Existence of tight simplices and other codes in compact spaces

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Joint work with Henry Cohn and Gregory Minton

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Coding theory

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Basic problem of coding theory: distribute N points on X in a "good" way (e.g. spaced far apart).

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Linear programming bounds were originally used by Delsarte for coding theory/association schemes. But later applied to spherical codes, sphere packings, energy minimization, etc.

Example: simplices in S^{n-1}

Proposition

Let N be a collection of $n+1$ unit vectors $v_1, \ldots, v_{n+1}$ in S^{n-1}, with $\langle v_i, v_j \rangle \leq \alpha$ for $i \neq j$. Then $\alpha \geq -1/n$, with equality iff they form a regular simplex.

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Proof.

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\begin{aligned}
0 &\leq \langle \sum v_i, \sum v_i \rangle \\
&= n + 1 + \sum_{i \neq j} \langle v_i, v_j \rangle \\
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This is the linear programming bound with the positive definite kernel $f(x) = x$.

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Many examples of tight codes on spheres and real and complex projective spaces were described by Levenshtein (some newer examples by others, too). Cohn-Kumar studied these from the perspective of energy minimization and universal optimality.

Projective spaces

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Recall: projective space $K\mathbb{P}^{n-1}$ over $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ can be thought of as lines in K^n: identify x and $x\alpha$ for $x \in K^d \setminus \{0\}$ and $\alpha \in K^\times$. For $\mathbb{OP}^2$ the description is more complicated (need the coordinates of x to associate).

The LP bounds for projective spaces

Points in projective space may be considered as projectors of rank 1 (namely $xx^\dagger$). The inner product can then be defined as $\langle A, B \rangle = \mathrm{Re}\,\mathrm{Tr}(AB)$, and the chordal distance as $\sqrt{2 - 2\langle A, B \rangle}$.

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Let e be the dimension of the algebra K over $\mathbb{R}$, i.e. $1, 2, 4, 8$ for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ respectively.

Proposition (Lemmens-Seidel)

A regular simplex in $K\mathbb{P}^{n-1}$ can have at most $n + e(n^2 - n)/2$ points. If it has N points, then the maximal inner product α satisfies

$$\alpha \geq \frac{N - n}{n(N-1)}.$$

LP bounds for projective spaces II

Proof idea: The bound on N comes from considering the dimension of the space of Hermitian $n \times n$ matrices over K. It can be shown that the projectors corresponding to the points of a regular simplex are linearly independent.

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The bound on α is a linear programming bound, just from the positive definite kernel $f(x) = x$. One can show that it provides the optimal LP bound for simplices. So we say a simplex is tight if it matches the bound.

Results

For $\mathbb{HP}^2$ through $\mathbb{HP}^5$, we find positive dimensional families of tight simplices for many values of N in the allowed range.

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Finding the codes: We find an *approximate* tight simplices in one of two ways:

- through energy minimization (gradient descent).
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Proving existence: through an effective version of the implicit function theorem. This also allows us to find the local dimension of the space of solutions to the system of equations, and therefore a lower bound for the dimension of the space of tight simplices.

Some highlights

For $\mathbb{HP}^2$, we find tight simplices for all the allowed values of N (i.e. between 1 and 15) except for $N = 14$. We conjecture that there does not exist a tight simplex of 14 points.

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For $\mathbb{OP}^2$, we find tight simplices for all the allowed values of N (i.e. between 1 and 27) except for $N = 26$. We conjecture that there does not exist a tight simplex of 26 points.

In particular, we settle the existence of a tight 27-point simplex (equivalently, a tight 2-design) in $\mathbb{OP}^2$ (conjectured by Hoggar).

Highlights II

Our methods can also be used to find tight codes with more than one distance.

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In particular, we find a maximal set of 13 mutually unbiased bases in $\mathbb{OP}^2$, i.e. 39 points in 13 triples, where the inner products within the triple are $1/3$ and between elements of different triples are 0.

Highlights II

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In particular, we find a maximal set of 13 mutually unbiased bases in $\mathbb{OP}^2$, i.e. 39 points in 13 triples, where the inner products within the triple are $1/3$ and between elements of different triples are 0.

We also rigorously show the existence of many tight simplices in Grassmannians $G(m, n, \mathbb{R})$, which were reported by Conway, Hardin and Sloane, based on numerical evidence.

Tight simplices discovered for $\mathbb{HP}^2$

N	$r(3, N, \mathbb{H})$	N	$r(3, N, \mathbb{H})$
5	0	10	10
6	4	11	9
7	7	12	2
8	9	13	2
9	10	15	14

Here $r(3, N, \mathbb{H})$ gives the local dimension of the space of solutions.

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Of these, 5 through 11 were found and proved through the same schema of equations. For tight simplices of 12, 13 and 15 points, we have to do some extra work (e.g. impose symmetry for 12 and 13) so their local dimensions do not fit the general pattern.

System of equations

We use the equations corresponding to the conditions in the following proposition.

Proposition

Suppose $x_1, \ldots, x_N \in \mathbb{H}^n$ $(n > 1)$ *and* $w_1, \ldots, w_N \in \mathbb{R}$ *satisfy the following conditions:*

- $|x_i|^2 = 1$ *for all* $i = 1, \ldots, N$;
- $|\langle x_i, x_j \rangle|^2 = |\langle x_{i'}, x_{j'} \rangle|^2$ *for all* $1 \leq i < j \leq N$ *and* $1 \leq i' < j' \leq N$; *and*
- $\sum_{i=1}^{N} w_i x_i x_i^{\dagger} = I$.

Then $w_1 = \cdots = w_N = \frac{n}{N}$ *and* $\{x_1, \ldots, x_N\}$ *is a tight simplex.*

Conclusion

We use the implicit function theorem in a geometric setting, allowing us to show the existence of many tight codes which were not previously known to exist, and which may not have simple algebraic constructions.

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It would be nice to rigorously show the non-existence of 14-point tight codes in $\mathbb{HP}^2$ and 26-point tight codes in $\mathbb{OP}^2$. Perhaps part of a larger pattern.

Thank you!