) and equating them leads to

$$c_3 = 9pc_{22}. \tag{7.19}$$

Combining all of the preceding requirements (7.15), (7.16), (7.18) and (7.19) leads to the strain energy function

$$W = c_0 + c_{22}\Big[-3p^2 i_2 + 9pi_3 + i_2^2\Big], \tag{7.20}$$

where

$$c_{22} > 0, \qquad p = \frac{1}{2}(\alpha^2 - 1). \tag{7.21}$$

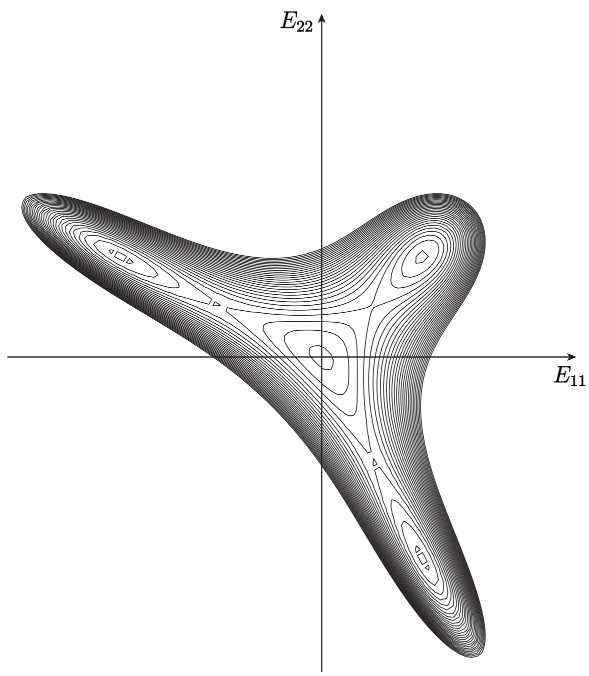

Figure 7.4: Constant energy contours on the E_{11}, E_{22}-plane for the strain energy function (7.20), (7.21). Observe the presence of the cubic phase energy-well at $\mathbf{E} = \mathbf{0}$ surrounded by the three tetragonal phase energy-wells at $\mathbf{E} = \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$.

Figure 7.4 shows the equal energy contours of the strain-energy function (7.20), (7.21) on the E_{11}, E_{22}-plane. It shows the cubic energy-well at the origin $\mathbf{E} = \mathbf{0}$ surrounded by the three tetragonal energy-wells at $\mathbf{E} = \mathbf{E}_1, \mathbf{E}_2$ and $\mathbf{E}_3$. The values of the material parameters

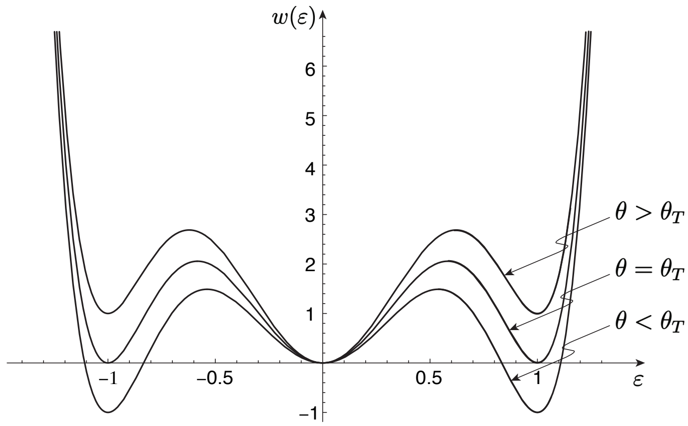

Figure 7.5: A one-dimensional cross-section of the energy (7.20), (7.21) along the path (7.22) in strain-space. The figure plots $w(\varepsilon) = W(\widehat{\mathbf{E}}(\varepsilon))$ versus strain ε at three constant temperatures. The analysis in these notes were limited to the transformation temperature θ_T where the wells have the same height. We have included the corresponding plots, both above and below the temperature θ_T, taken from [1]. The material parameters underlying this plot are the same as those associated with the previous figure.

associated with this figure were chosen solely on the basis of obtaining a fairly clear contour plot.

In order to draw a one-dimensional cross-section of the energy (7.20), (7.21) consider the following path in strain space

$$\mathbf{E} = \widehat{\mathbf{E}}(\varepsilon) = p\varepsilon^2 \mathbf{c}_1 \otimes \mathbf{c}_1 + \frac{1}{2}p\varepsilon(3-\varepsilon)\mathbf{c}_2 \otimes \mathbf{c}_2 - \frac{1}{2}p\varepsilon(3+\varepsilon)\mathbf{c}_3 \otimes \mathbf{c}_3, \qquad -1.5 < \varepsilon < 1.5, \quad (7.22)$$

where ε is a parameter. Observe that $\widehat{\mathbf{E}}(0) = \mathbf{0}$, $\widehat{\mathbf{E}}(-1) = \mathbf{E}_2, \widehat{\mathbf{E}}(1) = \mathbf{E}_3$ so that this path passes through the cubic well and two of the tetragonal wells. The energy along this path is $w(\varepsilon) := W(\widehat{\mathbf{E}}(\varepsilon))$ and Figure 7.5 shows a plot of $w(\varepsilon)$ versus ε (the curve labelled $\theta = \theta_T$). The calculations in this chapter were carried out at the transformation temperature ($\theta = \theta_T$) where the three energy-wells have the same height. Figure 7.5 taken from Abeyaratne et al. [1] shows the results for two other temperatures as well, one $> \theta_T$ and the other $< \theta_T$.

The strain-energy function (7.20) captures the key qualitative characteristics of a material that exists in cubic and tetragonal phases. However, due to the restrictive nature of the kinematic constraints (7.5), it fails to provide a *quantitatively* accurate model. The natural generalization of (7.20) is therefore to relax these constraints by using Lagrange multipliers

c_5 and c_{11} and to replace (7.12) by

$$W = c_0 + c_2 I_2 + c_3 I_3 + c_{22} I_2^2 + c_5 I_5 + c_{11} I_1^2. \tag{7.23}$$

R.D. James (unpublished[2]) has shown that the response predicted by this generalized form of W is in reasonable quantitative agreement with the observed behavior of In-Tl.

Remark: A strain energy function for a Cu-Al-Ni alloy can be found in Vedantam and Abeyaratne [8]. This alloy can exist in a cubic phase as well as *six* variants of a orthorhombic phase.

Remark: In Problem 2.35 you looked at the kinematics of a piecewise homogeneous deformation that involved the cubic phase on one side of a planar interface and a tetragonal variant on the other.

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[2]See Kloućek and Luskin [7].

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Chapter 8

A Micromechanical Constitutive Model

Continuum theory says that an elastic material is characterized by a strain energy function $W(\mathbf{C})$. If additional information on material symmetry is available, this can be reduced further, for example to the form $W(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}))$ for an isotropic material. However, that is as far as the theory goes. The examples of explicit functions W given in Section 4.7 (corresponding, for example, to the Blatz-Ko or Fung models) are "phenomenological models" of particular elastic materials, i.e. the functional form of W is laid out at the outset at the continuum level, and subsequent laboratory experiments are used to refine it.

On the other hand the macroscopic (or continuum) behavior of a material reflects its underlying microscopic behavior. If one could describe the processes at the microscopic scale, and knew how to homogenize them across scales, one could then infer the response at the macroscopic scale. When this is possible, the continuum model so developed captures the microscopic physics.

In this chapter we start at the atomistic scale and develop an explicit form for the strain energy function $W(\mathbf{C})$ for a crystalline solid. The atomistic model we use is the simplest conceivable one, and our purpose is *merely to illustrate* how one might develop continuum models from microscale models.

A second example that rightfully belongs here is the derivation of the strain energy function for rubber-like materials based on a polymer chain model. Unfortunately, the strain energy of such materials turns out to be dominated by its entropy, and since we have not considered thermodynamics in these notes (at least not beyond the brief discussion in Section 9.2), we are not able to describe those calculations here, not without a lot of preliminary work. The interested reader may look at Volume II in this series on notes.

8.1 Example: Lattice Theory of Elasticity.

The notes in this section closely follow the unpublished lecture notes of Professor Kaushik Bhattacharya of Caltech. I am most grateful to him for sharing them with me. The original calculations are due to Cauchy, see Love [3].

The aim of this section is to illustrate how a simple atomistic model of a crystalline solid can be used to *derive* explicit continuum scale constitutive response functions $\widehat{\mathbf{T}}$ and $\widehat{W}$ for the Cauchy stress and the strain energy function in terms of the deformation gradient tensor. We will see that the expressions to be derived *automatically satisfy the requirements of material frame indifference, material symmetry and the dissipation inequality.* Moreover we find that the traction - stress relation $\mathbf{t} = \mathbf{Tn}$ and the symmetry condition $\mathbf{T} = \mathbf{T}^T$ hold automatically. The expressions for $\widehat{\mathbf{T}}$ and $\widehat{W}$ that we derive are explicit in terms of the lattice geometry and the interatomic force potential; see (8.11) and (8.16).

8.1.1 A Bravais Lattice. Pair Potential.

A Bravais lattice $\mathcal{L}$ is an infinite set of points in $\mathbb{R}^3$ generated by translating a point $\mathbf{y}_o$ through three linearly independent vectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$: i.e.,

$$\mathcal{L}(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3) = \big\{ \mathbf{y} : \ \mathbf{y} \in \mathbb{R}^3, \ \mathbf{y} = \mathbf{y}_o + \nu_i \boldsymbol{\ell}_i \quad \text{for all integers } \nu_1, \nu_2, \nu_3 \ \big\} \ . \tag{8.1}$$

The *lattice vectors* $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ define a *unit cell*. Note the distinction between the lattice $\mathcal{L}$, which is an infinite set of periodically arranged points in space, and the lattice vectors. In particular, it is generally possible to generate the same lattice $\mathcal{L}$ from more than one set of lattice vectors, i.e., a given set of lattice vectors generates a unique lattice, but the converse is not necessarily true. More on this later. We shall take the orientation of the lattice vectors to be right-handed so that in particular, the volume of the unit cell is

$$\text{vol}\,(\text{unit cell}) = (\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2) \cdot \boldsymbol{\ell}_3 \ > \ 0. \tag{8.2}$$

The neighborhood of any lattice point, say $\mathbf{y}_A$, is identical to that of any other lattice point, say $\mathbf{y}_C$. To see this we simply note that if $\mathbf{y}_B$ is any third lattice point, then there necessarily is a fourth lattice point $\mathbf{y}_D$ such that $\mathbf{y}_D - \mathbf{y}_C = \mathbf{y}_B - \mathbf{y}_A$. Thus the position of $\mathbf{y}_B$ relative to $\mathbf{y}_A$ is the same as the position of $\mathbf{y}_D$ relative to $\mathbf{y}_C$. Any two lattice points $\mathbf{y}_A$ and $\mathbf{y}_C$ of a Bravais lattice are therefore *geometrically equivalent*. Bravais lattices can represent only monoatomic lattices; in particular, no alloy is a Bravais lattice[1].

We will ignore lattice vibrations and assume that the atoms are located at the lattice points. Therefore the calculations we carry out are valid at zero degrees Kelvin.

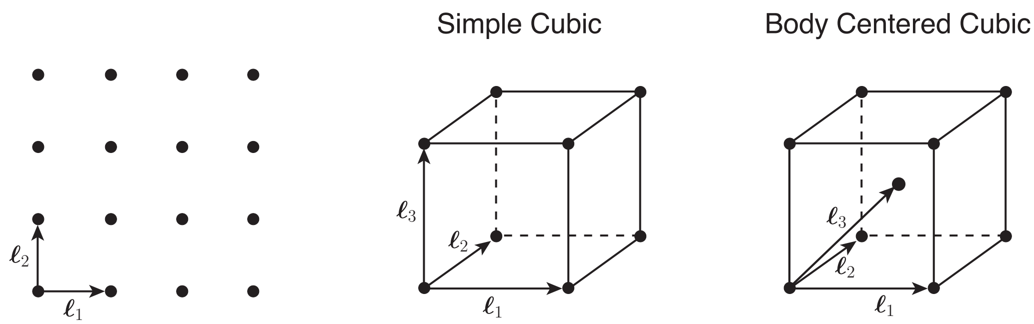

Figure 8.1: Examples of lattices in $\mathbb{R}^2$ and $\mathbb{R}^3$.

In the simplest model of interatomic interactions one assumes the existence of a *pair potential* $\phi(\rho)$ such that the force exerted by atom A on atom B, say $\mathbf{f}_{A,B}$, is the gradient of this potential:

$$\mathbf{f}_{A,B} = -\boldsymbol{\nabla}_y \phi(|\mathbf{y}|)\Big|_{\mathbf{y}=\mathbf{y}_B-\mathbf{y}_A} = -\phi'(|\mathbf{y}_B - \mathbf{y}_A|)\ \frac{\mathbf{y}_B - \mathbf{y}_A}{|\mathbf{y}_B - \mathbf{y}_A|}\ . \tag{8.3}$$

In this model the force exerted by one atom on the other depends solely on the relative positions of *those* two atoms and is independent of the positions of the surrounding atoms. Note that the force (8.3) is a *central force* in that it acts along the line joining those two atoms. Also observe that if the distance ρ between the atoms is such that $\phi'(\rho) < 0$, then the force between them is repulsive; if $\phi'(\rho) > 0$ it is attractive. Finally, observe from (8.3) that $\mathbf{f}_{A,B} = -\mathbf{f}_{B,A}$ so that the force exerted by atom A on atom B is equal in magnitude and opposite in direction to the force applied by atom B on atom A.

Figure 8.2 shows a graph of a typical pair-potential $\phi(\rho)$ versus the distance ρ between the pair of atoms. Note that the associated force is repulsive at short distances ($< \rho_o$) and attractive at large distances ($> \rho_o$).

[1]Even some monoatomic lattices – e.g. a hexagonal close-packed lattice – cannot be represented as a Bravais lattice.

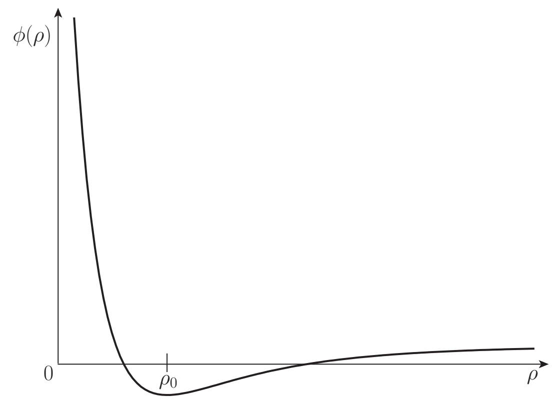

Figure 8.2: Typical graph of the pair-potential ϕ.

Several of the calculations to follow will involve infinite sums of terms involving ϕ, ϕ' and ϕ'' over all lattice points; it is necessary to ensure that these sums converge to finite values. This requires that $\phi(\rho) \to 0$ fast enough as $\rho \to \infty$. We assume that ϕ possesses the requisite[2] decay rate.

Because of the periodicity and symmetry of a Bravais lattice, if $\mathbf{y}_A$ and $\mathbf{y}_B$ are any two lattice points, there necessarily is a third lattice point $\mathbf{y}_C$ which is such that $\mathbf{y}_A - \mathbf{y}_B = -(\mathbf{y}_A - \mathbf{y}_C)$. Therefore according to the force law (8.3), the forces exerted on atom A by atoms B and C are equal in magnitude and opposite in direction. Consequently for each atom B that exerts a force on atom A, there is another atom C that exerts an equal and opposite force on A. Thus a Bravais lattice is *always in equilibrium.*

8.1.2 Homogenous Deformation of a Bravais Lattice.

Most, but not all, of the discussion to follow will be carried out entirely on the deformed lattice. There will however be a few occasions when we wish to consider a reference lattice

[2]To determine the required decay rate, one can consider a sphere of radius, say $\overline{\rho}$, and separate the infinite sum over the entire lattice into a finite sum over the finite number of lattice points in the interior of the sphere plus a sum over the infinite number of lattice points in the exterior of the sphere. An upper bound for the second term can then be written by replacing the sum by an integral (over the entire three dimensional region exterior to the sphere). Convergence of the integral guarantees convergence of the sum. For example the energy (8.16) will converge if the integral of $\rho^2\phi(\rho)$ over the interval $[\overline{\rho}, \infty)$ converges, which would be true if $\phi \to 0$ faster than ρ^{-3} as $\rho \to \infty$.

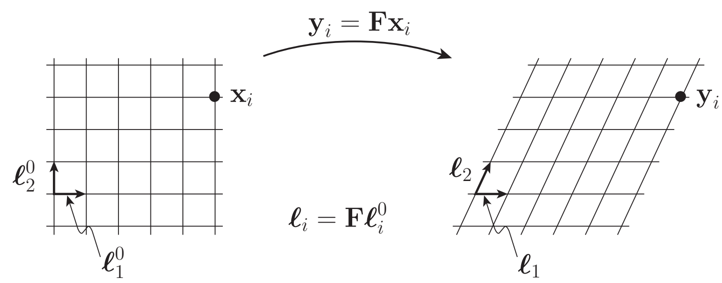

Figure 8.3: Homogeneous deformation of a lattice. The lattice vectors $\{\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o\}$ of the reference lattice are mapped by $\mathbf{F}$ into the lattice vectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2\}$ of the deformed lattice.

and for this purpose we consider a second Bravais lattice $\mathcal{L}_0$:

$$\mathcal{L}(\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o, \boldsymbol{\ell}_3^o) = \left\{\mathbf{x} : \ \mathbf{x} \in \mathbb{R}^3, \ \mathbf{x} = \mathbf{x}_o + \nu_i \boldsymbol{\ell}_i^o \quad \text{for all integers } \nu_1, \nu_2, \nu_3 \ \right\}$$

where the lattice vectors $\{\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o, \boldsymbol{\ell}_3^o\}$ define a unit cell of the reference lattice. Since each set of lattice vectors is linearly independent, there is a nonsingular tensor $\mathbf{F}$ that maps $\{\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o, \boldsymbol{\ell}_3^o\}$ $\to \{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$:

$$\boldsymbol{\ell}_i = \mathbf{F}\boldsymbol{\ell}_i^o, \quad i = 1, 2, 3. \tag{8.4}$$

This is illustrated in Figure 8.3.

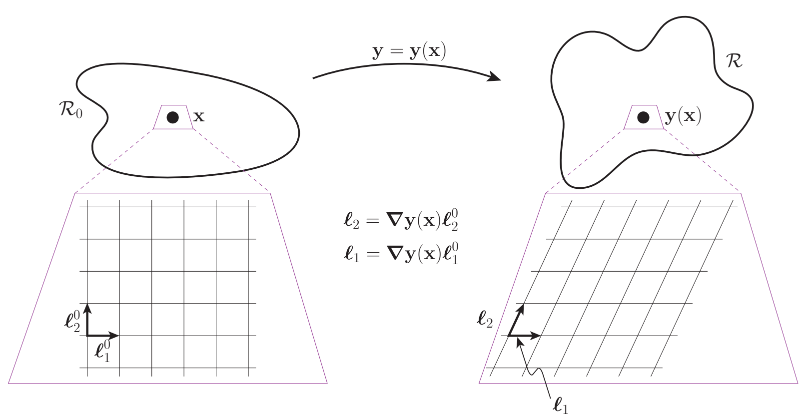

Figure 8.4: The deformation $\mathbf{y}(\mathbf{x})$ carries the three dimensional region $\mathcal{R}_0$ into $\mathcal{R}$. The figure shows blown-up views of infinitesimal neighborhoods of $\mathbf{x}$ and $\mathbf{y}(\mathbf{x})$. The mapping of the lattice vectors is assumed to be described by Grad $\mathbf{y}(\mathbf{x})$ ($= \nabla\mathbf{y}(\mathbf{x})$) as depicted in the figure.

Suppose that we associate a (continuum) body with the lattice. The lattices $\mathcal{L}_0$ and $\mathcal{L}$ are associated with two configurations of the body. Let $\mathbf{y}(\mathbf{x})$ be the deformation of the continuum that maps $\mathcal{R}_0$ into $\mathcal{R}$. The deformation gradient tensor is $\nabla\mathbf{y}(\mathbf{x})$. As discussed in Section 2.3, $\nabla\mathbf{y}(\mathbf{x})$ maps material fibers of the continuum from the reference to the deformed configurations. The tensor $\mathbf{F}$ introduced above maps the reference lattice vectors to the deformed lattice vectors through (8.4). The Cauchy-Born hypothesis (rule) states that the "continuum deforms with the lattice" in the sense that $\nabla\mathbf{y}(\mathbf{x}) = \mathbf{F}$. This is illustrated in Figure 8.4.

8.1.3 Traction and Stress.

We now establish a notion of traction and then derive an explicit expression for it in terms of the interatomic forces. Let $\mathcal{P}$ be an arbitrary plane through the lattice and let $\mathbf{n}$ denote a unit vector normal to $\mathcal{P}$. Let $\mathcal{L}^+$ and $\mathcal{L}^-$ denote the two subsets of the lattice $\mathcal{L}$ that are on, respectively, the side into which and the side away from which $\mathbf{n}$ points; see Figure 8.5. Let $\mathcal{A}$ be a subregion of the plane $\mathcal{P}$. Consider two lattice points $\mathbf{y}_+ \in \mathcal{L}^+$ and $\mathbf{y}_- \in \mathcal{L}^-$ such that the line joining them intersects the subregion $\mathcal{A}$; see Figure 8.5. By summing the forces between all such pairs of atoms, we can associate a force with the region $\mathcal{A}$. The traction $\mathbf{t}$ can then be defined as the normalization of this force by the area of $\mathcal{A}$:

$$\mathbf{t}(\mathcal{A}) = \frac{1}{\text{area}\,(\mathcal{A})}\sum \mathbf{f}_{i,j} \;=\; \frac{1}{\text{area}\,(\mathcal{A})}\sum -\phi'(|\mathbf{y}_- - \mathbf{y}_+|)\,\frac{\mathbf{y}_- - \mathbf{y}_+}{|\mathbf{y}_- - \mathbf{y}_+|}, \tag{8.5}$$

where the summation is carried out over all $\mathbf{y}_+ \in \mathcal{L}^+$ and $\mathbf{y}_- \in \mathcal{L}^-$ which are such that the line joining $\mathbf{y}_+$ to $\mathbf{y}_-$ intersects $\mathcal{A}$.

For (8.5) to be useful, we need to characterize the range of summation in a simpler form. First, since $\mathbf{y}_-$ and $\mathbf{y}_+$ are lattice points, it follows that there are integers $\{\nu_1, \nu_2, \nu_3\}$ for which $\mathbf{y}_- - \mathbf{y}_+ = \nu_i \boldsymbol{\ell}_i$. Conversely, given any three integers $\{\nu_1, \nu_2, \nu_3\}$ which are such that $(\nu_i \boldsymbol{\ell}_i)\cdot\mathbf{n} < 0$ (which simply means that the vector $\nu_i\boldsymbol{\ell}_i$ points in the $-\mathbf{n}$ direction), there exist pairs (note plural) of lattice points $\mathbf{y}_+ \in \mathcal{L}^+$ and $\mathbf{y}_- \in \mathcal{L}^-$ such that $\mathbf{y}_- - \mathbf{y}_+ = \nu_i\boldsymbol{\ell}_i$; of these, the number of pairs whose line of connection intersects $\mathcal{A}$ can be estimated to be

$$\begin{aligned} N &= \frac{\text{volume of the (non-prismatic) cylinder with base } \mathcal{A} \text{ and generator } \nu_i\boldsymbol{\ell}_i}{\text{volume of the unit cell}} \\ &= \frac{\text{area}\,(\mathcal{A})\,|(\nu_i\boldsymbol{\ell}_i)\cdot\mathbf{n}|}{(\boldsymbol{\ell}_1\times\boldsymbol{\ell}_2)\cdot\boldsymbol{\ell}_3} = -\text{area}\,(\mathcal{A})\frac{(\nu_i\boldsymbol{\ell}_i)\cdot\mathbf{n}}{(\boldsymbol{\ell}_1\times\boldsymbol{\ell}_2)\cdot\boldsymbol{\ell}_3}, \end{aligned} \tag{8.6}$$

when the area of $\mathcal{A}$ is sufficiently large[3]; see Figure 8.6. In the last step we have used the

[3] In a homogeneously deformed continuum, the traction on the plane $\mathcal{P}$ would be uniform, i.e. it would

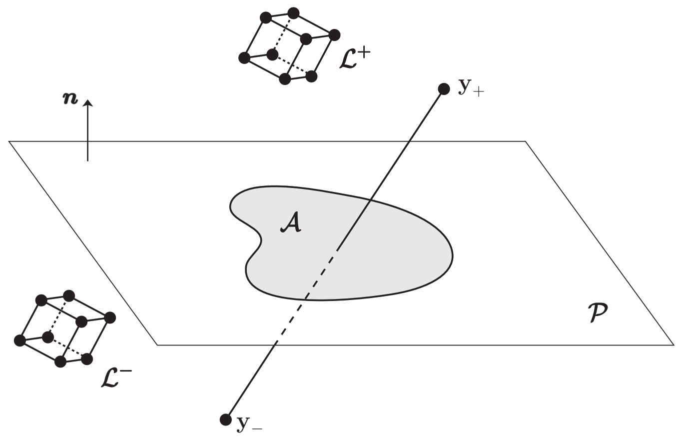

Figure 8.5: A plane $\mathcal{P}$ separating the lattice into two parts $\mathcal{L}^+$ and $\mathcal{L}^-$. The two lattice points $\mathbf{y}_+ \in \mathcal{L}^+$ and $\mathbf{y}_- \in \mathcal{L}^-$ are such that the line joining them intersects the subregion $\mathcal{A} \subset \mathcal{P}$.

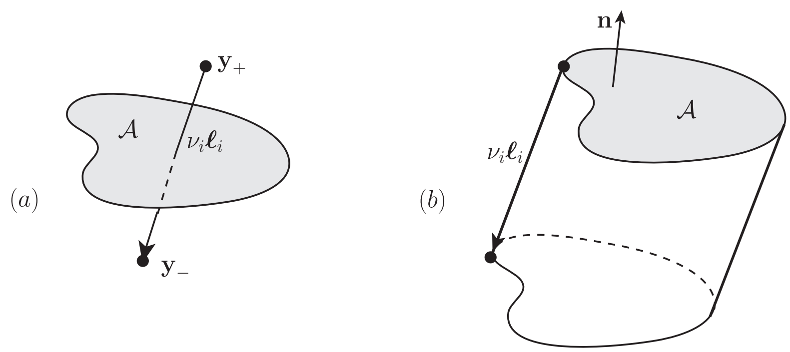

Figure 8.6: (a) Two lattice points $\mathbf{y}_+ \in \mathcal{L}^+$ and $\mathbf{y}_- \in \mathcal{L}^-$: $\mathbf{y}_- - \mathbf{y}_+ = \nu_i \boldsymbol{\ell}_i$ for some integers ν_1, ν_2, ν_3. (b) Non-prismatic cylinder whose base is $\mathcal{A}$ and generator is $\nu_i \boldsymbol{\ell}_i$.

fact that $(\nu_i \boldsymbol{\ell}_i) \cdot \mathbf{n} < 0$. Given the triplet of integers $\{\nu_1, \nu_2, \nu_3\}$, equation (8.6) gives the corresponding number of pairs of points whose line of connection intersects $\mathcal{A}$.

We can now evaluate the summation in (8.5) in two steps: first, for given $\{\nu_1, \nu_2, \nu_3\}$ with $(\nu_i \boldsymbol{\ell}_i) \cdot \mathbf{n} < 0$, we sum over all pairs of lattice points $\mathbf{y}_-$ and $\mathbf{y}_+$ which have $\mathbf{y}_- - \mathbf{y}_+ = \nu_i \boldsymbol{\ell}_i$ and where the line connecting them intersects $\mathcal{A}$. Then, we sum over all triplets of integers

be the same at all point on $\mathcal{P}$. The lattice at hand has a uniform geometry and we want (8.5) to be related to the continuum notion of traction. This requires that the right-hand side of (8.5) be independent of the size of $\mathcal{A}$. This in turn requires that the subregion $\mathcal{A}$ be sufficiently large.

$\{\nu_1, \nu_2, \nu_3\}$ obeying $(\nu_i \boldsymbol{\ell}_i) \cdot \mathbf{n} < 0$. This leads to

$$\mathbf{t}(\mathcal{A}) = \frac{1}{\text{area }(\mathcal{A})} \sum_{\substack{\{\nu_1,\nu_2,\nu_3\} \ni \\ (\nu_i \boldsymbol{\ell}_i)\cdot \mathbf{n} < 0}} -\phi'(|\nu_p \boldsymbol{\ell}_p|)\, \frac{\nu_i \boldsymbol{\ell}_i}{|\nu_k \boldsymbol{\ell}_k|}\; N \;. \tag{8.7}$$

Substituting (8.6) into this yields

$$\mathbf{t}(\mathcal{A}) = \frac{1}{(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2)\cdot \boldsymbol{\ell}_3} \sum_{\substack{\{\nu_1,\nu_2,\nu_3\} \ni \\ (\nu_i \boldsymbol{\ell}_i)\cdot \mathbf{n} < 0}} \phi'(|\nu_p \boldsymbol{\ell}_p|)\, \frac{\nu_i \boldsymbol{\ell}_i}{|\nu_k \boldsymbol{\ell}_k|}\, (\nu_j \boldsymbol{\ell}_j)\cdot \mathbf{n} \;. \tag{8.8}$$

Finally, observe that if we change $\{\nu_1, \nu_2, \nu_3\} \to \{-\nu_1, -\nu_2, -\nu_3\}$, the term within the summation sign remains unchanged though $(\nu_i \boldsymbol{\ell}_i) \cdot \mathbf{n}$ changes sign. Therefore, the sum above with the restriction $(\nu_i \boldsymbol{\ell}_i) \cdot \mathbf{n} < 0$ equals one-half the sum without this restriction. Therefore we obtain the following expression for the *traction* on the plane $\mathcal{P}$:

$$\mathbf{t}(\mathcal{A}) = \left[\frac{1}{2(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2)\cdot \boldsymbol{\ell}_3} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi'(|\nu_p \boldsymbol{\ell}_p|)\, \frac{(\nu_i \boldsymbol{\ell}_i)\otimes(\nu_j \boldsymbol{\ell}_j)}{|\nu_k \boldsymbol{\ell}_k|} \right] \mathbf{n} \tag{8.9}$$

where the summation is taken over all triplets of integers $\{\nu_1, \nu_2, \nu_3\}$.

Observe that the traction given by (8.9) depends linearly on the unit normal vector $\mathbf{n}$. This suggests that we define the *Cauchy stress tensor* $\mathbf{T}$ by

$$\mathbf{T} = \frac{1}{2(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2)\cdot \boldsymbol{\ell}_3} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi'(|\nu_p \boldsymbol{\ell}_p|)\, \frac{(\nu_i \boldsymbol{\ell}_i)\otimes(\nu_j \boldsymbol{\ell}_j)}{|\nu_k \boldsymbol{\ell}_k|} \;. \tag{8.10}$$

Note that $\mathbf{t} = \mathbf{T}\mathbf{n}$. Moreover $\mathbf{T} = \mathbf{T}^T$ as required by the balance of angular momentum. Given a Bravais lattice and a pair potential, equation (8.10) provides *an explicit formula for the stress*. It involves the geometry of the deformed lattice and the pair-potential.

Finally we provide a representation for $\mathbf{T}$ in terms of a referential lattice by replacing the deformed lattice vectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ in (8.10) by reference lattice vectors. To this end, consider a reference lattice defined by lattice vectors $\{\boldsymbol{\ell}^o_1, \boldsymbol{\ell}^o_2, \boldsymbol{\ell}^o_3\}$. The lattice vectors $\{\boldsymbol{\ell}^o_1, \boldsymbol{\ell}^o_2, \boldsymbol{\ell}^o_3\}$ of the reference lattice are related to the lattice vectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ of the deformed lattice through the nonsingular tensor $\mathbf{F}$ where $\boldsymbol{\ell}_i = \mathbf{F}\boldsymbol{\ell}^o_i$. The stress (in the deformed lattice) given by (8.10) can now be written in terms of the referential lattice vectors and $\mathbf{F}$ as

$$\boxed{\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F}) = \frac{1}{2(\mathbf{F}\boldsymbol{\ell}^o_1 \times \mathbf{F}\boldsymbol{\ell}^o_2)\cdot \mathbf{F}\boldsymbol{\ell}^o_3} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi'(|\nu_p\, F\boldsymbol{\ell}^o_p|)\frac{(\nu_i\, \mathbf{F}\boldsymbol{\ell}^o_i)\otimes(\nu_j\, \mathbf{F}\boldsymbol{\ell}^o_j)}{|\nu_k\, \mathbf{F}\boldsymbol{\ell}^o_k|} \;.} \tag{8.11}$$

This provides an explicit formula for the *stress response function* $\widehat{\mathbf{T}}$ in terms of the referential lattice, the tensor $\mathbf{F}$, and the pair potential. If we associate a continuum with this lattice and invoke the Cauchy-Born hypothesis, $\mathbf{F}$ would be the deformation gradient tensor.

8.1.4 Energy.

We begin by calculating the energy of a single atom located at a lattice point $\mathbf{y}$. The energy associated with the pair of atoms located at $\mathbf{y}$ and $\boldsymbol{\xi}$ is $\phi(|\mathbf{y}-\boldsymbol{\xi}|)$. Assume that this energy is equally shared by the two atoms. Then, the energy of the atom located at $\mathbf{y}$ due to its interaction with all other atoms of the lattice is

$$\frac{1}{2} \sum_{\substack{\boldsymbol{\xi} \in \mathcal{L} \\ \boldsymbol{\xi} \neq \mathbf{y}}} \phi(|\mathbf{y}-\boldsymbol{\xi}|) \;=\; \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \boldsymbol{\ell}_i|). \tag{8.12}$$

Observe that this energy does not depend on $\mathbf{y}$, reflecting the fact that the lattice is uniform and the energy of each atom is the same. Now consider the energy associated with some region $\mathcal{R}$ of three dimensional space. If $\mathcal{R}$ is sufficiently large, the number of lattice points in $\mathcal{R}$ is

$$\frac{\text{vol}\,(\mathcal{R})}{(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2)\cdot \boldsymbol{\ell}_3} \tag{8.13}$$

where the denominator denotes the volume of the unit cell. Therefore the energy associated with the region $\mathcal{R}$ is given by the product of the two preceding expressions:

$$\frac{\text{vol}\,(\mathcal{R})}{(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2)\cdot \boldsymbol{\ell}_3} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \boldsymbol{\ell}_i|). \tag{8.14}$$

On dividing by vol$\,(\mathcal{R})$, we get the energy per unit deformed volume. Thus, given a Bravais lattice and a pair potential, equation (8.14) provides *an explicit formula for the energy per unit deformed volume*. It involves the geometry of the deformed lattice and the pair-potential.

Finally we express this in terms of a referential lattice. Consider the lattice defined by lattice vectors $\{\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o, \boldsymbol{\ell}_3^o\}$ that are related to the deformed lattice vectors by $\boldsymbol{\ell}_i = \mathbf{F}\boldsymbol{\ell}_i^o$. Substituting $\boldsymbol{\ell}_i = \mathbf{F}\boldsymbol{\ell}_i^o$ and using the fact that the volumes of $\mathcal{R}$ and its pre-image $\mathcal{R}_o$ in the reference configuration are related by $\text{vol}(\mathcal{R}) = \det \mathbf{F}\, \text{vol}(\mathcal{R}_o)$ allows us to write (8.14) as

$$\frac{\text{vol}\,(\mathcal{R}_o)\, \det \mathbf{F}}{(\mathbf{F}\boldsymbol{\ell}_1^o \times \mathbf{F}\boldsymbol{\ell}_2^o)\cdot \mathbf{F}\boldsymbol{\ell}_3^o} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^o|). \tag{8.15}$$

Finally, on using the identity $(\mathbf{Aa} \times \mathbf{Ab}) \cdot \mathbf{Ac} = \det \mathbf{A}\ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ and dividing by vol$(\mathcal{R}_o)$, we obtain the following expression for the *energy per unit referential volume*:

$$\boxed{\widehat{W}(\mathbf{F}) \;=\; \frac{1}{(\boldsymbol{\ell}_1^o \times \boldsymbol{\ell}_2^o) \cdot \boldsymbol{\ell}_3^o} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F} \boldsymbol{\ell}_i^o|).} \tag{8.16}$$

This provides an explicit formula for the *strain energy response function* $\widehat{W}$ in terms of the referential lattice, the tensor $\mathbf{F}$, and the pair potential. If we associate a continuum with this lattice and invoke the Cauchy-Born hypothesis, $\mathbf{F}$ would be the deformation gradient tensor.

Note from (8.16) and (8.4) that the function $\widehat{W}$ and the tensor $\mathbf{F}$ both depend on the choice of reference lattice vectors. However, the energy of the deformed lattice does not depend on the choice of reference lattice vectors. Therefore the way in which $\mathbf{F}$ and $\widehat{W}$ depend on the reference lattice vectors must balance each other out such that the value of $\widehat{W}$ is independent of the choice of reference lattice vectors.

It is shown in Problem 8.2 that the stress response function (8.11) derived previously and the energy response function (8.16) are related *automatically* through the relation

$$\widehat{\mathbf{T}}(\mathbf{F}) = \frac{1}{\det \mathbf{F}} \frac{\partial \widehat{W}}{\partial \mathbf{F}}(\mathbf{F})\, \mathbf{F}^T \tag{8.17}$$

which is precisely what the continuum theory requires based on an elastic material being dissipation-free.

8.1.5 Material Frame Indifference.

It is shown in Problem 8.1 that the constitutive response function $\widehat{\mathbf{T}}(\mathbf{F})$ defined by (8.11) *automatically* obeys the relation

$$\widehat{\mathbf{T}}(\mathbf{QF}) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T$$

for all proper orthogonal tensors $\mathbf{Q}$ as would be required by material frame indifference in the continuum theory.

It can similarly be verified that the energy response function (8.16) has the property that

$$\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{QF})$$

for all proper orthogonal tensors $\mathbf{Q}$. This shows that $\widehat{W}$ is automatically consistent with material frame indifference.

8.1.6 Linearized Elastic Moduli. Cauchy Relations.

In Problem 8.3 we shall linearize the constitutive quantities (8.16) and (8.17) to the special case of infinitesimal deformations. This leads to the constitutive relation of linear elasticity with the material being characterized by an elasticity tensor $\mathbb{C}$. In fact, equation (8.35) of Problem 8.3 provides an explicit formula for the components $\mathbb{C}_{ijk\ell}$ of the elasticity tensor in terms of the referential lattice and the pair potential.

The elastic moduli obtained in this way exhibit the symmetries

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{k\ell ij} = \mathbb{C}_{jik\ell} = \mathbb{C}_{ij\ell k}, \tag{8.18}$$

just as required by the continuum theory; see (4.154). However *in addition*, $\mathbb{C}_{ijk\ell}$ given by (8.35) also possesses the symmetry

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{i\ell kj} \tag{8.19}$$

which is not required by the continuum theory. The symmetries (8.19) obtained from the present lattice model are known as the *Cauchy relations*. The Cauchy relations are known to be not obeyed by most elastic materials[4] and this is therefore a limitation of the lattice theory formulated here. This limitation is directly related to the use of a pair-potential to model interatomic interactions. More realistic interatomic interaction models remove this limitation.

8.1.7 Lattice and Continuum Symmetry.

Since the stress and strain energy response functions $\widehat{\mathbf{T}}$ and $\widehat{W}$ given by (8.11) and (8.16) were derived from lattice considerations, they inherit the appropriate invariance characteristics associated with the symmetry of the underlying lattice. In this section we address three issues:

1. We examine the geometric invariance characteristics of a Bravais lattice and construct its "lattice symmetry group".

2. We show that the lattice symmetry group plays the role of the material symmetry group for the response functions $\widehat{\mathbf{T}}$ and $\widehat{W}$ derived above.

[4]For example, for an isotropic material, the Cauchy relations imply that the Poisson ratio must always be 0.25.

3. We remark on the suitability of using the lattice symmetry group to characterize the symmetry of a continuum.

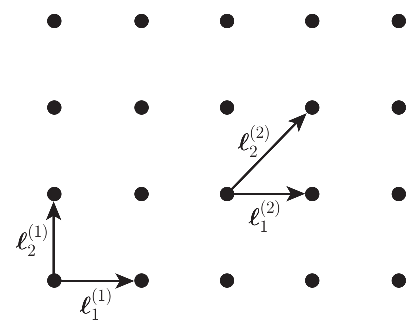

Figure 8.7: Two sets of lattice vectors that describe the same lattice.

Characterizing the symmetry of a Bravais lattice: First observe that because of its inherent symmetry, more than one set of lattice vectors may generate the same lattice. For example, the two-dimensional lattice shown in Figure 8.7 is generated by both $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}\}$ and $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}\}$. Observe that

$$\begin{aligned} \boldsymbol{\ell}_1^{(2)} &= \boldsymbol{\ell}_1^{(1)}, \\ \boldsymbol{\ell}_2^{(2)} &= \boldsymbol{\ell}_1^{(1)} + \boldsymbol{\ell}_2^{(1)}, \end{aligned} \tag{8.20}$$

so that the 2×2 matrix $[\mu]$, whose elements relate the two sets of lattice vectors through $\boldsymbol{\ell}_i^{(2)} = \mu_{ij}\boldsymbol{\ell}_j^{(1)}$, is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{8.21}$$

Note that the elements of $[\mu]$ are integers and that $\det[\mu] = 1$.

In general, let $\mathcal{L}(\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)})$ be the lattice generated by a given set of lattice vectors $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$. Suppose that $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$ is a second[5] set of lattice vectors that generates *this same lattice*, i.e.

$$\mathcal{L}(\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}) = \mathcal{L}(\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}).$$

One can show that two sets of lattice vectors generate the same lattice if and only if the matrix $[\mu]$, whose elements relate the two sets of lattice vectors through

$$\boldsymbol{\ell}_i^{(2)} = \mu_{ij}\boldsymbol{\ell}_j^{(1)}, \tag{8.22}$$

[5] We shall only consider lattice vector sets that have the same orientation.

has elements that are integers and whose determinant is 1.

An alternative more useful way in which to characterize symmetry is as follows: given a set of lattice vectors $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ and the associated Bravais lattice $\mathcal{L} = \mathcal{L}(\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)})$, let $\mathcal{G}(\mathcal{L})$ denote the set of all nonsingular tensors $\mathbf{H}$ that map $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ into a set of vectors $\{\mathbf{H}\boldsymbol{\ell}_1^{(1)}, \mathbf{H}\boldsymbol{\ell}_2^{(1)}, \mathbf{H}\boldsymbol{\ell}_3^{(1)}\}$ that generate the same lattice:

$$\mathcal{L}(\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}) = \mathcal{L}(\mathbf{H}\boldsymbol{\ell}_1^{(1)}, \mathbf{H}\boldsymbol{\ell}_2^{(1)}, \mathbf{H}\boldsymbol{\ell}_3^{(1)}) \qquad \text{for all } \mathbf{H} \in \mathcal{G}(\mathcal{L}).$$

It follows from (8.22) that $\mathcal{G}(\mathcal{L})$ admits the representation

$$\mathcal{G}(\mathcal{L}) = \left\{ \mathbf{H} : \ \mathbf{H}\boldsymbol{\ell}_i^{(1)} = \mu_{ij}\, \boldsymbol{\ell}_j^{(1)} \text{ for all } \mu_{ij} \text{ that are integers with } \det[\mu] = 1 \right\} . \tag{8.23}$$

Two sets of lattice vectors generate the same lattice if and only if

$$\boldsymbol{\ell}_i^{(2)} = \mathbf{H}\boldsymbol{\ell}_i^{(1)}, \quad i = 1, 2, 3, \tag{8.24}$$

where $\mathbf{H} \in \mathcal{G}(\mathcal{L})$. This is equivalent to (8.22). Despite the presence of the lattice vectors on the right hand side of (8.23), by its definition, $\mathcal{G}(\mathcal{L})$ depends on the lattice but not on the particular set of lattice vectors used to represent it. The set $\mathcal{G}(\mathcal{L})$ can be shown to be a group. It characterizes the symmetry of the lattice $\mathcal{L}$ and may be referred to as the "lattice symmetry group".

It is shown in Problem 8.5 that

$$\det\ \mathbf{H} = 1 \qquad \text{for all } \mathbf{H} \in \mathcal{G}(\mathcal{L}). \tag{8.25}$$

As a consequence, note that the volumes of the unit cells formed by lattice vectors $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ and $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$ are equal if the lattice vectors are related through a symmetry transformation:

$$(\boldsymbol{\ell}_1^{(2)} \times \boldsymbol{\ell}_2^{(2)}) \cdot \boldsymbol{\ell}_3^{(2)} = (\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)}) \cdot \boldsymbol{\ell}_3^{(1)} \tag{8.26}$$

provided

$$\boldsymbol{\ell}_i^{(2)} = \mathbf{H}\boldsymbol{\ell}_i^{(1)}, \qquad \mathbf{H} \in \mathcal{G}(\mathcal{L}). \tag{8.27}$$

Symmetry of the response functions $\widehat{\mathbf{T}}$ and $\widehat{W}$: As noted at the beginning of this subsection, since the stress and strain energy response functions $\widehat{\mathbf{T}}$ and $\widehat{W}$ given by (8.11) and (8.16) were derived from lattice considerations, they inherit the appropriate invariance characteristics associated with the symmetry of the underlying lattice. We shall now verify this claim and show, for example, that

$$\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{F}\mathbf{H}) \qquad \text{for all } \mathbf{H} \in \mathcal{G}(\mathcal{L}_o) \tag{8.28}$$

where $\mathcal{G}(\mathcal{L}_o)$ is the lattice symmetry group (8.23) of the reference lattice $\mathcal{L}_o$ and $\widehat{W}$ is the strain energy response function (8.16).

Recall from Section 4.4 that when examining symmetry in the continuum theory, we considered a deformed configuration χ and two reference configurations χ_1 and χ_2. We were interested in the special case when a symmetry transformation took $\chi_1 \to \chi_2$. In the lattice theory we analogously consider a deformed lattice $\mathcal{L}$ that is generated by lattice vectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ and two reference lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ that are generated by lattice vectors $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ and $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$. We are interested in the special case when a symmetry transformation takes $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ to $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$ in which case the reference lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are identical: $\mathcal{L}_1 = \mathcal{L}_2$.

Let $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ be a set of lattice vectors characterizing a reference lattice $\mathcal{L}_1$, and let $\widehat{W}_1$ be the stored energy response function with respect to this reference lattice:

$$\widehat{W}_1(\mathbf{F}) \;=\; \frac{1}{(\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)}) \cdot \boldsymbol{\ell}_3^{(1)}} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F} \boldsymbol{\ell}_i^{(1)}|). \tag{8.29}$$

Let $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$ be another set of lattice vectors characterizing a (possibly different) reference lattice $\mathcal{L}_2$, and let $\widehat{W}_2$ be the stored energy response function with respect to this reference lattice:

$$\widehat{W}_2(\mathbf{F}) \;=\; \frac{1}{(\boldsymbol{\ell}_1^{(2)} \times \boldsymbol{\ell}_2^{(2)}) \cdot \boldsymbol{\ell}_3^{(2)}} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F} \boldsymbol{\ell}_i^{(2)}|). \tag{8.30}$$

If the two sets of reference lattice vectors are related by (8.22), or equivalently by (8.24), then they generate the same reference lattice ($\mathcal{L}_1 = \mathcal{L}_2$) in which case

$$\widehat{W}_1(\mathbf{F}) = \widehat{W}_2(\mathbf{F}). \tag{8.31}$$

It then follows from (8.29) and (8.24) that

$$\begin{aligned}
\widehat{W}_1(\mathbf{F}\mathbf{H}) \;&=\; \frac{1}{(\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)}) \cdot \boldsymbol{\ell}_3^{(1)}} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F}\mathbf{H} \boldsymbol{\ell}_i^{(1)}|) \\
&=\; \frac{1}{(\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)}) \cdot \boldsymbol{\ell}_3^{(1)}} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F} \boldsymbol{\ell}_i^{(2)}|) \\
&=\; \frac{1}{(\boldsymbol{\ell}_1^{(2)} \times \boldsymbol{\ell}_2^{(2)}) \cdot \boldsymbol{\ell}_3^{(2)}} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F} \boldsymbol{\ell}_i^{(2)}|) \\
&=\; \widehat{W}_2(\mathbf{F})
\end{aligned}$$

where in the penultimate step we have used (8.26) and in the ultimate step we have used (8.30). It follows from this and (8.31) that

$$\widehat{W}_1(\mathbf{FH}) = \widehat{W}_1(\mathbf{F}) \qquad \text{for all nonsingular } \mathbf{F} \text{ and all } \mathbf{H} \in \mathcal{G}(\mathcal{L}_1).$$

Similarly one can show that

$$\widehat{\mathbf{T}}_1(\mathbf{F}) = \widehat{\mathbf{T}}_1(\mathbf{FH}) \qquad \text{for all } \mathbf{H} \in \mathcal{G}(\mathcal{L}_1)\ . \tag{8.32}$$

Thus the stress response function $\widehat{\mathbf{T}}$ and the energy response function $\widehat{W}$ derived from the present lattice theory, i.e. (8.11) and (8.16), are invariant under the group of transformations $\mathcal{G}(\mathcal{L}_o)$ that map the reference lattice back onto itself.

The lattice symmetry group and the symmetry of a continuum. Suppose that the lattice underlying the reference configuration of some elastic solid is a known Bravais lattice $\mathcal{L}_0$. However, suppose that one does not adopt the elementary pair potential model for interatomic interactions but arrives at a form for the strain energy function $\widehat{W}(\mathbf{F})$ by some other method, i.e. consider a strain energy response function $\widehat{W}(\mathbf{F})$ for the lattice that is *not* given by (8.16).

Even though the pair potential model for interatomic interactions was not used, the underlying lattice is (by assumption) a known Bravais lattice. Thus in particular the symmetry of the lattice is characterized by a known lattice symmetry group $\mathcal{G}(\mathcal{L}_o)$. Should one require that the continuum model exhibit all of the symmetries of the lattice? i.e. should we require

$$\widehat{W}(\mathbf{FH}) = \widehat{W}(\mathbf{F}) \quad \text{for all } \mathbf{H} \in \mathcal{G}(\mathcal{L}_o)? \tag{8.33}$$

The generally accepted answer is “no”: the material symmetry group of the continuum should be a suitable subgroup of $\mathcal{G}(\mathcal{L}_o)$. This is based on the fact that in addition to rotations and reflections, the lattice symmetry group $\mathcal{G}(\mathcal{L}_o)$ contains finite shears as well. For example consider (8.20), (8.21). In view of (8.21) one sees that the transformation from $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}\}$ to $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}\}$ is a simple shear. Such shears cause large distortions of the lattice and are usually associated with lattice slip and plasticity. It is natural therefore to exclude these large shears when modeling elastic materials.

Based on the work of Ericksen & Pitteri (see Bhattacharya) the appropriate material symmetry group of the continuum should be the subgroup of rotations in $\mathcal{G}(\mathcal{L}_o)$:

$$\mathsf{P}(\mathcal{L}_o) = \{\mathbf{R} :\ \mathbf{R} \in SO(3),\, \mathbf{R} \in \mathcal{G}(\mathcal{L}_o)\}\ . \tag{8.34}$$

Thus we would require $\widehat{W}(\mathbf{FR}) = \widehat{W}(\mathbf{F})$ for all $\mathbf{R} \in \mathsf{P}(\mathcal{L}_o)$ instead of the more stringent requirement (8.33). $\mathsf{P}(\mathcal{L}_o)$ is called the "point group" or " Laue group" of the lattice. It is the group of rotations which map the lattice[6] back into itself. The point group associated with any Bravais lattice is a finite group.

8.1.8 Worked Examples and Exercises.

Problem 8.1. Show that the stress response function $\widehat{\mathbf{T}}(\mathbf{F})$ given explicitly in (8.11) *automatically* satisfies the condition $\widehat{\mathbf{T}}(\mathbf{QF}) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T$ for all proper orthogonal tensors $\mathbf{Q}$. (Therefore this $\widehat{\mathbf{T}}(\mathbf{F})$ is automatically material frame indifferent.)

Solution: From (8.11),

$$\mathbf{T}(\mathbf{QF}) = \frac{1}{2(\mathbf{QF}\boldsymbol{\ell}_1^o \times \mathbf{QF}\boldsymbol{\ell}_2^o)\cdot\mathbf{QF}\boldsymbol{\ell}_3^o} \sum_{\{\nu_1,\nu_2,\nu_3\}} \left[\phi'(|\nu_p\, \mathbf{QF}\boldsymbol{\ell}_p^o|)\ .\ \frac{(\nu_i\, \mathbf{QF}\boldsymbol{\ell}_i^o)\otimes(\nu_j\, \mathbf{QF}\boldsymbol{\ell}_j^o)}{|\nu_k\, \mathbf{QF}\boldsymbol{\ell}_k^o|}\right]\ . \tag{a}$$

By using the vector identity $(\mathbf{Aa}\times\mathbf{Ab})\cdot\mathbf{Ac} = \det\mathbf{A}\ (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and the fact that $\det\mathbf{Q}=1$ we can write

$$(\mathbf{QF}\boldsymbol{\ell}_1^o \times \mathbf{QF}\boldsymbol{\ell}_2^o)\cdot\mathbf{QF}\boldsymbol{\ell}_3^o \ = \ (\mathbf{F}\boldsymbol{\ell}_1^o \times \mathbf{F}\boldsymbol{\ell}_2^o)\cdot\mathbf{F}\boldsymbol{\ell}_3^o\ . \tag{b}$$

Next, since $\mathbf{Q}$ is orthogonal, it preserves length, i.e. $|\mathbf{Qy}| = |\mathbf{y}|$ for all vectors $\mathbf{y}$, and consequently

$$|\nu_i\, \mathbf{QF}\boldsymbol{\ell}_i^o| = \ |\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^o|\ . \tag{c}$$

Finally, in view of the vector identity $(\mathbf{Aa})\otimes(\mathbf{Bb}) = \mathbf{A}(\mathbf{a}\otimes\mathbf{b})\mathbf{B}^T$ can write

$$(\nu_i\mathbf{QF}\boldsymbol{\ell}_i^o)\otimes(\nu_i\mathbf{QF}\boldsymbol{\ell}_i^o) = \mathbf{Q}\left((\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^o)\otimes(\nu_j\, \mathbf{F}\boldsymbol{\ell}_j^o)\right)\mathbf{Q}^T. \tag{d}$$

Therefore we can simplify (a) by using $(b), (c)$ and (d) to get

$$\begin{aligned}\mathbf{T}(\mathbf{QF}) \ &= \ \frac{1}{2(\mathbf{F}\boldsymbol{\ell}_1^o \times \mathbf{F}\boldsymbol{\ell}_2^o)\cdot\mathbf{F}\boldsymbol{\ell}_3^o}\,\mathbf{Q}\left(\sum_{\{\nu_1,\nu_2,\nu_3\}}\left[\ \phi'(|\nu_p\ F\boldsymbol{\ell}_p^o|)\frac{(\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^o)\otimes(\nu_j\, \mathbf{F}\boldsymbol{\ell}_j^o)}{|\nu_k\, \mathbf{F}\boldsymbol{\ell}_k^o|}\right]\right)\mathbf{Q}^T \\ &= \ \mathbf{QT}(\mathbf{F})\mathbf{Q}^T\ .\end{aligned}$$

Problem 8.2. Show that the Cauchy stress response function $\widehat{\mathbf{T}}(\mathbf{F})$ given explicitly by (8.11) and the strain energy response function $\widehat{W}(\mathbf{F})$ given explicitly by (8.16) are *automatically* related by

$$\widehat{\mathbf{T}}(\mathbf{F}) = \frac{1}{\det\mathbf{F}}\,\frac{\partial\widehat{W}}{\partial\mathbf{F}}(\mathbf{F})\ \mathbf{F}^T.$$

[6] For example, the point group of a simple cubic lattice consists of the 24 rotations that map the unit cube back into itself.

(Therefore the stress and strain energy response functions provided by the lattice theory automatically satisfy the relation imposed by the dissipation inequality; see Section 4.2.1 on page 346.)

Solution: Differentiating (8.16) with respect to $\mathbf{F}$ gives

$$2(\boldsymbol{\ell}_1^o \times \boldsymbol{\ell}_2^o) \cdot \boldsymbol{\ell}_3^o \left(\frac{\partial \widehat{W}}{\partial \mathbf{F}} \right) = \sum_{\{\nu_1,\nu_2,\nu_3\}} \left[\phi'(|\nu_p\, \mathbf{F}\boldsymbol{\ell}_p^o|) \left(\frac{\partial}{\partial \mathbf{F}} (|\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^o|) \right) \right] . \tag{a}$$

The following identity can be readily verified for an arbitrary vector $\mathbf{y}$:

$$\frac{\partial}{\partial \mathbf{F}}(|\mathbf{F}\mathbf{y}|) = \frac{1}{2|\mathbf{F}\mathbf{y}|} \frac{\partial}{\partial \mathbf{F}}(|\mathbf{F}\mathbf{y}|^2) \; = \; \frac{1}{2|\mathbf{F}\mathbf{y}|} \frac{\partial}{\partial \mathbf{F}}\Big(\mathbf{F}\mathbf{y} \cdot \mathbf{F}\mathbf{y}\Big) \; = \; \frac{1}{2|\mathbf{F}\mathbf{y}|} \Big(2\mathbf{F}\mathbf{y} \otimes \mathbf{y}\Big),$$

from which it follows that

$$\frac{\partial}{\partial \mathbf{F}}(|\nu_p\, \mathbf{F}\boldsymbol{\ell}_p^o|) = \frac{1}{|\nu_k\, \mathbf{F}\boldsymbol{\ell}_k^o|} \left[(\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^o) \otimes \nu_j\, \boldsymbol{\ell}_j^o \right] . \tag{b}$$

Substituting (b) into (a) yields

$$2(\boldsymbol{\ell}_1^o \times \boldsymbol{\ell}_2^o) \cdot \boldsymbol{\ell}_3^o \left(\frac{\partial \widehat{W}}{\partial \mathbf{F}} \right) = \sum_{\{\nu_1,\nu_2,\nu_3\}} \left[\phi'(|\nu_p\, \mathbf{F}\boldsymbol{\ell}_p^o|) \frac{(\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^o) \otimes \nu_j\, \boldsymbol{\ell}_j^o}{|\nu_k\, \mathbf{F}\boldsymbol{\ell}_k^o|} \right],$$

from which it follows that

$$2(\boldsymbol{\ell}_1^o \times \boldsymbol{\ell}_2^o) \cdot \boldsymbol{\ell}_3^o \left(\frac{\partial \widehat{W}}{\partial \mathbf{F}} \right) \mathbf{F}^T = \sum_{\{\nu_1,\nu_2,\nu_3\}} \left[\phi'(|\nu_p\, \mathbf{F}\boldsymbol{\ell}_p^o|) \frac{(\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^o) \otimes \nu_j\, \mathbf{F}\boldsymbol{\ell}_j^o}{|\nu_k\, \mathbf{F}\boldsymbol{\ell}_k^o|} \right] . \tag{c}$$

Finally, because of the identity $(\mathbf{A}\mathbf{a} \times \mathbf{A}\mathbf{b}) \cdot \mathbf{A}\mathbf{c} = \det \mathbf{A}\; (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, we see from (c) and (8.11) that the relation (8.17) between $\widehat{\mathbf{T}}$ and $\widehat{W}$ holds.

Problem 8.3. Derive an explicit expression for the elasticity tensor $\mathbb{C}$ of linear elasticity by linearization of the results of this chapter. Show that the resulting components $\mathbb{C}_{ijk\ell}$ posses the usual symmetries

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{k\ell ij} = \mathbb{C}_{jik\ell} = \mathbb{C}_{ij\ell k}, \tag{a}$$

as well as the additional symmetry

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{i\ell kj} \tag{b}$$

known as the Cauchy relations.

Solution: We first show that the energy response function $\widehat{W}$ given by (8.16) depends on $\mathbf{F}$ only through the Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$; thereafter we determine the components of the elasticity tensor $\mathbb{C}$ by recalling that

$$\mathbb{C}_{ijk\ell} = \left. \frac{\partial^2 W(\mathbf{C})}{\partial C_{ij}\, \partial C_{kl}} \right|_{\mathbf{C}=\mathbf{I}} .$$

The fact that $\widehat{W}$ depends on $\mathbf{F}$ only through $\mathbf{C}$ follows from

$$|\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^0| \; = \; \Big((\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^0) \cdot (\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^0) \Big)^{1/2} \; = \; \Big(\mathbf{F}^T\mathbf{F}(\nu_i\, \boldsymbol{\ell}_i^0) \cdot \nu_i\, \boldsymbol{\ell}_i^0 \Big)^{1/2} \; = \; \Big(\mathbf{C}(\nu_i\, \boldsymbol{\ell}_i^0) \cdot \nu_i\, \boldsymbol{\ell}_i^0 \Big)^{1/2}$$

whence we can write (8.16) as

$$W(\mathbf{C}) = \frac{1}{2(\boldsymbol{\ell}_1^0 \times \boldsymbol{\ell}_2^0) \cdot \boldsymbol{\ell}_3^0} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi\Big((\mathbf{C}(\nu_i\, \boldsymbol{\ell}_i^0) \cdot \nu_i\, \boldsymbol{\ell}_i^0)^{1/2} \Big) \ .$$

In order to calculate the elasticity tensor we must calculate the second derivative of W with respect to $\mathbf{C}$ and then evaluate it in the reference configuration where $\mathbf{C} = \mathbf{I}$. In order to simplify the writing it is convenient to introduce the notation

$$\alpha = 2(\boldsymbol{\ell}_1^0 \times \boldsymbol{\ell}_2^0) \cdot \boldsymbol{\ell}_3^0, \qquad \mathbf{y} = \nu_i\, \boldsymbol{\ell}_i^0, \qquad \rho(\mathbf{C}) = (\mathbf{C}\mathbf{y} \cdot \mathbf{y})^{1/2} \ ,$$

so that

$$W(\mathbf{C}) = \frac{1}{\alpha} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(\rho(\mathbf{C})).$$

It is straightforward to show that

$$\frac{\partial \rho(\mathbf{C})}{\partial C_{k\ell}} \ = \ \frac{y_k\, y_\ell}{2\rho(\mathbf{C})}.$$

Therefore the first derivative of W is

$$\frac{\partial W}{\partial C_{k\ell}}(\mathbf{C}) = \sum_{\{\nu_1,\nu_2,\nu_3\}} \frac{1}{\alpha}\, \phi'(\rho(\mathbf{C}))\, \frac{\partial \rho(\mathbf{C})}{\partial C_{k\ell}} \ = \sum_{\{\nu_1,\nu_2,\nu_3\}} \frac{1}{2\alpha\rho(\mathbf{C})}\, \phi'(\rho(\mathbf{C}))\, y_k\, y_\ell \ .$$

The second derivative can be calculated similarly by differentiating this once more, which leads after some calculation to

$$\frac{\partial^2 W(\mathbf{C})}{\partial C_{ij}\, \partial C_{k\ell}} = \sum_{\{\nu_1,\nu_2,\nu_3\}} \frac{1}{4\alpha\rho^2(\mathbf{C})} \left(\phi''(\rho(\mathbf{C})) - \frac{1}{\rho(\mathbf{C})} \phi'(\rho(\mathbf{C})) \right) y_i\, y_j\, y_k\, y_\ell \ .$$

In order to calculate the components of the elasticity tensor we set $\mathbf{C} = \mathbf{I}$, $\rho(\mathbf{C}) = \rho(\mathbf{I}) = |\mathbf{y}|$ in the preceding expression to obtain

$$\mathbb{C}_{ijk\ell} = \left. \frac{\partial^2 W(\mathbf{I})}{\partial C_{ij}\, \partial C_{k\ell}} \right|_{\mathbf{C}=\mathbf{I}} = \frac{1}{2(\boldsymbol{\ell}_1^0 \times \boldsymbol{\ell}_2^0) \cdot \boldsymbol{\ell}_3^0} \sum_{\{\nu_1,\nu_2,\nu_3\}} \frac{1}{4|\mathbf{y}|^2} \left(\phi''(|\mathbf{y}|) - \frac{1}{|\mathbf{y}|} \phi'(|\mathbf{y}|) \right) y_i\, y_j\, y_k\, y_\ell \qquad (8.35)$$

where the vector $\mathbf{y} = \nu_i \boldsymbol{\ell}_i$. The right-hand side of this is invariant with respect to the change of any pair of subscripts, and therefore so is the left-hand side. This establishes the symmetries (a) and (b).

Problem 8.4. Show that two sets of lattice vectors $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ and $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$ generate the same lattice if and only if the matrix $[\mu]$, whose elements relate the lattice vectors through

$$\boldsymbol{\ell}_i^{(2)} = \mu_{ij} \boldsymbol{\ell}_j^{(1)},$$

has elements that are integers and has determinant 1.

Problem 8.5. Let $\mathbf{H}$ be any member of the lattice symmetry group $\mathcal{G}(\mathcal{L})$ defined in (8.23). Show that $\det \mathbf{H} = 1$.

Solution: Substitute

$$\mathbf{H}\boldsymbol{\ell}_i^{(1)} = \mu_{ij}\boldsymbol{\ell}_j^{(1)}$$

into the vector identity

$$(\mathbf{H}\boldsymbol{\ell}_1^{(1)} \times \mathbf{H}\boldsymbol{\ell}_2^{(1)}) \cdot \mathbf{H}\boldsymbol{\ell}_3^{(1)} = \det \mathbf{H}\ (\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)}) \cdot \boldsymbol{\ell}_3^{(1)}$$

and expand out the result. This leads to

$$\det\ [\mu] = \det \mathbf{H}$$

after making use of the fact that

$$(\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)}) \cdot \boldsymbol{\ell}_3^{(1)} = (\boldsymbol{\ell}_2^{(1)} \times \boldsymbol{\ell}_3^{(1)}) \cdot \boldsymbol{\ell}_1^{(1)} = (\boldsymbol{\ell}_3^{(1)} \times \boldsymbol{\ell}_1^{(1)}) \cdot \boldsymbol{\ell}_2^{(1)}$$

where each of these expressions represents the volume of the unit cell. Finally, since $\det\ [\mu] = 1$ it follows that $\det \mathbf{H} = 1$.

References:

1. K. Bhattacharya, Unpublished lecture notes, California Institute of Technology,

2. K. Bhattacharya, *Microstructure of Martensite*, Chapter 3, Oxford, 2003.

3. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Appendix B, Cambridge University Press, 1927.

Chapter 9

Brief Remarks on Coupled Problems

While the theory and problems dealt with in these notes have been focused on the purely mechanical theory of an elastic solid, there are many settings in which elasticity is coupled with other physical phenomena. For example thermoelasticity involves the coupling of mechanical and thermal effects, while mechanical and electrical effects are coupled in piezoelectricity.

To adequately deal with such coupled phenomena would require several more chapters which is beyond the scope of these notes. However, given the current interest of many students in "coupled problems", I will provide a very brief introduction to **the formalism** by which one sets-up such theories. Common to each of these settings is

(a) identifying the additional fields involved;

(b) identifying the additional governing physical principles;

(c) deriving the additional field equations from (b);

(d) stating the (primitive) form of the constitutive relations; and

(e) simplifying the form of the constitutive relations using the dissipation inequality. Recall that in Section 4.2.1 we briefly touched on using the dissipation inequality to simplify a constitutive relation.

For example, modeling thermoelasticity requires that in addition to the deformation $\mathbf{y}(\mathbf{x}, t)$, the deformation gradient $\mathbf{F}(\mathbf{x}, t)$ and stress $\mathbf{S}(\mathbf{x}, t)$, one also consider the heat flux $\mathbf{q}(\mathbf{x}, t)$, internal energy $\epsilon(\mathbf{x}, t)$, temperature $\theta(\mathbf{x}, t)$ and entropy $\eta(\mathbf{x}, t)$, at least some of which would have to be determined when solving an initial-boundary value problem. The additional physical principles in this case are the first and second laws of thermodynamics.

In general, one cannot solve a mechanical problem to find $\mathbf{y}(\mathbf{x},t), \mathbf{F}(\mathbf{x},t)$ and $\mathbf{S}(\mathbf{x},t)$ and a separate thermal problem to find the remaining fields. This is because the mechanical and thermal fields are coupled: for example, the constitutive relation for stress depends on both the deformation and the temperature, and the first law of thermodynamics involves both the rate of working of the stress and the rate of heat flux.

As a second example consider a piezoelectric solid. Here one has the following additional fields (in electrostatics): the electric potential $\varphi(\mathbf{x},t)$, electric field $\mathbf{E}(\mathbf{x},t)$ and electric displacement $\mathbf{D}(\mathbf{x},t)$ and one must take into account Maxwell's equations as well as the laws of thermodynamics when setting up the theory.

In the rest of this chapter I will briefly illustrate the underlying mathematical formalism through two examples: hydrogels and thermoelasticity. In each example we will go through steps $(a) - (e)$. Piezoelectric materials will not be touched on since most such materials are ceramics that undergo very small strains. They are treated in Chapter 4 of Volume IV which concerns the linear(ized) theory of elasticity.

We will be using a referential formulation throughout. As usual, the body occupies a region $\mathcal{R}_{\rm R}$ in a reference configuration and an arbitrary part of the body occupies a region $\mathcal{D}_{\rm R} \subset \mathcal{R}_{\rm R}$. The position vector of a particle in the reference configuration is $\mathbf{x}$ and at time t during a motion is $\mathbf{y}(\mathbf{x},t)$. Whenever we say "per unit volume" or "per unit area" we mean "per unit reference volume" or "per unit reference area" unless explicitly stated otherwise. The deformation gradient tensor is $\mathbf{F} = \nabla\mathbf{y}$ and the Piola stress tensor is $\mathbf{S}$. A superior dot, such as in $\dot{\mathbf{F}}$, denotes the time derivative (with $\mathbf{x}$ held fixed – the material time derivative). All processes are assumed to be quasi-static in that inertial effects are neglected. Thus we do not account for linear and angular momentum and kinetic energy.

9.1 Hydrogels:

References:

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2 W. Hong, X. Zhao, J. Zhou and Z. Suo, *A theory of coupled diffusion and large deformation in polymeric gels*, J. Mech. Phys. Solids, , **56**(2008), pp. 1779-1793.

3. S.A. Chester and L. Anand, *A coupled theory of fluid permeation and large deformations for elastomeric materials*, J. Mech. Phys. Solids, **58** (2010), pp. 1879-1906.

4. F.P. Duda, A.C. Souza and E. Fried, *A theory for species migration in a finitely strained solid with application to polymer network swelling*, J. Mech. Phys. Solids, **58** (2010), pp. 515-529.

A hydrogel is essentially an elastic polymer through which a solvent, usually water, diffuses. In what follows we shall speak of a "polymer" (an elastic solid) and a "solvent" (an inviscid fluid) that can move in and out of the polymer. One speaks of the swelling of the hydrogel as it absorbs the solvent. We do not use mixture theory (at least not explicitly) and so speak only of a single effective continuum.

A part $\mathcal{P}$ of the body is "material" with respect to the elastic solid in the sense that the same set of polymer particles are associated with $\mathcal{P}$ at all times. Solvent particles however may diffuse in and out of $\mathcal{D}_{\mathrm{R}}$. Often, the reference configuration is taken to correspond to a dry stress-free state of the polymer, but that particular choice doesn't affect the development below.

In addition to the basic fields of deformation $\mathbf{y}(\mathbf{x},t)$, deformation gradient $\mathbf{F}(\mathbf{x},t)$ and stress $\mathbf{S}(\mathbf{x},t)$, the theory now involves the *concentration* $c_{\mathrm{R}}(\mathbf{x},t)$ of the solvent, the *flux* $\mathbf{j}_{\mathrm{R}}(\mathbf{x},t)$ of the solvent across a surface, and the *chemical potential* of the solvent $\mu(\mathbf{x},t)$. Let $c_{\mathrm{R}}(\mathbf{x},t)$ be the referential solvent concentration so that the number of solvent molecules in $\mathcal{D}_{\mathrm{R}}$ at time t is

$$\int_{\mathcal{D}_{\mathrm{R}}} c_{\mathrm{R}}(\mathbf{x},t)\, dV_x.$$

The number of solvent molecules crossing a unit area of the boundary $\partial\mathcal{D}_{\mathrm{R}}$ in unit time and entering $\mathcal{D}_{\mathrm{R}}$ is $-\mathbf{j}_{\mathrm{R}}\cdot\mathbf{n}_{\mathrm{R}}$ where $\mathbf{j}_{\mathrm{R}}(\mathbf{x},t)$ is the referential solvent flux vector (and $\mathbf{n}_{\mathrm{R}}$ is a unit outward pointing normal vector on $\partial\mathcal{D}_{\mathrm{R}}$). The total rate at which the solvent enters $\mathcal{D}_{\mathrm{R}}$ across $\partial\mathcal{D}_{\mathrm{R}}$ is thus

$$\int_{\partial\mathcal{D}_{\mathrm{R}}} -\mathbf{j}_{\mathrm{R}}\cdot\mathbf{n}_{\mathrm{R}}\, dA_x.$$

The *bulk supply* of solvent molecules per unit volume per unit time (from sources outside the body) is denoted by $r_{\mathrm{R}}(\mathbf{x},t)$, and so the rate at which solvent molecules directly enter the interior of $\mathcal{D}_{\mathrm{R}}$ (in contrast to diffusing across its boundary) is

$$\int_{\mathcal{D}_{\mathrm{R}}} r_{\mathrm{R}}(\mathbf{x},t)\, dV_x.$$

The role of r_{R} is similar to that of the mechanical body force $\mathbf{b}_{\mathrm{R}}$ in that it is usually prescribable. Finally, let $\mu(\mathbf{x},t)$ denote the chemical potential of the solvent. It represents the energy per solvent molecule and so the rate of increase of the chemical energy of $\mathcal{D}_{\mathrm{R}}$ due to the influx of solvent molecules is

$$\int_{\partial\mathcal{D}_{\mathrm{R}}} -\mu\,\mathbf{j}_{\mathrm{R}}\cdot\mathbf{n}_{\mathrm{R}}\,dA_x + \int_{\mathcal{D}_{\mathrm{R}}} \mu\,r_{\mathrm{R}}\,dV_x.$$

9.1.1 Basic mechanical equations. Balance laws and field equations.

Force and moment equilibrium require the usual balance laws

$$\int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{S}\mathbf{n}_{\mathrm{R}}\,dA_x + \int_{\mathcal{D}_{\mathrm{R}}} \mathbf{b}_{\mathrm{R}}\,dV_x = \mathbf{0}, \qquad \int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{y}\times(\mathbf{S}\mathbf{n}_{\mathrm{R}})\,dA_x + \int_{\mathcal{D}_{\mathrm{R}}} \mathbf{y}\times\mathbf{b}_{\mathrm{R}}\,dV_x = \mathbf{0},$$

that lead to the usual field equations

$$\mathrm{Div}\,\mathbf{S} + \mathbf{b}_{\mathrm{R}} = \mathbf{0}, \qquad \mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T. \tag{9.1}$$

9.1.2 Basic chemical equation. Balance law and field equation.

The conservation of solvent molecules requires that

$$\frac{d}{dt}\int_{\mathcal{D}_{\mathrm{R}}} c_{\mathrm{R}}\,dV_x = \int_{\partial\mathcal{D}_{\mathrm{R}}} -\mathbf{j}_{\mathrm{R}}\cdot\mathbf{n}_{\mathrm{R}}\,dA_x \;+\; \int_{\mathcal{D}_{\mathrm{R}}} r_{\mathrm{R}}\,dV_x. \tag{9.2}$$

The left-hand side represents the rate of increase of the number of solvent molecules in $\mathcal{D}_{\mathrm{R}}$. The first term on the right-hand side denotes the number of solvent molecules entering $\mathcal{D}_{\mathrm{R}}$ across its boundary while the second term is the number of solvent molecules added to its interior, both per unit time. Equation (9.2) must hold for all $\mathcal{D}_R$ and so localization leads to the field equation

$$\dot{c}_{\mathrm{R}} + \mathrm{Div}\,\mathbf{j}_{\mathrm{R}} = r_{\mathrm{R}}. \tag{9.3}$$

This must hold at each particle at each time.

9.1.3 Dissipation inequality.

The *dissipation inequality* states that the rate of increase of free energy of $\mathcal{D}_{\mathrm{R}}$ cannot exceed the rate at which mechanical work is done on $\mathcal{D}_{\mathrm{R}}$ plus the rate at which chemical energy is added to $\mathcal{D}_{\mathrm{R}}$. Thus the dissipation inequality requires that

$$\int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{S}\mathbf{n}_{\mathrm{R}}\cdot\mathbf{v}\,dA_x + \int_{\mathcal{D}_{\mathrm{R}}} \mathbf{b}_{\mathrm{R}}\cdot\mathbf{v}\,dV_x + \int_{\partial\mathcal{D}_{\mathrm{R}}} -\mu\mathbf{j}_{\mathrm{R}}\cdot\mathbf{n}_{\mathrm{R}}\,dA_x + \int_{\mathcal{D}_{\mathrm{R}}} \mu r_{\mathrm{R}}\,dV_x \geq \frac{d}{dt}\int_{\mathcal{D}_{\mathrm{R}}} \psi\,dV_x, \tag{9.4}$$

for all subregions $\mathcal{D}_{\mathrm{R}} \subset \mathcal{R}_{\mathrm{R}}$. Here $\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{y}}(\mathbf{x}, t)$ is the particle velocity. The first two terms represent the rate of working by the boundary traction and the body force, while the next two terms denote the influx of chemical energy. The *free energy* (per unit volume) has been denoted by ψ so that

$$\int_{\mathcal{D}_{\mathrm{R}}} \psi \, dV_x$$

is the total free energy of $\mathcal{D}_{\mathrm{R}}$ at time t. Its counterpart in the purely mechanical theory is the strain energy function W.

We can simplify (9.4) by converting the surface integrals to volume integrals using the divergence theorem and then making use of the field equations (9.1) and (9.3). This leads to

$$\int_{\mathcal{D}_{\mathrm{R}}} \Big(\mathbf{S} \cdot \dot{\mathbf{F}} - \mathbf{j}_{\mathrm{R}} \cdot \mathrm{Grad}\, \mu + \mu \dot{c}_{\mathrm{R}} - \dot{\psi} \Big) \, dV_x \geq 0. \tag{9.5}$$

Localization of (9.5) yields

$$\mathbf{S} \cdot \dot{\mathbf{F}} - \mathbf{j}_{\mathrm{R}} \cdot \nabla \mu + \mu \dot{c}_{\mathrm{R}} - \dot{\psi} \geq 0. \tag{9.6}$$

This is the local form of the dissipation inequality and it is required to hold at each point in the body at each time.

9.1.4 Constitutive equations:

Suppose that the material is characterized by the following set of constitutive relations[1]:

$$\psi = \psi(\mathbf{F}, c_{\mathrm{R}}), \qquad \mathbf{S} = \mathbf{S}(\mathbf{F}, c_{\mathrm{R}}), \qquad \mu = \mu(\mathbf{F}, c_{\mathrm{R}}), \qquad \mathbf{j}_{\mathrm{R}} = \mathbf{j}_{\mathrm{R}}(\mathbf{F}, c_{\mathrm{R}}, \nabla \mu). \tag{9.7}$$

This is the primitive form of the constitutive equations. It can be simplified (reduced) using the dissipation inequality as follows:

First note from $(9.7)_1$ that

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} + \frac{\partial \psi}{\partial c_{\mathrm{R}}} \dot{c}_{\mathrm{R}}.$$

Therefore we can write the dissipation inequality (9.6) as

$$\left[\mathbf{S}(\mathbf{F}, c_{\mathrm{R}}) - \frac{\partial \psi}{\partial \mathbf{F}}(\mathbf{F}, c_{\mathrm{R}}) \right] \cdot \dot{\mathbf{F}} + \left[\mu(\mathbf{F}, c_{\mathrm{R}}) - \frac{\partial \psi}{\partial c_{\mathrm{R}}}(\mathbf{F}, c_{\mathrm{R}}) \right] \dot{c}_{\mathrm{R}} - \mathbf{j}_{\mathrm{R}}(\mathbf{F}, c_{\mathrm{R}}, \nabla \mu) \cdot \nabla \mu \geq 0. \tag{9.8}$$

[1]See also Problem 9.1.1 on page 618.

Since (9.8) must hold in all processes, and therefore for all $\dot{\mathbf{F}}$ and $\dot{c}_{\mathrm{R}}$, and since the terms within the square brackets do not involve $\dot{\mathbf{F}}$ and $\dot{c}_{\mathrm{R}}$, it follows that those terms must vanish[2]. This leads to the following *constitutive relations* for $\mathbf{S}$ and μ:

$$\left.\begin{aligned} \mathbf{S} &= \frac{\partial \psi}{\partial \mathbf{F}}(\mathbf{F}, c_{\mathrm{R}}), \\ \mu &= \frac{\partial \psi}{\partial c_{\mathrm{R}}}(\mathbf{F}, c_{\mathrm{R}}). \end{aligned}\right\} \tag{9.9}$$

The dissipation inequality (9.8) now reduces to

$$\mathbf{j}_{\mathrm{R}}(\mathbf{F}, c_{\mathrm{R}}, \mathbf{g}) \cdot \mathbf{g} \leq 0 \tag{9.10}$$

for all vectors $\mathbf{g}$. The argument used in getting to (9.9) and (9.10) from (9.7) and (9.6) is known as the *Coleman-Noll argument*.

Thus a hydrogel is characterized by the free energy $\psi(\mathbf{F}, c_{\mathrm{R}})$ and the solvent flux law $\mathbf{j}_{\mathrm{R}}(\mathbf{F}, c_{\mathrm{R}}, \nabla\mu)$ where the latter must be consistent with (9.10). An example of the latter is

$$\mathbf{j}_{\mathrm{R}}(\mathbf{F}, c_{\mathrm{R}}, \nabla\mu) = -\mathbf{M}(\mathbf{F}, c_{\mathrm{R}})\, \nabla\mu, \tag{9.11}$$

where (9.10) requires the "mobility tensor" $\mathbf{M}$ to be positive semi-definite. This is the well-known Fick's law.

Exercise: Show that material frame indifference implies

$$\psi = \psi(\mathbf{C}, c_{\mathrm{R}}), \qquad \mu = \mu(\mathbf{C}, c_{\mathrm{R}}),$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and that

$$\mathbf{S} = 2\mathbf{F}\frac{\partial \psi}{\partial \mathbf{C}}(\mathbf{C}, c_{\mathrm{R}}).$$

As a consequence of this, moment balance, $(9.1)_2$, holds automatically.

Thus **in summary**, a boundary-initial value problem for a body composed of a hydrogel involves specifying the body force $\mathbf{b}_{\mathrm{R}}(\mathbf{x}, t)$, the solvent supply $r_{\mathrm{R}}(\mathbf{x}, t)$, the material characterizations $\psi(\mathbf{F}, c_{\mathrm{R}})$ and $\mathbf{j}_{\mathrm{R}}(\mathbf{F}, c_{\mathrm{R}}, \nabla\mu)$ and suitable initial and boundary conditions (both mechanical and chemical)[3]. Substituting $(9.9)_2$ and (9.11) into (9.3) gives a scalar partial differential equation involving $\mathbf{y}(\mathbf{x}, t)$ and $c_{\mathrm{R}}(\mathbf{x}, t)$, often referred to casually as the "diffusion equation". Similarly substituting $(9.9)_1$ into the equilibrium equation $(9.1)_1$ gives three scalar (or one vector) partial differential equation. These equations are to be solved for $\mathbf{y}(\mathbf{x}, t)$ and $c_{\mathrm{R}}(\mathbf{x}, t)$.

[2] See Section 4.2.1 for more details on this argument.

[3] See Problem 9.1 for an example.

9.1.5 Alternative form of the constitutive relation.

Exercise: Suppose that you wanted to express the constitutive relations as functions of $\mathbf{F}$ and μ (instead of $\mathbf{F}$ and $c_{\rm R}$) and therefore took the primitive form of the constitutive relations to be

$$\psi = \psi(\mathbf{F}, \mu), \qquad \mathbf{S} = \mathbf{S}(\mathbf{F}, \mu), \qquad c_{\rm R} = c_{\rm R}(\mathbf{F}, \mu), \qquad \mathbf{j}_{\rm R} = \mathbf{j}_{\rm R}(\mathbf{F}, \mu, \operatorname{Grad} \mu). \tag{9.12}$$

Can you use the dissipation inequality (9.6) to simplify these constitutive relations?

As you would have discovered from this exercise, one cannot directly use (9.6) to reduce constitutive response functions of the form (9.12). Since (9.6) involves $\dot{\mathbf{F}}$ and $\dot{c}_{\rm R}$ it favors constitutive characterizations in terms of $\mathbf{F}$ and $c_{\rm R}$. This suggests that if we want to consider constitutive relations that are functions of $\mathbf{F}$ and μ we should seek to rewrite the dissipation inequality in terms $\dot{\mathbf{F}}$ and $\dot{\mu}$.

This is achieved by introducing the function

$$\omega := \psi - \mu c_{\rm R}. \tag{9.13}$$

The transformation from $\psi \to \omega$ is called a Legendre transformation and ω is the *Legendre transform* of ψ with respect to μ and $c_{\rm R}$. In the present setting ω is referred to as the *grand canonical energy*. The dissipation inequality (9.6) can be written in terms of ω as

$$\mathbf{S} \cdot \dot{\mathbf{F}} - c_{\rm R}\dot{\mu} - \dot{\omega} - \mathbf{j}_{\rm R} \cdot \operatorname{Grad} \mu \geq 0. \tag{9.14}$$

Now consider constitutive relations of the primitive form

$$\omega = \omega(\mathbf{F}, \mu), \qquad \mathbf{S} = \mathbf{S}(\mathbf{F}, \mu), \qquad c_{\rm R} = c_{\rm R}(\mathbf{F}, \mu), \qquad \mathbf{j}_{\rm R} = \mathbf{j}_{\rm R}(\mathbf{F}, \mu, \nabla\mu). \tag{9.15}$$

Since,

$$\dot{\omega} = \frac{\partial \omega}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} + \frac{\partial \omega}{\partial \mu}\, \dot{\mu},$$

(9.14) can be written as

$$\left[\mathbf{S} - \frac{\partial \omega}{\partial \mathbf{F}}\right] \cdot \dot{\mathbf{F}} - \left[c_{\rm R} + \frac{\partial \omega}{\partial \mu}\right] \dot{\mu} - \mathbf{j}_{\rm R} \cdot \nabla\mu \geq 0.$$

The Coleman-Noll argument now tells us that

$$\mathbf{S} = \frac{\partial \omega}{\partial \mathbf{F}}(\mathbf{F}, \mu), \qquad c_{\rm R} = -\frac{\partial \omega}{\partial \mu}(\mathbf{F}, \mu),$$

together with $\mathbf{j}_{\rm R}(\mathbf{F}, \mu, \mathbf{g}) \cdot \mathbf{g} \leq 0$.

Problem 9.1.1. In the constitutive ansatz (9.15) we did not treat all of the constitutive response functions symmetrically: we allowed $\mathbf{j}_{\mathrm{R}}$ to depend on $\nabla\mu$ but not $\omega, \mathbf{S}$ and c_{R}. Carry out an analysis that starts from the following set of primitive constitutive relations:

$$\omega = \omega(\mathbf{F}, \mu, \nabla\mu), \qquad \mathbf{S} = \mathbf{S}(\mathbf{F}, \mu, \nabla\mu), \qquad c_{\mathrm{R}} = c_{\mathrm{R}}(\mathbf{F}, \mu, \nabla\mu), \qquad \mathbf{j}_{\mathrm{R}} = \mathbf{j}_{\mathrm{R}}(\mathbf{F}, \mu, \nabla\mu),$$

and use the Coleman-Noll argument to show that $\omega, \mathbf{S}$ and c_{R} must in fact be independent of $\nabla\mu$.

9.2 Thermoelasticity.

References:

1 R. Abeyaratne, *Continuum Mechanics*, Volume II in the series *Lecture Notes on The Mechanics of Solids*, http://web.mit.edu/abeyaratne/lecture_notes.html.

2 P. Chadwick, *Continuum Mechanics*, Dover, 1999.

3 M.E. Gurtin, E. Fried and L. Anand, *The Mechanics and Thermodynamics of Continua*, Cambridge University Press, 2010.

4 C. Truesdell and W. Noll, *The Non-Linear Field Theories of Mechanics*, in *Handbuch der Physik* III/3, edited by S. Flügge, Springer, Berlin, 1965.

In addition to the basic fields of deformation $\mathbf{y}(\mathbf{x}, t)$, deformation gradient $\mathbf{F}(\mathbf{x}, t)$ and stress $\mathbf{S}(\mathbf{x}, t)$, one now has to account for the following fields: *temperature* $\theta(\mathbf{x}, t)$, *heat flux* $\mathbf{q}_{\mathrm{R}}(\mathbf{x}, t)$, *entropy* $\eta(\mathbf{x}, t)$ and *internal energy* $\epsilon(\mathbf{x}, t)$. The additional physical principles to be considered are the first and second laws of thermodynamics. The amount of heat that crosses a unit area of the boundary $\partial\mathcal{D}_{\mathrm{R}}$ in unit time and enters $\mathcal{D}_{\mathrm{R}}$ is $-\mathbf{j}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}}$ where $\mathbf{j}_{\mathrm{R}}(\mathbf{x}, t)$ is the referential heat flux vector (and $\mathbf{n}_{\mathrm{R}}$ is a unit outward pointing normal vector on $\partial\mathcal{D}_{\mathrm{R}}$). The total rate at which heat enters $\mathcal{D}_{\mathrm{R}}$ across $\partial\mathcal{D}_{\mathrm{R}}$ is thus

$$\int_{\partial\mathcal{D}_{\mathrm{R}}} -\mathbf{q}_{\mathrm{R}} \cdot \mathbf{n}_{\mathrm{R}} \, dA_x.$$

The *bulk supply* of heat per unit volume per unit time (from sources outside the body) is denoted by $r_{\mathrm{R}}(\mathbf{x}, t)$ and so the rate at which heat directly enters the interior of $\mathcal{D}_{\mathrm{R}}$ is

$$\int_{\mathcal{D}_{\mathrm{R}}} r_{\mathrm{R}}(\mathbf{x}, t) \, dV_x.$$

Its role is similar to that of the mechanical body force $\mathbf{b}_{\mathrm{R}}$ in that we can usually take it to be prescribable. Let $\eta(\mathbf{x}, t)$ be the entropy per unit volume so that the total entropy in $\mathcal{D}_{\mathrm{R}}$

at time t is
$$\int_{\mathcal{D}_{\mathrm{R}}} \eta(\mathbf{x},t)\, dV_x.$$
Finally, let $\epsilon(\mathbf{x},t)$ denote the internal energy per unit volume. The total internal energy in $\mathcal{D}_{\mathrm{R}}$ is
$$\int_{\mathcal{D}_{\mathrm{R}}} \epsilon\, dV_x.$$

9.2.1 Basic mechanical equations.

As before, force and moment equilibrium require the usual balance laws
$$\int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{S}\mathbf{n}_{\mathrm{R}}\, dA_x + \int_{\mathcal{D}_{\mathrm{R}}} \mathbf{b}_{\mathrm{R}}\, dV_x = \mathbf{0}, \qquad \int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{y}\times(\mathbf{S}\mathbf{n}_{\mathrm{R}})\, dA_x + \int_{\mathcal{D}_{\mathrm{R}}} \mathbf{y}\times\mathbf{b}_{\mathrm{R}}\, dV_x = \mathbf{0},$$
that lead to the associated field equations
$$\mathrm{Div}\,\mathbf{S} + \mathbf{b}_{\mathrm{R}} = \mathbf{0}, \qquad \mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T. \tag{9.16}$$

9.2.2 First law of thermodynamics.

The *first law of thermodynamics* requires
$$\int_{\partial\mathcal{D}_{\mathrm{R}}} \mathbf{S}\mathbf{n}_{\mathrm{R}}\cdot\mathbf{v}\, dA + \int_{\mathcal{D}_{\mathrm{R}}} \mathbf{b}_{\mathrm{R}}\cdot\mathbf{v}\, dV \ + \int_{\partial\mathcal{D}_{\mathrm{R}}} -\mathbf{q}_{\mathrm{R}}\cdot\mathbf{n}_{\mathrm{R}}\, dA + \int_{\mathcal{D}_{\mathrm{R}}} r_{\mathrm{R}}\, dV \ = \ \frac{d}{dt}\int_{\mathcal{D}_{\mathrm{R}}} \epsilon\, dV\,, \tag{9.17}$$
where $\mathbf{v}(\mathbf{x},t) = \dot{\mathbf{y}}(\mathbf{x},t)$ is the particle velocity. The first two terms on the left-hand side represent the rate of mechanical working by the traction on $\partial\mathcal{D}_{\mathrm{R}}$ and the body force on $\mathcal{D}_{\mathrm{R}}$ respectively. The third and fourth terms quantify the heat supplied to $\mathcal{D}_{\mathrm{R}}$ across its boundary and to its interior respectively. The right-hand side is the rate of increase of internal energy. (Since we are considering quasi-static processes and accordingly neglected inertia in the equations of motion, we must neglect kinetic energy in the right-hand side of (9.17).)

We can localize (9.17) by converting the surface integrals to volume integrals using the divergence theorem and then making use of the field equations (9.16). This leads to the local form of the first law,
$$\mathbf{S}\cdot\dot{\mathbf{F}} - \mathrm{Div}\,\mathbf{q}_{\mathrm{R}} + r_{\mathrm{R}} \ = \ \dot{\epsilon}, \tag{9.18}$$
which is to hold at each point in the body and each time. Equation (9.18) is frequently referred to as the *energy equation*.

9.2.3 Dissipation inequality. The second law of thermodynamics.

Note that the temperature θ did not enter into the first law of thermodynamics.

Turning next to the second law, entropy accompanies the flow of heat. Specifically, we take the flow of entropy per unit area per unit time into $\mathcal{D}_{\rm R}$ across its boundary to be $(-\mathbf{q}_{\rm R}/\theta)\cdot\mathbf{n}_{\rm R}$ and the flow directly into the interior of $\mathcal{D}_{\rm R}$ per unit volume per unit time to be $r_{\rm R}/\theta$. Here $\theta > 0$ is the absolute temperature. The total entropy in $\mathcal{D}_{\rm R}$ is given by the volume integral of η. However the rate of increase of entropy and the inflow of entropy need not be balanced. The second law of thermodynamics states that the increase in entropy cannot be less than the inflow of entropy, i.e.

$$\frac{d}{dt}\int_{\mathcal{D}_{\rm R}} \eta \, dV_x \geq \int_{\partial\mathcal{D}_{\rm R}} \frac{(-\mathbf{q}_{\rm R}\cdot\mathbf{n}_{\rm R})}{\theta}\, dA_x + \int_{\mathcal{D}_{\rm R}} \frac{r_{\rm R}}{\theta}\, dV_x \ . \tag{9.19}$$

Thus (when the strict inequality holds) there is a net production of entropy. The entropy inequality plays the role here that the dissipation inequality played in the preceding section.

The entropy inequality (9.19) can be written after using the divergence theorem as

$$\int_{\mathcal{D}_{\rm R}} \left\{\dot\eta + \mathrm{Div}(\mathbf{q}_{\rm R}/\theta) - r_{\rm R}/\theta\right\}\, dV_x \geq 0. \tag{9.20}$$

This must hold for all $\mathcal{D}_{\rm R}$ and so may be localized to obtain

$$\theta\dot\eta - \frac{1}{\theta}(\mathbf{q}_{\rm R}\cdot\nabla\theta) + \mathrm{Div}\,\mathbf{q}_{\rm R} - r_{\rm R} \geq 0. \tag{9.21}$$

This local form of the entropy inequality must hold at each point in the body at each time.

9.2.4 Constitutive equations:

We now turn to the constitutive relations. In order to make use of (9.21) in our analysis we need to eliminate the heat supply term $r_{\rm R}$ (and ideally bring in the stress $\mathbf{S}$). This can be achieved by using the energy equation which leads us to

$$\theta\dot\eta + \mathbf{S}\cdot\dot{\mathbf{F}} - \dot\epsilon - \frac{1}{\theta}(\mathbf{q}_{\rm R}\cdot\nabla\theta) \geq 0. \tag{9.22}$$

Note the presence of $\dot{\mathbf{F}}$ and $\dot\eta$ in (9.22) and recall the remark in the first paragraph of Section 9.1.5. Accordingly, suppose that the material is characterized by the following set of constitutive relations:

$$\epsilon = \epsilon(\mathbf{F}, \eta), \qquad \mathbf{S} = \mathbf{S}(\mathbf{F}, \eta), \qquad \theta = \theta(\mathbf{F}, \eta), \qquad \mathbf{q}_{\mathrm{R}} = \mathbf{q}_{\mathrm{R}}(\mathbf{F}, \eta, \nabla\theta). \tag{9.23}$$

The entropy inequality (9.22) can now be used to reduce these relations to simpler forms using the Coleman-Noll argument. By $(9.23)_1$,

$$\dot{\epsilon} = \frac{\partial \epsilon}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} + \frac{\partial \epsilon}{\partial \eta} \dot{\eta},$$

and so we can write the entropy inequality as

$$\left[\theta(\mathbf{F}, \eta) - \frac{\partial \epsilon}{\partial \eta}(\mathbf{F}, \eta)\right] \dot{\eta} + \left[\mathbf{S}(\mathbf{F}, \eta) - \frac{\partial \epsilon}{\partial \mathbf{F}}(\mathbf{F}, \eta)\right] \cdot \dot{\mathbf{F}} - \frac{1}{\theta}(\mathbf{q}_{\mathrm{R}} \cdot \nabla\theta) \geq 0. \tag{9.24}$$

Since (9.24) must hold in all processes, and therefore for all $\dot{\mathbf{F}}$ and $\dot{\eta}$, and since the terms within the square brackets do not involve $\dot{\mathbf{F}}$ and $\dot{\eta}$, it follows that those terms must vanish. This leads to the following constitutive relations for θ and $\mathbf{S}$:

$$\theta = \frac{\partial \epsilon}{\partial \eta}(\mathbf{F}, \eta), \qquad \mathbf{S} = \frac{\partial \epsilon}{\partial \mathbf{F}}(\mathbf{F}, \eta), \tag{9.25}$$

and the entropy inequality (9.24) reduces to

$$\mathbf{q}_{\mathrm{R}}(\mathbf{F}, \eta, \mathbf{g}) \cdot \mathbf{g} \leq 0, \tag{9.26}$$

which is to hold for all vectors $\mathbf{g}$.

Thus a thermoelastic material is characterized by the internal energy $\epsilon(\mathbf{F}, \eta)$ and the heat flux law $\mathbf{q}_{\mathrm{R}}(\mathbf{F}, \eta, \nabla\theta)$ where the latter must be consistent with (9.26). An example of the latter is

$$\mathbf{q}_{\mathrm{R}}(\mathbf{F}, \eta, \nabla\theta) = -\mathbf{K}(\mathbf{F}, \eta)\, \nabla\theta, \tag{9.27}$$

where (9.26) requires the heat conductivity tensor $\mathbf{K}$ to be positive semi-definite.

9.2.5 Alternative form of the constitutive relation.

Suppose that we wish to express the constitutive relations as functions of $\mathbf{F}$ and θ (instead of $\mathbf{F}$ and η). In order to develop this form of the constitutive relations we must trade the term $\dot{\eta}$ in (9.22) for $\dot{\theta}$ and this is achieved by introducing the Legendre transform of ϵ with respect to η and θ defined by

$$\psi \;=\; \epsilon - \theta\eta; \tag{9.28}$$

ψ is called the *Helmholtz free energy* per unit volume. In terms of ψ, one may rewrite (9.22) as

$$-\eta\dot{\theta} + \mathbf{S}\cdot\dot{\mathbf{F}} - \dot{\psi} - \frac{1}{\theta}(\mathbf{q}_{\mathrm{R}}\cdot\nabla\theta) \geq 0.$$

Now suppose that we want to construct constitutive relations in the form

$$\psi = \psi(\mathbf{F},\theta), \qquad \mathbf{S} = \mathbf{S}(\mathbf{F},\theta), \qquad \eta = \eta(\mathbf{F},\theta), \qquad \mathbf{q}_{\mathrm{R}} = \mathbf{q}_{\mathrm{R}}(\mathbf{F},\theta,\nabla\theta).$$

Since

$$\dot{\psi} = \frac{\partial\psi}{\partial\mathbf{F}}\cdot\dot{\mathbf{F}} + \frac{\partial\psi}{\partial\theta}\dot{\theta},$$

the preceding entropy inequality yields

$$\left[-\eta - \frac{\partial\psi}{\partial\theta}\right]\dot{\theta} + \left[\mathbf{S} - \frac{\partial\psi}{\partial\mathbf{F}}\right]\cdot\dot{\mathbf{F}} - \frac{1}{\theta}(\mathbf{q}_{\mathrm{R}}\cdot\nabla\theta) \geq 0. \tag{9.29}$$

The Coleman-Noll argument thus allows us to write the constitutive relations as

$$\eta = -\frac{\partial\psi}{\partial\theta}(\mathbf{F},\theta), \qquad \mathbf{S} = \frac{\partial\psi}{\partial\mathbf{F}}(\mathbf{F},\theta), \tag{9.30}$$

and the entropy inequality reduces to

$$\mathbf{q}_{\mathrm{R}}(\mathbf{F},\theta,\mathbf{g})\cdot\mathbf{g} \leq 0. \tag{9.31}$$

Thus a thermoelastic material is characterized by the Helmholtz free energy $\psi(\mathbf{F},\theta)$ and the heat flux law $\mathbf{q}_{\mathrm{R}}(\mathbf{F},\theta,\nabla\theta)$ where the latter must be consistent with (9.31). An example of the latter is

$$\mathbf{q}_{\mathrm{R}}(\mathbf{F},\theta,\nabla\theta) = -\mathbf{K}(\mathbf{F},\theta)\,\nabla\theta, \tag{9.32}$$

where $\mathbf{K}$ is the positive semi-definite heat conductivity tensor. This is the familiar *Fourier's law*. The Helmholtz free energy function $\psi(\mathbf{F},\theta)$ here is the counterpart of the strain energy function $W(\mathbf{F})$ of the purely mechanical theory.

Thus **in summary**, a boundary-initial value problem for a thermoelastic body involves specifying the body force $\mathbf{b}_{\mathrm{R}}(\mathbf{x},t)$, the heat supply $r_{\mathrm{R}}(\mathbf{x},t)$, the material characterizations $\psi(\mathbf{F},\theta)$ and $\mathbf{q}_{\mathrm{R}}(\mathbf{F},\theta,\nabla\theta)$ and suitable initial and boundary conditions (both mechanical and thermal). One then solves the field equations (9.16), (9.18) subject to the constitutive relations (9.30), (9.32) in order to determine the deformation $\mathbf{y}(\mathbf{x},t)$ and the temperature $\theta(\mathbf{x},t)$.

9.2.6 Worked examples.

Problem 9.2.1. Consider a one-dimensional setting where a thermoelastic material is characterized by the Helmholtz free energy

$$\psi = \psi(\lambda, \theta), \tag{i}$$

and the heat conduction law

$$q = -K(\lambda, \theta)\frac{\partial \theta}{\partial x}, \tag{ii}$$

where $K > 0$ is a constant and for simplicity have dropped the R from q_{R}. Here λ is the stretch. Write down and simplify the one-dimensional counterparts of the general equations developed above.

Solution: Partial differentiation with respect to λ and θ will be denoted by subscripts while partial differentiation with respect to x and t will be displayed explicitly.

The counterparts of the constitutive equations $(9.30)_2$ and $(9.30)_1$ for stress and entropy are

$$\sigma = \psi_\lambda(\lambda, \theta), \tag{iii}$$

$$\eta = -\psi_\theta(\lambda, \theta), \tag{iv}$$

and the equilibrium equation and energy equation corresponding to $(9.16)_1$ and (9.18) read

$$\frac{\partial \sigma}{\partial x} + b = 0, \tag{v}$$

$$\sigma\frac{\partial \lambda}{\partial t} - \frac{\partial q}{\partial x} = \frac{\partial \epsilon}{\partial t}. \tag{vi}$$

For convenience we have taken the bulk heat supply to vanish, $r_{\mathrm{R}} = 0$, and have dropped the subscript R from b_{R}. Here ϵ is the internal energy and it is related to the Helmholtz free energy ψ by

$$\psi = \epsilon - \theta\eta. \tag{vii}$$

The one-dimensional counterpart of the entropy inequality (9.31) is

$$q\frac{\partial \theta}{\partial x} \leq 0. \tag{$viii$}$$

We first simplify the energy equation (vi):

$$\sigma\frac{\partial \lambda}{\partial t} - \frac{\partial q}{\partial x} \overset{(vi)}{=} \frac{\partial \epsilon}{\partial t} \overset{(vii)}{=} \frac{\partial \psi}{\partial t} + \eta\frac{\partial \theta}{\partial t} + \theta\frac{\partial \eta}{\partial t} \overset{(i)}{=} \psi_\lambda\frac{\partial \lambda}{\partial t} + \psi_\theta\frac{\partial \theta}{\partial t} + \eta\frac{\partial \theta}{\partial t} + \theta\frac{\partial \eta}{\partial t} =$$

$$\overset{(iii),(iv)}{=} \sigma\frac{\partial \lambda}{\partial t} - \eta\frac{\partial \theta}{\partial t} + \eta\frac{\partial \theta}{\partial t} + \theta\frac{\partial \eta}{\partial t},$$

which reduces to

$$-\frac{\partial q}{\partial x} = \theta \frac{\partial \eta}{\partial t}.$$

Substituting (ii) and (iv) into this gives

$$\frac{\partial}{\partial x}\left(K \frac{\partial \theta}{\partial x} \right) = -\theta \psi_{\theta\theta} \frac{\partial \theta}{\partial t} - \theta \psi_{\lambda\theta} \frac{\partial \lambda}{\partial t}. \qquad \square \qquad (ix)$$

The equilibrium equation (v) in view of the constitutive relation (iii) can be written as

$$\psi_{\lambda\lambda} \frac{\partial \lambda}{\partial x} + \psi_{\lambda\theta} \frac{\partial \theta}{\partial x} + b = 0. \qquad \square \qquad (x)$$

Equations $(ix), (x)$ are two partial differential equations for the stretch $\lambda(x,t)$ and the temperature $\theta(x,t)$.

Problem 9.2.2. Consider the particular thermoelastic material characterized by

$$\psi(\lambda, \theta) = \frac{1}{2}\mu(\lambda - 1)^2 - c\beta(\theta - \theta_0)(\lambda - 1) - c\,\theta \ln(\theta/\theta_0), \qquad (xi)$$

$$K(\lambda, \theta) = K \text{ (constant)}, \qquad (xii)$$

where θ_0 is a fixed ("reference") temperature and μ, c, β and K are constant material parameters. Further reduce the equations obtained in Problem 9.2.1. Interpret the four material parameters μ, c, K and β.

Solution: Observe that $\psi = 0$ when $\lambda = 1$ and $\theta = \theta_0$. From (iii) and (ix) the constitutive relation for stress is

$$\sigma = \mu(\lambda - 1) - c\beta(\theta - \theta_0). \qquad (xiii)$$

The parameter μ is therefore the *elastic modulus*. On writing $(xiii)$ as

$$\lambda - 1 = \frac{\sigma}{\mu} + \frac{c\beta}{\mu}(\theta - \theta_0)$$

we see that $c\beta/\mu$ is the *coefficient of thermal expansion*. Observe from $(xiii)$ that $\sigma = 0$ when $\lambda = 1$ and $\theta = \theta_0$.

Since $q = -K\,\partial\theta/\partial x$, K is the *thermal conductivity*.

The various second derivatives of $\psi(\lambda, \theta)$ can be calculated from (xi). They are

$$\psi_{\lambda\lambda} = \mu, \qquad \psi_{\lambda\theta} = -c\beta, \qquad \psi_{\theta\theta} = -c/\theta. \qquad (xiv)$$

Substituting (xiv) into the energy equation (ix) yields

$$K\frac{\partial^2\theta}{\partial x^2} = c\frac{\partial\theta}{\partial t} + c\beta\theta\frac{\partial\lambda}{\partial t}. \qquad \Box \tag{xv}$$

Observe that in general, the energy equation involves both the stretch and the temperature (and that the stretch drops out only if $\beta = 0$). If $\beta = 0$ this simplifies to

$$K\frac{\partial^2\theta}{\partial x^2} = c\frac{\partial\theta}{\partial t},$$

the so-called heat equation.

If one calculates the internal energy using $\epsilon = \psi + \theta\eta = \psi - \theta\psi_\theta$ and (i), one finds

$$\epsilon = \frac{1}{2}\mu(\lambda-1)^2 - c\beta\theta_0(\lambda-1) + c\,\theta.$$

Therefore $c = \partial\epsilon(\lambda,\theta)/\partial\theta$ and so c can be identified with the *specific heat* of the material (or more accurately the specific heat at constant stretch).

Substituting (xi) into (x) allows us to write the equilibrium equation as

$$\mu\frac{\partial\lambda}{\partial x} - c\beta\frac{\partial\theta}{\partial x} + b = 0. \qquad \Box \tag{xvi}$$

Finally, if we write the one-dimensional motion as

$$y = y(x,t) = x + u(x,t)$$

where $u(x,t)$ is the displacement, the stretch is

$$\lambda = 1 + \frac{\partial u}{\partial x}.$$

Therefore the energy equation (xv) and the equilibrium equation (xvi) can be written as

$$K\frac{\partial^2\theta}{\partial x^2} = c\frac{\partial\theta}{\partial t} + c\beta\theta\frac{\partial^2 u}{\partial t\partial x}, \qquad \mu\frac{\partial^2 u}{\partial x^2} - c\beta\frac{\partial\theta}{\partial x} + b = 0. \qquad \Box$$

This is a pair of coupled partial differential equations for $u(x,t)$ and $\theta(x,t)$.

9.3 Exercises.

Problem 9.1. Consider a one-dimensional setting where the body (the hydrogel) occupies the region $\mathcal{R}_{\rm R} = [0, L]$ in a reference configuration. Perhaps it is a slab of thickness L that is infinite in its other dimensions. On one side of the body, the region $x > L$, is an infinite reservoir of solvent at the fixed chemical potential μ_∞ and pressure p_∞. Therefore

$$\mu(L,t) = \mu_\infty, \qquad \sigma(L,t) = -p_\infty \qquad \text{for} \quad t > 0. \tag{i}$$

The left-hand end of the body is fixed and so the displacement there is zero; moreover it is impermeable to the solvent and so the solvent flux there vanishes:

$$u(0,t) = 0, \qquad j_{\rm R}(0,t) = 0 \qquad \text{for} \quad t > 0. \tag{ii}$$

Initially, the body is undeformed which tells us that the displacement vanishes; it is also dry which implies that the solvent concentration vanishes:

$$u(x,0) = 0, \qquad c_{\rm R}(x,0) = 0 \qquad \text{for} \quad 0 < x < L. \tag{iii}$$

Equations (i) and (ii) are the boundary conditions, (iii) are the initial conditions. (Question: do you also need to know whether the body is at rest at the initial instant so that $\dot{u}(x,0) = 0$?) Ignore any solvent supply and body force: $r_{\rm R} = 0, b_{\rm R} = 0$.

Write down the one-dimensional counterparts of the general equations for a hydrogel given in Section 9.1. Take the hydrogel to be characterized by a free energy function $\psi(\lambda, c_{\rm R})$ and a solvent flux law $j_{\rm R} = -M\,\frac{\partial \mu}{\partial x}$ where the mobility $M > 0$ is a constant. You may choose an explicit (not unreasonable) function $\psi(\lambda, c_{\rm R})$. Note: if you decide to use the function ψ given in Problem 9.5 keep in mind that it is for a material with the constraint $\det \mathbf{F} = 1 + \nu c_{\rm R}$.

Calculate the displacement and solvent concentration $u(x,t)$ and $c_{\rm R}(x,t)$ for $0 \le x \le L, t > 0$. What happens when $t \to \infty$?

Problem 9.2. Formulation of the equations for a hydrogel with respect to the *current configuration.* A part $\mathcal{P}$ of the body occupies a region $\mathcal{D}_t$ at time t and $\mathbf{y}$ is the position vector of a particle in that configuration.

(a) Let $c(\mathbf{y},t)$ denote the solvent concentration per unit current volume so that the number of solvent molecules in $\mathcal{D}_t$ at time t is

$$\int_{\mathcal{D}_t} c(\mathbf{y},t)\, dV_y.$$

Show that

$$c_{\rm R} = c{\rm J} \qquad \text{where } {\rm J} = \det \mathbf{F}.$$

(b) Let $\mathbf{j}(\mathbf{y},t)$ be the solvent flux vector characterizing the number of solvent molecules per unit current area crossing into $\mathcal{D}_t$ in unit time across its boundary so that the total solvent flux across $\partial\mathcal{D}_t$ is

$$\int_{\partial\mathcal{D}_t} -\mathbf{j}\cdot\mathbf{n}\,dA_y.$$

Here $\mathbf{n}$ is the unit outward-pointing normal vector on the boundary $\partial\mathcal{D}_t$. Show that

$$\mathbf{j}_{\mathrm{R}} = \mathrm{J}\,\mathbf{F}^{-1}\mathbf{j}.$$

(c) Let $r(\mathbf{y},t)$ be the number of solvent molecules per unit current volume directly entering the interior of $\mathcal{D}_t$ per unit time (from sources outside the body). Show that

$$r_{\mathrm{R}} = r\mathrm{J} \qquad \text{where } \mathrm{J} = \det\mathbf{F}.$$

(d) Show that the conservation of solvent molecules requires the balance law

$$\frac{d}{dt}\int_{\mathcal{D}_t} c\,dV_y = \int_{\partial\mathcal{D}_t} -\mathbf{j}\cdot\mathbf{n}\,dA_y \;+\; \int_{\mathcal{D}_t} r\,dV_y, \tag{9.33}$$

whose associated field equation is

$$\frac{\partial}{\partial t}c(\mathbf{y},t) + \mathrm{div}\,(c\mathbf{v}) + \mathrm{div}\,\mathbf{j} = r. \tag{9.34}$$

(e) If $\mu(\mathbf{y},t)$ is the chemical potential of a solvent molecule show that the rate of increase of chemical energy of $\mathcal{D}_t$ due to the influx of solvent molecules is

$$\int_{\partial\mathcal{D}_t} -\mu\mathbf{j}\cdot\mathbf{n}\,dA_y + \int_{\mathcal{D}_t} \mu r\,dV_y,$$

and thus show that the dissipation per unit current volume, $\Delta(\mathbf{y},t)$, obeys the following global inequality

$$\begin{aligned}\textstyle\int_{\mathcal{D}}\Delta\,dV_y \;\;\overset{\text{def}}{=}\;\; & \int_{\partial\mathcal{D}} \mathbf{T}\mathbf{n}\cdot\mathbf{v}\,dA_y + \int_{\mathcal{D}} \mathbf{b}\cdot\mathbf{v}\,dV_y + \\ + \;\; & \int_{\partial\mathcal{D}} -\mu\mathbf{j}\cdot\mathbf{n}\,dA_y + \int_{\mathcal{D}} \mu r\,dV_y \;-\; \frac{d}{dt}\int_{\mathcal{D}} \psi/\mathrm{J}\,dV_y \;\; \geq 0\end{aligned} \tag{9.35}$$

where $\mathbf{b}_{\mathrm{R}} = \mathrm{J}\mathbf{b}$ and $\mathbf{T}$ is the Cauchy stress tensor. Show that the local version of this inequality is

$$\mathbf{T}\cdot\mathbf{L} - \mathbf{j}\cdot\mathrm{grad}\,\mu + \mu\dot{c} + \mu c\,\mathrm{div}\,\mathbf{v} \geq \frac{1}{\mathrm{J}}\,\dot{\psi} \tag{9.36}$$

Problem 9.3. *Current polymer volume fraction.* Let ϕ be the volume of polymer per unit volume in the current configuration. Show that

$$\phi = 1 - \frac{\nu c_{\mathrm{R}}}{\det\mathbf{F}}, \tag{9.37}$$

where ν is the volume of a solvent molecule (assumed to be the same in any configuration).

Problem 9.4. Suppose that the polymeric matrix itself is incompressible (and the volume ν of a solvent molecule is the same in all configurations). This does *not* mean that the volume of a material region $\mathcal{D}_t$ does not increase, rather, that it increases solely due to the addition of solvent molecules. Show that the following relation between the deformation and solvent concentration,

$$\det \mathbf{F} = 1 + \nu c_{\mathrm{R}}, \tag{9.38}$$

is consistent with this constraint.

Problem 9.5. Recall that in the purely mechanical theory, the constitutive relation for stress,

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}},$$

had to be modified if the body could only undergo motions in which $\mathrm{J} = \det \mathbf{F} = 1$ (i.e. if it was incompressible). Carry out a similar modification to the constitutive relations for a hydrogel if it can only undergo processes in which $\det \mathbf{F} = 1 + \nu c_{\mathrm{R}}$ and show that

$$\mathbf{S} = \frac{\partial \psi}{\partial \mathbf{F}} - p\mathrm{J}\mathbf{F}^{-T}, \qquad \mu = \frac{\partial \psi}{\partial c_{\mathrm{R}}} + p\nu,$$

where the constant ν is the volume of a solvent molecule.

Remark 1: Note that the constraint $\det \mathbf{F} = 1 + \nu c_{\mathrm{R}}$ is not purely kinematic since it involves both the deformation and the solvent concentration.

Remark 2: The Helmholtz free energy function for a hydrogel is frequently taken to have the specific form

$$\psi(\mathbf{F}, c_{\mathrm{R}}) = W(\mathbf{F}) + \mu_{\mathrm{R}} c_{\mathrm{R}} + \psi_m(c_{\mathrm{R}}) \qquad \text{where} \quad \psi_m(c_{\mathrm{R}}) = k_B \theta c_{\mathrm{R}} \left[\ln\left(\frac{\nu c_{\mathrm{R}}}{1 + \nu c_{\mathrm{R}}} \right) + \frac{\chi}{1 + \nu c_{\mathrm{R}}} \right].$$

Here the constant parameter μ_{R} is the chemical potential of the pure solvent (in the absence of the polymer), $\psi_m(c_{\mathrm{R}})$ is the chemical potential "due to mixing" and the constant χ is known as the Flory-Huggins parameter. The separable form of ψ into one part that depends on $\mathbf{F}$ and not c_{R} and a second part that depends on c_{R} and not $\mathbf{F}$ might lead one to expect the mechanical and chemical problems to be decoupled. This is incorrect – they are in fact coupled through the constraint $\det \mathbf{F} = 1 + \nu c_{\mathrm{R}}$.

Chapter 10

Introduction to Variational Methods

> " ... nothing in all of the world will occur in which no maximum or minimum rule is somehow shining forth ...". Leonhard Euler, 1744.

10.1 Preliminary remarks.

Numerous problems in physics can be formulated as mathematical problems in optimization. For example in optics, Fermat's principle states that the path taken by a ray of light in propagating from one point to another is the path that minimizes the travel time. If a (non-planar) wire loop is dipped into soapy water, the soap film that forms across the loop is the one that minimizes the surface energy (which under most circumstances is equivalent to minimizing the surface area of the soap film).

Many equilibrium problems in mechanics involve finding a configuration that minimizes the potential energy of the system. For example a heavy elastic cable that hangs under gravity between two fixed pegs adopts the shape that, from among all possible shapes, minimizes the potential energy of the system (comprised of the sum of the gravitational and elastic potential energies). Or, if we subject a slender, straight, column to a compressive axial force, its deformed configuration is the one that minimizes the total energy of the system that, depending on the load, may be straight or bent (buckled).

This chapter is focused on the Principle of Minimum Potential Energy, a principle that is intimately related to the notion of stability. The principle is presented in Section 10.3 and several applications are described by the worked examples in Section 10.4. Some of these examples are concerned with obtaining approximate solutions to problems (e.g. nonsymmetric cavitation and end-effects in a bar in the spirit of Saint-Venant). We also revisit the stability of the Rivlin cube. In Section 10.5 we describe the Virtual Work Principle and the weak formulation of a problem. We do not address the Complementary Energy Principle; the reader can find a discussion of this in Section 5.4.3 of Ogden [3].

We start this chapter with a very brief and informal introduction to the calculus of variations, the main mathematical tool underlying minimization principles. We will not address several important issues. A somewhat more detailed treatment can be found in Chapter 7 of Volume I. The student unfamiliar with this topic could read a book dedicated to this subject, e.g. Gelfand and Fomin [1] or Troutman [5]. The appendix in Section 10.7 makes a few remarks on local versus global minimizers and weak versus strong minimizers.

10.2 A brief introduction to the calculus of variations.

Problems addressed by the calculus of variations involve a *scalar-valued quantity* Φ, such as the energy, that depends on a *function* $z(x)$ characterizing a possible configuration of the system, and from among all possible configurations $z(x)$ we want to find the particular one, say $y(x)$, that minimizes Φ. Note that the scalar-valued quantity Φ is defined on a *set of functions*; we shall denote this set by $\mathcal{A}$ ("A" for admissible). One refers to Φ as a *functional* and we write $\Phi\{z\}$. Thus $\Phi\{z\}$ is defined for all $z \in \mathcal{A}$ and takes values in $\mathbb{R}$. We are interested in the particular function $y \in \mathcal{A}$ that minimizes Φ over $\mathcal{A}$:

$$\Phi\{z\} \geq \Phi\{y\} \qquad \text{for all} \quad z \in \mathcal{A}.$$

As a specific example, consider the so-called Brachistochrone Problem. Two given points $(0, h)$ and $(1, 0)$ in the x, y-plane (with $h > 0$) are to be joined by a smooth wire as depicted in Figure 10.1. A bead is released from rest from the point $(0, h)$ and slides along the wire due to gravity. For what shape of wire is the travel time T from $(0, h)$ to $(1, 0)$ least? One can show that this travel time is given by

$$T\{w\} = \int_0^1 \sqrt{\frac{1 + (w'(x))^2}{2g(h - w(x))}}\, dx, \tag{10.1}$$

where $w(x)$, $0 \leq x \leq 1$, describes the shape of the wire; necessarily, $w(0) = h$ and $w(1) = 0$. Observe that $T\{w\}$ is scalar-valued and it is defined on a certain set of functions w: it is a functional. Our task is to find, from among all such functions w, the one that minimizes $T\{w\}$.

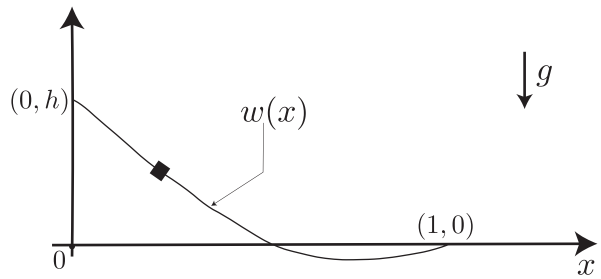

Figure 10.1: Curve joining $(0,h)$ to $(1,0)$ along which a bead slides under gravity.

This minimization takes place over a set of functions w, and in order to complete the formulation of the problem one must characterize this set of *admissible functions* (or *test functions*). A generic curve is described by a function $w(x)$, $0 \le x \le 1$. Since we are only interested in curves that pass through the points $(0,h)$ and $(1,0)$ we must require that $w(0) = h, w(1) = 0$. Finally, we do not want to consider curves that are discontinuous or have corners and so limit attention to functions that are continuous and have a continuous slope, i.e. w and w' are both continuous on $[0,1]$. Thus the set $\mathcal{A}$ of *admissible functions* that we wish to consider is[1]

$$\mathcal{A} = \left\{ \ w \in C^1[0,1] : \ w(0) = h, \ w(1) = 0 \right\}. \tag{10.2}$$

Our task is to minimize[2] $T\{w\}$ over the set $\mathcal{A}$.

One can consider various variants of the Brachistochrone problem. For example, the length of the curve joining the two points might be prescribed, in which case the minimization is to be carried out subject to the constraint that the length is given. Or perhaps the position of the left-hand end is prescribed as above but the right-hand end of the wire might be allowed to lie anywhere on the vertical line through $x = 1$. Or, there might be some prohibited region of the x, y-plane through which the path is disallowed from passing. And so on.

[1]A function $w(x)$ here is defined for $x \in [0,1]$ and its value is a real number. Thus w maps the interval $[0,1]$ into real numbers and this is written as $w : [0,1] \mapsto \mathbb{R}$. We have omitted writing this in (10.2).

[2]Since the path with the shortest *distance* between two points is the straight line that joins them, it is natural to wonder whether a straight line is also the curve that gives the minimum travel time. To investigate this consider (a) a straight line and (b) a family of circular arcs joining $(0,h)$ to $(1,0)$. Use (10.1) to calculate the travel time for each of these paths and show that the straight line is *not* the path that gives the least travel time.

Generalizing (10.1), the simplest problem in the *calculus of variations* involves finding a function $u(x)$ that minimizes a functional $F\{w\}$ of the form[3]

$$F\{w\} = \int_0^1 f(x, w, w')dx$$

over a set of admissible test functions $\mathcal{A}$. The admissible functions w are subject to certain conditions including: smoothness requirements, possibly (but not necessarily) boundary conditions at both ends $x = 0, 1$, and possibly (but not necessarily) side constraints of various forms. Our interest is in finding a function $u \in \mathcal{A}$ such that

$$F\{w\} \geq F\{u\} \qquad \text{for all } w \in \mathcal{A}.$$

10.2.1 Minimizing a functional.

Consider a function[4] $f(x, y, z)$ with continuous first and second partial derivatives with respect to its arguments. Let F be the functional

$$F\{w\} = \int_0^1 f(x, w(x), w'(x))\, dx, \tag{10.3}$$

defined for all functions w in the admissible set

$$\mathcal{A} = \{w \in C^1[0, 1] : \quad w(0) = a, w(1) = b\}. \tag{10.4}$$

Let $u(x) \in \mathcal{A}$ be a minimizer of (10.3) so that

$$F\{w\} \geq F\{u\} \qquad \text{for all } w \in \mathcal{A}.$$

Now consider the family of admissible functions $w(x; \epsilon) = u(x) + \epsilon\eta(x)$ for some sufficiently smooth function $\eta(x)$ and scalar $\epsilon \in [-\epsilon_0, \epsilon_0]$. We only consider functions $\eta(x)$ that do not

[3] From (10.1), the function $f(x, y, z)$ in the brachistochrone problem is

$$f(x, y, z) = \sqrt{\frac{1 + z^2}{2g(h - y)}}.$$

[4] For example the function f corresponding to the brachistochrone problem is, from (10.1),

$$f(x, y, z) = \sqrt{\frac{1 + z^2}{2g(h - y)}}.$$

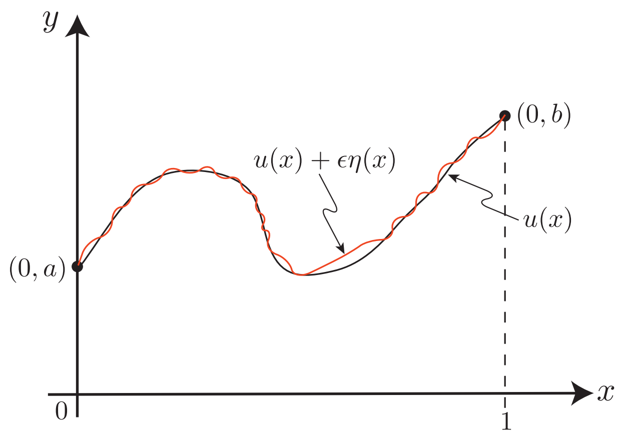

Figure 10.2: Two neighboring functions $u(x)$ and $u(x) + \epsilon\eta(x)$ passing through $(0, a)$ and $(1, b)$. The function $\epsilon\eta(x)$ corresponds to the difference between the two curves in the figure.

depend on ϵ. As a consequence, at each x, the values of the function $w(x, \epsilon)$ *and its first derivatives* approach those of $u(x)$ as $\epsilon \to 0$. Such a variation is said to be a weak variation and the extremum a weak extremum; see the Appendix in Section 10.7. The *difference* between the two curves in Figure 10.2 corresponds to the function $\epsilon\eta(x)$. Since $u(x)$ and $u(x) + \epsilon\eta(x)$ are both admissible, each obeys the boundary conditions in (10.4),

$$\left.\begin{aligned} u(0) = a, \qquad & u(1) = b, \\ u(0) + \epsilon\eta(0) = a, \qquad & u(1) + \epsilon\eta(1) = b, \end{aligned}\right\} \quad \text{for all } \epsilon \in [-\epsilon_0, \epsilon_0], \tag{10.5}$$

and so $\eta(x)$ satisfies the boundary conditions

$$\eta(0) = 0, \qquad \eta(1) = 0. \tag{10.6}$$

An *admissible variation* is any smooth enough function $\eta(x)$ conforming to (10.6) and the set of all admissible variations is

$$\mathcal{V} = \{\eta \in C^1[0, 1] : \eta(0) = \eta(1) = 0\}. \tag{10.7}$$

Since u is a minimizer, it is necessary that

$$F\{u + \epsilon\eta\} \geq F\{u\} \qquad \text{for all } \eta \in \mathcal{V},\ \epsilon \in [-\epsilon_0, \epsilon_0]. \tag{10.8}$$

On keeping $\eta \in \mathcal{V}$ fixed for the moment, the minimizer $u \in \mathcal{A}$ being of course fixed, we can transform this minimization problem to the (familiar) problem in calculus of minimizing a scalar-valued *function* of a scalar variable ϵ. To this end we introduce the function $\overline{F}(\epsilon)$ defined by

$$\overline{F}(\epsilon) := F\{u + \epsilon\eta\} = \int_0^1 f(x, u + \epsilon\eta, u' + \epsilon\eta')\, dx, \qquad -\epsilon_0 < \epsilon < \epsilon_0. \tag{10.9}$$

It follows from (10.8) and (10.9) that $\overline{F}(\epsilon) \geq \overline{F}(0)$ for $-\epsilon_0 < \epsilon < \epsilon_0$ and therefore that $\overline{F}(\epsilon)$ has a local minimum at $\epsilon = 0$. Thus we must have

$$\overline{F}\,'(0) = \frac{d}{d\epsilon}\overline{F}(\epsilon)\Big|_{\epsilon=0} = 0, \qquad \overline{F}\,''(0) = \frac{d^2}{d\epsilon^2}\overline{F}(\epsilon)\Big|_{\epsilon=0} \geq 0. \tag{10.10}$$

Explicitly calculating the first derivative,

$$\frac{d}{d\epsilon}\overline{F}(\epsilon) \;=\; \frac{d}{d\epsilon}\left(\int_0^1 f(x, u + \epsilon\eta, u' + \epsilon\eta')\, dx\right) = \int_0^1 \left(\frac{\partial f}{\partial u}\,\eta + \frac{\partial f}{\partial u'}\,\eta'\right) dx,$$

where the terms $\partial f/\partial u$ and $\partial f/\partial u'$ are evaluated at $(x, u + \epsilon\eta, u' + \epsilon\eta')$. By setting $\epsilon = 0$ and using $(10.10)_1$ we get the necessary condition

$$\frac{d}{d\epsilon}\overline{F}(\epsilon)\Big|_{\epsilon=0} = \int_0^1 \left[\frac{\partial f}{\partial u}(x, u, u')\,\eta + \frac{\partial f}{\partial u'}(x, u, u')\,\eta'\right] dx \;=\; 0, \tag{10.11}$$

where the terms $\partial f/\partial u$ and $\partial f/\partial u'$ have now been evaluated at (x, u, u') as shown.

Thus far we held $\eta \in \mathcal{V}$ fixed. We now take advantage of the fact that (10.11) must in fact hold for all $\eta \in \mathcal{V}$, and use this to eliminate η. To this end we aim to integrate (10.11) by parts so as to trade the term η' for η, and in preparation for this we write the preceding equation as

$$\int_0^1 \left[\frac{\partial f}{\partial u}\eta + \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\eta\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right)\eta\right] dx = 0.$$

Integrating the middle term gives

$$\int_0^1 \left[\frac{\partial f}{\partial u}\eta - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right)\eta\right] dx \;+\; \left[\frac{\partial f}{\partial u'}\eta\right]_0^1 = 0. \tag{10.12}$$

Since this holds for all admissible variations $\eta \in \mathcal{V}$, and since $\eta(0) = \eta(1) = 0$, we are thus led to

$$\int_0^1 \left[\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right)\right]\eta\, dx = 0 \qquad \text{for all } \eta \in \mathcal{V}. \tag{10.13}$$

Fundamental lemma of the calculus of variations: By specializing the localization result in Problem 1.42 to a function of a single scalar variable, we have the following result: let $p(x)$ be a *continuous* function on $[0, 1]$ and suppose that

$$\int_0^1 p(x)n(x)dx \;=\; 0$$

for *all* continuous functions $n(x)$ with $n(0) = n(1) = 0$. Then

$$p(x) = 0 \qquad \text{for} \quad 0 < x < 1;$$

see Lemma 1 in Chapter 1 of Gelfand and Fomin [1] for a proof.

Since (10.13) holds for all $\eta \in \mathcal{V}$, it follows from the fundamental lemma of the calculus of variations that the term in square brackets in (10.13) must vanish at each point in the interval over which the integral is being taken, i.e.

$$\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right) = 0 \qquad \text{for} \quad 0 < x < 1. \tag{10.14}$$

This is referred to as the **Euler-Lagrange equation** (or the Euler equation) associated with the functional (10.3). The minimizer $u(x)$ is found by solving this second order ordinary differential equation together with the boundary conditions $u(0) = a, u(1) = b$.

Observe that the functional (10.3) is well-defined for functions $w(x)$ that are continuous and have continuous first derivatives. This is the smoothness we insisted on in (10.4). However the Euler-Lagrange equation (10.14) involves the second derivative of the minimizing function $u(x)$. The fact that $\frac{d}{dx} f_{u'}$ exists is guaranteed by Lemma 4 in Chapter 1 of Gelfand and Fomin [1].

A word of caution about notation: In writing $\partial f / \partial u$ and $\partial f / \partial u'$ we treat u and u' as independent variables in $f(x, u, u')$. More precisely, with $f(x, y, z)$ being the underlying function, we are writing

$$\frac{\partial f}{\partial u} := \left.\frac{\partial f}{\partial y}\right|_{(x,y,z)=(x,u,u')} \qquad \text{and} \qquad \frac{\partial f}{\partial u'} := \left.\frac{\partial f}{\partial z}\right|_{(x,y,z)=(x,u,u')}.$$

On the other hand d/dx refers to differentiating $f(x, u(x), u'(x))$ with respect to x.

10.2.2 Worked examples.

Problem 10.2.1. Consider the system depicted in Figure 10.3 where an elastic string is stretched to some (large) tension T and its ends are attached to the fixed points $(0, 0)$ and $(L, 0)$. The x-axis is chosen as shown in the figure. A distributed force (per unit length[5]) $b(x)$ is applied along the string and the string rests on an elastic foundation of stiffness (per unit length) $k(x)$.

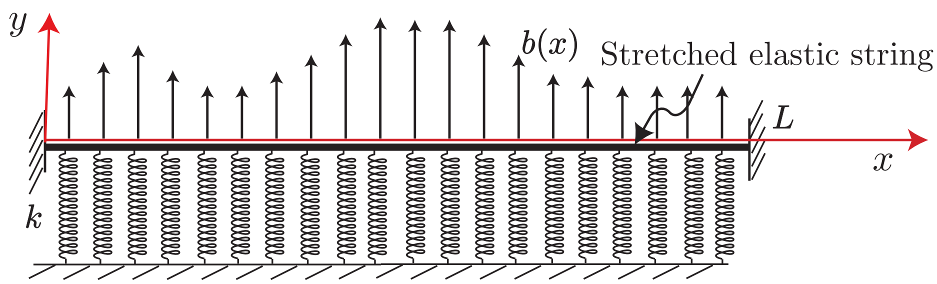

Figure 10.3: A stretched elastic string attached to an elastic foundation carrying a distributed force $b(x)$.

A geometrically (kinematically) possible displacement of the string is described by any sufficiently smooth function $w(x)$ that obeys the boundary conditions $w(0) = w(L) = 0$. Thus the set $\mathcal{A}$ of all kinematically admissible displacement fields is

$$\mathcal{A} = \quad \{w \in C^1[0, L] : \ w(0) = 0, w(L) = 0\}. \tag{i}$$

The total potential energy associated with any $w \in \mathcal{A}$ is comprised of (the integral over the string of) the elastic potential energy of the string[6] $\frac{1}{2}T(w'(x))^2$, plus the elastic potential energy of the foundation $\frac{1}{2}k(x)(w(x))^2$, plus the potential energy of the dead loading $-b(x)w(x)$:

$$\Phi\{w\} = \int_0^L \left[\frac{1}{2}T\big(w'(x)\big)^2 + \frac{1}{2}k(x)w^2(x) - b(x)w(x)\right] \, dx \qquad \text{for all } w \in \mathcal{A}. \tag{ii}$$

The displacement has been taken to be positive in the upward direction. Our aim is to find the particular displacement $u \in \mathcal{A}$ that minimizes the potential energy functional Φ over the set $\mathcal{A}$:.

When (ii) is written in the form (10.3) we have

$$f(x, u, u') = \frac{1}{2}Tu'^2 + \frac{1}{2}k(x)u - b(x)u, \tag{iii}$$

whence

$$\frac{\partial f}{\partial u} = ku - b, \qquad \frac{\partial f}{\partial u'} = Tu'; \tag{iv}$$

or more formally as

$$f(x, y, z) = \frac{1}{2}Tz^2 + \frac{1}{2}k(x)y - b(x)y, \qquad \frac{\partial f}{\partial y} = ky - b, \quad \frac{\partial f}{\partial z} = Tz,$$

so that

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y,z)=(x,u,u')} = ku - b =: \frac{\partial f}{\partial u}, \qquad \left.\frac{\partial f}{\partial z}\right|_{(x,y,z)=(x,u,u')} = Tu' =: \frac{\partial f}{\partial u'}.$$

[5] We shall be only concerned with small deflections of the string so we do not need to distinguish between force per unit reference length and force per unit deformed length.

[6] Exercise: Derive the expression $\frac{1}{2}T(w'(x))^2$ for the elastic energy in the string.

Substituting (iv) into the Euler-Lagrange equation (10.14) yields

$$ku - b - \frac{d}{dx}(Tu') = 0 \qquad \Rightarrow \qquad -Tu''(x) + k(x)u(x) - b(x) = 0, \qquad 0 < x < L.$$

The displacement field $u(x)$ that minimizes the potential energy can now be found by solving this second order ordinary differential equation together with the boundary conditions $u(0) = u(L) = 0$.

Problem 10.2.2. Minimize the functional

$$F\{w\} = \int_0^1 f(w(x), w'(x))\, dx$$

over the admissible set (10.4); note that the function f in the integrand here does not explicitly depend on x, i.e. $f = f(\cancel{x}, w(x), w'(x))$. Show that the Euler-Lagrange equation satisfied by the extremizer $u(x)$ has the first integral (i.e. can be integrated once to give)

$$f(u, u') - u' \frac{\partial f}{\partial u'}(u, u') = c \text{ (constant)}. \tag{10.15}$$

10.2.3 A formalism for deriving the Euler-Lagrange equation.

In order to expedite the steps involved in deriving the Euler-Lagrange equation, one usually uses the following formal procedure. First, one adopts the following *notation*: if H is a function or functional that depends on $u(x)$, then by δH we mean

$$\delta H \;:=\; H\{u + \epsilon\eta\} - H\{u\} \quad \text{up to terms linear in } \epsilon, \tag{10.16}$$

that is,

$$\delta H \;:=\; \epsilon \left[\frac{dH}{d\epsilon}\{u + \epsilon\eta\}\right]_{\epsilon=0}. \tag{10.17}$$

For example, by $\delta u(x)$ we mean

$$\delta u(x) = \big[u(x) + \epsilon\eta(x)\big] - \big[u(x)\big] = \epsilon\eta(x); \tag{10.18}$$

and, for example,

$$\delta u^2 = \delta(u^2) = \big[u + \epsilon\eta\big]^2 - \big[u\big]^2 \text{ up to linear terms} = 2u\epsilon\eta \overset{(10.18)}{=} 2u\,\delta u. \tag{10.19}$$

If $\eta(0) = \eta(1) = 0$, then

$$\delta u(0) = \delta u(1) = 0. \tag{10.20}$$

Similarly $\delta u' = \delta(u')$ denotes

$$\delta u' = \delta(u') = \big[u' + \epsilon\eta'\big] - \big[u'\big] = \epsilon\eta' \overset{(10.18)}{=} (\delta u)'; \tag{10.21}$$

and, for example,

$$\delta(u')^2 = \big[u' + \epsilon\eta'\big]^2 - \big[u'\big]^2 \text{ up to linear terms} = 2u'\epsilon\eta' \overset{(10.21)}{=} 2u'\,\delta u'. \tag{10.22}$$

Generalizing to a function $f(x,u,u')$, by δf we mean

$$\delta f = f(x,\ u+\epsilon\eta,\ u'+\epsilon\eta') - f(x,u,u') \quad \text{up to linear terms,} \tag{10.23}$$

which can be written as

$$\delta f = \frac{\partial f}{\partial u}(x,u,u')\,\epsilon\eta + \frac{\partial f}{\partial u'}(x,u,u')\,\epsilon\eta' \overset{(10.18),(10.21)}{=} \frac{\partial f}{\partial u}\,\delta u + \frac{\partial f}{\partial u'}\,\delta u'. \tag{10.24}$$

Finally, for a functional $F = \displaystyle\int_0^1 f\,dx$, by δF we mean

$$\begin{aligned}
\delta F \quad &= \quad \delta\int_0^1 f\,dx = F\{u+\epsilon\eta\} - F\{u\} = \\
&= \quad \int_0^1 f(x,u+\epsilon\eta,u'+\epsilon\eta')dx \; - \; \int_0^1 f(x,u,u')dx \quad \text{up to linear terms} \; = \\
&= \quad \int_0^1 \Big[f(x,u+\epsilon\eta,u'+\epsilon\eta') \; - \; f(x,u,u')\Big]\,dx \quad \text{up to linear terms} \; = \\
&\overset{(10.23)}{=} \quad \int_0^1 \delta f\,dx \overset{(10.24)}{=} \int_0^1 \left[\frac{\partial f}{\partial u}\,\delta u + \frac{\partial f}{\partial u'}\,\delta u'\right]\,dx.
\end{aligned} \tag{10.25}$$

Note that δF is linear with respect to the function $\delta u(x)$. Moreover, note by making use of (10.18), (10.21) and (10.24) that the steps involved in the calculation (10.25) can be written as

$$\delta F = \delta\int_0^1 f\,dx = \int_0^1 \delta f\,dx = \int_0^1 \left[\frac{\partial f}{\partial u}\,\delta u + \frac{\partial f}{\partial u'}\,\delta u'\right]\,dx. \tag{10.26}$$

Observe that the variation δ does not operate on x and so (as in (10.24)) in the last step we wrote $\delta f = f_u\delta u + f_{u'}\delta u'$ and *not* $\delta f = f_x\delta x + f_u\delta u + f_{u'}\delta u'$. This is because it is the function $u(x)$ that is being varied, not the independent variable x.

One refers to $\delta u(x)$ as an **admissible variation** and to δF as the **first variation** of the functional F. Note from (10.25) that the first variation δF depends on both u and δu: $\delta F = \delta F\{u, \delta u\}$. Observe also that

$$\delta F\{u,\delta u\} = F\{u+\delta u\} - F\{u\} \quad \text{up to linear terms.}$$

The first variation of F necessarily vanishes at a minimizer u:

$$\boxed{\delta F\{u, \delta u\} = 0 \quad \text{for all admissible variations } \delta u.} \tag{10.27}$$

The derivation of the Euler-Lagrange equation in Section 10.2.1 can be carried out expeditiously using the present notation[7]. Given a functional

$$F\{w\} = \int_0^1 f(x, w(x), w'(x))\, dx \tag{10.28}$$

defined for all functions w in some admissible set $\mathcal{A}$, the first variation of F vanishes at the minimizer u, and so from (10.26)

$$0 = \delta F = \delta \int_0^1 f\, dx = \int_0^1 \delta f\, dx = \int_0^1 \left[\frac{\partial f}{\partial u}\, \delta u + \frac{\partial f}{\partial u'}\, \delta u' \right] dx. \tag{10.29}$$

Integrating the term involving $\delta u'$ by parts, using the boundary conditions $\delta u(0) = \delta u(1) = 0$ and the fundamental lemma of the calculus of variations (page 635) yields the Euler-Lagrange equation (10.14).

10.2.4 Natural boundary conditions.

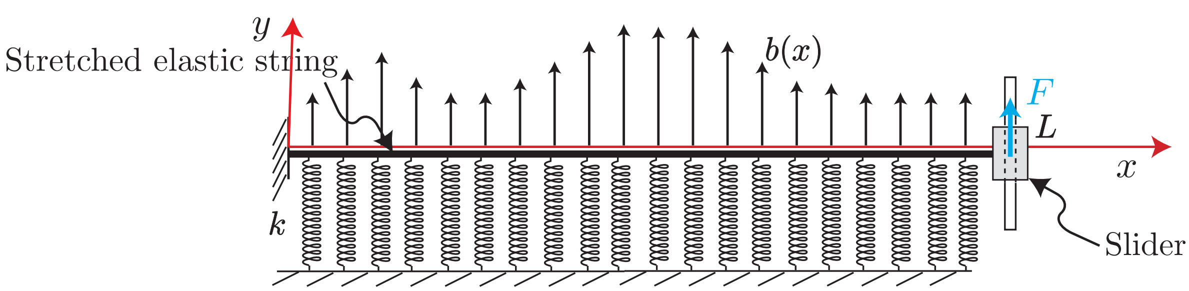

Figure 10.4: A stretched elastic string is attached to an elastic foundation and carries a distributed force $b(x)$. The right-hand end of the string is attached to a slider on which an externally applied force F acts. The slider can move on a frictionless vertical rail.

Reconsider the problem concerning the stretched elastic string looked at on page 635 but now suppose it is attached to a (massless) slider at its right-hand end. An externally applied upward force of magnitude F acts on the slider and it is free to move on a frictionless

[7] If ever in doubt about a particular step during a calculation, always go back to the meaning of the symbols δu, etc. or revert to using $\epsilon\eta$.

vertical rail as shown in Figure 10.4. The string continues to be fixed at its left-hand end. Therefore a kinematically admissible displacement $w(x)$ must obey the boundary condition $w(0) = 0$ but the value of $w(L)$ is not specified. Thus the set $\mathcal{A}$ of all geometrically admissible displacements is now

$$\mathcal{A} = \{w \in C^1[0, L] : \ w(0) = 0\}. \tag{i}$$

Note that the class of admissible functions $\mathcal{A}$ here is larger than before. The total potential energy of the system associated with any $w \in \mathcal{A}$ is

$$\Phi\{w\} = \int_0^L \left[\frac{1}{2}T\big(w'(x)\big)^2 + \frac{1}{2}kw^2(x) - b(x)w(x)\right] \, dx - Fw(L), \tag{ii}$$

where, again, the displacement is taken to be positive in the upward direction, and the last term in (ii) is the potential energy of the concentrated force on the slider.

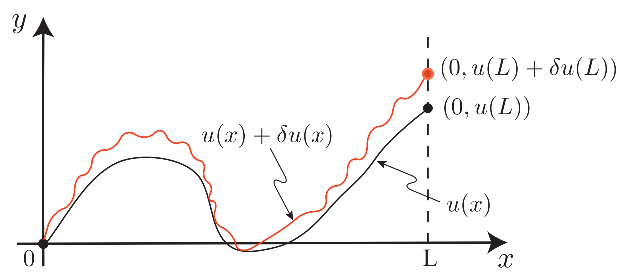

Figure 10.5: The minimizer $u(x)$ and the neighboring function $u(x)+\delta u(x)$ both pass through $(0, 0)$. Their values at $x = L$ are not prescribed and therefore observe that $\delta u(L)$ need not vanish. The function $\delta u(x)$ corresponds to the difference between the two curves in the figure and so $\delta u(0) = 0$ and $\delta u(L)$ is arbitrary.

Generalizing this, we now return to the functional

$$F\{w\} = \int_0^1 f(x, w(x), w'(x)) \, dx, \tag{10.30}$$

that now is defined for all functions w in the admissible set

$$\mathcal{A} = \{w \in C^1[0, 1] : \ w(0) = a\}. \tag{10.31}$$

Let $u(x) \in \mathcal{A}$ be a minimizer of (10.30) so that

$$F\{w\} \geq F\{u\} \qquad \text{for all } w \in \mathcal{A}.$$

The initial steps of the calculation we used previously when deriving the Euler-Lagrange equation, all the way until equation (10.12), continue to remain valid and so we are again led to

$$\delta F\{u, \delta u\} = \int_0^1 \left[\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right) \right] \delta u \, dx \; + \; \left[\frac{\partial f}{\partial u'} \delta u \right]_0^1 . \tag{10.32}$$

Since u and $u + \delta u$ are both in $\mathcal{A}$, it follows that $u(0) = a$ and $u(0) + \delta u(0) = a$ and therefore that $\delta u(0) = 0$. Thus the set of *admissible variations* is

$$\mathcal{V} = \{\delta u \in C^1[0,1] : \, \delta u(0) = 0\}. \tag{10.33}$$

Thus

$$\delta F\{u, \delta u\} = \int_0^1 \left[\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right) \right] \delta u \, dx \; + \; \left. \frac{\partial f}{\partial u'} \right|_{x=1} \delta u(1). \tag{10.34}$$

At a minimizer we have

$$\delta F\{u, \delta u\} = 0 \qquad \text{for all } \delta u \in \mathcal{V}. \tag{10.35}$$

First restrict attention to all variations with the additional property $\delta u(1) = 0$. Equation (10.35) must necessarily hold for all such variations. The boundary term in (10.34) now drops out and we are led to

$$\int_0^1 \left[\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right) \right] \delta u \, dx = 0 \tag{10.36}$$

which must hold for all variations with $\delta u(0) = 0$ and $\delta u(1) = 0$. By the fundamental lemma of the calculus of variations (page 635) it follows that

$$\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right) = 0 \qquad \text{for} \qquad 0 < x < 1. \tag{10.37}$$

We now return to (10.34), (10.35) which because of (10.37) reduces to

$$\left. \frac{\partial f}{\partial u'} \right|_{x=1} \delta u(1) = 0 \qquad \text{for all } \delta u \in \mathcal{V}. \tag{10.38}$$

Since the value $\delta u(1)$ is arbitrary it follows that

$$\left. \frac{\partial f}{\partial u'} \right|_{x=1} = 0. \tag{10.39}$$

Equation (10.37) is the Euler-Lagrange equation associated with the problem and (10.39) is the *natural boundary condition*. A natural boundary condition is generated by the minimization process. The particular displacement that minimizes the energy is found by solving the boundary-value problem (10.37), (10.39) and $u(0) = a$. The boundary condition $u(0) = a$ is sometimes referred to as the *essential boundary condition*.

When applied to the problem described in Figure 10.4, we first write (ii) as

$$\Phi\{u\} = \int_0^L \left[\frac{1}{2}T\big(u'(x)\big)^2 + \frac{1}{2}ku^2(x) - b(x)u(x) - Fu'(x)\right]\, dx,$$

and so the natural boundary condition reads

$$\left.\frac{\partial f}{\partial u'}\right|_{x=L} = \Big[Tu'(x) - F\Big]_{x=L} = Tu'(L) - F = 0. \qquad (iii)$$

It can be readily verified that the natural boundary condition (iii) characterizes vertical force balance for the slider, a condition we did not impose a priori. (In the special case $F = 0$, it states that the string must be horizontal at the right-hand end; the force applied on the slider by the rail is horizontal in this case.)

10.3 Principle of minimum potential energy.

We now return to the general setting of an elastic solid in three-dimensions. The body occupies a region $\mathcal{R}_R$ in a reference configuration and its boundary is $\partial R_R = \mathcal{S}_1 \cup \mathcal{S}_2$. The deformation $\widehat{\mathbf{y}}(\mathbf{x})$ is prescribed on $\mathcal{S}_1$, the Piola traction $\widehat{\mathbf{s}}(\mathbf{x})$ is prescribed on $\mathcal{S}_2$, and the body force $\mathbf{b}_R(\mathbf{x})$ is prescribed on $\mathcal{R}_R$. The body is composed of an elastic material whose strain energy function $W(\mathbf{F})$ is known.

A **kinematically admissible deformation** (virtual deformation) is any sufficiently smooth vector field $\mathbf{z}(\mathbf{x})$ defined for all $\mathbf{x} \in \mathcal{R}_{\mathrm{R}}$ that satisfies the kinematic boundary condition:

$$\mathbf{z}(\mathbf{x}) = \widehat{\mathbf{y}}(\mathbf{x}) \qquad \text{for} \quad \mathbf{x} \in \mathcal{S}_1. \qquad (10.40)$$

Let $\mathcal{A}$ denote the set of all kinematically admissible deformations:

$$\mathcal{A} = \{\mathbf{z} \in C^1(\mathcal{R}_R) :\ \mathbf{z}(\mathbf{x}) = \widehat{\mathbf{y}}(\mathbf{x}) \text{ for } \mathbf{x} \in \mathcal{S}_1\}. \qquad (10.41)$$

The potential energy associated with an admissible deformation $\mathbf{z}(\mathbf{x}) \in \mathcal{A}$ is

$$\Phi\{\mathbf{z}\} = \int_{\mathcal{R}_{\mathrm{R}}} W(\nabla\mathbf{z})\, dV_x - \int_{\mathcal{R}_{\mathrm{R}}} \mathbf{b}_R \cdot \mathbf{z}\, dV_x - \int_{\mathcal{S}_2} \widehat{\mathbf{s}} \cdot \mathbf{z}\, dA_x. \qquad (10.42)$$

If there are other kinematic constraints, those too must also be enforced. For example if the material is incompressible we would modify the kinematically admissible set of deformations to be

$$\mathcal{A} = \{\mathbf{z} \in C^1(\mathcal{R}_R) :\ \mathbf{z}(\mathbf{x}) = \widehat{\mathbf{y}}(\mathbf{x}) \text{ for } \mathbf{x} \in \mathcal{S}_1; \quad \det \nabla\mathbf{z} = 1 \text{ for } \mathbf{x} \in \mathcal{R}_R\}. \qquad (10.43)$$

We want to find the deformation $\mathbf{y}(\mathbf{x}) \in \mathcal{A}$ that minimizes the potential energy functional (10.42) from among all kinematically admissible deformations:

$$\Phi\{\mathbf{z}\} \geq \Phi\{\mathbf{y}\} \qquad \text{for all } \mathbf{z} \in \mathcal{A}. \tag{10.44}$$

Observe that the prescribed body force $\mathbf{b}_R(\mathbf{x})$, boundary traction $\widehat{\mathbf{s}}(\mathbf{x})$ and strain energy function $W(\mathbf{F})$ describing the material all appear in Φ, while the prescribed boundary deformation $\widehat{\mathbf{y}}(\mathbf{x})$ is part of the admissible set of functions $\mathcal{A}$. Moreover, note that we have said nothing about the equilibrium equations and in fact, not even introduced the notion of stress.

Since the minimizer $\mathbf{y}$ is in $\mathcal{A}$ we know that necessarily

$$\mathbf{y}(\mathbf{x}) = \widehat{\mathbf{y}}(\mathbf{x}) \qquad \text{for} \quad \mathbf{x} \in \mathcal{S}_1. \tag{10.45}$$

Now consider kinematically admissible deformations of the form

$$\mathbf{z}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) + \epsilon\, \boldsymbol{\eta}(\mathbf{x}), \qquad -\epsilon_0 \leq \epsilon \leq \epsilon_0, \tag{10.46}$$

where ϵ is a scalar parameter. We refer to $\boldsymbol{\eta}(\mathbf{x})$ as an **admissible variation**. In view of (10.40) and (10.45) it follows that

$$\boldsymbol{\eta}(\mathbf{x}) = \mathbf{o} \qquad \text{for} \quad \mathbf{x} \in \mathcal{S}_1. \tag{10.47}$$

This, and smoothness, are the only requirements of an admissible variation $\boldsymbol{\eta}(\mathbf{x})$. The set of all admissible variations is

$$\mathcal{V} = \{\boldsymbol{\eta} \in C^1(\mathcal{R}_R) : \boldsymbol{\eta}(\mathbf{x}) = \mathbf{o} \text{ for } \mathbf{x} \in \mathcal{S}_1\}. \tag{10.48}$$

On evaluating the potential energy (10.42) at a kinematically admissible deformation (10.46) we obtain

$$\Phi\{\mathbf{y} + \epsilon\, \boldsymbol{\eta}\} = \int_{\mathcal{R}_R} W(\nabla \mathbf{y} + \epsilon\, \nabla \boldsymbol{\eta})\, dV_x - \int_{\mathcal{R}_R} \mathbf{b}_R \cdot (\mathbf{y} + \epsilon \boldsymbol{\eta})\, dV_x - \int_{\mathcal{S}_2} \widehat{\mathbf{s}} \cdot (\mathbf{y} + \epsilon\, \boldsymbol{\eta})\, dA_x. \tag{10.49}$$

On keeping $\boldsymbol{\eta}(\mathbf{x}) \in \mathcal{V}$ fixed for the moment, we can view the potential energy $\Phi\{\mathbf{y} + \epsilon\, \boldsymbol{\eta}\}$ as a scalar valued function of the scalar parameter ϵ:

$$\Phi = \overline{\Phi}(\epsilon) := \Phi\{\mathbf{y} + \epsilon\, \boldsymbol{\eta}\}, \qquad -\epsilon_0 \leq \epsilon \leq \epsilon_0. \tag{10.50}$$

Since $\mathbf{y}(\mathbf{x})$ is a minimizer of the potential energy functional $\Phi\{\mathbf{z}\}$ it follows from (10.46) and (10.50) that $\epsilon = 0$ is a minimizer of the potential energy function $\overline{\Phi}(\epsilon)$. This requires

$$\left.\frac{\mathrm{d}\overline{\Phi}}{d\epsilon}\right|_{\epsilon=0} = 0, \qquad \left.\frac{\mathrm{d}^2\overline{\Phi}}{d\epsilon^2}\right|_{\epsilon=0} \geq 0. \tag{10.51}$$

We want to examine the implications of (10.51) and it will be convenient for this to introduce the functions $\mathbf{S}(\mathbf{F})$ and $\mathbb{A}(\mathbf{F})$, defined for all tensors $\mathbf{F}$ with $\det \mathbf{F} > 0$, whose cartesian components are

$$S_{ij}(\mathbf{F}) := \frac{\partial W}{\partial F_{ij}}(\mathbf{F}), \qquad \mathbb{A}_{ijk\ell}(\mathbf{F}) := \frac{\partial S_{ij}}{\partial F_{k\ell}}(\mathbf{F}) = \frac{\partial^2 W}{\partial F_{ij}\partial F_{k\ell}}(\mathbf{F}). \tag{10.52}$$

(a) We now calculate the first derivative of $\overline{\Phi}$ with respect to ϵ from (10.49), (10.50). This yields

$$\frac{d\overline{\Phi}}{d\epsilon} = \int_{\mathcal{R}_{\rm R}} \left[\frac{\partial W}{\partial F_{ij}}(\nabla\mathbf{y} + \epsilon\,\nabla\boldsymbol{\eta})\ \frac{\partial \eta_i}{\partial x_j} - \mathbf{b}_R\cdot\boldsymbol{\eta}\right] dV_x - \int_{\mathcal{S}_2} \widehat{\mathbf{s}}\cdot\boldsymbol{\eta}\, dA_x,$$

and so

$$\begin{aligned}\left.\frac{d\overline{\Phi}}{d\epsilon}\right|_{\epsilon=0} &= \int_{\mathcal{R}_{\rm R}} \left[\frac{\partial W}{\partial F_{ij}}(\nabla\mathbf{y})\ \frac{\partial \eta_i}{\partial x_j} - \mathbf{b}_R\cdot\boldsymbol{\eta}\right] dV_x - \int_{\mathcal{S}_2} \widehat{\mathbf{s}}\cdot\boldsymbol{\eta}\, dA_x = \\ &\overset{(10.52)_1}{=} \int_{\mathcal{R}_{\rm R}} \left[S_{ij}(\nabla\mathbf{y})\ \frac{\partial \eta_i}{\partial x_j} - \mathbf{b}_R\cdot\boldsymbol{\eta}\right] dV_x - \int_{\mathcal{S}_2} \widehat{\mathbf{s}}\cdot\boldsymbol{\eta}\, dA_x.\end{aligned}$$

Thus

$$\delta\Phi\{\mathbf{y},\boldsymbol{\eta}\} = \left.\frac{d\overline{\Phi}}{d\epsilon}\right|_{\epsilon=0} = \int_{\mathcal{R}_{\rm R}} \left[\mathbf{S}(\nabla\mathbf{y})\cdot\nabla\boldsymbol{\eta} - \mathbf{b}_R\cdot\boldsymbol{\eta}\right] dV_x - \int_{\mathcal{S}_2} \widehat{\mathbf{s}}\cdot\boldsymbol{\eta}\, dA_x.$$

Since the first variation $\delta\Phi\{\mathbf{y},\boldsymbol{\eta}\}$ vanishes for all variations $\boldsymbol{\eta}\in\mathcal{V}$,

$$\int_{\mathcal{R}_{\rm R}} \left[S_{ij}(\nabla\mathbf{y})\ \frac{\partial \eta_i}{\partial x_j} - \mathbf{b}_R\cdot\boldsymbol{\eta}\right] dV_x - \int_{\mathcal{S}_2} \widehat{\mathbf{s}}\cdot\boldsymbol{\eta}\, dA_x\ = 0, \qquad \boldsymbol{\eta}\in\mathcal{V}.$$

Writing this as

$$\int_{\mathcal{R}_{\rm R}} \left[\frac{\partial}{\partial x_j}\left(S_{ij}(\nabla\mathbf{y})\ \eta_i\right) - \frac{\partial S_{ij}}{\partial x_j}(\nabla\mathbf{y})\ \eta_i - \mathbf{b}_R\cdot\boldsymbol{\eta}\right] dV_x - \int_{\mathcal{S}_2} \widehat{\mathbf{s}}\cdot\boldsymbol{\eta}\, dA_x\ = 0$$

and using the divergence theorem leads to

$$\int_{\partial\mathcal{R}_{\rm R}} S_{ij}(\nabla\mathbf{y})\ \eta_i\, n_j^R\, dA_x - \int_{\mathcal{R}_{\rm R}} \left[\frac{\partial S_{ij}}{\partial x_j}(\nabla\mathbf{y})\ + b_i\right] \eta_i dV_x - \int_{\mathcal{S}_2} \widehat{s}_i\eta_i\, dA_x\ = 0.$$

Since $\boldsymbol{\eta}(\mathbf{x}) = \mathbf{o}$ on $\mathcal{S}_1$, the first integral can be written over $\mathcal{S}_2$ and combined with the last integral:

$$\int_{\mathcal{S}_2} \big[S_{ij}(\nabla\mathbf{y})n_j^R - \widehat{s}_i\big]\,\eta_i\,dA_x - \int_{\mathcal{R}_\mathrm{R}} \left[\frac{\partial S_{ij}}{\partial x_j}(\nabla\mathbf{y})\;+b_i\right]\eta_i dV_x = 0.$$

This must hold for all admissible variations $\boldsymbol{\eta}(\mathbf{x})$. Since $\boldsymbol{\eta}(\mathbf{x})$ is arbitrary on $\mathcal{R}_\mathrm{R}$ and on $\mathcal{S}_2$ we conclude that the following field equation and natural boundary condition must necessarily hold:

$$\frac{\partial S_{ij}}{\partial x_j}(\nabla\mathbf{y}(\mathbf{x}))\;+b_i = 0 \quad \text{for} \quad \mathbf{x}\in\mathcal{R}_\mathrm{R}, \qquad S_{ij}(\nabla\mathbf{y})n_j^R = \widehat{s}_i \quad \text{for} \quad \mathbf{x}\in\mathcal{S}_2.$$

i.e.

$$\mathrm{Div}\,\mathbf{S}(\nabla\mathbf{y}(\mathbf{x}))\;+\mathbf{b} = \mathbf{o} \quad \text{for} \quad \mathbf{x}\in\mathcal{R}_\mathrm{R}, \qquad \mathbf{S}(\nabla\mathbf{y}(\mathbf{x}))\mathbf{n}_R = \widehat{\mathbf{s}} \quad \text{for} \quad \mathbf{x}\in\mathcal{S}_2. \tag{10.53}$$

If we now define $\mathbf{S}(\nabla\mathbf{y})$ to be the Piola stress, we see that the preceding equations are the equilibrium equation on $\mathcal{R}_R$ and the traction boundary condition on $\mathcal{S}_2$.

(b) Next we turn to the second derivative condition $(10.51)_2$ and so differentiate $d\Phi/d\epsilon$ with respect to ϵ. This yields

$$\frac{d^2\overline{\Phi}}{d\epsilon^2} = \int_{\mathcal{R}_\mathrm{R}} \frac{\partial^2 W}{\partial F_{ij}\partial F_{k\ell}}(\nabla\mathbf{y}+\epsilon\,\nabla\boldsymbol{\eta})\frac{\partial\eta_i}{\partial x_j}\frac{\partial\eta_k}{\partial x_\ell}\,dV_x, \tag{10.54}$$

and so

$$\left.\frac{d^2\overline{\Phi}}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_\mathrm{R}} \frac{\partial^2 W}{\partial F_{ij}\partial F_{k\ell}}(\nabla\mathbf{y})\frac{\partial\eta_i}{\partial x_j}\frac{\partial\eta_k}{\partial x_\ell}\,dV_x. \tag{10.55}$$

Thus if $\mathbf{y}(\mathbf{x})$ is to be a minimizer it is also necessary that

$$\int_{\mathcal{R}_\mathrm{R}} \frac{\partial^2 W}{\partial F_{ij}\partial F_{k\ell}}(\nabla\mathbf{y})\,\frac{\partial\eta_i}{\partial x_j}\frac{\partial\eta_k}{\partial x_\ell}\,dV_x \geq 0 \qquad \text{for all } \boldsymbol{\eta}\in\mathcal{V}. \tag{10.56}$$

See Problem 10.4.8.

10.4 Worked examples.

Problem 10.4.1. *Deriving equilibrium equations.* We know from the principle of minimum potential energy that the Euler-Lagrange equation associated with an elastic solid subjected to dead loading is in fact the equilibrium equation. Make use of this to derive the non-trivial equilibrium equation associated with a radially symmetric deformation of a homogeneous isotropic body.

Solution: Let $r(R)$ be the radial component of a spherically symmetric deformation. The associated principal stretches are

$$\lambda_1 = \lambda_r = r'(R), \qquad \lambda_2 = \lambda_\theta = \lambda_3 = \lambda_\phi = \frac{r(R)}{R}. \tag{i}$$

Our interest is in deriving the Euler-Lagrange equation in a spherically symmetric setting and we are not concerned with the boundary conditions. Thus without loss of generality we assume the body to be a hollow sphere of inner and outer radii A and B in the reference configuration, and we take the deformations $r(A)$ and $r(B)$ be given so that then their variations $\delta r(A) = \delta r(B) = 0$. The potential energy functional is

$$\Phi\{r\} = \int_{\mathcal{R}_R} W\, dV_x = \int_0^B W\Big(r'(R),\, r(R)/R,\, r(R)/R\Big)\, 4\pi R^2\, dR. \tag{ii}$$

In the calculations to follow we shall use the notation

$$W_i \equiv \frac{\partial W}{\partial \lambda_i}(\lambda_1, \lambda_2, \lambda_3).$$

Calculating the first variation of Φ in the usual way:

$$\begin{aligned}
\delta\Phi\{r, \delta r\} &= \int_A^B [W_1\delta\lambda_1 + W_2\delta\lambda_2 + W_3\delta\lambda_3]\, 4\pi R^2\, dR = \int_A^B [W_1\delta\lambda_1 + 2W_2\delta\lambda_2]\, 4\pi R^2\, dR = \\
&= \int_A^B \left[W_1\delta r' + 2W_2\frac{\delta r}{R}\right] 4\pi R^2\, dR = 4\pi \int_A^B \left[R^2 W_1 \delta r' + 2RW_2\delta r\right]\, dR = \\
&= 4\pi \int_A^B \left[\frac{d}{dR}[R^2 W_1 \delta r] - \frac{d}{dR}(R^2W_1)\delta r + 2RW_2\delta r\right]\, dR = \\
&= 4\pi \int_A^B \left[-\frac{d}{dR}(R^2W_1) + 2RW_2\right] \delta r\, dR
\end{aligned}$$

where we used $\delta r(A) = \delta r(B) = 0$ in getting to the last line. Since $\delta\Phi\{r, \delta r\} = 0$ for all admissible δr, we arrive at the Euler-Lagrange equation

$$\frac{d}{dR}(R^2 W_1) - 2RW_2 = 0 \qquad \text{for} \quad 0 < R < B. \tag{iii}$$

From the general principle of minimum potential energy we know that this corresponds to (the radial component of) the equilibrium equation. Our final task is to write (iii) in the familiar form in terms of the principal Cauchy stress components.

Recall from the constitutive relation for an isotropic elastic material that the radial and circumferential components of the Cauchy stress tensor can be written as

$$T_{rr} = \frac{\lambda_1}{\lambda_1\lambda_2\lambda_3} W_1 = \frac{1}{\lambda_\theta^2} W_1 \overset{(i)}{=} \frac{R^2}{r^2} W_1, \qquad T_{\theta\theta} = T_{\phi\phi} = \frac{\lambda_2}{\lambda_1\lambda_2\lambda_3} W_2 = \frac{1}{\lambda_r\lambda_\theta} W_2 \overset{(i)}{=} \frac{R}{rr'} W_2.$$

This gives

$$R^2 W_1 = r^2 T_{rr}, \qquad RW_2 = rr' T_{\theta\theta}. \tag{iv}$$

We now simplify the left-hand side of (iii) as follows:

$$\begin{aligned}
\frac{d}{dR}(R^2W_1) - 2RW_2 &\overset{(iv)}{=} \frac{d}{dR}(r^2 T_{rr}) - 2rr'T_{\theta\theta} = \frac{d}{dr}(r^2T_{rr})\,\frac{dr}{dR} - 2rr'T_{\theta\theta} = \\
&= r'\left[\frac{d}{dr}(r^2T_{rr}) - 2rT_{\theta\theta}\right] = r'\left[2rT_{rr} + r^2\frac{dT_{rr}}{dr} - 2rT_{\theta\theta}\right] = \\
&= r^2 r'\left[\frac{dT_{rr}}{dr} + 2\frac{T_{rr} - T_{\theta\theta}}{r}\right]
\end{aligned}$$

Therefore the Euler-Lagrange equation (*iii*) can be written as

$$\frac{dT_{rr}}{dr} + 2\frac{T_{rr} - T_{\theta\theta}}{r} = 0. \qquad \square$$

Remark: A weakness of this derivation is that we assumed the material to be (*a*) elastic and (*b*) isotropic, and therefore don't know whether the equilibrium equation $\square$ holds for a general inelastic material. We know from Chapter 3 that it does! A derivation analogous to the one above but starting from the principle of virtual work avoids this issue.

Problem 10.4.2. *Elastic string subject to dead loading.* We are concerned with an elastic string modeled as a one-dimensional continuum. A generic particle of the string is located at $\mathbf{x} = x\,\mathbf{e}_1$, $0 \leq x \leq L$, in the reference configuration and at $\mathbf{y}(x) = y_1(x)\mathbf{e}_1 + y_2(x)\mathbf{e}_2, 0 \leq x \leq L$, in the deformed configuration. The left- and right-hand ends of the string are fixed at $(0,0)$ and (a,b) respectively:

$$(y_1(0), y_2(0)) = (0,0), \qquad (y_1(L), y_2(L)) = (a,b). \tag{i}$$

Thus in the reference and current configurations we have

$$\mathcal{R}_R = \{x \,:\, 0 \leq x \leq L\}, \qquad \mathcal{R} = \{(y_1(x), y_2(x)) \,:\, 0 \leq x \leq L\}.$$

The string is subjected to a body force distribution $\mathbf{b}_R(x)$ per unit undeformed length.

Consider the following set of kinematically admissible deformations $y_1(x), y_2(x)$:

$$\mathcal{A} = \{y_1, y_2 \in C^1[0,L] : \; y_1(0) = 0, y_2(0) = 0, y_1(L) = a, y_2(L) = b\}. \tag{ii}$$

The potential energy of the system associated with any admissible $y_1, y_2 \in \mathcal{A}$ is

$$\Phi\{y_1, y_2\} = \int_0^L W(\lambda(x))\,dx - \int_0^L \mathbf{b}_R \cdot \mathbf{y}\,dx, \tag{iii}$$

where $W(\lambda)$ is the stored elastic energy per unit undeformed length and λ is the stretch.

(*a*) Minimize Φ over $\mathcal{A}$ and derive a pair of differential equations governing $y_1(x), y_2(x)$.

(*b*) Show that these are the same equations we derived previously in Problem 4.34.

Solution: Since the values of $y_1(0), y_2(0) y_1(L)$ and $y_2(L)$ are prescribed, an admissible variation $\delta y_1(x), \delta y_2(x)$ obeys

$$\delta y_1(0) = \delta y_2(0) = \delta y_1(L) = \delta y_2(L) = 0. \tag{iv}$$

In what follows we will have to calculate $\delta\lambda$ and so recall that by the definition of stretch,

$$\lambda = s'(x) \tag{v}$$

where $s(x)$ is arc length along the deformed string. Since we will want to express δs in terms of δy_1 and δy_2 we also note by geometry that

$$(s')^2 = (y_1')^2 + (y_2')^2, \qquad \cos\phi = y_1'/s', \qquad \sin\phi = y_2'/s', \tag{vi}$$

where $\phi(x)$ denotes the slope of the string in the deformed configuration. In equation (vi) and what follows, a prime denotes d/dx except for $W'(\lambda)$.

We can now calculate $\delta\lambda$. From (v), $\delta\lambda = \delta s'$ and from $(vi)_1$,

$$s'\delta s' \overset{(iii)_1}{=} y_1'\delta y_1' + y_2'\delta y_2', \qquad \Rightarrow \qquad \delta s' = \frac{y_1'}{s'}\delta y_1' + \frac{y_2'}{s'}\delta y_2' \overset{(vi)_2,(vi)_3}{=} \cos\phi\,\delta y_1' + \sin\phi\,\delta y_2'.$$

Thus

$$\delta\lambda = \cos\phi\,\delta y_1' + \sin\phi\,\delta y_2'. \tag{vii}$$

We now calculate the first variation of Φ from (iii):

$$\begin{aligned}
\delta\Phi \quad &= \int_0^L \delta W(\lambda)\,dx - \int_0^L \mathbf{b}_R\cdot\delta\mathbf{y}\,dx = \\
&= \int_0^L W'(\lambda)\delta\lambda\,dx - \int_0^L (b_1^R\delta y_1 + b_2^R\delta y_2)\,dx = \\
&\overset{(vii)}{=} \int_0^L \Big[W'(\lambda)\cos\phi\,\delta y_1' + W'(\lambda)\sin\phi\,\delta y_2'\Big]\,dx - \int_0^L (b_1^R\delta y_1 + b_2^R\delta y_2)\,dx = \\
&\overset{(iv)}{=} \int_0^L -\frac{d}{dx}\left[W'(\lambda)\cos\phi\right]\delta y_1 - \frac{d}{dx}\left[W'(\lambda)\sin\phi\right]\delta y_2\,dx - \int_0^L (b_1^R\delta y_1 + b_2^R\delta y_2)\,dx \\
&= -\int_0^L \left[\left(\frac{d}{dx}[W'(\lambda)\cos\phi] + b_1^R\right)\delta y_1 + \left(\frac{d}{dx}[W'(\lambda)\sin\phi] + b_2^R\right)\delta y_2\right]\,dx.
\end{aligned}$$

where in getting to the penultimate line we integrated by parts and used the boundary conditions (iv).

Since since the first variation vanishes, $\delta\Phi\{y_1, y_2, \delta y_1, \delta y_2\} = 0$ for all admissible variations $\delta y_1, \delta y_2$, and so by the fundamental lemma of the calculus of variations (page 635), we conclude that

$$\frac{d}{dx}[W'(\lambda)\cos\phi] + b_1^R = 0, \qquad \frac{d}{dx}[W'(\lambda)\sin\phi] + b_2^R = 0. \qquad \square \tag{$viii$}$$

Keeping in mind that $\lambda = s' = \sqrt{y_1'^2 + y_2'^2}$, this is a pair of differential equations involving $y_1(x), y_2(x)$ to be solved together with the boundary conditions $y_1(0) = y_2(0) = 0, y_1(L) = a, y_2(L) = b$.

(b) To write $(viii)$ in the form we had previously we first combine $(viii)_1$ and $(viii)_2$ vectorially:

$$\left(\frac{d}{dx}[W'(\lambda)\cos\phi] + b_1^R\right)\mathbf{e}_1 + \left(\frac{d}{dx}[W'(\lambda)\sin\phi] + b_2^R\right)\mathbf{e}_2 = \mathbf{o},$$

which can be written as

$$\frac{d}{dx}\Big[W'(\lambda)\big(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2\big)\Big] + \big(b_1^R\mathbf{e}_1 + b_2^R\mathbf{e}_2\big) = \mathbf{o}.$$

The unit vector $\boldsymbol{\ell}$ tangent to the string in the deformed configuration is

$$\boldsymbol{\ell} = \cos\phi\ \mathbf{e}_1 + \sin\phi\ \mathbf{e}_2,$$

and so we have

$$\frac{d}{dx}\Big[W'(\lambda)\boldsymbol{\ell}\Big] + \mathbf{b}_R = \mathbf{o}.$$

Previously we used arc length s instead of the coordinate x and so had d/ds instead of d/dx. Thus we use the chain rule to write

$$\frac{d}{ds}\Big[W'(\lambda)\boldsymbol{\ell}\Big]\,\frac{ds}{dx} + \mathbf{b}_R = \mathbf{o} \quad \overset{(v)}{\Rightarrow} \quad \frac{d}{ds}\Big[W'(\lambda)\boldsymbol{\ell}\Big]\,\lambda + \mathbf{b}_R = \mathbf{o} \quad \Rightarrow \quad \frac{d}{ds}\Big[W'(\lambda)\boldsymbol{\ell}\Big] + \mathbf{b} = \mathbf{o}.$$

where $\mathbf{b} = \mathbf{b}_R/\lambda$ is the body force per unit *deformed* length; this follows from $\mathbf{b}\,ds = \mathbf{b}_R\,dx$. Finally setting $\sigma = W'(\lambda)$ for the force in the string we obtain

$$\frac{d}{ds}\big[\sigma\boldsymbol{\ell}\big] + \mathbf{b} = \mathbf{o}. \qquad \square$$

This is the form in which we wrote the equilibrium equation previously.

Problem 10.4.3. *Cavitation in an incompressible solid.* (See Problem 10.1 for cavitation in a compressible solid.) In an unstressed reference configuration the homogeneous, isotropic, incompressible elastic body is a solid sphere of radius B. A uniformly distributed radial tensile dead load (Piola traction) of magnitude $\sigma > 0$ is applied on the boundary $R = B$. The material is described by its strain energy function $W(\lambda_1, \lambda_2, \lambda_3)$. Examine the possibility of cavitation, where the deformed body contains a hole.

Reference: J.M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, *Philosophical Transactions of the Royal Society (London)*, **A306**, (1982), pp. 557-611.

Solution: Consider spherically symmetric kinematically admissible deformations of the form $r = r(R), \theta = \Theta, \phi = \Phi$ where (R, Θ, Φ) and (r, θ, ϕ) are spherical polar coordinates in the reference and current configurations respectively. We know from, e.g. Chapter 5.4, that the associated principal stretches are

$$\lambda_1 = r'(R), \qquad \lambda_2 = \lambda_3 = r(R)/R. \tag{i}$$

A kinematically admissible deformation is described by the function $r(R)$, defined and suitably smooth on $[0, B]$, satisfying the kinematic requirement of incompressibility:

$$\lambda_1\lambda_2\lambda_3 = 1 \qquad \Rightarrow \qquad r^2(R)r'(R) = R^2 \qquad \Rightarrow \qquad r(R) = [R^3 + a^3]^{1/3}, \tag{ii}$$

where $a \geq 0$ is an unknown constant.

Observe that the set of kinematically admissible deformations is in fact a one-parameter family of deformations $r(R; a) = [R^3 + a^3]^{1/3}$ parameterized by $a \geq 0$. If $a = 0$ then $r(R) = R$ and so the body is undeformed (but stressed); if $a > 0$ then $r(0^+) = a > 0$ and so a hole of deformed radius a has appeared in the center of the body.

The prescribed traction on the outer boundary is $\widehat{\mathbf{s}} = \sigma\mathbf{e}_r$, while the deformation there is $\mathbf{y} = r(B)\mathbf{e}_r$. Therefore the potential energy associated with the dead loading is

$$\int_{\partial\mathcal{R}_R} \widehat{\mathbf{s}} \cdot \mathbf{y}\, dA_x = 4\pi B^2\sigma\, r(B) \overset{(ii)}{=} 4\pi B^2\sigma\,[B^3 + a^3]^{1/3}. \tag{iii}$$

The elastic potential energy of the body due to deformation is

$$\int_{\mathcal{R}_R} W(\lambda_1, \lambda_2, \lambda_3)dV_x = \int_0^B w(\lambda)4\pi R^2\, dR \tag{iv}$$

where $w(\lambda)$ is the restriction of W to spherically symmetric, isochoric deformations:

$$w(\lambda) \coloneqq W(\lambda^{-2}, \lambda, \lambda) \qquad \text{for } \lambda > 0. \tag{v}$$

The total potential energy associated with an admissible deformation is therefore

$$\Phi(a) = \int_0^B 4\pi R^2 w(\lambda)\, dR - 4\pi B^2 \sigma\, [B^3 + a^3]^{1/3}, \tag{vi}$$

where

$$\lambda = \frac{r(R)}{R} = \frac{[R^3 + a^3]^{1/3}}{R}. \tag{vii}$$

Observe that in this problem the potential energy functional has reduced to a function $\Phi(a)$ of the parameter a since the kinematically admissible deformations considered here has an explicit form (ii). It will be useful for what follows to observe from (vii) that

$$\frac{\partial\lambda}{\partial a} = \frac{a^2}{R(R^3 + a^3)^{2/3}}, \qquad \frac{\partial\lambda}{\partial R} = \frac{-\,a^3}{R^2(R^3 + a^3)^{2/3}}, \qquad \lambda^3 - 1 = a^3/R^3, \qquad \lambda_b \coloneqq \frac{[B^3 + a^3]^{1/3}}{B}. \tag{$viii$}$$

In an equilibrium configuration we have $\Phi'(a) = 0$. Thus, differentiating (vi) with respect to a

$$\Phi'(a) = \int_0^B 4\pi R^2 w'(\lambda)\, \frac{d\lambda}{da}\, dR - \frac{4}{3}\pi B^2 \sigma\, \frac{3a^2}{[B^3 + a^3]^{2/3}}\,,$$

which because of $(viii)_1$ and $(viii)_4$ can be written as

$$\Phi'(a) = \int_0^B 4\pi R^2 \frac{a^2}{R(R^3 + a^3)^{2/3}}\, w'(\lambda)\, dR - \frac{4\pi a^2 \sigma}{\lambda_b^2}\,,$$

which in turn using $(viii)_2$ and $(viii)_3$ yields

$$\Phi'(a) = \int_0^B -4\pi a^2 \frac{R^3}{a^3} \frac{d\lambda}{dR}\, w'(\lambda)\, dR - \frac{4\pi a^2 \sigma}{\lambda_b^2} = 4\pi a^2 \int_{\lambda_b}^{\infty} \frac{w'(\lambda)}{\lambda^3 - 1}\, d\lambda - \frac{4\pi a^2 \sigma}{\lambda_b^2}. \tag{ix}$$

Finally, setting $\Phi'(a) = 0$ (with $a \neq 0$) leads to

$$\sigma = \lambda_b^2 \int_{\lambda_b}^{\infty} \frac{w'(\lambda)}{\lambda^3 - 1}\, d\lambda \qquad \text{where} \quad \lambda_b = \left[1 + \frac{a^3}{B^3}\right]^{1/3}. \tag{x}$$

(Assuming the convergence of the integral) equation $(x)_1$ is a relation between the applied stress σ and the radius a of the cavity in the deformed configuration; see Figure 10.6. To find the critical stress σ_{cr} at which the cavity just appears, we take the limit $a/B \to 0$ in (x) which yields

$$\sigma_{cr} = \int_1^{\infty} \frac{w'(\lambda)}{\lambda^3 - 1}\, d\lambda. \tag{xi}$$

The convergence of this integral, and therefore the existence of the cavitation phenomenon, was analyzed carefully by Ball (1982), and was touched on at the end of Chapter 5.4.

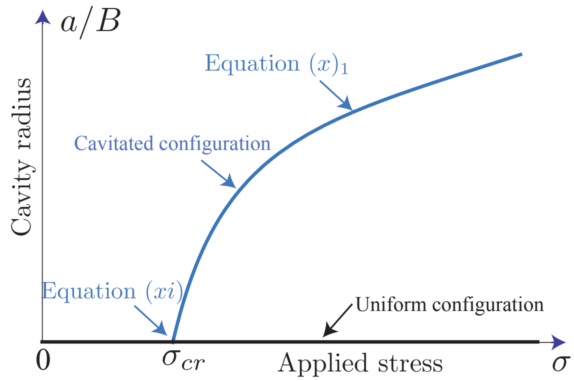

Figure 10.6: Schematic bifurcation diagram. Uniform configuration without a cavity for all $\sigma \geq 0$ (black). Configuration with a cavity of radius $r(0) = a > 0$ for $\sigma > \sigma_{cr}$ (blue).

Problem 10.4.4. *3D continuum subjected to a constant pressure loading.* A homogeneous elastic body occupies a region $\mathcal{R}_R$ in a reference configuration. The deformation is prescribed to be $\widehat{\mathbf{y}}(\mathbf{x})$ at each point $\mathbf{x}$ on some part $\mathcal{S}_1$ of the boundary $\partial\mathcal{R}_R$. A constant pressure p per unit *deformed* area is applied on the rest of the boundary, the pressure being normal to the *deformed* boundary of the body.

We know that pressure loading is not dead loading so the field equations and natural boundary conditions governing this system *cannot* be obtained by extremizing the potential energy functional (10.42).

In this problem we are asked to derive the functional $\Phi\{\mathbf{z}\}$ that, when extremized over the set

$$\mathcal{A} = \{\mathbf{z} \in C^1(\mathcal{R}) : \ \mathbf{z}(\mathbf{x}) = \widehat{\mathbf{y}}(\mathbf{x}) \text{ for } \mathbf{x} \in \mathcal{S}_1\} \tag{i}$$

of kinematically admissible deformations, leads to the fields equations on $\mathcal{R}_R$ and natural boundary conditions on $\mathcal{S}_2 = \partial\mathcal{R}_R - \mathcal{S}_1$.

Solution: Let $\mathcal{S}_2'$ be the image of $\mathcal{S}_2$ in the deformed configuration (where $\mathcal{S}_2 = \partial\mathcal{R}_R - \mathcal{S}_1$ is the part of the boundary $\partial\mathcal{R}_R$ on which the pressure is applied). We are told that the traction boundary condition is

$$\mathbf{Tn} = -p\mathbf{n} \qquad \text{for} \quad \mathbf{y} \in \mathcal{S}_2'.$$

We can write this boundary condition in terms of the Piola stress by first writing it as $\mathbf{Tn}\, dA_y = -p\mathbf{n}\, dA_y$, and then using $\mathbf{Tn}\, dA_y = \mathbf{Sn}_R\, dA_x$ and Nanson's formula $\mathbf{n}\, dA_y = J\mathbf{F}^{-T}\mathbf{n}_R\, dA_x$ to get $\mathbf{Sn}_R\, dA_x = -pJ\mathbf{F}^{-T}\mathbf{n}_R\, dA_x$. Thus we can write the traction boundary condition above in the equivalent form

$$\mathbf{Sn}_R = -pJ\mathbf{F}^{-T}\mathbf{n}_R \qquad \text{for} \quad \mathbf{x} \in \mathcal{S}_2,$$

or in terms of components,

$$S_{ij}n_j^R = -pJ\, F_{ji}^{-1}\, n_j^{\mathrm{R}} \qquad \text{for} \quad \mathbf{x} \in \mathcal{S}_2. \tag{ii}$$

It will simplify our writing if we denote the partial derivative with respect to x_j of some field by a subscript containing a comma followed by j so that for example we will write

$$y_{i,j} \equiv \frac{\partial y_i}{\partial x_j}, \qquad S_{ij,j} \equiv \frac{\partial S_{ij}}{\partial x_j}, \qquad \text{etc.}$$

Accordingly, we can write the equilibrium equation (in the absence of body forces) as

$$S_{ij,j} = 0 \qquad \text{for} \quad \mathbf{x} \in \mathcal{R}_R. \tag{iii}$$

The constitutive relation is

$$S_{ij} = \frac{\partial W}{\partial F_{ij}}. \tag{iv}$$

An admissible variation is any sufficiently smooth function $\delta\mathbf{y}(\mathbf{x})$ defined of $\mathcal{R}_R$ that vanishes on $\mathcal{S}_1$:

$$\delta y_i(\mathbf{x}) = 0 \qquad \text{for} \quad \mathbf{x} \in \mathcal{S}_1. \tag{v}$$

Multiplying (ii) by an arbitrary variation δy_i and integrating over $\mathcal{S}_2$ and multiplying (iii) by δy_i and integrating over $\mathcal{R}_R$ gives

$$\int_{\mathcal{S}_2} \left(S_{ij} + pJ\,F_{ji}^{-1}\right) n_j^{\mathrm{R}}\,\delta y_i\,dA_x = 0, \qquad \int_{\mathcal{R}_R} S_{ij,j}\,\delta y_i\,dV_x = 0, \tag{vi}$$

respectively. Subtracting the second from the first gives

$$\begin{aligned} 0 \;&= \int_{\mathcal{S}_2} \left(S_{ij} + pJ\,F_{ji}^{-1}\right) n_j^{\mathrm{R}}\,\delta y_i\,dA_x - \int_{\mathcal{R}_R} S_{ij,j}\,\delta y_i\,dV_x = \\ &\overset{(v)}{=} \int_{\partial\mathcal{R}_R} \left(S_{ij} + pJ\,F_{ji}^{-1}\right) n_j^{\mathrm{R}}\,\delta y_i\,dA_x - \int_{\mathcal{R}_R} S_{ij,j}\,\delta y_i\,dV_x = \\ &= \int_{\partial\mathcal{R}_R} S_{ij} n_j^{\mathrm{R}}\,\delta y_i\,dA_x - \int_{\mathcal{R}_R} S_{ij,j}\,\delta y_i\,dV_x + \int_{\partial\mathcal{R}_R} pJ\,F_{ji}^{-1} n_j^{\mathrm{R}}\,\delta y_i\,dA_x = 0 \end{aligned} \tag{vii}$$

First consider the two terms involving the stress S_{ij}. They can be simplified as follows:

$$\begin{aligned} \int_{\partial\mathcal{R}_R} S_{ij} n_j^{\mathrm{R}}\,\delta y_i\,dA_x - \int_{\mathcal{R}_R} S_{ij,j}\,\delta y_i\,dV_x \;&\overset{(a)}{=} \int_{\mathcal{R}_R} (S_{ij,j}\,\delta y_i + S_{ij}\delta y_{i,j})\,dV_x - \int_{\mathcal{R}_R} S_{ij,j}\,\delta y_i\,dV_x = \\ &= \int_{\mathcal{R}_R} S_{ij}\delta y_{i,j}\,dV_x \overset{(b)}{=} \int_{\mathcal{R}_R} S_{ij}\delta F_{ij}\,dV_x = \\ &\overset{(iv)}{=} \int_{\mathcal{R}_R} \frac{\partial W}{\partial F_{ij}} \delta F_{ij}\,dV_x = \int_{\mathcal{R}_R} \delta W(\mathbf{F})\,dV_x = \\ &= \delta \int_{\mathcal{R}_R} W(\mathbf{F})\,dV_x, \end{aligned} \tag{viii}$$

where we used the divergence theorem in step (a) and $F_{ij} = y_{i,j}$ in step (b). Next consider the term in (vii) that involves the pressure p. It can be simplified as follows:

$$\begin{aligned} \int_{\partial\mathcal{R}_R} pJ\,F_{ji}^{-1}\,n_j^{\mathrm{R}}\,\delta y_i\,dA_x \;&\overset{(c)}{=} \int_{\mathcal{R}_R} p(J\,F_{ji}^{-1}\delta y_i)_{,j}\,dV_x = \int_{\mathcal{R}_R} \left[p\big(J\,F_{ji}^{-1}\big)_{,j}\delta y_i + pJ\,F_{ji}^{-1}\delta y_{i,j}\right] dV_x = \\ &\overset{(d)}{=} \int_{\mathcal{R}_R} pJ\,F_{ji}^{-1}\,\delta y_{i,j}\,dV_x \overset{(e)}{=} \int_{\mathcal{R}_R} p\frac{\partial J}{\partial F_{ij}}\,\delta y_{i,j}\,dV_x = \\ &\overset{(f)}{=} \int_{\mathcal{R}_R} p\frac{\partial J}{\partial F_{ij}}\,\delta F_{ij}\,dV_x = \int_{\mathcal{R}_R} p\,\delta J\,dV_x = \\ &= \delta \int_{\mathcal{R}_R} pJ\,dV_x \overset{(g)}{=} \delta \int_{\mathcal{R}} p\,dV_y, \end{aligned} \tag{ix}$$

where in step (c) we used the divergence theorem, in step (d) the identity (Problem 2.24 , page 217)

$$\mathrm{Div}\,(J\mathbf{F}^{-T}) = \mathbf{o}, \qquad (JF_{ji}^{-1})_{,j} = 0; \tag{ix}$$

in step (e) the identity (Problem 1.47, page 108)

$$\frac{\partial J}{\partial \mathbf{F}} = J\,\mathbf{F}^{-T}, \qquad \frac{\partial J}{\partial F_{ij}} = J\,F_{ji}^{-1} \tag{x}$$

in step (f) we used $F_{ij} = y_{i,j}$ and in step (g) we used $J\,dV_x = dV_y$.

On substituting $(viii)$ and (ix) into (vii) we get

$$\delta \int_{\mathcal{R}_R} W(\mathbf{F})\,dV_x \;+\; \delta \int_{\mathcal{R}} p\,dV_y = 0.$$

Thus the potential energy functional that we minimize is

$$\Phi\{\mathbf{z}\} = \int_{\mathcal{R}_R} W(\nabla \mathbf{z})\,dV_x \;+\; \int_{\mathcal{R}} p\,dV_y, \qquad \mathbf{z} \in \mathcal{A}. \qquad \square$$

Problem 10.4.5. (Steigman) *Axi-symmetric membrane subjected to axial stretch at its two ends.* An unpressurized circular cylindrical membrane has length $2L$ and radius R. Its ends, $Z = \pm L$, are attached to two rigid rings of radius R. The membrane material can be modeled by the strain energy function $W = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3)$ per unit reference area. The rings are moved apart by the application of axial forces of magnitude F. If the distance between the rings in the deformed configuration is 2ℓ, calculate the relationship between F and ℓ.

Remark: You can reduce the problem to a pair of differential equations $\dot{\lambda}_1 = f(\lambda_1, \lambda_2)$, $\dot{\lambda}_2 = g(\lambda_1, \lambda_2)$ which you can then solve numerically (using MATHEMATICA, MATLAB, ... or writing your own code using say a shooting scheme). Since dimensional considerations tell us that $F/\mu R^2$ is a function of ℓ/L and R/L, choose a particular value for R/L.

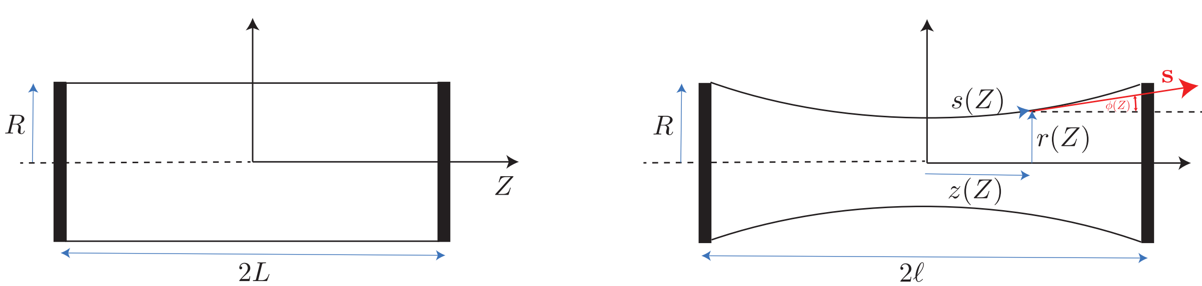

Figure 10.7: Left: Unstressed circular cylindrical membrane of radius R and length $2L$, attached to two rigid rings (of radius R) at its ends. Right: The rings have been moved apart by the application of axial forces of magnitude F. The deformation takes $(R, \Theta, Z) \mapsto (r(Z), \Theta, z(Z))$. Arc length along a meridional curve is $s(Z)$ and the unit tangent vector $\mathbf{s}(Z)$ makes an angle $\phi(Z)$ with the horizontal.

Reference: D.J. Steigmann, Finite Elasticity Theory, Oxford, 2017.

Solution: Let (R, Θ, Z) with $0 \le \Theta \le 2\pi, -L \le Z \le L$ be cylindrical polar coordinates of a particle in the reference configuration. If its image in the deformed configuration is (r, θ, z) then the axi-symmetric deformation can be described by

$$r = r(Z), \qquad \theta = \Theta, \qquad z = z(Z). \tag{i}$$

Since the rings are rigid,

$$r(\pm L) = R. \tag{ii}$$

Let the arc length along, and slope of, a meridional curve be $s(Z)$ and $\phi(Z)$. Then,

$$(s')^2 = (r')^2 + (z')^2, \qquad z' = s'\cos\phi, \quad r' = s'\sin\phi, \tag{iii}$$

where here, and below, a prime denotes differentiation with respect to Z. The principal stretches are

$$\lambda_1 = s', \qquad \lambda_2 = \frac{r}{R}, \tag{iv}$$

and so

$$\lambda_2' \overset{(iv)_2}{=} \frac{r'}{R} \overset{(iii)_3}{=} \frac{s'}{R}\sin\phi \overset{(iv)_1}{=} (\lambda_1/R)\sin\phi. \tag{v}$$

The set $\mathcal{A}$ of all kinematically admissible deformations $r(Z)$, $z(Z)$ is

$$\mathcal{A} = \{r, z \in C^1[-L, L] : r(\pm L) = R\}. \tag{vi}$$

The potential energy associated with a kinematically admissible deformation is

$$\Phi\{r, z\} = \int_{-L}^{L} 2\pi R W(\lambda_1, \lambda_2)\, dZ - Fz(L) + Fz(-L), \tag{vii}$$

where W is the strain energy function per unit referential area.

We are interested in the particular kinematically admissible deformation that extremizes the potential energy from among deformations in $\mathcal{A}$. Thus calculating the first variation of Φ and introducing the notation $W_i = \partial W/\partial\lambda_i, W_{ij} = \partial^2 W/\partial\lambda_i\partial\lambda_j$:

$$\begin{aligned} \delta\Phi &= \int_{-L}^{L} 2\pi R\Big(W_1\delta\lambda_1 + W_2\delta\lambda_2\Big)dZ - F\delta z(L) + F\delta z(-L) = \\ &\overset{(iv)}{=} \int_{-L}^{L} 2\pi R\Big(W_1\delta s' + W_2\delta r/R\Big)dZ - F\delta z(L) + F\delta z(-L). \end{aligned} \tag{$viii$}$$

However from $(iii)_1$,

$$s'\,\delta s' = r'\,\delta r' + z'\,\delta z' \qquad \overset{(iii)_{2,3}}{\Rightarrow} \qquad \delta s' = \sin\phi\,\delta r' + \cos\phi\,\delta z'. \tag{ix}$$

Therefore

$$\delta\Phi = \int_{-L}^{L} 2\pi R\Big(W_1\sin\phi\delta r' + W_1\cos\phi\delta z' + W_2\delta r/R\Big)dZ - F\delta z(L) + F\delta z(-L). \tag{x}$$

Integrating the first and second terms by parts and using $\delta r(\pm L) = 0$,

$$\begin{aligned} \delta\Phi &= \int_{-L}^{L} 2\pi R\left[\left(\frac{W_2}{R} - \frac{d}{dZ}\left(W_1\sin\phi\right)\right)\delta r - \frac{d}{dZ}(W_1\cos\phi)\delta z\right]dZ - \\ &\quad -(2\pi R W_1\cos\phi - F)\delta z(L) + (2\pi R W_1\cos\phi - F)\delta z(-L). \end{aligned} \tag{xi}$$

Since this must vanish for all admissible variations $\delta r(Z)$ and $\delta z(Z)$ it follows that the following field equations

$$\frac{W_2}{R} - \frac{d}{dZ}\left(W_1 \sin\phi\right) = 0 \qquad \text{for} \quad -L \le Z \le L, \tag{xii}$$

$$\frac{d}{dZ}(W_1 \cos\phi) = 0 \qquad \text{for} \quad -L \le Z \le L, \tag{xiii}$$

and natural boundary conditions

$$F = 2\pi R W_1 \cos\phi \qquad \text{for} \quad Z = \pm L, \tag{xiv}$$

must hold. Thus the functions $\lambda_1(Z), \lambda_2(Z)$ and $\phi(Z)$ are to be found by solving the differential equations $(v), (xii)$ and $(xiii)$, subject to the boundary conditions (ii) and (xv).

Remark 1: Note that with σ_1 and τ_1 denoting the Piola and Cauchy forces per unit length (conjugate to λ_1), we have $\sigma_1 = W_1$ and $\tau_1 = \sigma_1/\lambda_2$ (which follows from $2\pi r\tau_1 = 2\pi R\sigma_1$). The natural boundary condition (xv) then reads $F = 2\pi R\,\sigma_1 \cos\phi = 2\pi r\tau_1\, \cos\phi$ as expected since $\sigma_1\cos\phi$ and $\tau_1\cos\phi$ are the $\mathbf{e}_z$ components of the Piola and Cauchy traction respectively at $Z = L$.

Remark 2: The differential equation $(xiii)$ can be integrated with respect to Z, which together with (xv) gives

$$F = 2\pi R W_1 \cos\phi \qquad \text{for} \quad -L \le Z \le L. \tag{xv}$$

Thus the field equations to be solved are

$$\lambda_2' = \frac{\lambda_1}{R}\sin\phi, \qquad \frac{W_2}{R} - \frac{d}{dZ}\left(W_1 \sin\phi\right) = 0, \qquad 2\pi R W_1 \cos\phi = F. \tag{xvi}$$

In order to solve these equations numerically (e.g. using the shooting method) it is convenient to write them in the form $\lambda_1' = f(\lambda_1, \lambda_2, \phi)$, $\lambda_2' = g(\lambda_1, \lambda_2, \phi)$ and $h(\lambda_1, \lambda_2, \phi) = 0$. To this end one can combine the three equations in (xvi) to get

$$\lambda_1' = \frac{\sin\phi}{R}\,\frac{W_2 - W_{12}\lambda_1}{W_{11}}.$$

Thus the equations to be solved are

$$\lambda_1' = \frac{\sin\phi}{R}\,\frac{W_2 - W_{12}\lambda_1}{W_{11}}, \qquad \lambda_2' = \frac{\lambda_1}{R}\sin\phi, \qquad 2\pi R W_1\cos\phi = F \qquad \text{for} \quad -L \le Z \le L.$$

If the material is neo-Hookean we take $W^* = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3)$ where to convert this energy per unit reference volume to the energy W per unit reference area we (see Section 10.4 of Steigman).

Problem 10.4.6. *Non-spherically symmetric cavitation. Approximation solution.* In a reference configuration the body is a solid sphere of radius B. It is composed of a homogeneous, isotropic, incompressible material; when seeking explicit results, assume the material to be neo-Hookean. The boundary $\partial\mathcal{R}_R$ is subjected to a dead load Piola traction

$$\widehat{\mathbf{s}} = \boldsymbol{\Sigma}\mathbf{n}_R, \tag{i}$$

where $\boldsymbol{\Sigma}$ is the constant tensor

$$\boldsymbol{\Sigma} = \sigma_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_3 \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{ii}$$

and the unit outward normal vector on $\partial\mathcal{R}_R$ is

$$\mathbf{n}_R = \left.\frac{\mathbf{x}}{B}\right|_{|\mathbf{x}|=B}. \qquad (iii)$$

Thus from $(i),(ii)$ and (iii)

$$\widehat{\mathbf{s}} = \sigma_1 n_1^R \mathbf{e}_1 + \sigma_2 n_2^R \mathbf{e}_2 + \sigma_3 n_3^R \mathbf{e}_3 = \sigma_1 \frac{x_1}{B}\mathbf{e}_1 + \sigma_2 \frac{x_2}{B}\mathbf{e}_2 + \sigma_3 \frac{x_3}{B}\mathbf{e}_3 \qquad \text{for } \mathbf{x} \in \partial\mathcal{R}_R. \qquad (iv)$$

You are asked to investigate the possibility of cavitation under this loading. In the special case where $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ this is the spherically symmetric cavitation problem studied by Ball; see Chapter 5.4 and also Problem 10.4.3. Since it is unlikely that we can find a closed-form solution to the non-symmetric problem at hand, seek an approximate solution by minimizing the potential energy functional over a suitable subset of kinematically admissible deformations.

Reference: H-S Hou and R. Abeyaratne, Cavitation is elastic and elastic-plastic solids, Journal of the Mechanics and Physics of Solids, volume 40, issue 3, 1992, pp. 571-592. (This paper considers Cauchy traction loading.)

Solution: Consider deformations of the form

$$y_1 = f_1(R)x_1, \qquad y_2 = f_2(R)x_2, \qquad y_3 = f_3(R)x_3, \qquad (v)$$

where

$$R = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \qquad f_i(R) > 0. \qquad (vi)$$

In order that $(v),(vi)$ be admissible, it must satisfy the kinematic requirement $\det \mathbf{F} = 1$. Calculating the components $F_{ij} = \partial y_i/\partial x_j$ of the deformation gradient tensor associated with (v) leads to

$$F_{11} = f_1' x_1^2/R + f_1, \qquad F_{12} = f_1' x_1 x_2/R, \qquad F_{13} = f_1' x_1 x_3/R,$$

$$F_{21} = f_2' x_1 x_2/R, \qquad F_{22} = f_2' x_2^2/R + f_2, \qquad F_{23} = f_2' x_2 x_3/R,$$

$$F_{31} = f_3' x_3 x_1/R, \qquad F_{32} = f_3' x_3 x_2/R, \qquad F_{33} = f_3' x_3^2/R + f_3.$$

Since the deformation must be isochoric we require[8]

$$\det \mathbf{F} = f_1 f_2 f_3 + f_2 f_3 \frac{f_1'}{R} x_1^2 + f_1 f_3 \frac{f_2'}{R} x_2^2 + f_1 f_2 \frac{f_3'}{R} x_3^2 = 1$$

which can be written as

$$f_1 f_2 f_3 + R f_1 f_2 f_3' - 1 + \frac{x_1^2}{R} f_2 (f_1' f_3 - f_1 f_3') + \frac{x_2^2}{R} f_1 (f_2' f_3 - f_2 f_3') = 0.$$

This must hold at all points in the body and so

$$f_1 f_2 f_3 + R f_1 f_2 f_3' = 1, \qquad f_1' f_3 - f_1 f_3' = 0, \qquad f_2' f_3 - f_2 f_3' = 0 \qquad \text{for} \quad 0 \le R \le B.$$

[8]Many of the calculations were carried out using MATHEMATICA.

The second and third of these equations tell us that f_1 and f_2 are proportional to f_3 so that on writing $f_1 = \eta_1 f_3$ and $f_2 = \eta_2 f_3$ the first equation reduces to

$$Rf_3^2 f_3' + f_3^3 = \frac{1}{\eta_1\eta_2} \qquad \Rightarrow \quad f_3(R) = \left(\frac{1}{\eta_1\eta_2} + \frac{\eta_3}{R^3}\right)^{1/3} = \frac{1}{\eta_1^{1/3}\eta_2^{1/3}}\left(1 + \frac{\eta_1\eta_2\eta_3}{R^3}\right)^{1/3},$$

η_1, η_2 and η_3 being constants. The deformation (v), when isochoric, can therefore be written in the form

$$y_1 = \alpha_1 f(R) x_1, \qquad y_2 = \alpha_2 f(R) x_2, \qquad y_3 = \alpha_3 f(R) x_3, \tag{vii}$$

where

$$\alpha_1\alpha_2\alpha_3 = 1, \qquad f(R) = \left(1 + \frac{\beta^3}{R^3}\right)^{1/3}, \tag{viii}$$

with $\alpha_2 > 0, \alpha_3 > 0, \beta \geq 0$ being arbitrary constants.

Remark: It can be readily seen that the deformation (vii) takes a spherical surface $R = \mathsf{constant}$ in the reference configuration into the ellipsoidal surface

$$\frac{y_1^2}{\alpha_1^2} + \frac{y_2^2}{\alpha_2^2} + \frac{y_3^2}{\alpha_3^2} = (R^3 + \beta^3)^{2/3}$$

in the deformed configuration. Observe that α_2 and α_3 describe its aspect ratios ($\alpha_1 = \alpha_2^{-1}\alpha_3^{-1}$) and β its size. If $\beta > 0$ the origin maps into the ellipsoid corresponding to $R = 0$; if $\beta = 0$ the origin $R = 0$ remains at the origin.

In order to write the various equations below in symmetric form, we will continue to involve α_1 but make sure to enforce $\alpha_1 = 1/(\alpha_2\alpha_3)$ when necessary.

The class of kinematically admissible deformations $(vii), (viii)$ is in fact an explicit 3-parameter family of deformations where the parameters are $\alpha_2 > 0, \alpha_3 > 0$ and $\beta \geq 0$. Thus the potential energy functional will reduce to a potential energy *function*

$$\Phi(\alpha_2, \alpha_3, \beta) = \int_{\mathcal{R}_R} W \, dV_x - \int_{\partial\mathcal{R}_R} \widehat{\mathbf{s}} \cdot \mathbf{y} \, dA_x. \tag{ix}$$

Let the potential energies of the elastic body and the loading be denoted by

$$\mathcal{E}(\alpha_2, \alpha_3, \beta) = \int_{\mathcal{R}_R} W \, dV_x, \qquad \Psi(\alpha_2, \alpha_3, \beta) = \int_{\partial\mathcal{R}_R} \widehat{\mathbf{s}} \cdot \mathbf{y} \, dA_x, \tag{x}$$

so that $\Phi(\alpha_2, \alpha_3, \beta) = \mathcal{E}(\alpha_2, \alpha_3, \beta) - \Psi(\alpha_2, \alpha_3, \beta)$.

Having previously calculated the components of the deformation gradient tensor, we can now calculate the principal scalar invariants I_1 and I_2 and thus obtain

$$\mathcal{E}(\alpha_2, \alpha_3, \beta) = \int_{\mathcal{R}_R} W(I_1, I_2) \, dV_x. \tag{ix}$$

For example for a neo-Hookean material, $W = \frac{\mu}{2}(I_1 - 3) = \frac{\mu}{2}(\mathbf{F} \cdot \mathbf{F} - 3)$, this gives the explicit expression

$$\mathcal{E} = \frac{2\pi\mu}{3} B^3 \left[(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)(B^3 + 2\beta^3)(B^3 + \beta^3)^{-1/3} B^{-2} - 3\right]. \tag{x}$$

Now consider the potential energy Ψ of the loading. From $(x)_2, (iv)$ and (vii) we have

$$\Psi = \frac{1}{B}\int_{\partial\mathcal{R}_R} (\sigma_1\alpha_1 x_1^2 f(B) + \sigma_2\alpha_2 x_2^2 f(B) + \sigma_3\alpha_3 x_3^2 f(B))\, dA_x$$

which yields

$$\Psi = \frac{4\pi}{3}\, B^3 f(B)(\sigma_1\alpha_1 + \sigma_2\alpha_2 + \sigma_3\alpha_3). \tag{xi}$$

Thus

$$\left.\begin{aligned} \frac{\partial\Psi}{\partial\alpha_2} = \frac{4\pi}{3}B^3 f(B)\,\frac{(\sigma_2\alpha_2 - \sigma_1\alpha_1)}{\alpha_2}, \qquad \frac{\partial\Psi}{\partial\alpha_3} = \frac{4\pi}{3}B^3 f(B)\,\frac{(\sigma_3\alpha_3 - \sigma_1\alpha_1)}{\alpha_3},\\ \frac{\partial\Psi}{\partial\beta} = \frac{4\pi}{3}\,\frac{1}{f^2(B)}\,\beta^2\,(\sigma_1\alpha_1 + \sigma_2\alpha_2 + \sigma_3\alpha_3), \end{aligned}\right\} \tag{xii}$$

where we have used $\alpha_1 = \alpha_2^{-1}\alpha_3^{-1}$ and $f(B) = (1 + \beta^3/B^3)^{1/3}$.

Since minimizing $\Phi(\alpha_2, \alpha_3, \beta) = \mathcal{E}(\alpha_2, \alpha_3, \beta) - \Psi(\alpha_2, \alpha_3, \beta)$ requires

$$\frac{\partial\Psi}{\partial\alpha_2} = \frac{\partial\mathcal{E}}{\partial\alpha_2}, \qquad \frac{\partial\Psi}{\partial\alpha_3} = \frac{\partial\mathcal{E}}{\partial\alpha_3}, \qquad \frac{\partial\Psi}{\partial\beta} = \frac{\partial\mathcal{E}}{\partial\beta}, \tag{xiii}$$

we conclude from (xii) and $(xiii)$ that

$$\left.\begin{aligned} \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3 &= \frac{3}{4\pi}\,\frac{f^2(B)}{\beta^2}\,\frac{\partial\mathcal{E}}{\partial\beta},\\ \frac{\sigma_2\alpha_2 - \sigma_1\alpha_1}{\alpha_2} &= \frac{3}{4\pi B^3}\,\frac{1}{f(B)}\,\frac{\partial\mathcal{E}}{\partial\alpha_2},\\ \frac{\sigma_3\alpha_3 - \sigma_1\alpha_1}{\alpha_3} &= \frac{3}{4\pi B^3}\,\frac{1}{f(B)}\,\frac{\partial\mathcal{E}}{\partial\alpha_3}. \end{aligned}\right\} \tag{xiv}$$

The three equations (xiv) relate the unknown geometric parameters α_2, α_3 and β to the given stresses $\sigma_1, \sigma_2, \sigma_3$. At the onset of cavitation the cavity size $\beta \to 0^+$ and such a state is described by

$$\left.\begin{aligned} \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3 &= \frac{3}{4\pi}\,\left[\frac{1}{\beta^2}\,\frac{\partial\mathcal{E}}{\partial\beta}\right]_{\beta\to 0},\\ \frac{\sigma_2\alpha_2 - \sigma_1\alpha_1}{\alpha_2} &= \frac{3}{4\pi B^3}\,\left.\frac{\partial\mathcal{E}}{\partial\alpha_2}\right|_{\beta\to 0^+},\\ \frac{\sigma_3\alpha_3 - \sigma_1\alpha_1}{\alpha_3} &= \frac{3}{4\pi B^3}\,\left.\frac{\partial\mathcal{E}}{\partial\alpha_3}\right|_{\beta\to 0^+}, \end{aligned}\right\} \tag{xv}$$

having used $f(B) = (1 + \beta^3/B^3)^{1/3} = 1$ when $\beta = 0$.

For a neo-Hookean material we use (x) in (xv) to find the following explicit condition describing the onset of cavitation:

$$\left.\begin{aligned} \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3 &= \tfrac{5}{2}\mu\,(\alpha_1^2 + \alpha_2^2 + \alpha_3^2),\\ \alpha_2\sigma_2 - \alpha_1\sigma_1 &= \mu\,(\alpha_2^2 - \alpha_1^2),\\ \alpha_3\sigma_3 - \alpha_1\sigma_1 &= \mu\,(\alpha_3^2 - \alpha_1^2). \end{aligned}\right\} \tag{xvi}$$

On solving (xvi) we get

$$\sigma_1/\mu = \frac{3\alpha_1^2 + \alpha_2^2 + \alpha_3^2}{2\alpha_1}, \quad \sigma_2/\mu = \frac{\alpha_1^2 + 3\alpha_2^2 + \alpha_3^2}{2\alpha_2}, \quad \sigma_3/\mu = \frac{\alpha_1^2 + \alpha_2^2 + 3\alpha_3^2}{2\alpha_3}, \qquad \alpha_2 > 0, \alpha_3 > 0. \tag{xvii}$$

keeping in mind that $\alpha_1 = \alpha_2^{-1}\alpha_3^{-1}$. Observe that when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ this gives

$$\sigma_1 = \sigma_2 = \sigma_3 = \frac{5}{2}\mu$$

which is Ball's result for the case of spherically symmetric loading, see Problem 10.4.3.

Equation $(xvii)$ (with $\alpha_1 = \alpha_2^{-1}\alpha_3^{-1}$) provides a parametric description of a surface in $\sigma_1, \sigma_2, \sigma_3$-stress space, $\alpha_2 > 0, \alpha_3 > 0$ being the parameters. This surface characterizes the onset of cavitation. When the loading $(\sigma_1, \sigma_2, \sigma_3)$ lies on one side of this surface, the side on which the origin is located, the body does not involve a cavity. When the loading is on the other side, the body is in a cavitated configuration.

To visualize this prediction in a special case, consider ellipsoidal cavities with equal aspect ratios $\alpha_2 = \alpha_3$. Equations $(xvii)$ then specialize to $\sigma_2 = \sigma_3$ and

$$\sigma_1/\mu = \frac{3\alpha_1^2 + 2\alpha_2^2}{2\alpha_1}, \qquad \sigma_2/\mu = \frac{\alpha_1^2 + 4\alpha_2^2}{2\alpha_2}, \qquad \alpha_2 > 0, \qquad \alpha_1 = 1/\alpha_2^2. \tag{$xviii$}$$

This provides a parametric description of the cavitation curve in the σ_1, σ_2-plane as illustrated in Figure 10.8. This curve intersects the line $\sigma_1 = \sigma_2$ at the value $5\mu/2$ which corresponds to Ball's result for spherically symmetric cavitation. Cavitation occurs when (σ_1, σ_2) lies in the wedge-shaped region between the two branches of the cavitation curve.

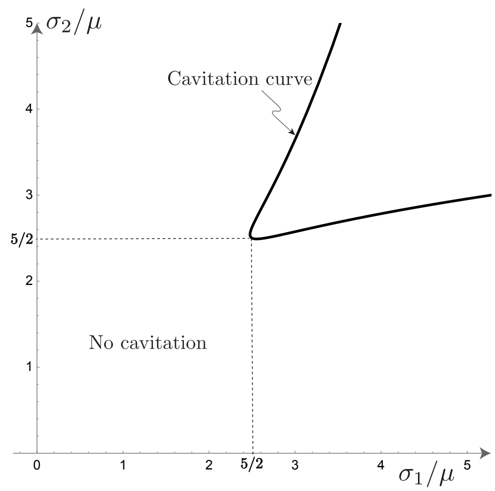

Figure 10.8: Cavitation curve on the σ_1, σ_2-plane in the special case of cavities where two semi-major axes have the same length ($\alpha_2 = \alpha_3$). The cavitation curve is characterized parametrically by equation $(xviii)$. The cavitation curve intersects the line $\sigma_1 = \sigma_2$ at the value $5\mu/2$ which corresponds to Ball's result for spherically symmetric cavitation. Cavitation occurs when (σ_1, σ_2) lies in the wedge-shaped region between the two branches of the cavitation curve.

Problem 10.4.7. *Approximate solution. End effects in a bar a la Saint-Venant.* Figure 10.9 shows a solid circular cylindrical bar that in an unstressed reference configuration has length L and radius R. It is firmly bonded to two rigid end plates. The left-hand plate (at $x_1 = 0$) is held fixed while the plate on the right-hand end (at $x_1 = L$) is displaced by $\Delta\,\mathbf{e}_1$ as shown in the figure. It will be useful to let $\lambda = 1 + \Delta/L > 1$. Thus the prescribed kinematic boundary conditions are

$$\mathbf{y} = \mathbf{x} \quad \text{on } x_1 = 0 \qquad \text{and} \qquad \mathbf{y} = \mathbf{x} + \underbrace{(\lambda - 1)L}_{=\Delta}\,\mathbf{e}_1 \quad \text{on } x_1 = L, \tag{i}$$

with the lateral boundary of the bar being traction-free.

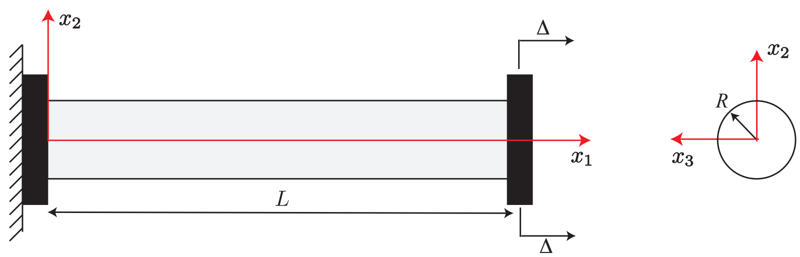

Figure 10.9: The bar is stretched after being attached to rigid end-plates. The end plates prevent "Poisson contraction" in the transverse direction at the two ends.

If the bar was stretched in this way *without* being attached to rigid end plates, it will contract in the transverse direction and undergo a homogeneous deformation of the form

$$y_1 = \lambda x_1, \qquad y_2 = \Lambda x_2, \qquad y_3 = \Lambda x_3 \qquad \text{for} \quad \mathbf{x} \in \mathcal{R}_R, \tag{ii}$$

(for some value Λ that we expected to be < 1 due to the "Poisson effect"). In the present problem however the bar is attached to rigid end plates and so cannot contract in the transverse direction at the two ends. Said differently, the deformation (ii) does not satisfy the kinematic boundary conditions (i) (except in some pathological case where the material shows no Poisson effect, i.e. $\Lambda = 1$).

Suppose you want to determine an approximate expression for the deformation using the minimum potential energy principle. If you minimize the potential energy functional over all admissible deformations you will simply obtain the usual equations of elasticity. To find an approximate solution you want to minimize the potential energy over some suitable subset of all admissible deformations.

To this end consider kinematically admissible deformations of the form

$$y_1 = \lambda x_1, \qquad y_2 = x_2 h(x_1), \qquad y_3 = x_3 h(x_1) \qquad \text{for} \quad \mathbf{x} \in \mathcal{R}_R, \tag{iii}$$

where the function $h(x_1)$ is to be determined. The bar is composed of an unconstrained, homogeneous, isotropic elastic material characterized by the strain energy function

$$W = \frac{\mu}{2}(I_1 - 3) + f(J), \qquad I_1 = \operatorname{tr}\mathbf{F}\mathbf{F}^T, \quad J = \det\,\mathbf{F}, \tag{iv}$$

where

$$f(1) = 0, \qquad f'(1) = -\mu, \qquad \mu > 0. \tag{v}$$

Equations $(v)_{1,2}$ ensure that the energy and stress vanish in the reference configuration.

(a) What restrictions must be placed on the function $h(x_1)$ in order that the deformation (iii) is kinematically admissible?

(b) (Based on your intuition,) sketch what you expect the graph of $h(x_1)$ versus x_1 to look like in the case $L \gg R$. Label key values.

(c) Specialize the general 3D potential energy functional of finite elasticity to the present setting and thus derive the potential energy functional $\Phi\{h\}$.

(d) Derive the Euler-Lagrange equation and natural boundary conditions (if any).

(e) Show that $h(x_1) = \Lambda$ for $0 \leq x_1 \leq L$ is a solution of the Euler-Lagrange but not the boundary conditions.

(f) Approximate the Euler-Lagrange equation to the case where $h(x_1) \approx \Lambda$ and solve the resulting linear boundary value problem. Does the graph of your solution $h(x_1)$ look like what you sketched in part (b)?

Solution:

(a) The displacement at the left-hand boundary vanishes and therefore we must have

$$(i)_1 \qquad \Rightarrow \qquad \left.\begin{aligned} y_1(0, x_2, x_3) &= 0, \\ y_2(0, x_2, x_3) &= x_2, \\ y_3(0, x_2, x_3) &= x_3, \end{aligned}\right\} \qquad \overset{(iii)}{\Rightarrow} \qquad h(0) = 1.$$

Similarly, the displacement at the right-hand boundary is $(\lambda - 1)L\mathbf{e}_1$ and therefore we must have

$$(i)_2 \qquad \Rightarrow \qquad \left.\begin{aligned} y_1(L, x_2, x_3) &= \lambda L, \\ y_2(L, x_2, x_3) &= x_2, \\ y_3(L, x_2, x_3) &= x_3, \end{aligned}\right\} \qquad \overset{(iii)}{\Rightarrow} \qquad h(L) = 1.$$

Thus a kinematically admissible function $h(x_1)$ must obey

$$h(0) = h(L) = 1, \qquad \square \tag{vi}$$

(together with an appropriate level of smoothness).

(b) If we *assume* Saint-Venant's principle to be valid in the nonlinear theory, then we would expect the central part of the bar, away from the two ends, to be in a state of uniaxial tension and therefore that the deformation will have the form (ii) in that region. Thus we expect $h(x_1) \approx \Lambda$ away from $x_1 = 0$ and $x_1 = L$. The boundary conditions (vi) must hold at the two ends and so we must have $h(0) = h(1) = 1$. Thus we expect that $h(x_1)$ will vary rapidly from $h = \Lambda$ to $h = 1$ as one approaches each end. Thus we expect the graph of $h(x_1)$ versus x_1 to be as depicted schematically in Figure 10.10.

If we have a state of uniaxial stress in the x_1-direction, the deformation will have the form (ii). Then

$$T_{22} = \frac{\lambda_2}{J}\frac{\partial W}{\partial \lambda_2} = \frac{\lambda_2}{J}\Big[\mu\lambda_2 + \lambda_1\lambda_3 f'(J)\Big] = \frac{\Lambda}{\lambda\Lambda^2}\Big[\mu\Lambda + \lambda\Lambda f'(\lambda\Lambda^2)\Big] = 0,$$

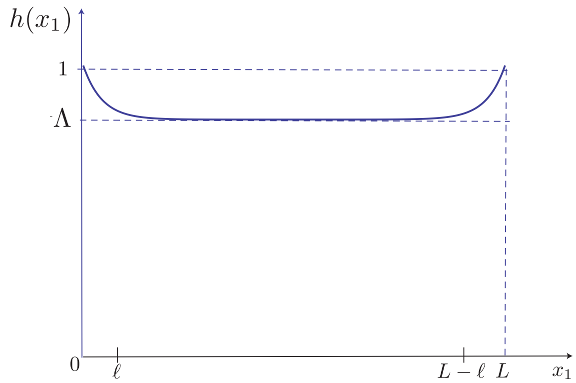

Figure 10.10: When $L \gg R$ we expect $h(x_1) \approx \Lambda$ in most of the bar except near the two ends. The parameter ℓ is the length scale over which the end effects are important; a formula for ℓ is calculated in part (f).

and so Λ will be given by

$$1 + \frac{\lambda f'(\lambda\Lambda^2)}{\mu} = 0. \qquad (vii)$$

(c) The part $\mathcal{S}_1$ of the boundary on which the deformation is prescribed is comprised of the two ends $x_1 = 0$ and $x_1 = L$; the part $\mathcal{S}_2$ on which the traction is prescribed is the lateral boundary. Since the prescribed traction on $\mathcal{S}_2$ vanishes, there is no potential energy associated with the loading. Thus the total potential energy is

$$\Phi = \int_{\mathcal{R}_R} W\, dV_x - \int_{\mathcal{R}_R} \mathbf{b}_R\, dV_x - \int_{\mathcal{S}_2} \widehat{\mathbf{s}} \cdot \mathbf{y} dA_x = \int_{\mathcal{R}_R} W\, dV_x = \int_{\mathcal{R}_R} \Big[\frac{\mu}{2}(I_1 - 3) + f(J)\Big]\, dV_x. \qquad (viii)$$

In order to calculate $I_1 = \mathrm{tr}\,(\mathbf{F}\mathbf{F}^T)$ and $J = \det \mathbf{F}$ we first use $F_{ij} = \partial y_i/\partial x_j$ to calculate the components of the deformation gradient tensor associated with (iii):

$$\begin{array}{lll} F_{11} = \lambda, & F_{12} = 0, & F_{13} = 0, \\ F_{21} = x_2 h'(x_1), & F_{22} = h(x_1), & F_{23} = 0, \\ F_{31} = x_3 h'(x_1), & F_{32} = 0, & F_{33} = h(x_1). \end{array} \qquad (ix)$$

Therefore

$$I_1 = \mathrm{tr}\,(\mathbf{F}\mathbf{F}^T) = F_{ij}F_{ij} \overset{(ix)}{=} \lambda^2 + 2h^2 + r^2(h')^2 \qquad \text{and} \qquad J = \det \mathbf{F} \overset{(ix)}{=} \lambda h^2, \qquad (x)$$

where

$$r^2 = x_2^2 + x_3^2.$$

Thus from $(viii)$ and (x):

$$\Phi\{h\} = \int_{\mathcal{R}_R} \Big[\frac{\mu}{2}(\lambda^2 + 2h^2 + r^2(h')^2 - 3) + f(\lambda h^2)\Big]\, dV_x. \qquad \square$$

(d) From (x) we have

$$\delta I_1(x_1) = 4h(x_1)\delta h(x_1) + 2r^2 h'(x_1)\delta h'(x_1) \qquad \text{and} \qquad \delta J(x_1) = 2\lambda h(x_1)\delta h(x_1). \tag{xi}$$

Therefore

$$\begin{aligned}
\delta\Phi\{h,\delta h\} &= \int_{\mathcal{R}_R} \delta W \, dV_x \overset{(viii)}{=} \int_{\mathcal{R}_R} \Big[\frac{\mu}{2}\,\delta I_1 + f'(J)\,\delta J\Big]\, dV_x = \\
&\overset{(xi)}{=} \int_{\mathcal{R}_R} \Big[\frac{\mu}{2}\big[4h\,\delta h + 2r^2 h'\,\delta h'\big] + 2\lambda f'(J) h\,\delta h\Big]\, dV_x = \\
&= \int_0^L \int_{\mathcal{D}} \Big[\frac{\mu}{2}\big[4h\,\delta h + 2r^2 h'\,\delta h'\big] + 2\lambda f'(J) h\,\delta h\Big] \underbrace{dA_x}_{dx_2 dx_3}\, dx_1,
\end{aligned} \tag{xii}$$

where $\mathcal{D}$ is a cross-section of the bar. Let

$$A := \int_{\mathcal{D}} dA_x, \qquad I_p := \int_{\mathcal{D}} r^2 dA_x, \tag{$xiii$}$$

which for a circular cross-section specializes to

$$A = \pi R^2, \qquad I_p = \frac{1}{2}\pi R^4. \tag{$xiiia$}$$

Therefore

$$\begin{aligned}
\delta\Phi\{h,\delta h\} &\overset{(xii)}{=} \int_0^L \Big[\frac{\mu}{2}\big[4Ah\,\delta h + 2I_p h'\,\delta h'\big] + 2\lambda A f'(J) h\,\delta h\Big]\, dx_1 = \\
&= \int_0^L \big[\mu\big[2Ah\,\delta h + I_p h'\,\delta h'\big] + 2\lambda A f'(J) h\,\delta h\big]\, dx_1 = \\
&= \int_0^L \big[\mu\big[2Ah\,\delta h - I_p h''\,\delta h\big] + 2\lambda A f'(J) h\,\delta h\big]\, dx_1 = \\
&= \int_0^L \Big[\mu\big[2Ah - I_p h''\big] + 2\lambda A f'(J) h\Big]\,\delta h\, dx_1.
\end{aligned} \tag{xiv}$$

Since $\delta\Phi\{h,\delta h\} = 0$ for all admissible δh, it follows from the fundamental lemma of the calculus of variations that $2\mu A h - \mu I_p h'' + 2\lambda A f'(J) h = 0$:

$$\frac{I_p}{2A} h''(x_1) - \Big[1 + \lambda\frac{f'(J(x_1))}{\mu}\Big] h(x_1) = 0, \qquad 0 < x_1 < L, \qquad \square \tag{xv}$$

where $J \overset{(x)_2}{=} \lambda h^2(x_1)$. Thus $h(x_1)$ is to be found by solving the second-order, nonlinear, ordinary differential equation (xv) subject to the boundary conditions (vi). Remark: for a circular cross-section, from ($xiiia$) we have

$$\frac{I_p}{A} = \frac{R^2}{2};$$

the calculations below are not limited to a circular cross-section.

(e) Observe because of (vii) and $J = \lambda h^2$ that

$$h(x_1) = \Lambda, \qquad 0 \le x_1 \le L, \tag{xvi}$$

is a solution of the differential equation (xv) (but not the boundary conditions (vi)).

(*f*) Now suppose that $h(x_1) \approx \Lambda$. It is then convenient to let

$$g(x_1) = h(x_1) - \Lambda \qquad (xvii)$$

where we assume $g(x_1)$ and its derivatives to be small. Then $J \overset{(x)_2}{=} \lambda h^2 = \lambda(\Lambda + g)^2 \doteq \lambda\Lambda^2 + 2\lambda\Lambda g$ and so

$$f'(J) = f'(\lambda h^2) \doteq f'(\lambda\Lambda^2 + 2\lambda\Lambda g) \doteq f'(\lambda\Lambda^2) + 2\lambda\Lambda f''(\lambda\Lambda^2)g.$$

Thus the term involving the square bracket in the Euler-Lagrange equation (xv) can be approximated as

$$\left[1 + \frac{\lambda}{\mu} f'(J)\right] h \doteq \left[1 + \frac{\lambda}{\mu} f'(\lambda\Lambda^2) + \frac{\lambda}{\mu} 2\lambda\Lambda f''(\lambda\Lambda^2)g\right](\Lambda + g) \overset{(vii)}{=} \frac{2\lambda^2\Lambda^2}{\mu} f''(\lambda\Lambda^2)g.$$

The differential equation (xv) therefore linearizes to

$$\frac{I_p}{2A} g''(x_1) - \frac{2\lambda^2\Lambda^2}{\mu} f''(\lambda\Lambda^2)g = 0, \qquad 0 < x_1 < L. \qquad (xviii)$$

Since f has the same dimensions as μ (see (iv)) it follows that the coefficient of g is dimensionless. Moreover the term I_p/A has the dimension of length squared. Thus we can introduce the length scale

$$\ell := \sqrt{\frac{\mu I_p/A}{4\lambda^2\Lambda^2 f''(\lambda\Lambda^2)}}. \qquad (xix)$$

Note: this assumes $f''(\lambda\Lambda^2) > 0$. We can write $(xviii)$ as

$$\ell^2 g''(x_1) - g(x_1) = 0, \qquad 0 < x_1 < L. \qquad (xx)$$

Thus we are to find $g(x_1)$ by solving the linear differential equation (xx) subject to the boundary conditions

$$g(0) = g(L) = 1 - \Lambda. \qquad (xxi)$$

We now turn to determining g.

Equation (xx) has solutions $\mathrm{e}^{-x_1/\ell}$ and $\mathrm{e}^{x_1/\ell}$. For reasons that will become apparent shortly it is convenient to write the second solution as $\mathrm{e}^{-(L-x_1)/\ell}$. Thus the general solution of (xx) can be written as

$$g(x_1) = C_1 \exp\left(-\frac{x_1}{\ell}\right) + C_2 \exp\left(-\frac{(L-x_1)}{\ell}\right).$$

By enforcing the boundary conditions (xxi) we find the constants C_1 and C_2 to be

$$C_1 = C_2 = \frac{1-\Lambda}{1 + \exp(-L/\ell)} \ .$$

Therefore

$$g(x_1) = (1-\Lambda)\,\frac{\exp(-x_1/\ell) + \exp(-(L-x_1)/\ell)}{1 + \exp(-L/\ell)}$$

and so by $(xvii)$,

$$h(x_1) = \Lambda + (1-\Lambda)\,\frac{\exp(-x_1/\ell) + \exp(-(L-x_1)/\ell)}{1 + \exp(-L/\ell)}. \qquad \square \qquad (xxii)$$

Note that at any point x_1 away from the two ends we have

$$\frac{x_1}{\ell} \gg 1 \qquad \text{and} \qquad \frac{(L-x_1)}{\ell} \gg 1 \ .$$

Therefore the exponential terms in the numerator of $(xxii)$ are very small in the interior region of the bar and we have $h(x_1) \approx \Lambda$. Therefore away from the two ends, the displacement field $(iii), (xxii)$ has the form (ii). On the other hand at each end, we have $h(0) = h(L) = 1$. The term $\exp(-x_1/\ell)$ is important near the left-hand end and the term $\exp(-(L-x_1)/\ell)$ is important near the right-hand end. A plot of $h(x_1)$ versus x_1 according to $(xxii)$ is shown in Figure 10.10. The length scale ℓ represents the *decay length* of the end-effects.

Problem 10.4.8. *Second variation. Stability of the "Rivlin Cube" with respect to arbitrary perturbations.* Reconsider the stability of the "Rivlin cube" studied in Section 5.3. There, we first determined the various pure homogeneous deformations the body could undergo, and second, investigated whether these deformations minimized the potential energy. In this latter calculation, we limited attention to virtual deformations that were homogeneous and coaxial with the pure homogeneous deformations we were studying. In the present problem, you are asked to consider all virtual deformations.

In the "Rivlin cube" problem the unit cube is subjected to the dead loading $\mathbf{s} = \overline{\mathbf{S}}\mathbf{n}_{\rm R}$ on $\partial\mathcal{R}_{\rm R}$ where $\overline{\mathbf{S}}$ is a given constant tensor. The associated deformation whose stability we want to study is

$$\mathbf{y}(\mathbf{x}) = \mathbf{F}\mathbf{x} \qquad \text{for} \quad \mathbf{x} \in \mathcal{R}_{\rm R}, \tag{i}$$

where the constant tensor $\mathbf{F}$ has $\det \mathbf{F} = 1$ and

$$\overline{\mathbf{S}} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) - q\,\mathbf{F}^{-T}. \tag{ii}$$

In order to study the stability of a deformation (i), consider virtual deformations of the form

$$\mathbf{z}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) + \epsilon\boldsymbol{\eta}(\mathbf{x}) = \mathbf{F}\mathbf{x} + \epsilon\boldsymbol{\eta}(\mathbf{x}) \qquad \text{for} \quad \mathbf{x} \in \mathcal{R}_{\rm R}. \tag{iii}$$

Here $\mathbf{z}(\mathbf{x})$ is the virtual deformation, $\mathbf{y}(\mathbf{x})$ is the deformation whose stability we wish the study, and $\epsilon\boldsymbol{\eta}(\mathbf{x})$ is the virtual displacement. The associated virtual deformation gradient tensor is

$$\mathbf{G} = \nabla_x \mathbf{z} = \mathbf{F} + \epsilon \nabla_x \boldsymbol{\eta}. \tag{iv}$$

Here and in what follows, a subscript, e.g. x, on ∇ indicates that the gradient is being taken with respect to the position, e.g. $\mathbf{x}$. In (iii), ϵ is a scalar parameter and $\boldsymbol{\eta}(\mathbf{x})$ is an arbitrary smooth function subject only to the incompressibility requirement

$$\det \mathbf{G} = 1. \tag{v}$$

The potential energy associated with a virtual deformation $\mathbf{z}(\mathbf{x})$ is

$$\Phi = \int_{\mathcal{R}_{\rm R}} W(\nabla_x \mathbf{z}) dV_x - \int_{\partial\mathcal{R}_{\rm R}} \overline{\mathbf{S}}\mathbf{n}_{\rm R} \cdot \mathbf{z}\, dA_x.$$

It is convenient to incorporate the kinematic constraint (v) into the potential energy through a Lagrange multiplier q, and to therefore consider

$$\Phi = \int_{\mathcal{R}_\mathrm{R}} \big(W(\nabla_x \mathbf{z}) - q \det(\nabla_x \mathbf{z})\big)\, dV_x - \int_{\partial\mathcal{R}_\mathrm{R}} \overline{\mathbf{S}}\mathbf{n}_\mathrm{R} \cdot \mathbf{z}\, dA_x. \qquad (vi)$$

On substituting the virtual deformation (iii) into the potential energy (vi), and keeping $\boldsymbol{\eta}(\mathbf{x})$ fixed for the moment, we can view the potential energy as a function of the scalar parameter ϵ:

$$\Phi = \Phi(\epsilon). \qquad (vii)$$

Since $\mathbf{z}(\mathbf{x}) = \mathbf{y}(\mathbf{x})$ when $\epsilon = 0$, see (iii), it follows that if $\mathbf{y}(\mathbf{x})$ is a minimizer of the potential energy then $\epsilon = 0$ is a minimizer of $\Phi(\epsilon)$. This requires

$$\left.\frac{\mathrm{d}\Phi}{d\epsilon}\right|_{\epsilon=0} = 0, \qquad \left.\frac{\mathrm{d}^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} \geq 0. \qquad (viii).$$

It will be convenient in what follows to let

$$\mathbf{H} := \nabla_y \boldsymbol{\eta} = \nabla_x \boldsymbol{\eta}\, \mathbf{F}^{-1}. \qquad (ix)$$

(a) Show that

$$\det \mathbf{G} = 1 + \operatorname{tr}\mathbf{H} + \mathrm{O}(\epsilon^2) \qquad \text{as} \quad \epsilon \to 0, \qquad (x)$$

so that the incompressibility requirement (v) tells us that $\operatorname{tr}\mathbf{H} = 0 + \mathrm{O}(\epsilon)$ as $\epsilon \to 0$.

(b) Evaluate $d\Phi/d\epsilon$ and show that, in view of (ii), the first requirement $(viii)_1$ holds automatically.

(c) Show that

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_\mathrm{R}} \left[\frac{\partial^2 W}{\partial F_{ij}\partial F_{k\ell}}(\mathbf{F})\eta_{i,j}\eta_{k,\ell} - q(H_{ii}H_{jj} - H_{ij}H_{ji})\right] dV_x. \qquad (xi)$$

(d) Next consider a neo-Hookean material:

$$W = \frac{\mu}{2}(\mathbf{F}\cdot\mathbf{F} - 1), \qquad (xii)$$

and show that (xi) now specializes to

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_\mathrm{R}} [\mu B_{kj} H_{ij} H_{ik} + q H_{ij} H_{ji}]\, dV_x, \qquad (xiii)$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$.

(e) Now consider the stability of the cubic solution $\mathbf{F} = \mathbf{I}$. Recall that the loading is in fact an equi-triaxial dead loading, i.e. $\overline{\mathbf{S}} = S\,\mathbf{I}$. In this case $(ii), (xii)$ gives $q = \mu - S$. Show that

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_\mathrm{R}} \Big[(2\mu - S)\varepsilon_{ij}\varepsilon_{ij} + S\omega_{ij}\omega_{ij}\Big] dV_x, \qquad (xiv)$$

where we have set $\varepsilon_{ij} := \frac{1}{2}(\eta_{i,j} + \eta_{j,i})$ and $\omega_{ij} := \frac{1}{2}(\eta_{i,j} - \eta_{j,i})$.

Thus far we kept $\boldsymbol{\eta}(\mathbf{x})$ fixed. But in fact it is arbitrary, subject only to the requirement stemming from incompressibility. Thus, for stability, it is necessary that the expression in the previous equation be non-negative for all such $\eta_{i,j}$. Show from this that the cubic deformation is stable for $0 < S < 2\mu$. And unstable for $S > 2\mu$ and $S < 0$. What is the nature of a virtual deformation that makes the cubic configuration unstable in the case $S < 0$?

References:

– R, Hill, On uniqueness and stability in the theory of finite elastic strain, *Journal of the Mechanics and Physics of Solids*, 5 (1957), pp. 229–241.
– R.S. Rivlin, Stability of pure homogeneous deformations of an elastic cube under dead loading, *Quarterly Journal of Applied Mathematics*, 1974, pp. 265-271.

Solution:

(a) Observe from (iv) that

$$\mathbf{G}|_{\epsilon=0} = \mathbf{F}, \qquad \frac{d\mathbf{G}}{d\epsilon} = \nabla_x \boldsymbol{\eta}.$$

and from (v) that

$$1 = \det\, \mathbf{G} = \det\left(\mathbf{F} + \epsilon\nabla_x\boldsymbol{\eta}\right) = \det\left[(\mathbf{I} + \epsilon\nabla_x\boldsymbol{\eta}\mathbf{F}^{-1})\mathbf{F}\right] = \det(\mathbf{I} + \epsilon\mathbf{H}) = 1,$$

where we have used (ix). In view of the identity $\det(\mathbf{I} + \mathbf{A}) = 1 + I_1(\mathbf{A}) + I_2(\mathbf{A}) + I_3(\mathbf{A})$ we can write

$$\det \mathbf{G} = 1 + I_1\big(\epsilon\mathbf{H}\big) + I_2\big(\epsilon\mathbf{H}\big) + \mathrm{O}(\epsilon^3) = 1 + \epsilon H_{ii} + \frac{\epsilon^2}{2}(H_{ii}H_{jj} - H_{ij}H_{ji}) + \mathrm{O}(\epsilon^3),$$

from which (x) follows. The requirement (v) now yields

$$\mathrm{tr}\, \mathbf{H} = 0 + \mathrm{O}(\epsilon). \qquad (xv)$$

(b) By using the divergence theorem, the potential energy associated with the loading device can be written as

$$\int_{\partial\mathcal{R}_\mathrm{R}} -\overline{\mathbf{S}}\mathbf{n}_\mathrm{R} \cdot \mathbf{z}\, dA_x = \int_{\mathcal{R}_\mathrm{R}} -\overline{\mathbf{S}} \cdot \mathbf{G}\, dV_x,$$

and therefore (vi) can be rewritten as

$$\Phi = \int_{\mathcal{R}_\mathrm{R}} [W(\mathbf{G}) - \overline{\mathbf{S}} \cdot \mathbf{G} - q \det \mathbf{G}] dV_x.$$

On calculating the first derivative of Φ with respect to ϵ one gets

$$\frac{d\Phi}{d\epsilon} = \int_{\mathcal{R}_\mathrm{R}} \left[\frac{\partial W}{\partial \mathbf{F}}(\mathbf{G}) \cdot \nabla_x\boldsymbol{\eta} - \overline{\mathbf{S}} \cdot \nabla_x\boldsymbol{\eta} - q\left[H_{ii} + \epsilon(H_{ii}H_{jj} - H_{ij}H_{ji})\right]\right] dV_x + \mathrm{O}(\epsilon^2)$$

and so

$$\left.\frac{d\Phi}{d\epsilon}\right|_{\epsilon=0} = \int_{\mathcal{R}_\mathrm{R}} \left[\frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) \cdot \nabla_x\boldsymbol{\eta} - q\, \mathrm{tr}\mathbf{H} - \overline{\mathbf{S}} \cdot \nabla_x\boldsymbol{\eta}\right] dV_x.$$

But $\operatorname{tr}\mathbf{H} = \mathbf{F}^{-T}\cdot\nabla_x\boldsymbol{\eta}$ and so this yields

$$\left.\frac{d\Phi}{d\epsilon}\right|_{\epsilon=0} = \int_{\mathcal{R}_{\rm R}} \left[\frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) - q\,\mathbf{F}^{-T} - \overline{\mathbf{S}}\right]\cdot\nabla_x\boldsymbol{\eta}\,dV_x$$

which vanishes because of (ii).

(c) On calculating the second derivative of Φ with respect to ϵ one gets

$$\frac{d^2\Phi}{d\epsilon^2} = \int_{\mathcal{R}_{\rm R}} \left[\frac{\partial^2 W}{\partial F_{ij}\partial F_{k\ell}}(\mathbf{G})\eta_{i,j}\eta_{k,\ell} - q(H_{ii}H_{jj} - H_{ij}H_{ji})\right]dV_x + \mathrm{O}(\epsilon),$$

and so

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_{\rm R}} \left[\frac{\partial^2 W}{\partial F_{ij}\partial F_{k\ell}}(\mathbf{F})\eta_{i,j}\eta_{k,\ell} - q(H_{ii}H_{jj} - H_{ij}H_{ji})\right]dV_x. \qquad (xvi)$$

(d) Now consider a neo-Hookean material:

$$W = \frac{\mu}{2}(\mathbf{F}\cdot\mathbf{F} - 1),$$

for which

$$\frac{\partial W}{\partial F_{ij}} = \mu F_{ij}, \qquad \frac{\partial^2 W}{\partial F_{ij}\partial F_{k\ell}} = \mu\delta_{ik}\delta_{j\ell}.$$

Thus (ii) gives

$$\overline{\mathbf{S}} = \mu\mathbf{F} - q\mathbf{F}^{-T} \qquad \Rightarrow \qquad q\mathbf{I} = \mu\mathbf{B} - \overline{\mathbf{T}}, \qquad (xvii)$$

where $\overline{\mathbf{T}} = \overline{\mathbf{S}}\mathbf{F}^T$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. Equation (xvi) specializes for the neo-Hookean material to

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_{\rm R}} \left[\mu\eta_{i,j}\eta_{i,j} - q(H_{ii}H_{jj} - H_{ij}H_{ji})\right]dV_x,$$

which further specializes because of (xv) to

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_{\rm R}} \left[\mu\,\eta_{i,j}\eta_{i,j} + qH_{ij}H_{ji}\right]dV_x.$$

In view of (ix),

$$\nabla_x\boldsymbol{\eta}\cdot\nabla_x\boldsymbol{\eta} = \mathbf{HF}\cdot\mathbf{HF} = \mathbf{HB}\cdot\mathbf{H},$$

and so

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_{\rm R}} \left[\mu B_{kj}H_{ij}H_{ik} + qH_{ij}H_{ji}\right]dV_x. \qquad (xviii)$$

We can use this to study the stability of any equilibrium deformation $\mathbf{y}(\mathbf{x}) = \mathbf{F}\mathbf{x}$ subject to a uniform dead loading $\mathbf{s} = \overline{\mathbf{S}}\mathbf{n}_{\rm R}$.

(e) Now consider the stability of the undeformed configuration. Here $\mathbf{F} = \mathbf{B} = \mathbf{I}$ and so $(xviii)$ simplifies to

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_{\rm R}} \left[\mu H_{ij}H_{ij} + qH_{ij}H_{ji}\right]dV_x.$$

Moreover $(xvii)$ with $\overline{\mathbf{S}} = S\mathbf{I}$ and $\mathbf{F} = \mathbf{I}$ gives

$$q = \mu - S. \qquad (xix)$$

If we set $\varepsilon_{ij} := \frac{1}{2}(\eta_{i,j} + \eta_{j,i})$ and $\omega_{ij} := \frac{1}{2}(\eta_{i,j} - \eta_{j,i})$ we can write (xix) as

$$\left.\frac{d^2\Phi}{d\epsilon^2}\right|_{\epsilon=0} = \int_{\mathcal{R}_\mathrm{R}} \Big[(2\mu - S)\varepsilon_{ij}\varepsilon_{ij} + S\omega_{ij}\omega_{ij})\Big] dV_x. \tag{xx}$$

Thus far we held $\boldsymbol{\eta}(\mathbf{x})$ fixed subject only to the incompressibility requirement (xv) that, in view of $ix), (xv)$ and $\mathbf{F} = \mathbf{I}$ reads

$$\eta_{i,i} = 0. \tag{xxi}$$

The second necessary condition $(xviii)_2$ requires that the expression in (xx) be non-negative for all $\boldsymbol{\eta}(\mathbf{x})$ obeying (xxi).

Note that $\varepsilon_{ij}\varepsilon_{ij}$ and $\omega_{ij}\omega_{ij}$ are both non-negative. If $0 < S < 2\mu$ then both coefficients in (xx) are positive and so the inequality holds and the equilibrium state is stable. If either coefficient is negative, i.e. $S > 2\mu$ or $S < 0$, then one can find some $\eta_{i,j}$ that violates this inequality and so the equilibrium state is unstable.

Remark: The instability in the case $S < 0$ involves choosing $\eta_{i,j}$ such that $\varepsilon_{ij} = 0$ and so the instability is with respect to infinitesimal rotations.

10.5 Virtual Work. Weak formulation.

Consider again a body that occupies a region $\mathcal{R}_R$ in a reference configuration whose boundary is $\partial R_R = \mathcal{S}_1 \cup \mathcal{S}_2$. The deformation $\widehat{\mathbf{y}}(\mathbf{x})$ is prescribed on $\mathcal{S}_1$, the Piola traction $\widehat{\mathbf{s}}(\mathbf{x})$ is prescribed on $\mathcal{S}_2$, and the body force $\mathbf{b}_R(\mathbf{x})$ is prescribed on $\mathcal{R}_R$. The virtual work principle provides an alternative statement of the equilibrium equations and traction boundary conditions.

The set of kinematically admissible deformations is

$$\mathcal{A} = \{\mathbf{z} \in C^1(\mathcal{R}_R) : \mathbf{z}(\mathbf{x}) = \widehat{\mathbf{y}}(\mathbf{x}) \text{ for } \mathbf{x} \in \mathcal{S}_1\}, \tag{10.57}$$

and the set of admissible variations is

$$\mathcal{V} = \{\mathbf{w} \in C^1(\mathcal{R}_R) : \mathbf{w}(\mathbf{x}) = \mathbf{o} \text{ for } \mathbf{x} \in \mathcal{S}_1\}. \tag{10.58}$$

The principle of virtual work states that

$$\int_{\mathcal{R}_R} \mathbf{S} \cdot \nabla\mathbf{w}\, dV_x \;=\; \int_{\mathcal{S}_2} \widehat{\mathbf{s}} \cdot \mathbf{w}\, dA_x + \int_{\mathcal{R}_R} \mathbf{b}_R \cdot \mathbf{w}\, dV_x \quad \text{for all } \mathbf{w} \in \mathcal{V}, \tag{10.59}$$

if and only if

$$\mathbf{S}\mathbf{n}_R = \widehat{\mathbf{s}} \qquad \text{on } \mathcal{S}_2, \tag{10.60}$$

$$\text{Div}\,\mathbf{S} + \mathbf{b}_R = \mathbf{o} \qquad \text{on } \mathcal{R}_R. \tag{10.61}$$

Note that this does not require the body to be elastic. The proof is straightforward and merely involves using the divergence theorem and the fundamental lemma of the calculus of variations. If the material is elastic, we can write (10.59) as

$$\int_{\mathcal{R}_R} W_{\mathbf{F}} \cdot \nabla\mathbf{w}\, dV_x \;=\; \int_{\mathcal{S}_2} \widehat{\mathbf{s}} \cdot \mathbf{w}\, dA_x + \int_{\mathcal{R}_R} \mathbf{b}_R \cdot \mathbf{w}\, dV_x \quad \text{for all } \mathbf{w} \in \mathcal{V}, \tag{10.62}$$

where $W_{\mathbf{F}}$ is evaluated at $\nabla\mathbf{y}(\mathbf{x})$.

The mathematical problem associated with the problem described in the first paragraph of this section, in the case when the material is elastic, can now be stated in three alternative forms: the strong form, weak form and variational form.

<u>Strong form</u>: Find the deformation $\mathbf{y}(\mathbf{x}) \in C^2(\mathcal{R}_R)$ such that

$$\left.\begin{array}{c} \text{Div}\left[W_{\mathbf{F}}(\nabla\mathbf{y})\right] + \mathbf{b}_R(\mathbf{x}) = \mathbf{o} \qquad \text{for} \quad \mathbf{x} \in \mathcal{R}_R, \\[2mm] \mathbf{y}(\mathbf{x}) = \widehat{\mathbf{y}}(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \mathcal{S}_1, \qquad W_{\mathbf{F}}(\nabla\mathbf{y})\mathbf{n}_R = \widehat{\mathbf{s}}(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \mathcal{S}_2. \end{array}\right\} \tag{10.63}$$

<u>Weak form</u>: Find the deformation $\mathbf{y}(\mathbf{x}) \in \mathcal{A}$ such that

$$\int_{\mathcal{R}_R} W_{\mathbf{F}}(\nabla\mathbf{y}) \cdot \nabla\mathbf{w}\, dV_x = \int_{\mathcal{S}_2} \widehat{\mathbf{s}} \cdot \mathbf{w}\, dA_x + \int_{\mathcal{R}_R} \mathbf{b}_R \cdot \mathbf{w}\, dV_x \qquad \text{for all } \mathbf{w} \in \mathcal{V}, \tag{10.64}$$

where $\mathcal{A}$ and $\mathcal{V}$ were defined in (10.57) and (10.58). Note that (10.64) is in fact the virtual work statement (10.62).

<u>Variational form</u>: The potential energy functional is defined for all kinematically admissible deformations $\mathbf{z} \in \mathcal{A}$ by

$$\Phi\{\mathbf{z}\} := \int_{\mathcal{R}_R} W(\nabla\mathbf{z}) dV_x - \int_{\mathcal{R}_R} \mathbf{b}_R \cdot \mathbf{z} dA_x - \int_{\mathcal{S}_2} \widehat{\mathbf{s}} \cdot \mathbf{z} dA_x \qquad \text{for all} \quad \mathbf{z} \in \mathcal{A}.$$

Find $\mathbf{y}(\mathbf{x}) \in \mathcal{A}$ such that

$$\delta\Phi\{\mathbf{y}, \delta\mathbf{y}\} = 0 \quad \text{for all } \delta\mathbf{y} \in \mathcal{V} \qquad \text{where} \qquad \delta\Phi\{\mathbf{y}, \delta\mathbf{y}\} := \left.\frac{d\Phi}{d\epsilon}\{\mathbf{y} + \epsilon\boldsymbol{\eta}\}\right|_{\epsilon=0}.$$

This is related to finding $\mathbf{y}(\mathbf{x}) \in \mathcal{A}$ such that

$$\Phi\{\mathbf{z}\} \geq \Phi\{\mathbf{y}\} \qquad \text{for all } \mathbf{z} \in \mathcal{A}.$$

10.6 Worked examples.

Problem 10.5.1. Show that force and moment balance of a part $\mathcal{D}$,

$$\int_{\partial\mathcal{D}} \mathbf{t}\, dA_y + \int_{\mathcal{D}} \mathbf{b}\, dV_y = \mathbf{o}, \tag{i}$$

$$\int_{\partial\mathcal{D}} \mathbf{y}\times\mathbf{t}\, dA_y + \int_{\mathcal{D}} \mathbf{y}\times\mathbf{b}\, dV_y = \mathbf{o}, \tag{ii}$$

hold if and only if the rate of working (of the tractions and body forces on $\mathcal{D}$) vanishes in all steady rigid motions.

Solution We will show at the end of this solution that the velocity field in a steady rigid motion is

$$\mathbf{v}(\mathbf{y}) = \mathbf{c} + \boldsymbol{\omega}\times\mathbf{y}, \tag{iii}$$

where the constant vectors $\boldsymbol{\omega}$ and $\mathbf{c}$ represent the angular and translational velocities respectively.

Substituting (iii) into the expression for the rate of working gives

$$\mathbb{P}_{\text{rigid}} = \int_{\partial\mathcal{D}} \mathbf{t}\cdot\mathbf{v}\, dA_y + \int_{\mathcal{D}} \mathbf{b}\cdot\mathbf{v}\, dV_y = \int_{\partial\mathcal{D}} \mathbf{t}\cdot(\mathbf{c}+\boldsymbol{\omega}\times\mathbf{y})\, dA_y + \int_{\mathcal{D}} \mathbf{b}\cdot(\mathbf{c}+\boldsymbol{\omega}\times\mathbf{y})\, dV_y,$$

which, since $\mathbf{c}$ is a constant vector, we can write as

$$\mathbb{P}_{\text{rigid}} = \mathbf{c}\cdot\left[\int_{\partial\mathcal{D}} \mathbf{t}\, dA_y + \int_{\mathcal{D}} \mathbf{b}\, dV_y\right] + \left[\int_{\partial\mathcal{D}} \mathbf{t}\cdot(\boldsymbol{\omega}\times\mathbf{y})\, dA_y + \int_{\mathcal{D}} \mathbf{b}\cdot(\boldsymbol{\omega}\times\mathbf{y})\, dV_y\right].$$

This can be rewritten using the vector identity $\mathbf{p}\cdot(\mathbf{q}\times\mathbf{r}) = \mathbf{q}\cdot(\mathbf{r}\times\mathbf{p})$ as

$$\mathbb{P}_{\text{rigid}} = \mathbf{c}\cdot\left[\int_{\partial\mathcal{D}} \mathbf{t}\, dA_y + \int_{\mathcal{D}} \mathbf{b}\, dV_y\right] + \left[\int_{\partial\mathcal{D}} \boldsymbol{\omega}\cdot(\mathbf{y}\times\mathbf{t})\, dA_y + \int_{\mathcal{D}} \boldsymbol{\omega}\cdot(\mathbf{y}\times\mathbf{b})\, dV_y\right],$$

from which, since $\boldsymbol{\omega}$ is a constant vector, we conclude that

$$\mathbb{P}_{\text{rigid}} = \mathbf{c}\cdot\left[\int_{\partial\mathcal{D}} \mathbf{t}\, dA_y + \int_{\mathcal{D}} \mathbf{b}\, dV_y\right] + \boldsymbol{\omega}\cdot\left[\int_{\partial\mathcal{D}} \mathbf{y}\times\mathbf{t}\, dA_y + \int_{\mathcal{D}} \mathbf{y}\times\mathbf{b}\, dV_y\right]. \tag{iv}$$

Therefore when force and moment balance, (i) and (ii), hold, it follows from (iv) that the rate of working vanishes: $\mathbb{P}_{\text{rigid}} = 0$. Conversely if the rate of working vanishes is every steady rigid motion, then $\mathbb{P}_{\text{rigid}} = 0$ for all vectors $\mathbf{c}$ and $\boldsymbol{\omega}$, and so it follows from (iv) that each term in square brackets must vanish, and therefore that force and moment balance necessarily hold.

A steady rigid motion: Such a motion is described by $\mathbf{y}(\mathbf{x},t) = \mathbf{Q}(t)\mathbf{x} + \mathbf{d}(t)$ where the proper orthogonal tensor $\mathbf{Q}(t)$ characterizes the rotation and the vector $\mathbf{d}(t)$ the translation. Differentiating with respect to time t gives the corresponding velocity

$$\mathbf{v} = \dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{d}} = \dot{\mathbf{Q}}(\mathbf{Q}^T\mathbf{y} - \mathbf{Q}^T\mathbf{d}) + \dot{\mathbf{d}} = \dot{\mathbf{Q}}\mathbf{Q}^T\mathbf{y} + (\dot{\mathbf{d}} - \dot{\mathbf{Q}}\mathbf{Q}^T\mathbf{d}) =: \mathbf{W}\mathbf{y} + \mathbf{c}. \tag{v}$$

where we have set $\mathbf{c} = \dot{\mathbf{d}} - \dot{\mathbf{Q}}\mathbf{Q}^T\mathbf{d}$ and $\mathbf{W} = \dot{\mathbf{Q}}\mathbf{Q}^T$. However, since $\mathbf{Q}$ is orthogonal at each instant t we have $\mathbf{Q}(t)\mathbf{Q}^T(t) = \mathbf{I}$. Differentiating this with respect to time gives $\mathbf{0} = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = \dot{\mathbf{Q}}\mathbf{Q}^T + \left(\dot{\mathbf{Q}}\mathbf{Q}^T\right)^T = \mathbf{W} + \mathbf{W}^T$. Therefore $\mathbf{W}$ is a skew-symmetric tensor. Let $\boldsymbol{\omega}$ be the associated axial vector in which case $\mathbf{W}\mathbf{z} = \boldsymbol{\omega}\times\mathbf{z}$ for all vectors $\mathbf{z}$. It follows that (v) can be written as (iii).

10.7 Appendix: some remarks.

Existence of minimizers: It is worth recalling that even in the case of a familiar *function* $\varphi(x)$ of a scalar variable x, the function need not have a minimizer. For example, the function $\varphi_1(x) = x$ defined on $\mathcal{A}_1 = (-\infty, \infty)$ is unbounded as $x \to \pm\infty$. Another example is the function $\varphi_2(x) = x$ defined on $\mathcal{A}_2 = (-1, 1)$ noting that $\varphi_2 \geq -1$ on $\mathcal{A}_2$; however, while the value of φ_2 can get arbitrarily close to -1, it cannot actually achieve the value -1 since there is no $x \in \mathcal{A}_2$ at which $\varphi_2(x) = -1$; note that $-1 \notin \mathcal{A}_2$. Finally, consider the function $\varphi_3(x)$ defined on $\mathcal{A}_3 = [-1, 1]$ where $\varphi_3(x) = 1$ for $-1 \leq x \leq 0$ and $\varphi_3(x) = x$ for $0 < x \leq 1$; the value of φ_3 can get arbitrarily close to 0 but cannot achieve it since $\varphi_3(0) = 1$. In the first example $\mathcal{A}_1$ was unbounded. In the second, $\mathcal{A}_2$ was bounded but open. And in the third example $\mathcal{A}_3$ was bounded and closed but the function was discontinuous on $\mathcal{A}_3$. In order for a minimizer to exist, $\mathcal{A}$ must be bounded and closed (i.e. "compact"). It can be shown that if $\mathcal{A}$ is compact and if φ is continuous on $\mathcal{A}$ then φ assumes both maximum and minimum values on $\mathcal{A}$. The corresponding questions regarding the existence of minimizers in the calculus of variations are not addressed in these notes.

Local vs. global and weak vs. strong minimizers: In the calculus of variations we are given a functional F defined on a function space $\mathcal{A}$ with $F : \mathcal{A} \mapsto \mathbb{R}$, and we are asked to find a function $u \in \mathcal{A}$ that minimizes F over $\mathcal{A}$: i.e. to find $u \in \mathcal{A}$ for which

$$F\{w\} \geq F\{u\} \qquad \text{for all} \quad w \in \mathcal{A}.$$

Often we will be looking for a *local (or relative) minimizer*, i.e. for a function u that minimizes F relative to all "nearby functions" w. This requires that we select a norm so that the distance between two functions can be quantified. For example, for a function w in the set of functions that are continuous on an interval $[0, 1]$, i.e. for $w \in C[0, 1]$, one can define a norm by, say,

$$||w||_0 = \max_{0 \leq x \leq 1} |w(x)|.$$

As a second example, for a function w in the set of functions that are continuous and have continuous first derivatives on $[0, 1]$, i.e. for $w \in C^1[0, 1]$, one can define a norm by, say,

$$||w||_1 = \max_{0 \leq x \leq 1} |w(x)| \ + \ \max_{0 \leq x \leq 1} |w'(x)|;$$

and so on. (Of course the norm $||w||_0$ can also be used on $C^1[0, 1]$.)

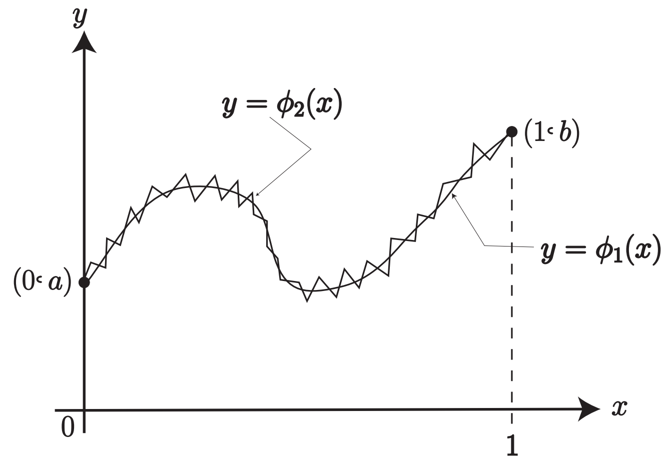

Figure 10.11: Two functions ϕ_1 and ϕ_2 that are "close" in the sense of the norm $||\cdot||_0$ but not in the sense of the norm $||\cdot||_1$.

Suppose we are working with functions that are $C^1[0,1]$. When seeking a local minimizer of F we might say we want to find u for which

$$F\{w\} \geq F\{u\} \qquad \text{for all admissible } w \text{ such that} \quad ||w-u||_0 < r,$$

for some $r > 0$. In this case the minimizer u is being compared with all admissible functions w whose values are close to those of u for all $0 \leq x \leq 1$. Such a local minimizer is called a *strong minimizer*. On the other hand, we may want to find u such that

$$F\{w\} \geq F\{u\} \qquad \text{for all admissible } w \text{ such that} \quad ||w-u||_1 < r,$$

for some $r > 0$. In this case the minimizer is being compared with all functions whose values *and whose first derivatives* are close to those of u for all $0 \leq x \leq 1$. Such a local minimizer is called a *weak minimizer*. A strong minimizer is automatically a weak minimizer.

In our analysis when determining the minimizer $u \in \mathcal{A}$ we consider the one-parameter family of admissible functions

$$w(x;\epsilon) = u(x) + \epsilon\,\eta(x) \tag{10.65}$$

that were close to u. Here ϵ is a real variable in some range $-\epsilon_0 < \epsilon < \epsilon_0$, $\eta(x) \in \mathcal{V}$, and $u + \epsilon\eta \in \mathcal{A}$ for each $\epsilon \in (-\epsilon_0, \epsilon_0)$ and $\eta \in \mathcal{V}$.

We took η to be independent of ϵ. In this case the functions $u(x)$ and $w(x;\epsilon)$, and their derivatives, are close to each other for small ϵ. Thus we were concerned with *weak local minima*.

Suppose we had chosen for example the functions $u(x)$ and $w(x;\epsilon) = u(x) + \epsilon \sin(x/\epsilon)$. In this case the functions u and w are close to each other for small ϵ but their derivatives are not close.

10.8 Exercises.

Problem 10.1. *Cavitation in a compressible solid.* (See Problem 10.4.3 for cavitation in a compressible solid.) In an unstressed reference configuration, the body occupies a solid sphere of radius B. The boundary $R = B$ is given a purely radial displacement such that its radius in the deformed configuration is ΛB where $\Lambda > 1$. The resulting spherically symmetric deformation has the form $r = r(R), \theta = \Theta, \phi = \Phi$ where (R, Θ, Φ) and (r, θ, ϕ) are spherical polar coordinates in the reference and current configurations respectively. The homogeneous, isotropic, unconstrained, elastic material is described by its strain energy function $W(\lambda_1, \lambda_2, \lambda_3)$.

A kinematically admissible radial deformation is described by a function $\mathfrak{r}(R)$ defined and suitably smooth on $[0, B]$ satisfying the kinematic boundary condition $\mathfrak{r}(B) = \Lambda B$. Note that we impose no requirements on $\mathfrak{r}(0)$. The admissible set $\mathcal{A}$ of all such functions is

$$\mathcal{A} = \{\mathfrak{r} \in C^1(0, B) : \quad \mathfrak{r}(B) = \Lambda B; \quad \mathfrak{r}(0) \geq 0; \quad \mathfrak{r}^2(R)\mathfrak{r}'(R)/R^2 > 0 \text{ for } 0 < R < B\}, \qquad (i)$$

where the requirement $\mathfrak{r}^2(R)\mathfrak{r}'(R)/R^2 > 0$ ensures that the associated Jacobian determinant is positive and $\mathfrak{r}(0) \geq 0$ prevents interpenetration. The potential energy functional[9] is defined as

$$\Psi\{\mathfrak{r}\} = \int_{\mathcal{R}_R} W\, dV_x = \int_0^B W\Big(\mathfrak{r}'(R),\, \mathfrak{r}(R)/R,\, \mathfrak{r}(R)/R\Big)\, 4\pi R^2\, dR \qquad \text{for all } \mathfrak{r} \in \mathcal{A}. \qquad (ii)$$

(a) If $r(R)$ is an extremizer of the potential energy functional, derive the Euler-Lagrange equation on $(0, B)$ and the natural boundary at $R = 0$ that $r(R)$ must obey. Interpret the natural boundary condition.

(b) Determine the extremizing deformation(s) of (ii) for a compressible Varga material described by the strain energy function

$$W = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3) + F(J), \qquad J = \lambda_1\lambda_2\lambda_3, \qquad \mu > 0. \qquad (iii)$$

Assume $F(J)$ to have the following characteristics: the strain energy and stress vanish in the reference configuration whence

$$F(1) = 0, \qquad F'(1) = -2\mu < 0; \qquad (iv)$$

the Baker-Ericksen inequalities require $\mu > 0$; the shear and bulk moduli at infinitesimal deformations are μ and $\kappa = F''(1) - 4\mu/3$ so that requiring $\kappa > 0$ demands

$$F''(1) > \frac{4}{3}\mu > 0. \qquad (v)$$

Moreover take $F(J)$ to have a local minimum at some $J = J_0 > 1$,

$$F'(J_0) = 0, \quad F''(J_0) > 0, \qquad J_0 > 1; \qquad (vi)$$

---

[9]Since we used the symbol Φ to denote a spherical polar angle, we are using Ψ here to denote the potential energy.

and assume that $F''(J) > 0$ on some interval $[1, J_*]$ that contains J_0 in its interior:

$$F''(J) > 0 \qquad \text{for} \quad 1 \le J \le J_* \quad \text{for some } J_* > J_0. \tag{vii}$$

Thus $F(J)$ has a unique local minimum in the interval $[1, J_*]$ and it occurs at J_0. Figure 10.12 shows a schematic sketch of $F(J)$. We will only be concerned with deformations in which the Jacobian determinant takes values in the interval $[1, J_*]$.

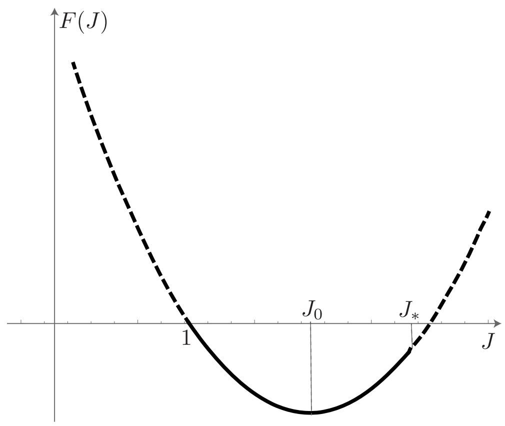

Figure 10.12: Schematic sketch of the constitutive function $F(J)$ for the compressible Varga material (iii) where $F(1) = 0, F'(1) < 0, F''(1) > 0$; $F'(J_0) = 0$; $F''(J) > 0$ for $1 \le J \le J_*$ where $J_* > J_0 > 1$. We will only be concerned with deformations on the interval $[1, J_*]$.

(b1) Under what conditions does a solution with $r(0) = 0$ exist?

(b2) Under what conditions (if any) does a (cavitation) solution with $r(0) > 0$ exist?

(b2) When both types of solutions exist, which has less potential energy?

Solution:

References

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(a) Using

$$\lambda_1 = \lambda_r = r'(R), \qquad \lambda_2 = \lambda_\theta = \lambda_3 = \lambda_\phi = \frac{r(R)}{R}, \tag{viii}$$

and the notation

$$W_i \equiv \frac{\partial W}{\partial \lambda_i}(\lambda_1, \lambda_2, \lambda_3), \tag{ix}$$

we calculate the first variation of Ψ in the usual way:

$$\begin{aligned}
\delta\Psi\{r, \delta r\} &= \int_0^B [W_1\delta\lambda_1 + W_2\delta\lambda_2 + W_3\delta\lambda_3]\, 4\pi R^2\, dR = \int_0^B [W_1\delta\lambda_1 + 2W_2\delta\lambda_2]\, 4\pi R^2\, dR = \\
&= \int_0^B \left[W_1\delta r' + 2W_2\frac{\delta r}{R}\right] 4\pi R^2\, dR = 4\pi \int_0^B \left[R^2 W_1 \delta r' + 2RW_2\delta r\right]\, dR = \\
&= 4\pi \int_0^B \left[\frac{d}{dR}[R^2 W_1 \delta r] - \frac{d}{dR}(R^2 W_1)\delta r + 2RW_2\delta r\right]\, dR = \\
&= 4\pi \Big[R^2 W_1 \delta r\Big]_0^B + 4\pi \int_0^B \left[-\frac{d}{dR}(R^2 W_1) + 2RW_2\right]\delta r\, dR = \\
&= -4\pi R^2 W_1\Big|_{R=0} \delta r(0) + 4\pi \int_0^B \left[-\frac{d}{dR}(R^2 W_1) + 2RW_2\right] 4\pi R^2 \delta r\, dR,
\end{aligned}$$

where we used $\delta r(B) = 0$ in getting to the last line. Since $\delta\Psi\{r, \delta r\} = 0$ for all admissible δr, we arrive at the Euler-Lagrange equation

$$\frac{d}{dR}(R^2 W_1) - 2RW_2 = 0 \qquad \text{for} \quad 0 < R < B, \qquad \square \tag{x}$$

and the natural boundary condition

$$R^2 W_1 = 0 \qquad \text{for} \quad R = 0. \qquad \square \tag{xi}$$

We showed in Problem 10.4.1 that the Euler-Lagrange equation (x) is in fact the radial equilibrium equation.

To interpret the natural boundary condition, we recall that the radial component of the Cauchy stress tensor anywhere in the body can be written as

$$T_{rr} = \frac{\lambda_1}{\lambda_1\lambda_2\lambda_3} W_1 = \frac{1}{\lambda_\theta^2} W_1 \overset{(viii)}{=} \frac{R^2}{r^2(R)} W_1.$$

Therefore the natural boundary condition (xi) can be written equivalently as

$$r^2(R) T_{rr}(R) = 0 \qquad \text{for} \quad R = 0. \tag{xii}$$

Observe that this boundary condition can be satisfied in two ways: one, $r(0) = 0$ (with $T_{rr}(0)$ arbitrary); and two, $r(0) > 0$ with $T_{rr}(0) = 0$. When the former holds, the body remains a solid sphere in the deformed configuration. In the event that the second alternative holds, a traction-free cavity of radius $r(0)$ has appeared in the body. $\square$

(b) For the compressible Varga material (iii) we have

$$W_1 = 2\mu + \lambda_\theta^2 F'(J), \qquad W_2 = 2\mu + \lambda_r\lambda_\theta F'(J). \tag{xiii}$$

– First consider the equilibrium equation (x). Substituting $(xiii)$ into it and simplifying leads to

$$r^2 \frac{d}{dR} F'(J) = 0 \qquad \text{for} \quad 0 < R < B,$$

which can be solved as follows:

$$F''(J)\frac{dJ}{dR} = 0 \quad \overset{(vii)}{\Rightarrow} \quad \frac{dJ}{dR} = 0 \quad \Rightarrow \quad J(R) = k_1 = \text{constant} \quad \text{for} \quad 0 < R < B, \tag{xiv}$$

where k_1 is a positive constant to be determined. On using $J = \lambda_r \lambda_\theta^2 \overset{(viii)}{=} r' r^2/R^2$, equation (xiv) yields

$$\frac{r^2}{R^2} r'(R) = k_1 \qquad \Rightarrow \qquad r(R) = \left[k_1 R^3 + a^3\right]^{1/3} \qquad \text{for} \quad 0 < R < B, \tag{xv}$$

where a is a second constant parameter to be determined. Note that

$$r(0) = a \geq 0. \tag{xvi}$$

– Next consider the essential boundary condition $r(B) = \Lambda B$. Enforcing this on (xv) gives $\Lambda^3 B^3 = k_1 B^3 + a^3$ from which we find

$$k_1 = \Lambda^3 - a^3/B^3. \tag{$xvii$}$$

The deformation (xv) can thus be written as

$$r(R) = \left[\left(\Lambda^3 - a^3/B^3\right) R^3 + a^3\right]^{1/3}, \qquad 0 < R \leq B, \tag{$xviii$}$$

where $a \geq 0$ remains to be determined.

– Finally we turn to the natural boundary condition $R^2 W_1 = 0$ at $R = 0$. This requires that we examine W_1 as $R \to 0$ where, from $(xiii)_1$ and (xiv),

$$W_1 = 2\mu + \lambda_\theta^2(R) F'(k_1). \tag{xix}$$

Observe that

$$\lambda_\theta(R) = \frac{r}{R} \overset{(xviii)}{=} \left[\left(\Lambda^3 - a^3/B^3\right) + a^3/R^3\right]^{1/3}. \tag{xx}$$

Therefore as $R \to 0$,

$$\lambda_\theta = \Lambda \text{ if } a = 0 \qquad \text{and} \qquad \lambda_\theta \sim a/R \text{ if } a > 0. \tag{xxi}$$

$(b1)$ Case $a = 0$: since $\lambda_\theta = \Lambda$ in this case we see from (xix) that W_1 remains bounded as $R \to 0$ and so the natural boundary condition $R^2 W_1 = 0$ as $R \to 0$ holds automatically. The deformation associated with this solution is given by $(xviii)$ to be

$$r(R) = \Lambda R \qquad \text{for} \quad 0 \leq R \leq B. \qquad \square \tag{$xxii$}$$

This solution, corresponding to

$$a = 0, \tag{$xxiii$}$$

is possible for all values of the applied stretch $\Lambda > 0$. The bold black horizontal line in Figure 10.13 depicts this solution.

$(b2)$ Case $a > 0$: since $\lambda_\theta \sim a/R$ as $R \to 0$ in this case, it follows from (xix) that

$$W_1 \sim \lambda_\theta^2 F'(k_1) = a^2 F'(k_1)/R^2. \tag{$xxiv$}$$

Therefore $R^2 W_1 = a^2 F'(k_1)$ as $R \to 0$ and so the boundary condition $R^2 W_1 = 0$ reduces to

$$F'(k_1) = 0. \tag{xxv}$$

In view of (vi) this tells us that

$$k_1 = J_0 \tag{xxvi}$$

where J_0 is where the minimum of $F(J)$ occurs; see Figure 10.12. From $(xvii)$ we now have

$$a = B\Big[\Lambda^3 - J_0\Big]^{1/3}. \tag{xxvii}$$

The associated deformation is given by $(xviii)$ and $(xxvii)$:

$$r(R) = \big[J_0 R^3 + B^3(\Lambda^3 - J_0)\big]^{1/3} \qquad \text{for} \quad 0 \le R \le B. \qquad \Box \tag{xxviii}$$

Since $a > 0$ it follows from $(xxvii)$ that this solution exists only for

$$\Lambda > J_0^{1/3}. \tag{xxix}$$

Equation $(xxvii)$ describes the relationship between the cavity radius $r(0) = a$ and the applied stretch Λ, which is depicted by the bold blue curve in Figure 10.13.

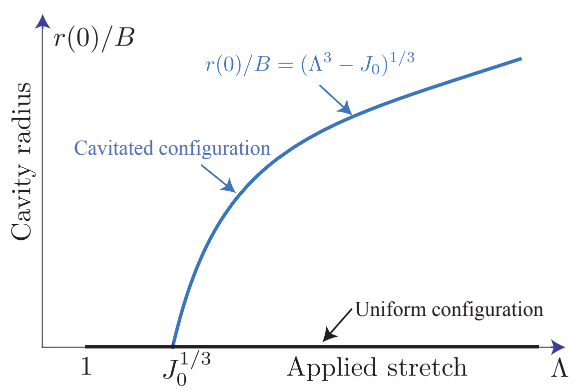

Figure 10.13: Bifurcation diagram. Uniform configuration for all $\Lambda \ge 1$ (black). Configuration with a cavity of radius $r(0) > 0$ for stretches $\Lambda > J_0^{1/3}$ (blue).

$(b3)$ We have the uniform solution $(xxii)$ for all $\Lambda > 1$. In addition, for $\Lambda > J_0^{1/3}$, we have the solution $(xxviii)$ involving a cavity. Both solutions exist for $J_0^{1/3} < \Lambda \le J_*^{1/3}$. To compare their energies we calculate

$$\begin{aligned}
\Psi \ &\overset{(ii),(iii)}{=} \int_0^B 4\pi W R^2 dR = 4\pi \int_0^B \Big[2\mu(\lambda_r + 2\lambda_\theta) + F(J)\Big] R^2 dR = \\
&\overset{(viii)}{=} 4\pi \int_0^B \left[2\mu\left(r'(R) + 2\frac{r(R)}{R}\right) + F(J)\right] R^2 dR = 4\pi \int_0^B \left[\frac{2\mu}{R^2}\frac{d}{dR}(R^2 r) + F(J)\right] R^2 dR = \\
&= 8\pi\mu\Lambda B^3 + 4\pi \int_0^B F(J) R^2 dR,
\end{aligned} \tag{xxx}$$

having used $r(B) = \Lambda B$ in the last step. Observe from this that

$$\delta\Psi = 4\pi \int_0^B F'(J)\,\delta J\, R^2 dR, \qquad \delta^2\Psi = 4\pi \int_0^B F''(J)\,(\delta J)^2\, R^2 dR, \tag{xxxi}$$

Remark: If we restricted attention to the compressible Varga material from the start, we could have simplified the analyses in parts $(b1)$ and $(b2)$ by starting from $(xxxi)_1$. Since $J = r^2 r'/R^2$ we have

$$\delta J = \frac{r^2\delta r' + 2rr'\delta r}{R^2}$$

and so

$$\begin{aligned}
\delta\Psi &= 4\pi \int_0^B F'(J)(r^2\delta r' + 2rr'\delta r)\,dR = 4\pi\int_0^B \Big[\frac{d}{dR}(F'(J)r^2\delta r) - \frac{d}{dR}(F'(J)r^2)\delta r + 2rr'F'(J)\delta r\Big]\,dR = \\
&= -4\pi\Big[r^2F'(J)\Big]_{R=0}\delta r(0) - 4\pi\int_0^B r^2\frac{d}{dR}F'(J)\,\delta r\,dR
\end{aligned}$$

Thus at a minimizer we have the Euler-Lagrange equation

$$r^2\frac{d}{dR}F'(J) = 0 \qquad \text{for} \quad 0 < R < B$$

and the natural boundary condition

$$r^2(R)F'(J(R)) = 0 \qquad \text{as} \quad R \to 0,$$

which recovers what we had before.

In the uniform and cavitated solutions we have $J = \Lambda^3$ and $J = J_0$ respectively. Observe from $(xxxi)_2$ that $\delta^2\Psi \geq 0$ at both the uniform and cavitated solutions since $F''(\Lambda^3) > 0$ and $F''(J_0) > 0$. To compare their energies we evaluate (xxx) at each solution which gives

$$\Psi\Big|_{\text{uniform}} \overset{(xxii)}{=} 8\pi\mu\Lambda B^3 + 4\pi\int_0^B F(\Lambda^3)R^2dR = 8\pi\mu\Lambda B^3 + \frac{4}{3}\pi B^3 F(\Lambda^3),$$

$$\Psi\Big|_{\text{cavitated}} \overset{(xxviii)}{=} 8\pi\mu\Lambda B^3 + 4\pi\int_0^B F(J_0)R^2dR = 8\pi\mu\Lambda B^3 + \frac{4}{3}\pi B^3 F(J_0).$$

Since J_0 is the unique minimum of $F(J)$ on the interval of interest, see Figure 10.12, we have $F(\Lambda^3) > F(J_0)$ for $\Lambda > J_0^{1/3}$ and so

$$\Psi\Big|_{\text{uniform}} > \Psi\Big|_{\text{cavitated}}.$$

when both solutions exist.

Problem 10.2. *Elastic string subjected to pressure loading.* (Remark: The sign of the pressure term in this and Problem 10.4.4 are different. This is because the pressure p points into $\mathcal{R}_R$ in the 3D problem while it points out of the area A in the string problem.) In a reference configuration, a generic particle of an elastic string is located at $\mathbf{x} = x\,\mathbf{e}_1$, $0 \le x \le L$. Its location in a deformed configuration is $\mathbf{y} = y_1(x)\mathbf{e}_1 + y_2(x)\mathbf{e}_2$. The left-hand end of the string in fixed at $(0,0)$ while the right-hand end is displaced to a location (a,b):

$$y_1(0) = y_2(0) = 0, \quad y_1(L) = a, \quad y_2(L) = b. \qquad (i)$$

The string is subjected to a *constant* pressure p per unit deformed length that acts in a direction normal to the deformed string.

Consider the following set of kinematically admissible deformations $y_1(x), y_2(x)$:

$$\mathcal{A} = \{y_1, y_2 \in C^1[0,L] : \; y_1(0) = y_2(0) = 0,\; y_1(L) = a,\; y_2(L) = b\}. \qquad (ii)$$

Let $s(x)$ and $\phi(x)$ denote arc length and slope along the string in a kinematically admissible deformation so that

$$\big(s'\big)^2 = \big(y_1'\big)^2 + \big(y_2'\big)^2, \qquad \cos\phi = y_1'/s', \qquad \sin\phi = y_2'/s', \qquad (iii)$$

and let $\lambda(x)$ be the associated stretch:

$$\lambda = s'(x). \qquad (iv)$$

The potential energy of the system is

$$\Phi\{y_1, y_2\} = \int_0^L W(\lambda(x))\,dx - p\,A\{y_1, y_2\}. \qquad (v)$$

where $W(\lambda)$ is the stored elastic energy per unit undeformed length. If the pressure was not applied, the deformed configuration of the string would be the straight line joining $(0,0)$ to (a,b). The value of the functional A in (v) is the area between the curve describing the deformed string (with the pressure applied) and this straight line.

Minimize the potential energy (v) over the set $\mathcal{A}$ and derive the associated Euler-Lagrange equations.

Solution: An admissible variation $\delta y_1(x), \delta y_2(x)$ obeys

$$\delta y_1(0) = \delta y_2(0) = \delta y_1(L) = \delta y_2(L) = 0, \qquad (va)$$

and from (iv) the variations δs and $\delta\lambda$ are related to an admissible variation by

$$s'\delta s' \overset{(iii)_1}{=} y_1'\delta y_1' + y_2'\delta y_2' \qquad \Rightarrow \qquad \delta\lambda \overset{(iv)}{=} \delta s' \overset{(iii)_{2,3}}{=} \cos\phi\,\delta y_1' + \sin\phi\,\delta y_2'. \tag{vi}$$

The area A can be written as

$$A\{y_1, y_2\} = \int_0^L y_2 y_1'\,dx - \frac{1}{2}ab,$$

and so its first variation is

$$\delta A = \int_0^L (y_1'\,\delta y_2 + y_2\,\delta y_1')\,dx.$$

Integrating the second term by parts and using (va) yields

$$\delta A = \int_0^L (y_1'\,\delta y_2 - y_2'\,\delta y_1)\,dx. \tag{vii}$$

Thus the first variation of Φ is

$$\begin{aligned}
\delta\Phi &= \int_0^L \delta W(\lambda)\,dx - p\,\delta A = \int_0^L W'(\lambda)\delta\lambda\,dx - p\,\delta A = \\
&\overset{(vi)}{=} \int_0^L W'(\lambda)\delta s'\,dx - p\,\delta A = \\
&\overset{(vi)}{=} \int_0^L \Big[W'(\lambda)\cos\phi\,\delta y_1' + W'(\lambda)\sin\phi\,\delta y_2'\Big]\,dx - p\,\delta A = \\
&\overset{(va)}{=} \int_0^L -\frac{d}{dx}\,[W'(\lambda)\cos\phi]\,\delta y_1 - \frac{d}{dx}\,[W'(\lambda)\sin\phi]\,\delta y_2\,dx - p\,\delta A
\end{aligned} \tag{$viii$}$$

where in getting to the final expression we integrated by parts and used (va). Substituting (vii) into $(viii)$ yields

$$\delta\Phi = \int_0^L \left[\left(-\frac{d}{dx}[W'(\lambda)\cos\phi] + py_2'\right)\delta y_1 - \left(\frac{d}{dx}[W'(\lambda)\sin\phi] + py_1'\right)\delta y_2\right]\,dx = 0.$$

Since this must vanish for all admissible variations $\delta y_1, \delta y_2$, we conclude that

$$\frac{d}{dx}[W'(\lambda)\cos\phi] - py_2' = 0, \qquad \frac{d}{dx}[W'(\lambda)\sin\phi] + py_1' = 0. \tag{$\square$}$$

Remark: By setting $t(x) = W'(\lambda(x))$ and using (iii) one can rewrite $(\square)$ as

$$\frac{d}{dx}(t\cos\phi) - ps'\sin\phi = 0, \qquad \frac{d}{dx}(t\sin\phi) + ps'\cos\phi = 0.$$

which we can write as

$$\frac{d}{ds}(t\cos\phi) - p\sin\phi = 0, \qquad \frac{d}{ds}(t\sin\phi) + p\cos\phi = 0.$$

Equilibrium of a differential element of the string can be readily shown to lead to these equation with t being the tension in the string, the former by force balance in the direction $\mathbf{e}_1$, the latter in the direction $\mathbf{e}_2$. In fact

$$\left(\frac{d}{ds}(t\cos\phi) - p\sin\phi\right)\mathbf{e}_1 + \left(\frac{d}{ds}(t\sin\phi) + p\cos\phi\right)\mathbf{e}_2 = \mathbf{o}.$$

$$\frac{d}{ds}\big(t(\cos\phi\mathbf{e}_1 + \sin\phi\mathbf{e}_2)\big) + p(-\sin\phi\mathbf{e}_1 + \cos\phi\mathbf{e}_2) = \mathbf{o}.$$

$$\frac{d}{ds}(t\boldsymbol{\ell}) + p\mathbf{n} = \mathbf{o}.$$

Problem 10.3. *Stability of the "Rivlin cube" for an arbitrary isotropic material.* In Section 5.3 we examined the stability of a neo-Hookean cube subjected to an equi-triaxial dead loading (Piola traction). Generalize that analysis to a cube composed of an arbitrary isotropic (unconstrained) elastic material by extremizing

$$\Phi(\mathbf{F}) = W(\mathbf{F}) - \mathbf{S}\cdot\mathbf{F}, \tag{i}$$

over all geometrically admissible homogeneous deformations. Assume, in keeping with the equi-triaxial dead loading, that

$$\mathbf{S} = \sum_{i=1}^{3} S_i\mathbf{e}_i\otimes\mathbf{e}_i, \tag{ii}$$

and consider only deformations of the form $\mathbf{y} = \mathbf{F}\mathbf{x}$ where

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i\mathbf{e}_i\otimes\mathbf{e}_i, \tag{iii}$$

i.e. where $\mathbf{F}$ and $\mathbf{S}$ are coaxial.

How would your analysis change if the material is incompressible?

Solution: For a loading of the form (ii) and deformation gradient of the form (iii), the potential energy $\Phi(\mathbf{F})$ can be written as

$$\Phi(\lambda_1,\lambda_2,\lambda_3) = W(\lambda_1,\lambda_2,\lambda_3) - \sum_{i=1}^{3} S_i\lambda_i \tag{iv}$$

Therefore at some nearby deformation gradient tensor characterized by (iii) with stretches $\lambda_1 + \delta\lambda_1, \lambda_2 + \delta\lambda_2, \lambda_3 + \delta\lambda_3$ the potential energy is

$$\Phi(\lambda_1 + \delta\lambda_1, \lambda_2 + \delta\lambda_2, \lambda_3 + \delta\lambda_3) = W(\lambda_1 + \delta\lambda_1, \lambda_2 + \delta\lambda_2, \lambda_3 + \delta\lambda_3) - \sum_{i=1}^{3} S_i(\lambda_i + \delta\lambda_i). \quad (v)$$

The first variation of the potential energy is defined by

$$\delta\Phi := \Phi(\lambda_1 + \delta\lambda_1, \lambda_2 + \delta\lambda_2, \lambda_3 + \delta\lambda_3) - \Phi(\lambda_1, \lambda_2, \lambda_3) \qquad \text{up to terms linear in the } \delta\lambda_i's$$

which by (iv) and (v) can be written as

$$\delta\Phi = \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i}(\lambda_1, \lambda_2, \lambda_3)\, \delta\lambda_i - \sum_{i=1}^{3} S_i \delta\lambda_i = \sum_{i=1}^{3} \left(\frac{\partial W}{\partial \lambda_i}(\lambda_1, \lambda_2, \lambda_3) - S_i \right) \delta\lambda_i. \qquad (vi)$$

The second variation of the potential energy is

$$\delta^2\Phi = \sum_{j=1}^{3} \sum_{i=1}^{3} \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}(\lambda_1, \lambda_2, \lambda_3)\, \delta\lambda_i \delta\lambda_j. \qquad (vii)$$

Equilibrium requires the first variation $\delta\Phi$ to vanish for all $\delta\lambda_i$'s and so gives

$$S_i = \frac{\partial W}{\partial \lambda_i}(\lambda_1, \lambda_2, \lambda_3). \qquad (viii)$$

Taking for granted that stability of an equilibrium configuration requires the second variation $\delta^2\Phi$ to be positive for all non-vanishing $\delta\lambda_i$'s and $\delta\lambda_j$'s we find that for stability

$$\text{The matrix whose } i, j \text{ element is } \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}(\lambda_1, \lambda_2, \lambda_3) \qquad \text{must be positive definite} \qquad (ix)$$

The matrix referred to in (ix) is called the Hessian matrix and it is to be evaluated at the equilibrium configurations given by $(viii)$.

Remark on the incompressible case: The preceding analysis was for an unconstrained material. We considered all $\delta\lambda_i$'s there. For an incompressible material, the allowable equilibrium configurations must obey the kinematic constraint $\lambda_1\lambda_2\lambda_3 = 1$ and this imposes a relationship between the three $\delta\lambda_i$'s. This can be accounted for in the usual way by introducing a Lagrange multiplier q into the potential energy and writing

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_2, \lambda_3) - \sum_{i=1}^{3} S_i \lambda_i - q(\lambda_1\lambda_2\lambda_3 - 1). \qquad (x)$$

The first variation of Φ is now

$$\begin{aligned}
\delta\Phi &= \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i}\,\delta\lambda_i - \sum_{i=1}^{3} S_i\delta\lambda_i - q(\delta\lambda_1\lambda_2\lambda_3 + \lambda_1\delta\lambda_2\lambda_3 + \lambda_1\lambda_2\delta\lambda_3) + \delta q(\lambda_1\lambda_2\lambda_3 - 1) = \\
&= \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i}\,\delta\lambda_i - \sum_{i=1}^{3} S_i\delta\lambda_i - q\sum_{i=1}^{3}\frac{\delta\lambda_i}{\lambda_i} + \delta q(\lambda_1\lambda_2\lambda_3 - 1) = \\
&= \sum_{i=1}^{3}\left(\frac{\partial W}{\partial \lambda_i} - S_i - q\frac{1}{\lambda_i}\right)\delta\lambda_i + \Big(\lambda_1\lambda_2\lambda_3 - 1\Big)\delta q.
\end{aligned} \tag{xi}$$

The coefficients of $\delta\lambda_i$ and δq must vanish in equilibrium and so one obtains

$$S_i = \frac{\partial W}{\partial \lambda_i} - q\frac{1}{\lambda_i}, \qquad \lambda_1\lambda_2\lambda_3 = 1. \tag{xii}$$

Alternatively one can use the incompressibility constraint to eliminate λ_3 and write the potential energy as

$$\widehat{\Phi}(\lambda_1,\lambda_2) := \Phi(\lambda_1,\lambda_2,\lambda_1^{-1}\lambda_2^{-1}) = W(\lambda_1,\lambda_2,\lambda_1^{-1}\lambda_2^{-1}) - (S_1\lambda_1 + S_2\lambda_2 + S_3\lambda_1^{-1}\lambda_2^{-1}). \tag{xiii}$$

Taking the second variation of $(xiii)$ leads to an expression of the form

$$\delta^2\Phi = M_{11}\delta\lambda_1^2 + 2M_{12}\delta\lambda_1\delta\lambda_2 + M_{22}\delta\lambda_2^2$$

where the $M_{\alpha\beta}$'s depend on the λ_i's, the second partial derivatives of $\widehat{W}(\lambda_1,\lambda_2) := W(\lambda_1,\lambda_2,\lambda_1^{-1}\lambda_2^{-1})$ with respect to λ_1 and λ_2 and the stress S_3.

Problem 10.4. *Buckling of a slender elastic beam.* A particle (on the centerline of the beam) is at the position $\mathbf{x} = x\mathbf{e}_1$, $0 \le x \le L$, in the reference configuration and at $\mathbf{y}(x) = y_1(x)\mathbf{e}_1 + y_2(x)\mathbf{e}_2$, $0 \le x \le L$, in the deformed configuration. Use x to identify a particle.

The left-hand end of the beam is fixed and the right-hand end is constrained to move on the horizontal axis:

$$y_1(0) = y_2(0) = 0, \qquad y_2(L) = 0.$$

A force $\widehat{\mathbf{s}} = -P\mathbf{e}_1$ is applied at the right-hand end. The beam is attached to an elastic foundation of stiffness k per unit reference length.

Extremize the potential energy functional $\Phi\{y_1, y_2\}$ over an admissible set of functions to find the differential equations and boundary conditions satisfied by $y_1(x)$ and $y_2(x)$. Specialize your model to the case where the centerline is inextensible, and further specialize it to the case where $|y_2'(x)| \ll 1$ at each x.

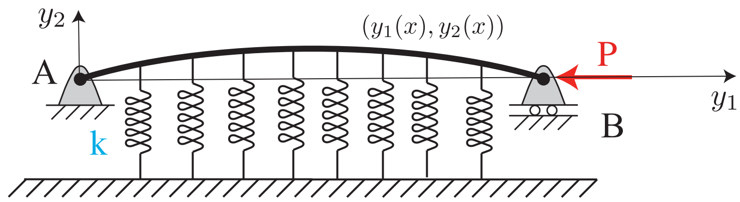

Figure 10.14: A pinned-pinned beam attached to an elastic foundation and subjected to a compressive force P. The particle x is located at $(y_1(x), y_2(x))$ in the deformed configuration.

Solution: Since a force $\widehat{\mathbf{s}} = -P\mathbf{e}_1$ is applied at the right-hand end where $x = L$ and $\mathbf{y}(L) = y_1(L)\mathbf{e}_1$, the potential energy associated with this dead loading is

$$-\widehat{\mathbf{s}} \cdot \mathbf{y}(L) = -(-P\mathbf{e}_1) \cdot (y_1(L)\mathbf{e}_1) = Py_1(L). \tag{i}$$

Let $s(x)$ and $\kappa(x)$ denote the arc length and curvature along the deformed beam; they are related to $y_1(x)$ and $y_2(x)$ by[10]

$$s'(x) = \Big[\big(y_1'(x)\big)^2 + \big(y_2'(x)\big)^2\Big]^{1/2}, \qquad \kappa(x) = \frac{y_1'(x)y_2''(x) - y_2'(x)y_1''(x)}{[\big(y_1(x)'\big)^2 + \big(y_2'(x)\big)^2]^{3/2}}. \tag{ii}$$

The stretch λ along the beam is

$$\lambda(x) = s'(x) = \Big[\big(y_1'(x)\big)^2 + \big(y_2'(x)\big)^2\Big]^{1/2}. \tag{iii}$$

With EI being the constant bending stiffness, the elastic energy due to bending per unit deformed length is

$$\frac{1}{2}EI\kappa^2. \tag{iv}$$

With EA being the constant axial stiffness, the elastic energy due to stretching per unit undeformed length is

$$\frac{1}{2}EA(\lambda(x) - 1)^2. \tag{v}$$

Finally, suppose the beam is attached to an elastic foundation of constant stiffness k such that the energy stored in the foundation per unit undeformed length is

$$\frac{1}{2}ky_2^2(x). \tag{vi}$$

From $(i), (iv), (v)$ and (vi) the total potential energy is

$$\Phi\{y_1, y_2\} = \int_0^\ell \frac{1}{2}EI\kappa^2\, ds \; + \int_0^L \frac{1}{2}EA(\lambda - 1)^2\, dx \; + \int_0^L \frac{1}{2}ky_2^2\, dx + Py_1(L) - PL,$$

[10] The formula $(ii)_2$ for κ is derived in the appendix.

where ℓ is the length of the deformed beam. With no loss of generality we have included an additive constant $-PL$ since then $\Phi = 0$ when the beam is undeformed. We can write this using $(iii)_1$ as

$$\Phi\{y_1, y_2\} = \int_0^L \frac{1}{2} EI\kappa^2 \, \lambda \, dx \; + \int_0^L \frac{1}{2} EA(\lambda - 1)^2 \, dx \; + \int_0^L \frac{1}{2} k y_2^2 \, dx + P y_1(L) - PL, \quad (vii)$$

where κ and λ are given in terms of y_1 and y_2 by $(ii)_2$ and (iii). The potential energy is defined for all admissible functions in

$$\mathcal{A} = \{y_1, y_2 \in C^2[0, L] : \; y_1(0) = y_2(0) = 0, y_2(L) = 0\}.$$

Extremizing Φ over $\mathcal{A}$ by setting

$$\delta\Phi\{y_1, y_2, \delta y_1, \delta y_2\} = 0$$

for all variations $\delta\mathbf{y}(x)$ (with $\delta y_1(0) = \delta y_2(0) = \delta y_2(0) = 0$) leads to two differential equations on $(0, L)$, one natural boundary condition at $x = 0$ and two natural boundary conditions at $x = L$. (Keep in mind that $\delta y_1(L), \delta y_2'(0)$ and $\delta y_2'(L)$ are arbitrary.)

(b) *Case: inextensible centerline.* In this case we have

$$s(x) = x, \qquad \lambda(x) = 1. \qquad (viii)$$

Equation (vii) now reduces to

$$\Phi = \int_0^L \frac{1}{2} EI\kappa^2 \, dx \; + \int_0^L \frac{1}{2} k y_2^2 \, dx + P y_1(L) - PL. \qquad (ix)$$

We can use $(viii)$ to eliminate $y_1(x)$ and rewrite the potential energy functional in the form $\Phi\{y_2\}$. To this end we want to eliminate $y_1(x)$ from κ and $y_1(L)$ from the last term in (ix). Since, from (iii) and $(viii)$,

$$(y_1'(x))^2 + (y_2'(x))^2 = 1, \qquad (x)$$

we find by differentiation that

$$y_1'' = -y_2' y_2''/y_1'. \qquad (xi)$$

Therefore

$$\kappa \overset{(ii)_2}{=} \frac{y_1' y_2'' - y_2' y_1''}{[(y_1')^2 + (y_2')^2]^{3/2}} \overset{(x)}{=} y_1' y_2'' - y_2' y_1'' \overset{(xi)}{=} y_1' y_2'' - y_2'(-y_2' y_2''/y_1') \overset{(x)}{=} \frac{y_2''}{y_1'} \overset{(x)}{=} \frac{y_2''}{\sqrt{1 - (y_2')^2}}. \quad (xii)$$

Moreover, rearranging (x) and integrating gives

$$y_1(x) = \int_0^x \sqrt{1 - (y_2'(\xi))^2} \, d\xi, \qquad (xiii)$$

and so in particular,

$$y_1(L) = \int_0^L \sqrt{1-(y_2'(x))^2}\, dx. \tag{xiv}$$

We can now use (xii) and (xiv) in (ix) to obtain explicitly

$$\Phi\{y_2\} = \int_0^L \frac{1}{2} EI \frac{(y_2'')^2}{1-(y_2')^2}\, dx \; + \int_0^L \frac{1}{2} k y_2^2\, dx + P \int_0^L \sqrt{1-(y_2'(x))^2}\, dx - PL.$$

To write this in more conventional notation let

$$v(x) \equiv y_2(x),$$

so that the potential energy functional then reads

$$\Phi\{v\} = \int_0^L \frac{1}{2} EI \frac{(v'')^2}{1-(v')^2}\, dx \; + \int_0^L \frac{1}{2} k v^2\, dx + P \int_0^L \sqrt{1-(v'(x))^2}\, dx \; - PL, \tag{xv}$$

for all

$$v \in \mathcal{A} = \{v \in C^2[0,L] :\ v(0) = 0, v(L) = 0\}. \tag{xvi}$$

Setting the first variation $\delta\Phi\{v, \delta v\} = 0$ for all admissible variations $\delta v(x)$ (with $\delta v(0) = \delta v(L) = 0$) leads to a differential equation on $(0, L)$ and one natural boundary condition at each end. Once $v(x)$ has been determined, we can determine $y_1(x)$ from $(xiii)$.

(c) *Special case:* When $|v'(x)|$ is small at each x we can approximate

$$\frac{(v'')^2}{1-(v')^2} = (v'')^2 \left[1-(v')^2\right]^{-1} = (v'')^2 + \ldots,$$

$$\sqrt{1-(v'(x))^2} = \left[1-(v'(x))^2\right]^{1/2} = 1 - \frac{1}{2}(v'(x))^2 + \ldots$$

and so we can replace (xv) by

$$\Phi\{v\} = \int_0^L \frac{1}{2} EI\, (v'')^2\, dx \; + \int_0^L \frac{1}{2} k v^2\, dx - P \int_0^L \frac{1}{2}(v'(x))^2\, dx\,. \tag{xvii}$$

Extremizing $(xvii)$ over (xvi) leads to

$$\delta\Phi\{v, \delta v\} = EIv''(L)\delta v'(L) - EIv''(0)\delta v'(0) + \int_0^L \Big[EIv'''' + kv - Pv''\Big]\delta v dx = 0.$$

Since this vanishes for all admissible variations $\delta v(x)$ we conclude that the following differential equation must hold:

$$EIv'''' + kv - Pv'' = 0 \qquad \text{for} \quad 0 < x < L. \tag{xviii}$$

Moreover, since $\delta v'(0)$ and $\delta v'(L)$ are arbitrary we also arrive at the natural boundary conditions

$$v''(0) = v''(L) = 0. \qquad (xix)$$

The deflected shape of the beam is thus to be determined by solving the 4th order differential equation $(xviii)$ subject to the boundary conditions $v(0) = v(L) = L$ and (xix).

Observe that $v(x) = 0$ for $0 \leq x \leq L$ is a solution of this problem no matter what the value of P. In addition, at certain values of P it will also have non-vanishing solutions $v(x)$. These are the values of P at buckling. (The mathematical problem here is an eigenvalue problem).

(d) Multiplying $(xviii)$ by $v(x)$, integrating over $[0, L]$ and using the boundary conditions yields

$$P = P\{v\} := \frac{\displaystyle\int_0^L \frac{1}{2} EI\big(v''\big)^2 dx + \int_0^L \frac{1}{2} k v^2 dx}{\displaystyle\int_0^L \frac{1}{2}\big(v'\big)^2 dx}. \qquad (xx)$$

If P_{crit} is the smallest buckling load one can show that

$$P_{crit} \leq P\{w\} \quad \text{for all} \quad w \in \mathcal{A} = \{w \in C^2[0, L] : \, w(0) = w(L) = 0\}. \qquad (xxi)$$

Equation (xxi) provides an upper bound on the smallest buckling load P_{crit}. We now calculate an explicit upper bound by choosing a particular test function $w(x)$. Since $w(x) \in \mathcal{A}$ our choice must satisfy the kinematic boundary conditions $w(0) = w(L) = 0$. Since the minimizer also satisfies the natural boundary conditions (xix), we pick a test function that also satisfies $w''(0) = w''(L) = 0$ since such a function would be closer to the actual deflection than would be a function that didn't have this characteristic. This suggests that we consider test functions of the form, e.g. $w(x) = w_0 \sin n\pi/L$. Finally, since we are interested in the *smallest* buckling load we expect the associated buckling mode to be more similar to $w(x) = w_0 \sin \pi/L$ than to, say, $w(x) = w_0 \sin 2\pi/L$. Accordingly we choose

$$w(x) = w_0 \, \sin \pi x/L.$$

Substituting this into (xx) and (xxi) and evaluating the integrals leads to the upper bound

$$P_{crit} \leq P\{w\}\Big|_{w(x) = w_0 \, \sin \pi x/L} = EI(\pi/L)^2 + k(L/\pi)^2.$$

Appendix: Curvature of a curve in the plane. Let $\mathcal{C}$ be a curve in the y_1, y_2-plane, described parametrically by

$$\mathbf{y}(x) = y_1(x)\mathbf{e}_1 + y_2(x)\mathbf{e}_2, \qquad x_1 \leq x \leq x_2, \qquad (i)$$

x being the parameter and $\{\mathbf{e}_1, \mathbf{e}_2\}$ a fixed orthonormal basis. For any function $h(x)$ we let a prime denote

$$h' = \frac{dh}{dx}(x). \tag{ii}$$

Let

$$\lambda = \sqrt{(y_1')^2 + (y_2')^2} \tag{iii}$$

denote the stretch along $\mathcal{C}$. The arc length $s(x)$ is found by integrating

$$s' = \lambda \tag{iv}$$

with respect to x. The unit tangent vector on $\mathcal{C}$ (in the direction of increasing arc length) is

$$\boldsymbol{\ell} = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2, \tag{v}$$

where the angle $\theta(x)$ that the tangent makes with the y_1-axis is given by

$$\cos\theta = y_1'/\lambda, \qquad \sin\theta = y_2'/\lambda. \tag{vi}$$

The unit normal vector, obtained by counter clockwise rotation of $\boldsymbol{\ell}$, is

$$\mathbf{n} = \mathbf{e}_3 \times \boldsymbol{\ell} = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2. \tag{vii}$$

The curvature of $\mathcal{C}$, by definition, is

$$\kappa := \frac{d\theta}{ds} = \theta'/\lambda, \tag{viii}$$

having used (i); it is the reciprocal of the radius of curvature $ds/d\theta$. It follows from (v), (vii) and $(viii)$ that

$$\mathbf{n}' = -\kappa\lambda\boldsymbol{\ell}, \qquad \boldsymbol{\ell}' = \kappa\lambda\mathbf{n}. \tag{ix}$$

Differentiating each equation in (iii) with respect to x leads to the respective equations

$$-\sin\theta\,\theta' = y_1''/\lambda - \cos\theta\lambda'/\lambda, \qquad \cos\theta\,\theta' = y_2''/\lambda - \sin\theta\lambda'/\lambda.$$

Multiplying the first of these by $\sin\theta$, the second by $\cos\theta$ and then subtracting the first from the second gives

$$\theta' = \cos\theta\,y_2''/\lambda - \sin\theta\,y_1''/\lambda \overset{(vi)}{=} (y_1'\,y_2'' - y_2'\,y_1'')/\lambda^2.$$

On combining this with (iii) and $(viii)$, we obtain the following formula for the curvature:

$$\kappa = \frac{y_1'y_2'' - y_2'y_1''}{\left[(y_1')^2 + (y_2')^2\right]^{3/2}}.$$

This is the formula for the curvature which we used above.

Problem 10.5. *Non-smooth minimizers.* From among all curves passing through the given points $A = (0, 1)$ and $B = (a, b)$, $b \geq 1$, you want to find the curve that generates the surface of revolution of least area when the curve is rotated rigidly about the x-axis.

(a) Assume that the curve can be described by $y = y(x), 0 \leq x \leq a$, and formulate this problem, i.e. derive the functional that is to be minimized and characterize the set of all admissible functions over which this minimization is to be carried out.

(b) Derive the associated Euler-Lagrange equation.

(c) Solve the Euler-Lagrange equation.

(d) What do the boundary conditions require? Can they be satisfied?

(e) Optional: If $b < a - 1$ show that there is no solution of the Euler-Lagrange equation that satisfies the boundary conditions $y(0) = 1, y(a) = b$. What is the area minimizing curve in this case?

Solution: We shall assume that the minimizer is a smooth curve joining A and B. This assumption will be revisited in part (e).

(a) When an infinitesimal line segment of length ds is rotated about the x-axis in a circle of radius $y(x)$, the area of the surface it generates is $2\pi y(x)ds$. Thus the total area of the surface of revolution is

$$\mathsf{A}\{y\} = \int_A^B 2\pi y(x)\, ds = \int_0^a 2\pi y(x)\sqrt{1 + (y'(x))^2}\, dx, \qquad (i)$$

where the set of admissible test functions is

$$\mathcal{A} = \{y \in C^1[0, 1] : \ y(0) = 1, \ y(a) = b\}. \qquad (ii)$$

At an extremizer y, the first variation of A must vanish,

$$\delta\mathsf{A}\{y, \delta y\} = 0 \quad \text{for all admissible } \delta y, \qquad (iii)$$

where $\delta y(x)$ is in the set of admissible variations

$$\mathcal{V} = \{\delta y(x) \in C^1[0, 1] : \ \delta y(0) = 0, \ \delta y(a) = 0\}. \qquad (iv)$$

(b) Functionals of the form (i) were examined in Section 10.2.1. In fact, (i) is of the particular form considered in Problem 10.2.2 and so the Euler-Lagrange equation can be written as (see (10.15))

$$y[1+y'^2]^{1/2} - \frac{yy'^2}{[1+y'^2]^{1/2}} = c_1, \tag{v}$$

where c_1 is a constant. The curve that minimizes the area of revolution is found by solving the differential equation (v) subject to the boundary conditions

$$y(0) = 1, \qquad y(a) = b. \tag{vi}$$

(c) Equation (v) can be simplified to read

$$y'^2 = \frac{y^2}{c_1^2} - 1 \qquad \Rightarrow \qquad \int \frac{c_1}{\sqrt{y^2 - c_1^2}}\, dy = \int dx. \tag{vii}$$

The integral on the left-hand side can be evaluated by making the substitution $y = c_1 \cosh \xi$. This leads to

$$y(x) = c_1 \cosh\left(\frac{x}{c_1} + c_2\right), \qquad 0 \leq x \leq a, \tag{$viii$}$$

where c_2 is a constant.

(d) The constants c_1 and c_2 are to be determined (if possible) using the boundary conditions $y(0) = 1, y(a) = b$. The first boundary condition gives $c_1 = 1/\cosh c_2$ and so the second boundary condition requires

$$b = \frac{\cosh(a \cosh c_2 + c_2)}{\cosh c_2}. \tag{ix}$$

Given $a > 0, b > 0$, if this equation can be solved for c_2 then there is a solution of the assumed (smooth) form.

(e) From Section 7.5 of Troutman [5]: For all z one has

$$\cosh z > |z| \geq z, \qquad \cosh z > 0, \qquad z \geq -|z|. \tag{x}$$

Therefore it follows that:

$$b = \frac{\cosh(a\cosh c_2 + c_2)}{\cosh c_2} \overset{(x)_1}{>} \frac{a\cosh c_2 + c_2}{\cosh c_2} = a + \frac{c_2}{\cosh c_2} \overset{(x)_3}{\geq} a - \frac{|c_2|}{\cosh c_2} \overset{(x)_{1,2}}{>} a - 1.$$

Therefore it follows that if a root c_2 of (ix) is to exist, then it is necessarily that $b > a - 1$. Or equivalently, if $b < a - 1$ then equation (ix) has no root c_2, and therefore there is no smooth curve that minimizes the surface area functional. (This argument does not tell us that there is a solution for $b > a - 1$.)

Figure 10.15 shows schematic plots of the minimizing curve $y = y(x)$ for three different values of a (at fixed b). As suggested by the rightmost figure, the minimizer approaches the degenerate three segment curve approached by the red curve. One can view the corresponding surface of revolution as a circle of radius 1 at $x = 0$, a circle of radius b at $x = a$, connected by the straight line segment $0 \leq x \leq a$; the area of this surface is $\pi 1^2 + \pi b^2$. Such non-smooth minimizers can be allowed for in the analysis by characterizing the curve parametrically in the form $x = x(t), y = y(t), 0 < t < 1$.

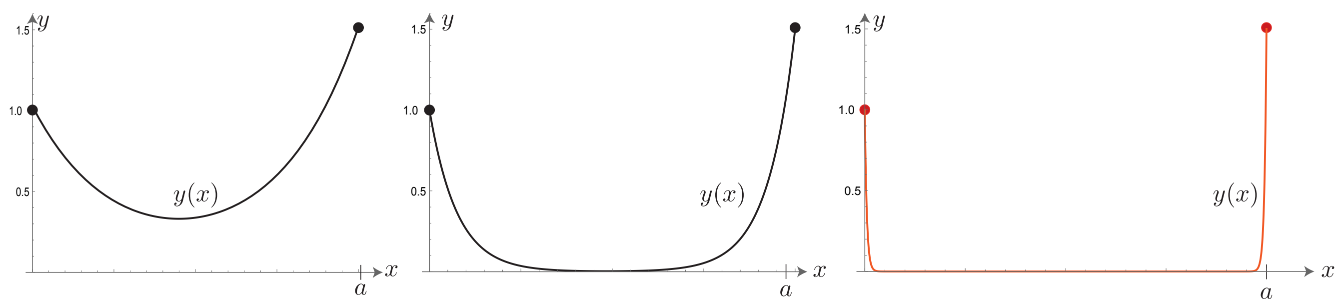

Figure 10.15: Schematic depiction of the curve $y = y(x)$ joining $(0,1)$ and $(a, 1.5)$ that gives the minimum surface area of revolution; the three figures correspond to three values of a with a slightly less than $b+1$ in the rightmost figure. As suggested by the rightmost figure, the minimizer shown in red approaches the degenerate three segment curve comprised of the segments $\{(x,y) : x = 0, 0 \leq y \leq 1\} \cup \{(x,y) : y = 0, 0 \leq x \leq a\} \cup \{(x,y) : x = a, 0 \leq y \leq b\}$.

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