K3 surfaces of high rank

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Abstract

In this dissertation we investigate the structure and moduli spaces of some algebraic K3 surfaces with high rank. In particular, we study K3 surfaces X which have a Shioda-Inose structure, that is, such that X has an involution ι which fixes any regular 2-form, and the quotient $X/\{1, \iota\}$ is birational to a Kummer surface.

We can specify the moduli spaces of K3 surfaces with Shioda-Inose structures by identifying them as lattice-polarized K3 surfaces for the lattice $E_8(-1)^2$, with the additional data of an ample divisor class. Similarly, we can give the quotient Kummer surface the structure of an $E_8(-1) \oplus N$-lattice polarized K3 surface, with the additional data of an ample divisor class.

One of the main results is that there is an isomorphism of the moduli spaces of these two types of lattice-polarized K3 surfaces.

When X is an elliptic K3 surface with reducible fibers of types E_8 and E_7, we describe the Nikulin involution and quotient map explicitly, and identify the quotient K3 surface as a Kummer surface of a Jacobian of a curve of genus 2. Our second main result gives the algebraic identification of the moduli spaces explicitly in this case.

Contents

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Chapter 1

Introduction

1.1 Motivation and overview

The classification of surfaces was carried out by Enriques, Castelnuovo, Kodaira, Zariski, Bombieri and Mumford. Two of the important classes of surfaces are complex tori and K3 surfaces. The algebraic versions, abelian surfaces and algebraic K3 surfaces, have rich algebraic as well as geometric content.

An important part of the geometry and arithmetic of a K3 surface is its Néron-Severi group, or Picard group, which can be considered as the group of line bundles on the surface up to isomorphism, or as the group of divisors modulo linear equivalence. This group has the structure of a lattice, where the bilinear form comes from intersection theory on the surface. The Picard number or rank of the K3 surface is the rank of the Néron-Severi lattice.

In this work, we study some K3 surfaces of high rank. Precisely, we consider K3 surfaces which have a Shioda-Inose structure: that is, an involution which preserves a global 2-form, such that the quotient is a Kummer surface. Kummer surfaces are a special class of K3 surfaces which are quotients of abelian surfaces. The Kummer surface thus carries algebro-geometric information about the abelian surface. It has rank at least 17, and so therefore do the K3 surfaces with Shioda-Inose structure.

These surfaces were first studied by Shioda and Inose [SI], who give a description of *singular* K3 surfaces, i.e. those with rank 20, the maximum possible for a K3 surface over a field of characteristic zero. They prove that there is a natural one-to-one correspondence between the set of singular K3 surfaces up to isomorphism and the set of equivalence classes of positive definite even integral binary quadratic forms. The result follows that of Shioda and Mitani [SM] who show that the set of singular abelian surfaces (that is, those having Picard number 4) is also in one-to-one correspondence with the equivalence classes of positive definite even integral binary quadratic forms. The construction of Shioda and Inose produces a singular K3 surface by taking a double cover of a Kummer surface associated to a singular abelian surface and with a specific type of elliptic fibration. The resulting K3 surface has an involution such that the quotient is the original Kummer surface. It also turns out that the lattice of transcendental cycles on the K3 surface (i.e. the orthogonal complement of the Néron-Severi group in the second singular cohomology group) is isomorphic to the lattice of transcendental cycles on the abelian surface.

Morrison studied Shioda-Inose structures more extensively in [M1], and gave other necessary and sufficient conditions for a K3 surface to have Shioda-Inose structure, in terms of the Néron-Severi group of the K3 surface. In this thesis, we study the moduli spaces of the K3 surfaces with Shioda-Inose structure, and of the resulting quotient surfaces. It is the aim of this and subsequent work to relate moduli spaces of K3 surfaces with Shioda-Inose structure with moduli spaces of abelian surfaces. It is also desirable to show that the relevant moduli spaces, which are quasi-projective varieties, are related by a morphism or correspondence over some number field. Such a correspondence would connect the non-trivial (transcendental) part of the Galois representation on the étale cohomology of the K3 surface with Shioda-Inose structure to the non-trivial part of the Galois representation on the abelian surface. We note that such an identification was made by Shioda and Inose, who first noted that singular abelian surfaces are isogenous to a product of a CM elliptic curve with itself. Therefore singular K3 surfaces also have models over a number field and, for instance, their Hasse-Weil zeta functions are related to the Hecke L-function of the Grössencharacter coming from the CM elliptic curve. It follows that the moduli of singular K3 surfaces are discrete.

In the second half of this thesis, we look at specific K3 surfaces where we can attempt to explicitly construct the Shioda-Inose structure, involution, and identify the quotient Kummer. For instance, one may consider (as Shioda and Inose did) elliptic K3 surfaces with certain specified bad fibers. If, in addition, the elliptic surface has a 2-torsion section, the translation by 2-torsion defines an involution, and the quotient elliptic surface is a Kummer surface. We may then ask for a description of an associated abelian surface (in general, there may be more than one).

We describe the construction for a family of elliptic K3 surface of rank 17 with Néron-Severi lattice of discriminant 2. This family has bad fibers of type II^* (or E_8) and III^* (or E_7) in the Néron-Kodaira notation. It turns out that we can give an alternative elliptic fibration with a 2-torsion section. The quotient elliptic surface is the Kummer of a unique principally polarized abelian surface (generically, the Jacobian of a curve of genus 2). We relate the invariants of genus 2 curve to the moduli of the original K3 elliptic surface, thus giving an explicit description of the map on moduli spaces. The description of the relation between the genus 2 curve and the K3 surface is geometric. We illustrate a technique used by Elkies [E] that allows us to verify computationally the correspondence between the elliptic K3 surface and the Jacobian of the genus 2 curve by using the Grothendieck-Lefschetz trace formula. This technique may also be used with other families of K3 surfaces with Shioda-

Inose structure to conjecture the associated abelian surface. A number theoretic application of elliptic K3 surfaces with high rank is to produce elliptic curves over $\mathbb{Q}(t)$ and $\mathbb{Q}$ of high rank. For instance, if the Néron-Severi group is all defined over $\mathbb{Q}$, then one tries to find a vector of norm 0 in the lattice that gives an elliptic fibration, such that there are no reducible fibers. By Shioda's formula, this will imply that the resulting elliptic surface has large Mordell-Weil rank. Elkies has used his techniques from [E] to construct an elliptic curve of rank 18 over $\mathbb{Q}(t)$ and elliptic curves of rank at least 28 over $\mathbb{Q}$.

1.2 Preliminaries on lattices

Definition 1.1. *A* **lattice** *will denote a finitely generated free abelian group* Λ *equipped with a symmetric bilinear form* $B : \Lambda \times \Lambda \to \mathbb{Z}$*.*

We abbreviate the data (Λ, B) to Λ sometimes, when the form is understood, and we interchangeably write $u \cdot v = \langle u, v\rangle = B(u, v)$ and $u^2 = \langle u, u\rangle = B(u, u)$ for $u \in \Lambda$.

The **signature** of the lattice is the real signature of the form B, written (r_+, r_-, r_0) where r_+, r_- and r_0 are the number of positive, negative and zero eigenvalues of B, counted with multiplicity. We say that the lattice is **non-degenerate** if the form B has zero kernel, i.e. $r_0 = 0$. In that case, the signature is abbreviated to (r_+, r_-). We say Λ is **even** if $x^2 \in 2\mathbb{Z}$ for all $x \in \Lambda$.

Let Λ be a non-degenerate lattice. The **discriminant** of the lattice is $|\det(B)|$. The lattice is said to be **unimodular** if its discriminant is 1.

A stronger invariant of the lattice is its **discriminant form** q_Λ, which is defined as follows. Let Λ be an even non-degenerate lattice. Let $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$ be the dual lattice of Λ. The form B on Λ induces a form on Λ^*, and there is a natural embedding of lattices $\Lambda \hookrightarrow \Lambda^*$. The finite abelian group Λ^*/Λ is called the **discriminant group** A_Λ, and the form B induces a quadratic form on A_Λ as follows. We have an induced form $\Lambda^* \times \Lambda^* \to \mathbb{Z}$ which takes $\Lambda \times \Lambda$ into $\mathbb{Z}$ and the diagonal of $\Lambda \times \Lambda$ to $2\mathbb{Z}$. Therefore we get an induced symmetric form $b : A_\Lambda \times A_\Lambda \to \mathbb{Q}/\mathbb{Z}$ and a quadratic form $q : A_\Lambda \to \mathbb{Q}/2\mathbb{Z}$ such that for all $n \in \mathbb{Z}$ and all $a, b \in \Lambda$, we have

$$q(na) = n^2 q(a)$$

$$q(a + a') - q(a) - q(a') \equiv 2b(a, a') \mod 2\mathbb{Z}$$

This data (A_Λ, b, q) will be abbreviated to q_Λ. Note that the discriminant of the lattice is just the size of the discriminant group. We shall let $l(A)$ denote the minimum number of generators of an abelian group A. Note that $l(A_\Lambda) \leq \mathrm{rank}(\Lambda^*) = \mathrm{rank}(\Lambda)$. For a unimodular lattice Λ, we have $l(A_\Lambda) = 0$.

The discriminant form of a unimodular lattice is trivial, and if $M \subset L$ is a **primitive** embedding of non-degenerate even lattices (that is, L/M is a free abelian group), with L unimodular, then we have

$$q_{M^\perp} = -q_M$$

For a lattice Λ and a real number α, we denote by $\Lambda(\alpha)$ the lattice which has the same underlying group but with the bilinear form scaled by α. The lattice of rank one with a generator of norm α will be denoted $\langle \alpha \rangle$.

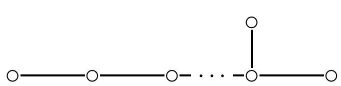
By a **root** of a positive definite lattice, we will mean an element x such that $x^2 = 2$, whereas for a negative-definite or indefinite lattice, we will mean an element x such that $x^2 = -2$.

We note here some theorems about the structure and embeddings of lattices.

A **root lattice** is a lattice that is spanned by its roots. First, let us introduce some familiar root lattices, through their Dynkin diagrams. The subscript in the name of the lattice is the dimension of the lattice, which is also the number of nodes in the Dynkin diagram.

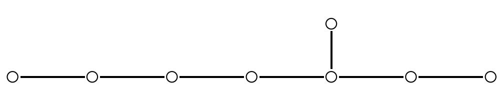
$A_n (n \geq 1)$, signature $(n, 0)$, discriminant $n + 1$.

This is the positive definite lattice with n generators $v_1, \ldots, v_n$ with $v_i^2 = 2$ and $v_i \cdot v_j = -1$ if the vertices i and j are connected by an edge, and 0 otherwise. It may be realized as the set of integral points on the hyperplane $\{x \in \mathbb{R}^{n+1} \big| \sum x_i = 0\}$.



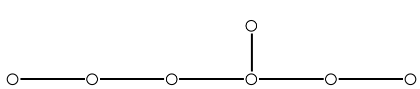
$D_n (n \geq 4)$, signature $(n, 0)$, discriminant 4.

D_n can be realized as $\{x \in \mathbb{Z}^n \big| \sum_{i=1}^{n} x_i \equiv 0 \mod 2\}$. It has $2n(n-1)$ roots.



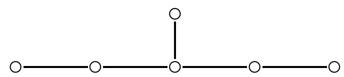
E_8, signature $(8, 0)$, discriminant 1.

One realization of E_8 is as the span of D_8 and the all-halves vector $(1/2, \ldots, 1/2)$. It has 240 roots.



E_7, signature $(7, 0)$, discriminant 2.

Taking the orthogonal complement of any root in E_8 gives us E_7. It has 126 roots.



E_6, signature $(6, 0)$, discriminant 3.

Taking the orthogonal complement of e_1 and e_2 in E_8, where e_1, e_2 are roots such that $e_1 \cdot e_2 = -1$, gives us E_6. It has 72 roots.

We let the **Nikulin lattice** N be the lattice generated by $v_1, \ldots, v_8$ and $\frac{1}{2}(v_1 + \ldots + v_8)$, with $v_i^2 = -2$ and $v_i \cdot v_j = 0$ for $i \neq j$. It is isomorphic to $D_8^*(-2)$.

Lemma 1.2. *N has 16 roots, namely $\pm v_i$.*

Proof. Let $\sum c_i v_i$ be a root. Then $\sum (-2c_i)^2 = -2$, so $\sum c_i^2 = 1$. Also, the c_i are either all integers or all half-integers. One checks easily that the only possibilities are $c_i = \pm\delta_{i,j}$ for some j. $\square$

Note that, in particular, N is not a root lattice.

Let U be the hyperbolic plane, i.e. the indefinite rank 2 lattice whose matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that $U(-1) \cong U$.

Theorem 1.3. (Milnor [Mi]) *Let Λ be an indefinite unimodular lattice. If Λ is even, then $\Lambda \cong E_8(\pm 1)^m \oplus U^n$ for some m and n. If Λ is odd, then $\Lambda \cong \langle 1 \rangle^m \oplus \langle -1 \rangle^n$ for some m and n.*

Theorem 1.4. (Kneser [Kn], Nikulin [N3]) *Let L be an even lattice with signature (s_+, s_-) and discriminant form q_L such that*

1. $s_+ > 0$
2. $s_- > 0$
3. $l(A_L) \leq \mathrm{rank}(L) - 2$.

Then L is the unique lattice with that signature and discriminant form, up to isometry.

Theorem 1.5. (Morrison [M1]) *Let M_1 and M_2 be even lattices with the same signature and discriminant-form, and let L be an even lattice which is uniquely determined by its signature and discriminant-form. If there is a primitive embedding $M_1 \hookrightarrow L$, then there is a primitive embedding $M_2 \hookrightarrow L$.*

Proposition 1.6. (Nikulin [N3] Theorem 1.14.4) *Let M be an even lattice with invariants (t_+, t_-, q_M) and let L be an even unimodular lattice of signature (s_+, s_-). Suppose that*

1. $t_+ < s_+$.
2. $t_- < s_-$.
3. $l(A_M) \leq rank(L) - rank(M) - 2$.

Then there exists a unique primitive embedding of M into L, up to automorphisms of L.

In fact, a stronger statement is the following.

Proposition 1.7. (Nikulin [N3] Theorem 1.14.4) *Let M be an even lattice with invariants (t_+, t_-, q_M) and let L be an even unimodular lattice of signature (s_+, s_-). Suppose that*

1. $t_+ < s_+$.
2. $t_- < s_-$.
3. $l(A_{M_p}) \leq rank(L) - rank(M) - 2$ *for* $p \neq 2$.
4. *If* $l(A_{M_2}) = rank(L) - rank(M)$, *then* $q_M \cong u_2^+(2) \oplus q'$ *or* $q_M \cong v_2^+(2) \oplus q'$ *for some* q'.

Then there exists a unique primitive embedding of M into L.

Here $u_2^+(2)$ is the discriminant form of the 2-adic lattice whose matrix is

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

and $v_2^+(2)$ is the discriminant form of the 2-adic lattice whose matrix is

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

Corollary 1.8. *There is a unique primitive embedding of N into $E_8(-1) \oplus U^3$, up to automorphisms of the ambient lattice. There is a unique primitive embedding of $N \oplus E_8(-1)$ into $E_8(-1)^2 \oplus U^3$.*

Proof. This follows from the proposition above, using the fact that the discriminant form of N is given by $(q_{U(2)})^3 = (u_2^+(2))^3$. □

If $N \hookrightarrow E_8(-1) \oplus U^3$ is any primitive embedding, then one can verify that the orthogonal complement of N is isomorphic to $U(2)^3$. We have a uniqueness result for the embeddings of $U(2)^3$ into $E_8(-1) \oplus U^3$ as well.

Proposition 1.9. (van Geemen - Sarti [vGS], Lemma 1.10) *There exists a unique primitive embedding of $U(2)^3$ into $E_8(-1) \oplus U^3$, such that the orthogonal complement is isomorphic to N, up to automorphisms. There exists a unique primitive embedding of $U(2)^3 \oplus E_8(-1)$ into $E_8(-1)^2 \oplus U^3$, such that the orthogonal complement is isomorphic to N, up to automorphisms.*

Remark 1.10. It is necessary to specify that the orthogonal complement is isomorphic to N. The lattice $U(2)^3$ does not have a unique primitive embedding into $E_8(-1) \oplus U^3$, up to automorphisms of the ambient lattice.

An elementary result, which we shall have occasion to use repeatedly, is the following.

Lemma 1.11. *Let M be a unimodular lattice. Then if $M \subset L$, we have an orthogonal decomposition $L = M \oplus M^{\perp}$.*

Proof. For any vector $v \in L$, we would like to define the projections v_M and $v_{M^\perp}$ and show they lie in M. The map $M \to \mathbb{Z}$ given by $u \mapsto \langle u, v \rangle$ is represented by some unique $v_M \in M^* = M$, and we let $v_{M^\perp} = v - v_M$. □

Chapter 2

K3 surfaces and Torelli theorems

2.1 K3 surfaces: Basic notions

Let X be a smooth projective surface over $\mathbb{C}$.

Definition 2.1. *We say that X is a* **K3 surface** *if $H^1(X, \mathcal{O}_X) = 0$ and the canonical bundle of X is trivial, i.e. $K_X \cong O_X$.*

We recall here a few facts about K3 surfaces which we need in the sequel. More extensive references are [BHPV], [K], [M2].

K3 surfaces are simply connected. From Noether's formula

$$\chi(\mathcal{O}_X) = \frac{1}{12}((K^2) + c_2) \tag{2.1}$$

it follows that the Euler characteristic of X is 24. Here $c_1 = 0$ and $c_2 = \chi(X)$ are the Chern classes of (the tangent bundle) of X. Therefore the middle cohomology $H_X := H^2(X, \mathbb{Z})$ is 22-dimensional. Identifying $H^4(X, \mathbb{Z})$ with $\mathbb{Z}$, we also know that H_X endowed with the cup-product pairing is a unimodular lattice (that is, the associated quadratic form has discriminant 1) because of Poincaré duality. Finally, we also know that H_X is an even lattice; this follows from Wu's formula involving Stiefel-Whitney classes. Finally, using the index theorem

$$\tau(X) = \frac{1}{3}(c_1^2 - 2c_2) \tag{2.2}$$

we see that H_X has signature $(3, 19)$.

Using the classification of Milnor, Theorem 1.3, we use there properties to conclude that $H_X \cong E_8(-1)^2 \oplus U^3$ as lattices. In the sequel, we will set $L := E_8(-1)^2 \oplus U^3$.

A useful and deep fact about K3 surfaces is the following theorem of Siu.

Theorem 2.2. (Siu [Siu]) *Every K3 surface is Kähler.*

Definition 2.3. *A* **marking** *of X will denote a choice of isomorphism $\phi : H_X \xrightarrow{\sim} L$.*

Definition 2.4. *If X is an algebraic K3 surface, a* **polarization** *of X is a choice of ample line bundle $\mathcal{L}$ on X.*

Now, the exact sequence of analytic sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{z \mapsto e^{2\pi i z}} \mathcal{O}_X^* \to 0$$

gives, upon taking the long exact sequence of cohomology, an injective map (the first Chern class map)

$$H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}).$$

Thus, an analytic line bundle is determined by its image in $H^2(X, \mathbb{Z})$. For an algebraic K3 surface, we can also say that an algebraic line bundle is determined by class in $H^2(X, \mathbb{Z})$. Linear equivalence, algebraic equivalence, and numerical equivalence all agree for an algebraic K3 surface. The image of $H^1(X, \mathcal{O}_X^*)$ in $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$ is a sublattice of $H^2(X, \mathbb{Z})$, which we call the Néron-Severi group of X, and denote by $NS(X)$ or S_X or $\mathrm{Pic}(X)$. In fact, it can be shown that

$$NS(X) = H^{1,1}_{\mathbb{Z}}(X).$$

Also,

$$NS(X) = \{z \in H^2(X, \mathbb{Z}) \mid \langle z, H^{2,0}(X) \rangle = 0\}.$$

Since the canonical bundle of X is trivial, there exists a regular $(2, 0)$ form on X, and we have $h^{2,0} = 1$. It follows that the Hodge diamond of X looks like

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The lattice $NS(X)$ lies in $H^{1,1}_{\mathbb{Z}}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}_{\mathbb{R}}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$, so in fact the data of $\phi : H_X \xrightarrow{\sim} L$ will impose upon $L \otimes \mathbb{C}$ a Hodge structure, i.e. a splitting

$$L \otimes \mathbb{C} = L^{0,2} \oplus L^{1,1} \oplus L^{2,0}$$

where we have written $L^{p,q} = \phi \otimes 1_{\mathbb{C}}(H^{q,p})$.

Note that the entire Hodge structure can be characterized by specifying the 1-dimensional subspace $L^{2,0}$, because $L^{0,2} = \overline{L^{2,0}}$ and $L^{1,1} = (L^{0,2} \oplus L^{2,0})^{\perp}$.

If ω is a non-vanishing $(2,0)$ form on X, then we have $\langle\omega,\omega\rangle = 0$ because there are no $(4,0)$ forms on X, whereas $\langle\omega,\overline{\omega}\rangle > 0$.

We identify $L \otimes \mathbb{C}$ with its dual using the scalar product on L.

Definition 2.5. *The period space Ω is defined by*

$$\Omega = \{\omega \in \mathbb{P}(L \otimes \mathbb{C}) \,|\, \langle\omega,\omega\rangle = 0, \langle\omega,\overline{\omega}\rangle > 0\}.$$

Then any marked K3 surface (X,α) defines a point of Ω, namely the image of $\omega \in H^{2,0}(X)$ in $\mathbb{P}(L \otimes \mathbb{C})$. The Torelli theorem says that "in some sense", X is determined by its image under the period map. We shall give various versions of the Torelli theorem.

2.2 Torelli theorems

We now state the Torelli theorem for algebraic K3 surfaces, in the form given by Piatetski-Shapiro and Shafarevich [PS]. Recall that a compact complex variety is algebraic iff it has a Hodge metric, i.e. a Kähler metric such that the associated $(1,1)$ differential form belongs to an integral cohomology class. For a K3 surface X, this means that $H^{1,1}_{\mathbb{Z}}(X) = H^{1,1}_X \cap H^2(X,\mathbb{Z})$ has signature $(1,k)$ for some k, i.e. it has a vector v such that $v^2 = \langle v,v\rangle > 0$. We can make a choice of v that will correspond to the class of a very ample line bundle on X.

Fix a vector $v \in L$ with $v^2 > 0$.

Definition 2.6. *A* **marked v-polarized K3 surface** *is a triple (X,ϕ,ξ) where X is a K3 surface, $\phi : H_X \to L$ is an isometry, and $\xi \in H_X$ is the class corresponding to a very ample line bundle on X and such that $\phi(\xi) = v$. An isomorphism of marked v-polarized K3 surfaces (X,ϕ,ξ) and (X',ϕ',ξ') is an isomorphism of surfaces $X \to X'$ such that $\phi f^* = \phi'$.*

Let the moduli space of marked v-polarized K3 surfaces up to isomorphism be $\mathcal{M}_v$. The corresponding period domain will be denoted Ω_v.

$$\Omega_v = \{\omega \in \Omega | \langle\omega, v\rangle = 0\}.$$

The period point of a marked v-polarized K3 surface X lies in Ω_v because the cup product $\langle\omega,v\rangle \in H^{3,1}(X) = 0$.

Theorem 2.7. *[PS](Torelli for algebraic K3) The period map $\tau : M_v \to \Omega_v$ is an imbedding.*

In other words, a marked v-polarized K3 surface is uniquely determined by its periods. To generalize the Torelli theorem to non-algebraic K3 surfaces, we need to replace the notion of ample divisor class by a suitable structure on H_X that carries enough information about X.

Definition 2.8. *Let X, X' be K3 surfaces. A* **Hodge isometry** $\phi : H_X \to H_{X'}$ *is an isometry of lattices such that* $(\phi \otimes 1_{\mathbb{C}})(H_X^{2,0}) = H_{X'}^{2,0}$.

Note that this automatically implies $\phi \otimes 1_{\mathbb{C}}(H_X^{0,2}) = H_{X'}^{0,2}$ by taking complex conjugation, and $\phi \otimes 1_{\mathbb{C}}(H_X^{1,1}) = H_{X'}^{1,1}$ by taking orthogonal complements. Hence, a Hodge isometry transports the entire Hodge structure.

Theorem 2.9. *(Weak Torelli theorem) Two K3 surfaces X and X' are isomorphic if and only if there exists a Hodge isometry $\phi : H_X \to H_{X'}$.*

In other words, two K3 surfaces are isomorphic if and only if there exist markings for them such that the corresponding period points are equal.

We state a useful fact which is used in the proof of the strong Torelli theorem, and is an easy corollary of it.

Lemma 2.10. *Let f be an automorphism of a K3 surface X. If the induced map f^* on $H^2(X, \mathbb{Z})$ is the identity, then f is the identity map.*

To state the strong form of the Torelli theorem, we need a few more definitions. For X a K3 surface, we have a complex conjugation on $H^2(X, \mathbb{Z}) \otimes \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$, induced from $\mathbb{C}$. Let $H^{1,1}(X, \mathbb{R})$ be the fixed subspace of $H^{1,1}$ under complex conjugation. It can also be described as $\{w \in H^2(X, Z) \otimes \mathbb{R} | \langle w, \omega \rangle = 0\}$ where ω is a holomorphic non-vanishing $(2, 0)$ form on X. The signature of the intersection form on $H^{1,1}(X, \mathbb{R})$ is $(1, h^{1,1} - 1)$ (this follows from the Hodge index theorem[1]).

Therefore the set $V(X) = \{x \in H^{1,1}(X, \mathbb{R}) | x^2 > 0\}$ consists of two disjoint connected cones. We will call the cone which contains the Kähler form c, the **positive cone** $V^+(X)$.

Definition 2.11. *Let $\Delta(X) = \{x \in H_{\mathbb{Z}}^{1,1} | x^2 = -2\}$ be the set of roots. The Kähler class c determines a partition $\Delta(X) = \Delta^+(X) \cup \Delta^-(X)$ into positive and negative roots, by setting* $\Delta^{\pm}(X) = \{x \in \Delta | \pm x \cdot c > 0\}$.

An **effective cycle** in H_X is one that is the class of an effective divisor. The **Kähler cone** of X is the set of elements in the positive cone that have positive inner product with any nonzero effective class.

Theorem 2.12. (Burns-Rapoport [BR]) *Let X and X' be two K3 surfaces and suppose that there is an isomorphism $\phi^* : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ that*

[1] Indeed, we have that the signature on the complementary subspace $(H^{2,0} \oplus H^{0,2}) \cap H^2(X, \mathbb{R})$ is $(2, 0)$, because a typical element of this subspace can be written as $\omega + \bar{\omega}$, and then its square is $2\omega\bar{\omega} > 0$.

1. *sends* $H^{2,0}(X', \mathbb{C})$ *to* $H^{2,0}(X, \mathbb{C})$,
2. *sends* $V^+(X')$ *to* $V^+(X)$, *and*
3. *sends an effective cycle with square -2 to an effective cycle.*

Then ϕ^* *is induced by a unique isomorphism* $X \to X'$.

Condition (2) in the statement of the theorem can be replaced by various equivalent conditions, some of which are stronger implications. In particular, it is a weakening of the condition that ϕ^* maps $\Delta^+(X')$ to $\Delta^+(X)$.

Proposition 2.13. *Let* X *and* X' *be algebraic K3 surfaces and let* $\phi^* : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ *be a Hodge isometry. Then the following are equivalent.*

1. ϕ^* *preserves effective classes;*
2. ϕ^* *preserves ample classes;*
3. ϕ^* *maps the Kähler cone of* X' *onto the Kähler cone of* X*;*
4. ϕ^* *maps an element of the Kähler cone of* X' *to the Kähler cone of* X'.

The strong Torelli theorem can also be viewed as stating that a particular period mapping from a suitable moduli space is an imbedding.

Let $\mathcal{M}$ be the moduli space of marked K3 surfaces up to isomorphism. To construct the period space, we proceed as follows. Let $\widetilde{\Omega}$ be the functor which to an analytic space S associated the collection of the following data:

1. A holomorphically varying Hodge structure H on $L \otimes \mathbb{C}$ parametrized by S,
2. A continuously varying choice of one of the two connected components $V_s^+ \subset V_s \subset H^{1,1}_{s\mathbb{R}}$,
3. For every point $s \in S$, a partition $P_s : \Delta_s = \Delta_s^+ \cup \Delta_s^-$,

such that if $\delta_1, \ldots, \delta_n \in \Delta^+$ and $\delta = \sum_{i=1}^n r_i \delta_i \in \Delta$ ($r_i > 0$, integers), then $\delta \in \Delta^+$, and such that the data (3) satisfy the following continuity condition:

For every point $s_0 \in S$ and every $c_0 \in V^+_{P_{s_0}}$, there exists an open neighborhood K of c_0 in $L \otimes \mathbb{R}$ and an open neighborhood U of s_0 in S such that for every $s \in U$ we have

$$\Delta_s^+ = \{\delta \in \Delta_s | \delta \cdot c > 0 \text{ for all } c \in K\}.$$

Then the forgetful morphism of functors $\pi : \widetilde{\Omega} \to \Omega$ is relatively representable by an étale morphism of analytic spaces. In particular, $\widetilde{\Omega}$ is representable by a smooth 20-dimensional complex-analytic space.

Intuitively, the fiber over any point of Ω consists of a discrete set of points, one for each possible choice of positive cone and Kähler cone (or positive roots). The fibers are not finite.

We have a period map $\tau : \mathcal{M} \to \widetilde{\Omega}$ which takes a marked $K3$ surface X and associates to it the period point which consists of the Hodge structure on H_X along with the positive cone and the choice of positive roots of X.

Now we can restate the strong Torelli theorem as follows:

Theorem 2.14. *The period map $\tau : \mathcal{M} \to \widetilde{\Omega}$ is an embedding.*

In fact, the period map is a bijection, due to the following theorem of Todorov.

Theorem 2.15. (Todorov [To]) *The period map $\mathcal{M} \to \widetilde{\Omega}$ is surjective.*

We collect some more definitions for future reference.

Definition 2.16. *For $\delta \in \Delta$, let $s_\delta : H_X \to H_X$ be given by $s_\delta(x) = x + (x \cdot \delta)\delta$. This is an automorphism of the lattice H_X, called the* **Picard-Lefschetz reflection** *associated to δ. The* **Weyl group** *of X is the subgroup of H_X generated by the Picard-Lefschetz reflections.*

2.3 Curves on a K3 surface

We state some basic theorems relating to divisors on an algebraic $K3$ surface. Recall the following basic theorems.

Theorem 2.17. *(Adjunction formula) If C is a non-singular curve on a surface X, then $\omega_C \cong \omega_X \otimes \mathcal{L}_C \otimes \mathcal{O}_C$.*

Theorem 2.18. *(Genus formula). Let D be an effective divisor on an algebraic surface X. Then its arithmetic genus is given by the formula*

$$2p_a - 2 = D \cdot (D + K).$$

Theorem 2.19. *(Riemann-Roch) Let D be a divisor on the algebraic surface X. Then we have*

$$h^0(X, \mathcal{L}(D)) - h^1(X, \mathcal{L}(D)) + h^2(X, \mathcal{L}(D)) = \frac{1}{2} D \cdot (D - K) + \chi(\mathcal{O}(X)).$$

Here $\mathcal{L}(D) = \mathcal{O}_X(D)$. Recall that $K = 0$ and $\chi(O_X) = 2$ for a K3 surface X.

Lemma 2.20. *For a smooth rational curve C on a K3 surface X, we have $C^2 = -2$ and $h^0(X, \mathcal{L}(C)) = 1$. That is, the only effective divisor linearly equivalent to C is C itself.*

Proof. The adjunction formula gives $C^2 = -2$. We have the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L}(C) \to \mathcal{L}(C) \otimes \mathcal{O}_C \to 0$$

The sheaf $\mathcal{L}(C) \otimes \mathcal{O}_C$ is $\mathcal{O}_C(-2)$ because $C^2 = -2$. Therefore the long exact sequence of cohomology immediately shows that $h^0(X, \mathcal{L}(C)) = h^0(X, \mathcal{O}_X) = 1$.

Alternatively, we can consider the following argument. If C' is an effective divisor linearly equivalent to C but distinct from it, then C cannot be a component of C' (otherwise $C' - C$ would be effective and linearly equivalent to zero, which is absurd). Therefore, we must have $C' \cdot C \geq 0$, but also $C' \cdot C = C^2 < 0$, a contradiction. □

Lemma 2.21. *Let $D \in NS(X)$ be such that $D^2 = -2$. Then there is an effective divisor (perhaps reducible) equivalent to D or to $-D$. If in addition the effective divisor is an irreducible curve, then it is a smooth rational curve.*

Proof. The existence follows from Riemann-Roch:

$$\ell(D) - s(D) + \ell(K - D) = \frac{1}{2}D(D - K) + \chi(O_X) = 1.$$

So $\ell(D) + \ell(-D) \geq 1$.

Now if C is effective with $C^2 = -2$, then the adjunction formula gives $p_a(C) = 0$. If C were not smooth, then for the normalization $\tilde{C}$ we would have $p_g(\tilde{C}) = p_a(\tilde{C}) < p_a(C) = 0$, which is absurd. □

Theorem 2.22. [PS] *Let D be an effective divisor on a K3 surface X such that $D^2 = 0$ and $D \cdot E \geq 0$ for any effective divisor E. Then the linear system $\ell(D)$ defines a pencil of elliptic curves $X \to \mathbb{P}^1$.*

Note that the terminology "pencil of elliptic curves" means "curves of genus 1", which are elliptic curves if we consider them over an algebraically closed field. However, if we also have an irreducible curve C with $C \cdot D = 1$, then it defines a section of the pencil, and then we really have an elliptic surface over a field of definition of X, D and C.

The rough idea of the proof of the theorem is that if D were nonsingular, then by the adjunction formula it would have genus 1. One then shows that the generic element of the linear system is irreducible and smooth, using the fact that we are in characteristic zero.

Corollary 2.23. *A K3 surface X can be fibered as a pencil of elliptic curves if and only if there exists an element $x \in NS(X), x \neq 0$ such that $x^2 = 0$.*

Proof. If we can find such an x, we can find an element w of the Weyl group such that $y = w(x)$ lies in the Kähler cone. Then y satisfies the conditions of the theorem. □

2.4 Kummer surfaces

In this section we will define Kummer surfaces and mention some of their properties. Proofs and more background may be found in [N2] and [PS]. We recall that an abelian surface A over $\mathbb{C}$ is a complex torus $\mathbb{C}^2/L$ such that there is a Riemann form on L. It is an abelian group under addition, and also a projective algebraic variety of dimension 2. We build a K3 surface from A as follows. Let ι be the multiplication by -1 on A. Then ι has 16 fixed points on A, namely the 2-torsion $A[2] \cong \frac{1}{2}L/L$. Consequently the quotient $A/\{1, \iota\}$ is an algebraic surface with 16 singular points. Resolving the singularities, we get a nonsingular K3 surface with 16 special rational curves. This is called the Kummer surface of A, or $\mathrm{Km}(A)$.

We indicate why $\mathrm{Km}(A)$ is a K3 surface (the argument may be found in [K]). Locally around the 2-torsion points of A, ι maybe written as $(\alpha, \beta) \mapsto (-\alpha, -\beta)$ and the invariants are $\alpha^2, \alpha\beta, \beta^2$. The quotient A/ι has a corresponding ordinary double point, since $\mathbb{C}[\alpha^2, \alpha\beta, \beta^2] \cong \mathbb{C}[u, v, w]/(uw - v^2)$. Blowing up the 16 points of $A[2]$, we get a surface $\tilde{A}$. Locally on a point of the blowup $\pi : \tilde{A} \to A$, the map may be written $(x, y) \mapsto (xy, y)$. The involution $\tilde{\iota}$ on A takes (x, y) to $(x, -y)$. The quotient $X = \tilde{A}/\tilde{\iota}$ is smooth and the quotient map $\tilde{A} \to X$ takes (x, y) to (x, y^2). Now we may compute what happens to the regular 2-forms.

$$\pi^*(d\alpha \wedge d\beta) = d(xy) \wedge dy = y dx \wedge dy = \frac{1}{2} dx \wedge d(y^2)$$

Hence the global form on A descends to give one on Y. A simple computation now shows that the Euler characteristic of X is 24, and so we get from Noether's formula that $h^1(X, \mathcal{O}_X) = 0$. Therefore $X = \mathrm{Km}(A)$ is a K3 surface.

The Néron-Severi lattice of a Kummer surface has 16 linearly independent divisor classes coming from the sixteen rational curves above. These generate a negative definite lattice, and there is also a class of a polarization on $\mathrm{Km}(A)$, since it is projective. Therefore its signature is $(1, r)$ for some $r \geq 16$. In fact, the Néron-Severi lattice always contains a particular lattice of signature $(0, 16)$ and discriminant 2^6, called the Kummer lattice K. We describe its structure.

The set $I = A[2] \cong (\mathbb{Z}/2)^4 \cong \mathbb{F}_2^4$ of 16 elements has a natural structure of an affine space of dimension 4 over $\mathbb{F}_2$. Choose a labeling $I = \{1, 2, \ldots, 16\}$ and let $e_1, \ldots, e_{16}$ be the classes of the rational curves corresponding to the blowups at the 2-torsion points. Let Q

be the set of 32 elements consisting of 30 affine hyperplanes (considered as subsets of I) as well as the empty set and all of I. This set has the structure of a vector space over $\mathbb{F}_2$, the addition operation being symmetric difference of sets. For every $M \in Q$, we have an element $e_M = \frac{1}{2}\sum_{i\in M} e_i$ of $\sum \mathbb{Q}e_i \in NS(\mathrm{Km}(A)) \otimes \mathbb{Q}$. These vectors actually lie in $NS(\mathrm{Km}(A))$. The lattice spanned by the $e_i, i = 1, \ldots, 16$ and the e_M has discriminant $2^{16}/(2^5)^2 = 2^6$ and it is the Kummer lattice.

We state a few results characterizing Kummer surfaces, due to Nikulin. The Kummer surfaces mentioned below are not necessarily algebraic.

Theorem 2.24. (Nikulin [N2]) *Let X be a K3 surface containing* 16 *nonsingular rational curves $E_1, \ldots, E_{16}$ which do not intersect each other. Then there exists a unique (up to isomorphism) complex torus T such that X and $E_1, \ldots, E_{16}$ are obtained from T by the above construction. In particular, X is a Kummer surface.*

Theorem 2.25. (Nikulin [N2]) *Let X be a K3 surface. Then X is a Kummer surface if and only if $NS(X) \supset K$ as a primitive sublattice.*

Recall that a primitive (or saturated) sublattice M_1 of M_2 is a sublattice such that M_2/M_1 is free.

Proposition 2.26. (Nikulin [N2]) *There exists a unique primitive embedding $K \subset L$ up to isomorphism.*

Here L is the K3 lattice $E_8(-1)^2 \oplus U^3$. The proposition also follows from Theorem 1.6, after using the fact that $q_K \cong (q_{U(2)})^3$, which is easily checked.

Chapter 3

Shioda-Inose structures

3.1 Lattice polarized K3 surfaces

We will closely follow Nikulin's paper [N1] here. As before, let $L = E_8(-1)^2 \oplus U^3$ be the $K3$ lattice. Let $M \subset L$ be a fixed primitive sublattice.

Definition 3.1. *A* **marked M-polarized K3 surface** *is a pair (X, ϕ) such that (X, ϕ) is a marked K3 surface and $\phi^{-1}(M) \subset NS(X)$. An isomorphism of two marked M-polarized K3 surfaces is an isomorphism as marked K3 surfaces.*

Let $\mathcal{M}_M$ be the moduli space of marked M-polarized K3 surfaces up to isomorphism.

Let $T = \{e_i | i \in I\}$ be a finite collection of roots of M, i.e. vectors such that $e_i^2 = -2$ for each i.

Definition 3.2. *A* **marked M-polarized K3 surface with rational curves in** T *is a marked M-polarized K3 surface (X, ϕ) such that for each i, $\phi^{-1}(e_i)$ is the class of a nonsingular rational curve on X. An isomorphism of two marked M-polarized K3 surfaces with rational curves in T is an isomorphism as marked K3 surfaces.*

Let $\mathcal{M}_{M,T}$ be the subset of the moduli space $\mathcal{M}$ of marked K3 surfaces consisting of the marked M-polarized K3 surfaces with rational curves in T. The condition of having $\phi^{-1}(M) \subset NS(X)$ is a closed condition, and the condition of having $\phi^{-1}(e_i)$ be a smooth rational curve is an open condition on the closed subset obtained. Thus, it is easy to see that $\mathcal{M}_{M,T}$ is an open subset of a closed smooth complex subspace of $\mathcal{M}$.

Recall the complex analytic space $\widetilde{\Omega}$ and the period map $\tau : \mathcal{M} \to \widetilde{\Omega}$ constructed in the previous chapter. Let $\widetilde{\Omega}_{M,T}$ consist of the points $\tilde{s} \in \widetilde{\Omega}$ such that

1. $M \subset H^{1,1}_{\tilde{s}}$ and
2. $\{e_i | i \in I\} \subset \Delta^+_{\tilde{s}}$, and all of the e_i are irreducible elements of $\Delta^+_{\tilde{s}}$; that is, $e_i \neq \sum_1^R k_r \delta_r$, where $R \geq 2$, $k_r > 0$ and $\delta_r \in \Delta^+_{\tilde{s}}$.

The following is clear.

Proposition 3.3. $(X, \alpha) = m \in \mathcal{M}_{M,T}$ *if and only if* $\tau(m) \in \widetilde{\Omega}_{M,T}$.

Let $\widetilde{\Omega}_M$ denote the subset of $\widetilde{\Omega}$ consisting of those points which just satisfy condition (1). For $\widetilde{\Omega}_M$ to be nonempty, it is necessary and sufficient that one of the following hold:

1. M has signature $(1, k)$, where $k \leq 19$.
2. The form on M has a one-dimensional kernel, and the quotient by it is negative definite; M has rank at most 19.
3. M is negative definite, and of rank at most 19 (this is inclusive of the case $M = \{0\}$).

Let $\widetilde{\Omega}^+$ and $\widetilde{\Omega}^-$ be the two connected components of $\widetilde{\Omega}$ (corresponding to the choice of the positive cone). Let $\Delta(M) = \{x \in M | x^2 = -2\}$ and let $P : \Delta^+(M) \cup \Delta^-(M) = \Delta(M)$ be a partition of $\Delta(M)$ satisfying the following property:

$$\text{If } \delta_1, \dots, \delta_n \in \Delta^+ \text{ and } \delta = \sum_1^n r_i \delta_i \in \Delta, r_i \in \mathbb{Z}_{>0}, \text{ then } \delta \in \Delta^+.$$

We set $\widetilde{\Omega}^{(\pm)P}_M = \{\tilde{s} \in \widetilde{\Omega}_M \cap \widetilde{\Omega}^{(\pm)}, P_{\tilde{s}} \text{ induces the partition } P \text{ on } \Delta(M)\}$ and also let $\widetilde{\Omega}^{(\pm)P}_T = \widetilde{\Omega}_T \cap \widetilde{\Omega}^{(\pm)P}_M$.

Proposition 3.4. (Nikulin [N1]). *Suppose that M satisfies one of the conditions above. Then $\widetilde{\Omega}_M$ is a closed smooth complex subspace of $\widetilde{\Omega}$ of dimension* $20 - \operatorname{rank} M$ *whose connected components are $\widetilde{\Omega}^{(\pm)P}_M$. Furthermore, $\widetilde{\Omega}^{(\pm)P}_M - \widetilde{\Omega}^{(\pm)P}_T$ is a closed subset of $\widetilde{\Omega}^{(\pm)P}_M$ which is a union of at most countably many closed complex subspaces of $\widetilde{\Omega}^{(\pm)P}_M$. In particular, $\widetilde{\Omega}^{(\pm)P}_T$ is connected.*

Let $\mathcal{M}^{(\pm)P}_M$ be the moduli space of marked M-polarized K3 surfaces (X, ϕ) with $\phi(\Delta^+(X)) \cap M = \Delta^+(M)$ (where $\Delta^+(M)$ is the data provided by P) up to isomorphism. It is easy to check that $\mathcal{M}^{(\pm)P}_M = \tau^{-1}(\widetilde{\Omega}^{(\pm)P}_M)$. Similarly, we define $\mathcal{M}^{(\pm)P}_{M,T}$ to be the subset of $\mathcal{M}^{(\pm)P}_M$ consisting of surfaces with rational curves representing the roots of T. Then $\mathcal{M}^{(\pm)P}_{M,T} = \tau^{-1}(\widetilde{\Omega}^{(\pm)P}_{M,T})$.

Theorem 3.5. (Nikulin [N1]) *Suppose M satisfies one of the conditions above and in addition* $\operatorname{rank} M \leq 18$ *in cases* 2 *and* 3*. Then $\mathcal{M}_M \subset \mathcal{M}$ is a closed smooth complex subspace of $\mathcal{M}$. If $\mathcal{M}_M \neq \phi$, then the connected components of $\mathcal{M}_M$ are $\mathcal{M}_M^{(\pm)P}$, where $\mathcal{M}_M^{(\pm)P}$ is open and everywhere dense in $\widetilde{\Omega}_M^{(\pm)P}$, and dim $\mathcal{M}_M^{(\pm)P}$ is* $20 - \operatorname{rank} M$*. Furthermore, $\mathcal{M}_M^{(\pm)P} - \mathcal{M}_T^{(\pm)P}$ is a closed subset of $\mathcal{M}_M^{(\pm)P}$ which is a union of at most countably many closed complex subspaces of $\mathcal{M}_M^{(\pm)P}$. In particular, $\mathcal{M}_T^{(\pm)P}$ is connected.*

3.2 Nikulin involutions

Definition 3.6. *An involution ι on a K3 surface X is called a* **Nikulin involution** *if $\iota^*(\omega) = \omega$ for every $\omega \in H^{2,0}(X)$.*

We first recall a few general facts about groups of automorphisms of K3 surfaces.

As usual, we let $S_X = NS(X) = H_{\mathbb{Z}}^{1,1} = \{x \in H^2(X, \mathbb{Z}) | \, x \cdot H^{2,0}(X) = 0\}$ be the algebraic cycles. Then $T_X = S_X^{\perp}$ in $H^2(X, \mathbb{Z})$ are the transcendental cycles.

There are three possibilities for S_X.

1. S_X is non-degenerate and has signature $(1, k)$, with $0 \leq k \leq 19$.
2. S_X has one-dimensional kernel, and the quotient by this kernel is negative definite, $\operatorname{rank}(S_X) \leq 19$.
3. S_X is non-degenerate and negative definite; $\operatorname{rank}(S_X) \leq 19$.

The first case is the one which corresponds to an algebraic K3 surface.

Lemma 3.7. (Nikulin [N1]) *Let G be a group of automorphisms of the K3 surface X, let $g \in G$. Then g acts trivially on T_X if and only if it fixes $\omega \in H^{2,0}(X)$.*

This lemma shows that we might have imposed the equivalent condition that the Nikulin involution fixes the transcendental cycles pointwise.

Lemma 3.8. (Nikulin [N1]) *Every Nikulin involution has 8 isolated fixed points. The rational quotient of X by a Nikulin involution is a K3 surface.*

Recall the definition of the Nikulin lattice: it is an even sublattice N of rank 8 generated by vectors $c_1, \ldots, c_8$ and $\frac{1}{2} \sum c_i$, with the form induced by $c_i \cdot c_j = -2\delta_{ij}$. It has discriminant 2^6, discriminant group $(\mathbb{Z}/2\mathbb{Z})^6$ and discriminant form $(q_{U(2)})^3$.

Nikulin proves the following theorems regarding finite automorphism groups acting on K3 surfaces.

Definition 3.9. *Let G be a finite group of automorphisms of a K3 surface X. We say that G acts as a group of algebraic automorphisms on X if each $g \in G$ fixes $H^{2,0}(X)$ pointwise.*

By the above lemma, this is equivalent to fixing T_X pointwise. Let $\mathbf{G}^{\mathbf{alg}}_{\mathbf{K3}}$ be the set of abstract finite groups which can act on some K3 surface as a group of algebraic automorphisms. Let $\mathbf{G}^{\mathbf{alg,ab}}_{\mathbf{K3}}$ be the collection of abelian groups in $\mathbf{G}^{\mathbf{alg}}_{\mathbf{K3}}$.

A faithful action of G on X gives rise to a faithful action of G on $H^2(X,\mathbb{Z}) \cong L$, by 2.10. Thus every automorphism of X determines a class of conjugate subgroups in the group $O(L)$ of isometries of L.

Theorem 3.10. (Nikulin [N1]) *A subgroup $G \subset O(L)$ gives a class of conjugate subgroups determined by some finite algebraic automorphism group of a Kähler K3 surface if and only if the following conditions hold:*

- *$S_G = (L^G)^\perp$ in L is negative definite,*
- *S_G has no elements of square -2.*

We explain the necessity of these conditions. If G acts by algebraic automorphisms on X, then let $T_{X,G} = (H_X)^G$ and $S_{X,G} = T^\perp_{X,G}$. Let Y be the minimal resolution of singularities of X/G. Then Y is a K3 surface, and analysis of the singular points on X/G shows that the resolution of the singular points introduces a negative definite lattice M_Y in H_Y. Then $N_Y = M_Y^\perp$ has signature $(3,k)$ for some k, and we have an injective map $N_Y \to T_{X,G}$. Therefore $S_{X,G}$ is negative definite. This proves the first condition. For the second, suppose $S_{X,G}$ has a root d. Then d or $-d$ is the class of an effective divisor D on X. Then we can take $\sum_{g\in G} g^*D$ which on the one hand is effective, and on the other hand lies in $S_{X,G} \cap T_{X,G} = \{0\}$. This contradiction proves the second necessary condition.

We say that $G \in \mathbf{G}^{\mathbf{alg}}_{\mathbf{K3}}$ has a unique action on the two-dimensional integral cohomology of K3 surfaces, if given any 2 embeddings $i : G \hookrightarrow \text{Aut } X$ and $i' : G \hookrightarrow \text{Aut } X'$, there is an isomorphism $\psi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ which connects the 2 actions, i.e. $i'(g)^* = \psi \cdot i(g)^* \cdot \psi^{-1}$, for any $g \in G$.

Theorem 3.11. (Nikulin [N1]) *Any abelian group $G \in \mathbf{G}^{\mathbf{alg,ab}}_{\mathbf{K3}}$ has a unique action on the two-dimensional integral cohomology of K3 surfaces.*

Theorem 3.12. (Nikulin [N1]) *A K3 surface X admits a finite abelian group G as an algebraic automorphism group iff $G \in \mathbf{G}^{\mathbf{alg,ab}}_{\mathbf{K3}}$ and S_G (well-defined by 3.11) is embedded in S_X as a primitive sublattice.*

The above theorems imply the following theorem, stated in [M1].

Theorem 3.13. *Let X be a K3 surface, let $G \cong \mathbb{Z}/2\mathbb{Z}$ be a subgroup of $O(H^2(X,\mathbb{Z}))$ and let $S_G = (H^2(X,\mathbb{Z})^G)^\perp$. Suppose that*

- *the lattice S_G is negative definite,*
- *no element of S_G has square length -2,*
- *$S_G \subset NS(X)$.*

Then there is a Nikulin involution ι on X and an element $w \in W(X)$ such that $\iota^* = wgw^{-1}$, where g is the generator of G.

The Weyl group shows up in the proof of Theorem 3.12 since we know that G will be conjugate in $O(L)$ to some group of automorphisms of X. Then we can conjugate by a Weyl group element w to make every element of G preserve the Kähler cone. By the Torelli theorem, every wgw^{-1} will then come from an actual automorphism of X.

Now we mention some consequences of existence of a Nikulin involution ι on a K3 surface X. Every Nikulin involution has eight isolated fixed points. The desingularized quotient $Y = \widetilde{X/\{1,\iota\}}$ is a K3 surface. The minimal primitive sublattice of $H^2(Y,\mathbb{Z})$ containing the classes of the eight exceptional curves on Y is isomorphic to N. (The eight roots $c_1,\ldots,c_8$ are the classes of the exceptional curves, and $\frac{1}{2}\sum c_i \in NS(Y) \subset H^2(Y,\mathbb{Z})$ because Y has a double cover branched on the union of the exceptional curves).

The next theorem of Morrison establishes the existence of a Nikulin involution in a special case.

Theorem 3.14. *Let X be a K3 surface such that $E_8(-1)^2 \hookrightarrow NS(X)$. Then there is a Nikulin involution ι on X such that if $\pi : X \to Y$ is the rational quotient map, then*

1. *there is a primitive embedding $N \oplus E_8(-1) \hookrightarrow NS(Y)$,*
2. *π_* induces a Hodge isometry $T_X(2) \to T_Y$.*

Proof. We briefly outline Morrison's proof here, since we will adapt it to other situations. The idea is to define with an involution of $H^2(X,\mathbb{Z})$ such that Theorem 3.13 may be applied to get an honest involution of X.

Let $e_1,\ldots,e_8$ and $f_1,\ldots,f_8$ be generators of the two copies of $E_8(-1)$, which we may choose to be roots, such that the Gram matrix of the $\{e_i\}$ is the same as that of the $\{f_i\}$.

Let $\phi : E_8(-1)^2 \hookrightarrow H^2(X,\mathbb{Z})$ be the given embedding. We define an involution on the cohomology group by

$$g(\phi(e_i)) = \phi(f_i), \quad g(\phi(f_i)) = \phi(e_i),$$

$$g(x) = x, \text{ for all } x \in \phi(E_8(-1)^2)^\perp.$$

This defines the action on all of $H^2(X,\mathbb{Z})$ because $E_8(-1)^2$ is unimodular, and so $\phi(E_8(-1)^2) \oplus \phi(E_8(-1)^2)^\perp = H^2(X,\mathbb{Z})$. We see that $S_G = (H^2(X,\mathbb{Z})^G)^\perp$ is generated by $\{\phi(e_i) - \phi(f_i)\}$, so $S_G \subset NS(X)$ and $S_G \cong E_8(-2)$, ensuring that it is negative definite and has no roots. Therefore, by Theorem 3.13 there is a Nikulin involution ι on X and an element w of the Weyl group of X such that $\iota^* = wgw^{-1}$.

Let $\psi = w\phi$, then we have that $\psi : E_8(-1)^2 \hookrightarrow NS(X)$ is a different embedding (since the Weyl group preserves $NS(X)$). It follows from the definition that i^* switches the $\psi(e_i)$ and $\psi(f_i)$ and is the identity on the orthogonal complement of $\psi(E_8(-1)^2)$. The quotient map $\pi : X \to Y$ defines a push-forward map $\pi_* : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$, as well as a pullback $\pi^* : H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$. It is an easy computation that $\pi_*(\psi(e_i)) = \pi_*(\psi(f_i))$ and $\langle \pi_*(\psi(e_i)), \pi_*(\psi(e_j))\rangle = \langle \psi(e_i), \psi(e_j)\rangle = \langle e_i, e_j\rangle$, and that the two copies of $E_8(-1)$ are therefore taken to one copy of $E_8(-1) \subset NS(Y)$. So we get the lattice $N \oplus E_8(-1)$ inside $NS(Y)$, the lattice N as above being orthogonal to the push-forward of $H^2(X,\mathbb{Z})$.

Also, for any $x \in \psi(E_8(-1)^2)^\perp$, we get $\langle \pi_* x, \pi_* x\rangle = 2\langle x, x\rangle$. Calculating discriminants, we see that π_* is an isometry between $\psi(E_8(-1)^2)^\perp$ and $(N \oplus E_8(-1))^\perp$, both lattices of discriminant 2^6. Since $T_X \subset \psi(E_8(-1)^2)^\perp$, we get the Hodge isometry $T_X(2) \cong T_Y$ induced by π_*. The fact that it is a Hodge isometry follows since the pullback of a global holomorphic $(2,0)$ form on Y certainly gives one on X. □

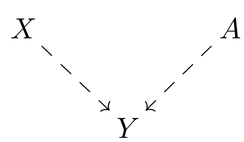
Remark 3.15. The proof shows an equality of discriminant forms

$$q_{N\oplus E_8(-1)} = -q_{\pi_*(\psi(E_8(-1)^2)^\perp)} = (q_{U(2)})^3.$$

3.3 Shioda-Inose structures

Definition 3.16. *We say that X admits a* **Shioda-Inose structure** *if there is a Nikulin involution ι on X with rational quotient map $\pi : X \to Y$ such that Y is a Kummer surface and π_* induces a Hodge isometry $T_X(2) \cong T_Y$.*

If X has a Shioda-Inose structure, let A be the complex torus whose Kummer surface is Y. Then we have a diagram



of rational maps of degree 2, and Hodge isometries $T_X(2) \cong T_Y \cong T_A(2)$, thus inducing a Hodge isometry $T_X \cong T_A$.

Theorem 3.17. (Morrison [M1]) *Let X be an algebraic K3 surface. The following are equivalent:*

1. *X admits a Shioda-Inose structure.*
2. *There exists an abelian surface A and a Hodge isometry $T_X \cong T_A$.*
3. *There is a primitive embedding $T_X \hookrightarrow U^3$.*
4. *There is an embedding $E_8(-1)^2 \hookrightarrow NS(X)$.*

Proof. Again we follow Morrison's proof.

(1) $\implies$ (2): This follows from remark above. A is an abelian surface since the Hodge structure $T_A \cong T_X$ is polarized. In other words, since S_X has signature $(1, 16+r)$, T_X has signature $(2, 3-r)$ and so does T_A. Therefore $T_A^{\perp}$ has a vector of positive norm, which is enough to ensure that there is an ample line bundle on A.

(2) $\implies$ (3): $T_X \cong T_A \hookrightarrow H^2(A, \mathbb{Z}) \cong U^3$ induces a primitive embedding $T_X \hookrightarrow U^3$.

(3) $\implies$ (4): We extend the given primitive embedding $T_X \hookrightarrow U^3$ to an embedding

$$\phi \oplus 0 : T_X \hookrightarrow U^3 \oplus E_8(-1)^2 \cong L$$

Since X is algebraic and $\rho(X) \geq 17$, we have $l(A_{T_X}) \leq \text{rank}(T_X) \leq 5 < 15 \leq \text{rank}(L) - \text{rank}(T_X) - 2$. So there is a unique primitive embedding of T_X into L. So $\phi \oplus 0$ is isomorphic to the usual embedding which comes from $T_X \subset H^2(X, \mathbb{Z}) \to L$ from a marking, and we get by taking orthogonal complements that

$$E_8(-1)^2 \hookrightarrow T_X^{\perp} = NS(X).$$

(4) $\implies$ (1): This is the most important part of the proof for us. By Theorem 3.14, we get a Nikulin involution on X such that the rational quotient map is $\pi : X \to Y$, then π_* induces a Hodge isometry $T_X(2) \cong T_Y$, and $N \oplus E_8(-1) \hookrightarrow NS(Y)$. Since Y is an algebraic $K3$ surface (because it is the quotient of an algebraic K3 surface by an automorphism), and $\rho(Y) \geq 17$, we have by similar reasoning in the previous paragraph that $NS(Y)$ is uniquely determined by its signature and discriminant form. Now we have that the Kummer lattice K has the same signature and discriminant form as $N \oplus E_8(-1)$, which is imbedded primitively in $NS(Y)$. Therefore by Theorem 1.5 K imbeds in $NS(Y)$ primitively, so it follows from Theorem 2.25 that Y is a Kummer surface. ∎

Chapter 4

More on Shioda-Inose structures

4.1 The double cover construction

We would like to start with an algebraic Kummer surface Y and construct a K3 surface X with a Shioda-Inose such that Y is the quotient of X by the Nikulin involution.

Proposition 4.1. *Let Y be an algebraic K3 surface. Then there is an embedding $K \hookrightarrow NS(Y)$ if and only if there is an embedding $N \oplus E_8(-1) \hookrightarrow NS(Y)$.*

Proof. Since Y is an algebraic K3 surface with $\rho(Y) \geq 17$, we have (c.f [M1]) that $NS(Y)$ is uniquely determined by its signature and discriminant form. To see this, we apply Theorem 1.4 where we only have to notice that $NS(Y)$ is indefinite and primitive inside the even unimodular lattice $H^2(Y, \mathbb{Z})$, and its rank is ≥ 17, so $l(A_{NS(Y)}) = l(A_{NS(Y)^\perp}) \leq \mathrm{rank}(NS(Y)^\perp) \leq 5 \leq 17 - 2$. Now the Kummer lattice K and $N \oplus E_8(-1)$ have the same signature $(0, 16)$ and discriminant form $(q_{U(2)})^3$. The proposition follows by using Theorem 1.5. □

Now, from Theorem 2.25 we know that a K3 surface Y is a Kummer surface if and only if there is a primitive embedding $K \hookrightarrow NS(Y)$. We get the following corollary, whose proof is immediate.

Corollary 4.2. *An algebraic surface Y is a Kummer surface if and only if it is a K3 surface and there is an embedding $N \oplus E_8(-1) \hookrightarrow NS(Y)$.*

Let us fix an embedding $j : N \oplus E_8(-1) \hookrightarrow L$. We would like to consider marked $j(N \oplus E_8(-1))$ polarized surfaces. In addition we would like, for technical convenience, to ensure that some specific roots in $j(N \oplus E_8(-1))$ correspond to smooth rational curves on our marked K3 surfaces. Let us choose a system of simple roots $b_1, \ldots, b_8$ for $E_8(-1)$, and eight roots $c_1, \ldots, c_8$ of N such that $1/2 \sum c_i \in N$. We shall let the c_i be the simple roots for N. One can easily check that they are indecomposable: for any c_i, there do not exist roots z and w of N such that $z + w = c_i$.

Now we consider marked $j(N \oplus E_8(-1))$-polarized surfaces with rational curves in $T = \{b_1, \ldots, b_8, c_1, \ldots, c_8\}$.

Theorem 4.3. *Let Y be an algebraic K3 surface containing rational curves $B_1, \ldots, B_8$, $C_1, \ldots, C_8$ such that the classes b_i of B_i are the standard simple root vectors which generate a copy of the E_8 lattice, the classes c_i of C_i satisfy $c_i.c_j = -2\delta_{ij}$, and $\frac{1}{2}\sum c_i \in NS(X)$, and such that $b_i \cdot c_j = 0$ for all i, j. This data canonically gives rise to a K3 surface X with a Shioda-Inose structure such that the quotient map under the involution is Y.*

Proof. Take the double cover $\pi : X' \to Y$ branched on the divisor $\bigsqcup C_i$. This is possible since $\frac{1}{2}\sum C_i$ exists in the Néron-Severi group of Y. Then, on X, we have that the curves $D_i = \pi^{-1}(C_i)$ satisfy $D_i^2 = -1$, i.e. they are exceptional curves. Blowing them down, we get the rational map $X \to Y$, where X has a non-vanishing $(2,0)$ form pulled back from Y. Also, the Euler characteristic of X is $2(24 - 2 \cdot 8) + 8 = 24$. Therefore X is a K3 surface. It inherits an involution from the involution on X' which interchanges the two sheets of the double cover $X' \to Y$. The involution on X' fixes the (2,0) form pulled back from Y, and therefore so does the involution on X. This provides the Nikulin involution ι. We need some additional information before we can show the Shioda-Inose structure on X.

Lemma 4.4. *In the situation above, there is an embedding $E_8(-1)^2 \hookrightarrow NS(X)$ such that the two copies of $E_8(-1)$ are interchanged by the Nikulin involution ι.*

Proof. For each rational curve B_i, we consider its pre-image on X, D_i. We have that $D_i^2 = \deg(\pi)B_i^2 = -4$. Note that the intersection numbers don't change from X' to X, because the B_i are orthogonal to the branch locus $\sqcup C_i$. This shows that D_i can't be irreducible, since otherwise from the adjunction formula, the arithmetic genus $g(D_i) = 1 + D_i(D_i + K_X)/2 = -1$ and the geometric genus can only be smaller. Since $X \to Y$ is 2 to 1 away from the branch locus, we see that D_i is a union of two rational curves E_i and F_i, which are interchanged by the Nikulin involution. Note that $(E_i + F_i) \cdot (E_j + F_j) = 2B_i \cdot B_j$. If $B_i \cdot B_j = 0$, then we get $E_i \cdot E_j = E_i \cdot F_j = F_i \cdot E_j = F_i \cdot F_j = 0$ because all four terms are nonnegative (no two of these curves coincide, by lemma 2.20). On the other hand, if $B_i \cdot B_j = 1$ we see that we must have either $E_i \cdot E_j = 1$ and $F_i \cdot F_j = \iota^*(E_i) \cdot \iota^*(E_j) = 1$ or $E_i \cdot F_j = 1$ and $F_i \cdot E_j = 1$. It is clear that we can make a choice of the E_i so $E_i \cdot E_j = F_i \cdot F_j = B_i \cdot B_j$ for all i, j. This proves the lemma. □

We come back to the proof of the theorem. All that remains to be shown is that π_* induces an isomorphism $T_X(2) \cong T_Y$. It will be a Hodge isometry because the $(2,0)$ form on X is pulled back from Y. The rest of the argument is identical to [SI] - we merely reproduce it here for convenience.

Let Γ_1 and Γ_2 be the two copies of $E_8(-1)$ constructed above. The lattice $\Gamma_1 \oplus \Gamma_2$ is unimodular, so we have the orthogonal decomposition

$$H^2(X, \mathbb{Z}) = \Gamma_1 \oplus \Gamma_2 \oplus Q.$$

where Q is an even unimodular lattice of signature $(3, 3)$ (and is therefore isometric to U^3, but we shall not need this fact).

Also, let Γ denote the copy of $E_8(-1) \subset NS(Y)$ that we started with. Clearly, we have

$$T_X \subset Q,\ \pi_*(\Gamma_1) = \pi_*(\Gamma_2) = \Gamma.$$

Now π_* maps $H^2(X, \mathbb{Z})$ into the orthogonal complement $N^\perp$ of N in $H^2(Y, \mathbb{Z})$. We have a natural map $\pi^* : N^\perp \to H^2(X, \mathbb{Z})$ such that

$$(\pi^* y_1 \cdot \pi^* y_2) = 2(y_1 \cdot y_2) \qquad (y_1, y_2 \in N^\perp),$$
$$\pi^* \pi_* x = x + \iota^* x \qquad (x \in H^2(X, \mathbb{Z})).$$

We also have $\pi_*(S_X) \subset S_Y$ and $\pi^*(S_Y \cap N^\perp) \subset S_X$ (from orthogonality to the $(2, 0)$ forms), and that $\iota^*(\Gamma_1) = \Gamma_2$ and $\iota^*(\Gamma_2) = \Gamma_1$.

Lemma 4.5. *The action of ι^* on Q is the identity. The map π_* induces a bijection of Q onto its image in $H^2(Y, \mathbb{Z})$, with $(\pi_* x_1 \cdot \pi_* x_2) = 2(x_1 \cdot x_2)$ for $x_1, x_2 \in Q$.*

Proof. The involution ι has 8 fixed points (namely, the curves D_i blown down). Using the Lefschetz fixed point formula, we get that

$$8 = 1 + 0 + \mathrm{Tr}(\iota^* | H^2(X, \mathbb{Z})) + 0 + 1.$$

so that $\mathrm{Tr}(\iota^* | H^2(X, \mathbb{Z})) = 6$. Now $H^2(X, \mathbb{Z}) = \Gamma_1 \oplus \Gamma_2 \oplus Q$ and $\iota^*(\Gamma_1) = \Gamma_2$ gives that $\mathrm{Tr}(\iota^* | \Gamma_1 \oplus \Gamma_2) = 0$. This shows $\mathrm{Tr}(\iota^* | Q) = 6$. But ι^* has eigenvalues ± 1 and Q has rank 6, so it follows that ι^* is the identity on Q. So $\pi^* \pi_* x = 2x$ for $x \in Q$, and it follows that $\pi_* | Q$ is injective. Finally,

$$(\pi_* x_1 \cdot \pi_* x_2) = \frac{1}{2}(\pi^* \pi_* x_1 \cdot \pi^* \pi_* x_2) = \frac{1}{2}(2x_1 \cdot 2x_2) = 2(x_1 \cdot x_2).$$

□

Now notice that we have $\pi_*(H^2(X, \mathbb{Z})) = \pi_*(Q \oplus \Gamma_1 \oplus \Gamma_2) = \pi_* Q \oplus \Gamma$. To see that $\pi_* Q$ and Γ are orthogonal, let $y_1 = \pi_* x_1$ and $y_2 = \pi_* x_2$ where $x_1 \in Q$, $y_2 \in \Gamma = \pi_* \Gamma_1$ and $x_2 \in \Gamma_1$ a pre-image of y_2. Then

$$(y_1 \cdot y_2) = \frac{1}{2}(\pi^* y_1 \cdot \pi^* y_2) = \frac{1}{2}(2x_1 \cdot (x_2 + \iota^* x_2)) = 0.$$

Next, let we claim that N is primitive in $H^2(Y,\mathbb{Z})$. Recall that N is generated by $C_1,\dots,C_8$ and $\frac{1}{2}\sum_{i=1}^{8} C_i$. Let $y=\sum a_iC_i$ ($a_i\in\mathbb{Q}$) be in $H^2(Y,\mathbb{Z})$. Because $(y\cdot C_i)\in\mathbb{Z}$ we must have that a_i are half-integers. We may assume they are in $\{0,\frac{1}{2}\}$. Let the number of non-zero a_i be m. From the fact that $y^2=-m/2$ is even, we see that $m=4$ or 8. Since $m=8$ corresponds to $\frac{1}{2}\sum_{i=1}^{8} C_i$ which is already in N, we need to rule out $m=4$. But if $m=4$ we see that $y^2=-2$, so that a divisor G with class y must have that G or $-G$ is effective, by Riemann-Roch. Since $2G=\sum_{\mu=1}^{4}(2a_{i_\mu})C_{i_\mu}$ is effective, we must have G effective. Now $G\cdot C_{i_\mu}=y\cdot C_{i_\mu}=-1$, so that C_{i_μ} is a component of G, i.e. $0\le x-\sum_\mu C_{i_\mu}=-\frac{1}{2}\sum_\mu C_{i_\mu}$, a contradiction.

Therefore we have $\det(N)=\det(\oplus\mathbb{Z}C_i)/4=2^8/4=2^6$ and $\det(N^\perp)=2^6$ as well.

Now we have $\pi_*(H^2(X,\mathbb{Z}))\subset N^\perp$. But also

$$\det\pi_*(H^2(X,\mathbb{Z}))=\det(\pi_*(Q)\oplus\Gamma))=\det(\pi_*Q)=2^6\det(Q)=2^6=\det(N^\perp).$$

Therefore $\pi_*H^2(X,\mathbb{Z})=N^\perp$. Finally we want to show $\pi_*T_X=T_Y$. Since $T_X\subset Q$ and π_* acts on Q by scaling the product by 2, we will be done. First, we need to see $\pi_*T_X\subset T_Y$, i.e. for $x\in T_X$ and $y\in S_Y$ we have $(\pi_*x\cdot y)=0$. This is clear if $y\in N$ since N is orthogonal to all of $\pi_*H^2(X,\mathbb{Z})$. On the other hand if $y\in N^\perp\cap S_Y$, we have that π^*y is defined and in S_X. Therefore

$$(\pi_*x\cdot y)=\frac{1}{2}(\pi^*\pi_*x\cdot\pi^*y)=\frac{1}{2}(2x\cdot\pi^*y)=0.$$

Therefore, π_*T_X is orthogonal to N and to $N^\perp\cap S_Y$ and since the rational span of these two sublattices is all of $S_Y\otimes\mathbb{Q}$, we have that $\pi_*T_Y\subset S_Y^\perp=T_Y$.

Next, we need to see $\pi_*T_X\supset T_Y$. Let $y\in T_Y\subset N^\perp=\pi_*H^2(X,\mathbb{Z})$, so that $y=\pi_*z$ (since $N\subset S_Y$) and $x\in S_X$. Then

$$\begin{aligned}\pi^*y\cdot x &= \pi^*\pi_*z\cdot x=(z+\iota^*z)\cdot x=\frac{1}{2}(z+\iota^*z)(x+\iota^*x)=\frac{1}{2}\pi^*\pi_*z\cdot\pi^*\pi_*x\\ &= \frac{1}{2}2(\pi_*z\cdot\pi_*x)=(y\cdot\pi_*x)=0.\end{aligned}$$

This shows $\pi^*T_Y\subset T_X$. Next, we have $T_Y=S_Y^\perp\subset N^\perp=\pi_*H^2(X,\mathbb{Z})=\pi_*Q\oplus\Gamma$. But $T_Y\perp\Gamma$, so that $T_Y\subset\pi_*Q$. For any $y\in T_Y$, we can find $x\in Q$ wit $y=\pi_*x$. Then $\pi^*y=2x\in T_X$ and since T_X is primitive in $H^2(X,\mathbb{Z})$ we have $x\in T_X$. Therefore $T_Y\subset\pi_*T_X$.

This completes the proof of Theorem 4.3. □

4.2 Relation between periods

In this section, we wish to elucidate the relation between the periods of a K3 surface X which has a Shioda-Inose structure, and the quotient (Kummer) surface Y.

We will consider X as a marked $E_8(-1)^2 + \mathbb{Z}u$ polarized K3 surface with rational curves in T, which is a certain set of 16 simple roots of $E_8(-1)^2$. Here u will be the class of an ample line bundle or polarization on X. The two copies of $E_8(-1)$ will be switched under the Nikulin involution, which will fix u (this avoids the conjugation by the Weyl group element).

On the other hand, the quotient Y will be considered naturally as an $N \oplus E_8(-1) + \mathbb{Z}v$ polarized marked K3 surface. Here v will be the class of an ample line bundle on Y, and the N naturally arises from the blowup of the eight singular points on the quotient, whereas $E_8(-1)$ comes from the image of the $E_8(-1)^2$ in $NS(X)$.

It turns out that we can define the marking on the double cover X of Y unambiguously, to make the following diagram compatible.

$$\begin{array}{ccccc}
H^2(X,\mathbb{Z}) & \xrightarrow{\phi} & L & = & E_8(-1) \oplus E_8(-1) \oplus U^3 \\
\downarrow{\scriptstyle \pi_*} & & & & \downarrow{\scriptstyle \xi} \\
H^2(Y,\mathbb{Z}) & \xrightarrow{\rho} & L & \longleftarrow & E_8(-1) \oplus N \oplus U(2)^3
\end{array}$$

That is, there is an underlying map of lattices $\xi : L \to L$ such that the two copies of $E_8(-1)$ go (via the identity) to the single copy of $E_8(-1)$ and U^3 goes to $U(2)^3$ again via the map which is the identity on the underlying abelian group (and scaled the form by 2). For any marked lattice-polarized K3 surfaces X and Y as above, the copies of $E_8(-1)$ on $NS(X)$ both go to the $E_8(-1)$ in $NS(Y)$ under π_*, and the orthogonal complement U^3 goes to the orthogonal complement of $N \oplus E_8(-1)$ (which is $\cong U(2)^3$) under π_*.

The holomorphic 2-form on X is pulled back from that on Y. So, the period points of X and Y considered as elements of $\Omega \in \mathbb{P}(L \otimes \mathbb{C})$ are the same. Since X and Y are naturally lattice-polarized, the period point of X may be naturally considered as an element of

$$\Omega_\mu = \{\omega \in \mathbb{P}(M_\mu^\perp \otimes \mathbb{C}) \,|\, \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}$$

with $M_\mu = E_8(-1)^2 + \mathbb{Z}u$ and the orthogonal complement is taken in L, as usual.

Similarly, the period point of Y falls in

$$\Omega_\nu = \{\omega \in \mathbb{P}(M_\nu^\perp \otimes \mathbb{C}) \,|\, \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}$$

with $M_\nu = E_8(-1) \oplus N + \mathbb{Z}v$.

The map from the period point of Y to that of its double cover X is a linear invertible map of (open subsets of) projective spaces of the same dimension, i.e. it is induced by a linear isomorphism of vector spaces.

4.3 Moduli space of K3 surfaces with Shioda-Inose structure

Let us choose once and for all a fixed system of simple roots $e_1, \ldots, e_8$ of $E_8(-1)$ whose dot products give the Dynkin diagram of E_8. Let p be the half the sum of the positive roots of $E_8(-1)$ (i.e. all the roots which are non-negative linear combinations of the simple roots). Now, $L = E_8(-1)^2 \oplus U^3$. For convenience of notation (to distinguish between the two copies of $E_8(-1)$), we will sometimes write this as $L = \Gamma_1 \oplus \Gamma_2 \oplus \Xi$, where $\Gamma_1 = \Gamma_2 = E_8(-1)$ and $\Xi = U^3$.

Let us choose vectors $c_i, i = \ldots 8$ of N such that $\langle c_i, c_j \rangle = -2\delta_{ij}$ and N is generated by $\{c_i\}_{i=1}^8 \cup \{\frac{1}{2} \sum c_i\}$.

Recall that the Nikulin lattice N embeds uniquely in $E_8(-1) \oplus U^3$ (up to automorphisms of $E_8(-1) \oplus U^3$), and also that there is a unique embedding of $E_8(-1) \oplus N$ into L. So let us write $E_8(-1) \oplus N \subset E_8(-1) \oplus E_8(-1) \oplus U^3 = L$, which is the the identity on the first factors of both lattices, and a fixed embedding of N into $E_8(-1) \oplus U^3$. We will alternatively write $L \supset \Gamma \oplus \Upsilon \oplus \Theta$, with $\Gamma = E_8(-1), \Upsilon = N$ and $\Theta = U(2)^3$. Here we have used that the orthogonal complement to N in $E_8(-1) \oplus U^3$ is isomorphic to $U(2)^3$, and chosen a fixed isomorphism.

Let g_i be the copy of e_i in Γ_1, and h_i be the copy of e_i in Γ_2.

Let f_i be the copy of e_i in Γ and d_i be the copy of c_i in Υ.

Let μ denote the following data: let $u = (u_1, u_2, u_3) \in L = \Gamma_1 \oplus \Gamma_2 \oplus \Xi$ be such that

1. It is symmetric with respect to Γ_1 and Γ_2 and equal to negative half the sum of the positive roots. $u_1 = u_2 = -p \in E_8(-1)$.
2. The vector $u_3 \in \Xi$ is primitive.
3. It has positive norm: $\langle u, u \rangle > 0$.

Lemma 4.6. *The data μ satisfies the following properties.*

1. *The roots g_i and h_i, for $1 \leq i \leq 8$ are all positive with respect to the polarization defined by u. That is, $g_i \cdot u > 0$ and $h_i \cdot u > 0$.*

2. *The roots g_i and h_i, $1 \leq i \leq 8$, are all simple with respect to the polarization on $\mathcal{M}_\mu = \Gamma_1 \oplus \Gamma_2 + \mathbb{Z}u$ defined by u, i.e. there are no $z_1, z_2 \in \mathcal{M}_\mu$ such that $u \cdot z_i > 0$ and $g_i = z_1 + z_2$ (similarly for h_i).*

Here, by saying that a root δ is simple with respect to the polarization defined by u, we mean that we cannot write $\delta = \sum_{i=1}^{r} \lambda_i \delta_i$, where $r \geq 2$, and for all i, $\lambda_i > 0$ and $\delta_i^2 \geq -2$ with $\delta_i \cdot u > 0$. In particular, the above lemma implies that for any marked K3 surface with which has an ample line bundle with class u, the g_i and h_i are actually classes of *irreducible* rational curves on X.

Proof. We know that half the sum of the positive roots p satisfies $p \cdot e_i = -1$ for all the simple roots e_i. Therefore we get $u \cdot g_i = u_1 \cdot g_i = 1$ and similarly for the $u \cdot h_i$. This proves the first part.

Suppose we had $g_i = z_1 + z_2$. Then $1 = u \cdot z_1 + u \cdot z_2$. But we assumed $u \cdot z_i > 0$ and we also know $u \cdot z_i \in \mathbb{Z}$, since the lattice M_μ is integral. Since we cannot have two positive integers adding to 1, we have a contradiction. Note that in fact, the proof shows that g cannot decompose into positive classes in L, not just M_μ. □

The lemma justifies the following definition.

Definition 4.7. *Let $\mathcal{M}_\mu$ be the space of marked $\Gamma_1 \oplus \Gamma_2$-polarized K3 surfaces with rational curves in $\{g_i, h_i | 1 \leq i \leq 8\}$, and such that u is the class of an ample line bundle on X (via the marking), up to isomorphism.*

Explicitly, let

$$\mathcal{M}_\mu = \{(X, \phi) \mid \quad \begin{array}{l} \phi : H^2(X, \mathbb{Z}) \to L \text{ such that } \phi^{-1}(\Gamma_1 \oplus \Gamma_2) \subset NS(X) \\ \phi^{-1}(u) \text{ is the class of an ample line bundle on } X \\ \phi^{-1}(g_i), \phi^{-1}(h_i) \text{ are classes of smooth rational curves on } X\}/\sim \end{array}$$

Here, an isomorphism of (X, ϕ) with (X', ϕ') consists of a map $f : X' \to X$ such that $\phi = \phi' \circ f^*$.

Let X be a K3 surface with sixteen smooth rational curves $G_i, H_i, 1 \leq i \leq 8$, on X, and an ample line bundle $\mathcal{L}$ with intersection properties as above (namely, the G_i intersection numbers are the negative of those in the Cartan matrix for E_8, similarly for the H_i, and $\mathcal{L}$ corresponds to a divisor D such that $D \cdot G_i = 1, D \cdot H_i = 1$, $D^2 > 0$, and furthermore $D + p(G_i) + p(H_i)$ is primitive).

Let u_3 be any vector in $\Xi \cong U^3$ with norm $D^2 - 2p^2$ (recall that $p^2 < 0$ since $p \in E_8(-1)$) and let $u = (-p, -p, u_3) \in L$. Then we can put a marking ϕ on X so that (X, ϕ) is an element of $\mathcal{M}_\mu$. The proof is as follows: we map the classes of the G_i to g_i, the classes of the H_i to h_i. The lattice $H^2(X,\mathbb{Z})$ decomposes as $\sum \mathbb{Z}[G_i] \oplus \sum \mathbb{Z}[H_i] \oplus K$, where $K = (\sum \mathbb{Z}[G_i] \oplus \sum \mathbb{Z}[H_i])^\perp \cong U^3$. We have a decomposition $D = D_1 + D_2 + D_3$. Now, we need to find a map of $K \cong U^3$ to $\Xi \cong U^3$ such that the vector D_3 maps to u_3. But this is possible, because the the norm of D_3 is by construction the same as the norm of u_3, and the automorphism group of U^3 is transitive on primitive vectors of a fixed norm. Recall that M_μ is the lattice $\Gamma_1 \oplus \Gamma_2 + \mathbb{Z}u = \sum \mathbb{Z}g_i \oplus \sum \mathbb{Z}h_i + \mathbb{Z}u$. We let G_μ be the subgroup of $O(L)$ that fixes M_μ pointwise. Now, if we have any two markings ϕ and ϕ' of the given X with the rational curves and ample line bundle, then $\phi' \circ \phi^{-1}$ is an element of G_μ. We see that $\mathcal{M}_\mu / G_\mu$ is a coarse moduli space for X with the above data.

Now we would like to define an appropriate moduli spaces for the K3 surfaces Y which are obtained from the above K3 surfaces as a quotient by a Nikulin involution. We will define an ample line bundle on Y whose class, through the marking, will be related to the class of the ample line bundle on X.

Thus, let ν be the following data. Letting $u = (u_1, u_2, u_3)$ be fixed as above, such that the norm of u is $B \geq 8$. We will let $k \geq 14$ be a positive integer and let $v = v_1 + v_2 + v_3 \in \Gamma \oplus \Upsilon \oplus \Theta$ be such that

1. $v_1 = -2kp \in E_8(-1) \cong \Gamma$.
2. $v_2 = -\frac{1}{2}\sum d_i \in \Upsilon$.
3. $v_3 = ku_3(2) \in U(2)^3 \cong \Theta$.

We have a similar lemma.

Lemma 4.8. *The data ν satisfies the following condition.*

1. *It has positive norm: $\langle v, v\rangle > 0$.*
2. *The roots $f_i, 1 \leq i \leq 8$, are positive for the polarization defined by v, i.e. $f_i \cdot v > 0$. Similarly, the $d_i, 1 \leq i \leq 8$, are positive.*
3. *The roots f_i and d_i are all simple with respect to the polarization on the saturation M_ν^{sat} of $M_\nu = \Gamma \oplus \Upsilon + \mathbb{Z}v$ defined by v.*

Proof. For the first part, we have

$$\begin{aligned} v^2 &= (-2kp)^2 + (-\frac{1}{2}\sum d_i)^2 + (ku_3(2))^2 \\ &= 2k^2(2p^2 + u_3^2) - 4 \\ &= 2k^2u^2 - 4 \end{aligned}$$

$$= \quad 2k^2B - 4 > 0$$

since $B \geq 8$ and $k \geq 14$.

Next, we have $v \cdot f_i = (-2kp) \cdot e_i = 2k > 0$ and $v \cdot d_i = (-\frac{1}{2}\sum d_i) \cdot d_i = 1 > 0$. Therefore the roots mentioned are all positive.

The third statement is harder to show. To show that the d_i are simple with respect to the polarization defined by v, the same proof works as for Lemma 4.6, since $v \cdot d_i = 1$, which cannot be a sum of two positive integers. Now assume f_i is not simple. We use the lemmas 4.20 and 4.21 of a later section, which translate to the following statements in our context. Let M_ν^{sat} be the saturation of M_ν in L, that is, the smallest primitive lattice of L containing M_ν. It equals $L \cap (M_\nu \otimes \mathbb{Q})$, and is contained in $\Gamma \oplus \frac{1}{2}\Upsilon \oplus \frac{1}{2}\mathbb{Z}u_3(2)$ (since N^*/N is 2-elementary, and $u_3(2)$ is primitive in Θ).

1. The root δ is simple if and only if there is no decomposition $\delta = z_1 + z_2$ in M_ν^{sat} with $z_1^2 \geq -2, \delta \cdot z_1 < 0$, and $0 < v \cdot z_1 < u \cdot \delta$.
2. If x is any element of the signature $(1, 16)$ lattice M_ν^{sat}, we have $(x^2)(v^2) < (v \cdot x)^2$.

Accordingly, let us assume we have $f_i = z_1 + z_2$, where now we have $z_1^2 \geq -2$, $f_i \cdot z_1 < 0$ and $0 < z_1 \cdot v < f_i \cdot v = 2k$. Then, since z_1 is in the smallest primitive sublattice of L lattice containing Γ, Υ and $v = v_1 + v_2 + v_3$, we can write $z_1 = a + b + Mu_3(2)$, where $a \in \Gamma \cong E_8(-1), b \in \frac{1}{2}\Upsilon \cong \frac{1}{2}N$, and $M \in \frac{1}{2}\mathbb{Z}$. We then have

$$0 < z_1 \cdot v = a \cdot (-2kp) + b \cdot \Big(-\frac{1}{2}\sum d_i\Big) + 2kMu_3^2 < 2k$$

Therefore

$$\left| a \cdot (-2kp) + b \cdot (-\frac{1}{2}\sum d_i) \right| > 2k|M|u_3^2 - 2k$$

We also have

$$z_1^2 < \frac{(z_1 \cdot v)^2}{v^2} < \frac{(2k)^2}{v^2} = \frac{4k^2}{2k^2(u_3^2 + 2p^2) - 4} = \frac{4k^2}{2k^2B - 16} = \frac{2}{B - 2/k^2}$$

which is less than 2 since $k \geq 14$ and $B \geq 10$. Therefore, since the lattice L is even, we have $z_1^2 = 0$ or -2. That is,

$$a^2 + b^2 + 2M^2u_3^2 = 0 \text{ or } -2$$

Therefore

$$|a^2 + b^2| \leq 2M^2u_3^2 + 2$$

Finally, since $\Gamma \oplus \Upsilon$ is negative definite, we have

$$\left| a \cdot (-2kp) + b \cdot \Big(-\frac{1}{2}\sum d_i\Big) \right|^2 \leq |a^2 + b^2| \cdot \left| (2kp)^2 + \Big(\frac{1}{2}\sum d_i\Big)^2 \right|$$

Therefore, by the above inequalities, we get

$$(2k|M|u_3^2 - 2k)^2 < (2M^2u_3^2 + 2)|(4k^2p^2 + 4)|$$

and using $p^2 = -620$ for the $E_8(-1)$ lattice, and $u_3^2 = -2p^2 + B = 1240 + B$, we have

$$(2k|M|(1240 + B) - 2k)^2 < (2M^2(1240 + B) + 2)(4k^2 \cdot 620 + 16)$$

We claim that if $M \neq 0$, this inequality fails. Without loss of generality, $M > 0$, and we divide both sides by $(2kM(B + 1240))^2$ to get

$$\left(1 - \frac{1}{M(B + 1240)}\right)^2 < \left(1 + \frac{1}{M^2(B + 1240)}\right)\left(\frac{1240}{B + 1240} + \frac{8}{4k^2(B + 1240)}\right)$$

using the inequalities $k \geq 14, M \geq 1/2, B \geq 8$, we get a contradiction.

Now we deal with the case that $M = 0$. Then we can write $z_1 = a + b$ with the stricter constraint $a \in \Gamma, b \in \Upsilon$. We have $-2 \leq z_1^2 = a^2 + b^2$, and we also know that every non-zero vector of $\Gamma \cong E_8(-1)$ and $\Upsilon \cong N$ has square at most -2. Therefore $a = 0$ or $b = 0$ is forced. If $b = 0$ then we have the decomposition $f_i = z_1 + z_2$ in Γ which contradicts that f_i is a simple root. If $a = 0$ then $z_1 \in \Upsilon$ and $f_i \cdot z_i = 0$, whereas we assumed $f_i \cdot z_1 < 0$. This contradiction completes the proof of the lemma. □

Let $\mathcal{M}_\nu$ be the space of marked $\Gamma \oplus \Upsilon$-polarized K3 surfaces with condition T for the f_i and d_i and such that v is the class of an ample line bundle on X, up to isomorphism. Explicitly, we let

$$\begin{aligned} \mathcal{M}_\nu = \{(Y, \rho)\,| \quad & \rho : H^2(X, \mathbb{Z}) \to L \text{ such that } \rho^{-1}(\Gamma \oplus \Upsilon) \subset NS(X) \\ & \rho^{-1}(v) \text{ is the class of an ample line bundle on } X \\ & \rho^{-1}(f_i), \rho^{-1}(d_i) \text{ are classes of smooth rational curves on } X\}/\sim \end{aligned}$$

Let Y be a K3 surface with sixteen smooth rational curves $F_i, D_i, 1 \leq i \leq 8$, on Y, a line bundle $\mathcal{E}$ and an ample line bundle $\mathcal{L}$ with intersection properties as above (namely, the F_i intersection numbers are the negative of those in the Cartan matrix for E_8, the D_i are all orthogonal to each other and to the F_i, we have $\mathcal{E}^2 \cong O_X(D_1 + \ldots + D_8)$, and $\mathcal{L}$ corresponds to a divisor D such that $D \cdot F_i = 2k, D \cdot D_i = 1$, $D^2 = 2k^2B - 4 > 0$, and furthermore, $D + 2kp(F_i) + \frac{1}{2}\sum D_i$ is k times a primitive divisor).

Let u_3 be a primitive vector in U^3 with norm $B - 2p^2$, and let $v = -2kp - \frac{1}{2}\sum d_i + ku_3(2)$ as above. Then we claim that there exists a marking ρ on Y such that (Y, ρ) is an element of $\mathcal{M}_\nu$. The proof is as follows: we map the classes of the F_i to f_i. Now, let $J = (\sum \mathbb{Z}[F_i] \oplus \sum \mathbb{Z}[D_i])^\perp \cong U(2)^3$. Let $D_3 = kD_3'$, where D_3' is primitive in $NS(X)$ and therefore in J. So we can choose an isomorphism $J \cong U(2)^3$ to take D_3' to $u_3(2)$.

The lattice $U(2)^3$ has a unique primitive embedding in $E_8(-1) \oplus U^3$, up to automorphisms of $E_8(-1) \oplus U^3$. Take an identification $(\sum \mathbb{Z}[F_i])^\perp \cong E_8(-1) \oplus U^3 \cong \Gamma^\perp$, and modify the embedding to make $J \to \Theta \cong U(2)^3$ be the above isomorphism.

Now we have that the D_i map to roots of Υ, namely the $\pm d_i$. If D_i maps to some $-d_{\tau(i)}$, we can change the marking by composing with a Weyl reflection in $d_{\tau(i)}$ to make D_i map to $+d_{\tau(i)}$, the images of the other D_j being unchanged, and not affecting the image of any vector in $\sum \mathbb{Z}[F_i]$ or J. Thus we may assume that the D_i map to some $d_{\tau(i)}$. We see that we have obtained an element of $\mathcal{M}_\nu$.

If we have two markings ρ and ρ' of the given Y with the specified rational curves and ample line bundle, then we see that $\rho' \circ \rho^{-1}$ fixes the f_i and acts by some permutation of the d_i (because we did not specify above that $D_i \mapsto d_i$ under the marking, only to some permutation $d_{\tau(i)}$). Also, we have that $\rho' \circ \rho^{-1}$ fixes the polarization class v. Therefore we have that $\rho' \circ \rho^{-1} \in G_0$ which is defined as follows.

$$G_0 = \{g \in O(L) \,|\, gx = x \text{ for } x \in \Gamma + \mathbb{Z}v, g(\Upsilon) = \Upsilon\}$$

Note that $gv = v$ guarantees that gd_i is a positive root of Γ. We see that $\mathcal{M}_\nu / G_0$ is a coarse moduli space for Y with the above data.

4.4 Map between moduli spaces

The main theorem of this section describes an identification of the moduli spaces corresponding to the reciprocal constructions of quotient by the Nikulin involution and taking the double cover. The data μ and ν of the polarization vectors u and v are related by a simple lattice theoretic condition, which stems from the construction of an ample line bundle on the quotient K3 surface.

For $x \in U^3$, we denote by $x(2)$ the same vector in $U(2)^3$; likewise, for $y \in U(2)^3$ we denote by $y(\frac{1}{2})$ the same vector in U^3.

Theorem 4.9. *Let $k \geq 14$ be a positive integer. Let μ, ν be as above, i.e. such that $u = (u_1, u_2, u_3)$ with $u_1 = u_2 = -p$, and $u_3 \in U^3$ primitive such that $u^2 = B \geq 10$, and $v = v_1 + v_2 + v_3$ with $v_1 = -2kp, v_2 = -\frac{1}{2}\sum d_i$ and $v_3 = ku_3(2)$ for some integer $k \geq 14$. Then there is a holomorphic isomorphism of complex spaces $\eta_{\nu,\mu} : \mathcal{M}_\nu \to \mathcal{M}_\mu$.*

Proof. The idea of the proof is as follows: the map $\mathcal{M}_\nu \to \mathcal{M}_\mu$ is given by the double cover construction. Injectivity will follow from the Torelli theorem, whereas surjectivity will be proved using the Shioda-Inose structure to quotient by a Nikulin involution.

In detail, we proceed as follows.

Step 1: Construction of the double cover and marking on it.

Suppose we are given a pair $(Y,\rho) \in \mathcal{M}_\nu$ of a K3 surface Y with a marking $\rho : H^2(Y,\mathbb{Z}) \to L$ satisfying the properties above. Then we let $D_1,\dots,D_8$ be the smooth rational curves whose classes are given by $\rho^{-1}(d_i)$. These are unique by Lemma 2.20. Let $F_1,\dots,F_8$ be the smooth rational curves whose classes are given by $\rho^{-1}(f_i)$. Theorem 4.3 shows that the existence of X which is a K3 surface with a Shioda-Inose structure, and which is a double cover of Y branched on $\cup C_i$ (with 8 exceptional curves blown down). We get from the construction smooth rational curves G_i and H_i, $i = 1,\dots,8$ such that the Nikulin involution on X interchanges G_i and H_i, the intersection numbers are compatible $G_i.G_j = H_i.H_j = F_i.F_j$ and such that the image of G_i or H_i under the quotient map is F_i. We define $\phi([G_i]) = g_i$ and similarly $\phi([H_i]) = h_i$ (we are making a choice here, but we shall soon see that the two possibilities give rise to isomorphic marked K3 surfaces). This is done to make the following diagram commute

$$\begin{array}{ccccccc} H^2(X,\mathbb{Z}) & \xrightarrow{\phi} & L & = & \Gamma_1 \oplus \Gamma_2 \oplus \Xi & = & E_8(-1) \oplus E_8(-1) \oplus U^3 \\ \downarrow{\scriptstyle \pi_*} & & & & & & \downarrow{\scriptstyle \xi} \\ H^2(Y,\mathbb{Z}) & \xrightarrow{\rho} & L & \longleftarrow & \Gamma \oplus \Upsilon \oplus \Theta & = & E_8(-1) \oplus N \oplus U(2)^3 \end{array}$$

Also, π_* defines an isomorphism of $(\sum \mathbb{Z}[G_i] \oplus \sum \mathbb{Z}[H_i])^\perp(2)$ (the orthogonal complement is taken inside $H^2(X,\mathbb{Z})$) with $(\sum \mathbb{Z}[F_i] \oplus \sum \mathbb{Z}[D_i])^\perp$ (the orthogonal complement is taken inside $H^2(Y,\mathbb{Z})$). Thus, we get

$$(\sum \mathbb{Z}[G_i] \oplus \sum \mathbb{Z}[H_i])^\perp(2) \xrightarrow{\pi_*} (\sum \mathbb{Z}[F_i] \oplus \sum \mathbb{Z}[D_i])^\perp \xrightarrow{\rho} \Theta = U(2)^3 = \Xi(2).$$

This defines by composition the map ϕ on the orthogonal complement of $\sum \mathbb{Z}[G_i] \oplus \sum \mathbb{Z}[H_i]$. Since $\sum \mathbb{Z}[G_i] \oplus \sum \mathbb{Z}[H_i] \cong E_8(-1)^2$ is a unimodular lattice, it and its orthogonal complement generate all of $H^2(X,\mathbb{Z}) \cong L$, so we have defined the map ϕ on all of $H^2(X,\mathbb{Z})$ unambiguously.

Step 2: Polarization

We need to show that $\phi^{-1}(u)$ defines a polarization of the K3 surface X .

Lemma 4.10. *$\phi^{-1}(u)$ is the class of an ample line bundle on X.*

Proof. For this, we will use the ampleness criterion of Nakai-Moishezon, which states that a divisor D on a surface X is ample if and only if $D^2 > 0$ and $D.C > 0$ for all irreducible curves C on X. Via the markings ϕ and ρ, we will identify vectors of L with vectors of $H^2(X,\mathbb{Z})$ or $H^2(Y,\mathbb{Z})$.

First, notice that $u = \frac{1}{2k}\pi^*(v_1, 0, v_3)$ is algebraic, since it is an integral class by construction, and $NS(X)$ is a primitive lattice inside $H^2(X, \mathbb{Z})$, containing $\pi^*(v_1, 0, v_3)$. Since $u^2 > 0$ by hypothesis, we see from the Riemann-Roch theorem that u or $-u$ must be effective. But $-u$ cannot be effective, since the inner product with the divisor class of one of the G_i is negative: $-u \cdot g_i = -\frac{1}{2}(\pi_*(u) \cdot \pi_*(g_i)) = -\frac{1}{2k}(v_1, 0, v_3) \cdot f_i = -\frac{1}{2k} v \cdot f_i < 0$. Therefore u is an effective class.

Now, for the class u to satisfy $u \cdot w > 0$ for all classes of effective divisors w on X, it is enough to show that u lies in the Kähler cone of X. We know that the Weyl group of X acts on the positive cone with the closure of the Kähler cone being a fundamental domain. So it is enough to show that $u \cdot w > 0$ for every nodal class w (that is, the class of a smooth rational curve on X). Let u be the class of an effective divisor U on X, and let W be the smooth rational curve whose class is w. Now let $z_1, \ldots, z_8$ be the eight fixed points of the Nikulin involution, and let $\tilde{X}$ be the blowup of X at these points, with $Z_1, \ldots, Z_8$ the exceptional curves. Let $\tilde{U}, \tilde{W}$ denote the proper transform of V and W. Let us assume W passes through the points $z_i, i \in I \subset \{1, \ldots, 8\}$. Note that $\tilde{W}$ is a smooth rational curve, since it is birational to W. Let us denote the map $\tilde{X} \to Y$ by π, by abuse of notation, and the blowup $\tilde{X} \to X$ by σ. Now we can move the divisor U (i.e. replace it by a linearly equivalent divisor, which may not be effective) so that it doesn't contain any of the points z_i and is transverse to W. Then we compute

$$
\begin{aligned}
U.W &= \sigma^* U.\sigma^* W \\
&= \tilde{U}.(\tilde{W} + \sum_{i \in I} Z_i) \text{ since the multiplicity } \mu_{z_i}(W) = 1 \text{ because } W \text{ is smooth.} \\
&= \tilde{U}.\tilde{W} \text{ since } \tilde{V} \text{ doesn't intersect the } Z_i \\
&= \frac{1}{2}\pi(\tilde{U}).\pi(\tilde{W})
\end{aligned}
$$

Now, $\pi(\tilde{W})$ is an irreducible rational curve on Y, since it's the image of an irreducible rational curve on $\tilde{X}$. Let $\tilde{w}$ be its class in $NS(Y)$. The class of $\pi(\tilde{U})$ is given by $u' = \frac{1}{k}(v_1, 0, v_3)$. Then we compute $u'.\tilde{w} = \frac{1}{k}(v_1, 0, v_3).\tilde{w} = \frac{1}{k}(v_1, v_2, v_3).\tilde{w} - \frac{1}{k}v_2.\tilde{w}$. The first term is positive because v is ample on Y and $\tilde{w}$ is the class of an irreducible curve on Y. The second term equals $\frac{1}{2k}\sum d_i \cdot \tilde{w}$. Since $\tilde{W}$ is distinct from the Z_i, we see that the second term is nonnegative and therefore the intersection number $\tilde{U}.\tilde{W}$ is indeed positive. □

The lemma shows that we have obtained an element of (X, ϕ) of $\mathcal{M}_\mu$. The ambiguity alluded to above in the choice of labeling g_i as the class of G_i and h_i as the class of H_i instead of the other way round, is inconsequential, since the other choice would give rise to an isomorphic marked surface $(X, \phi \circ \iota^*)$, where ι is the Nikulin involution. It is also clear that if we start from two isomorphic marked surfaces (Y, ρ) and (Y', ρ'), then we shall end up with isomorphic marked surfaces (X, ϕ) and (X', ϕ'). Therefore we get a well-defined map on moduli spaces $\eta_{\nu,\mu} : \mathcal{M}_\nu \to \mathcal{M}_\mu$.

Proposition 4.11. *The map $\eta_{\nu,\mu}$ defined above is surjective.*

Proof. For each $(X, \phi) \in \mathcal{M}_\mu$ we need to find a $(Y, \rho) \in \mathcal{M}_\nu$ which maps to it via $\eta_{\nu,\mu}$.

Step 3: Constructing the quotient with a marking.

We proceed in a similar manner as before. First, we find the Nikulin involution on X which follows from the data of ϕ. We have the marking $\phi : H^2(X, \mathbb{Z}) \cong L = E_8(-1) \oplus E_8(-1) \oplus U^3$. We will identify classes of divisors on X with elements of L using this marking.

Let $\gamma : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ be the map of lattices which interchanges the two copies of $E_8(-1)$ and fixes U^3. In other words, $\gamma(\phi^{-1}(g_i)) = \phi^{-1}(h_i), \gamma(\phi^{-1}(h_i)) = \phi^{-1}(g_i)$ and $\gamma|_{\phi^{-1}(U^3)}$ is the identity. From the definition of v, it follows that γ fixes v as well. Therefore it fixes the Kähler cone of X. Therefore, by the strong Torelli theorem, γ is induced by a unique involution $\iota : X \to X$. It is clear that ι is a Nikulin involution which interchanges the curves G_i and H_i, where g_i is the class of G_i and h_i is the class of H_i (these curves are uniquely defined by Lemma 2.20).

Now, let Y be the desingularized quotient of X by the Nikulin involution. The images of the curves G_i (or H_i) give curves F_i on Y, which are irreducible and therefore smooth rational curves. It is clear that we need to define ρ so that $\rho([F_i]) = f_i$. Let $D_1, \dots, D_8$ be the eight rational curves introduced as blowups of the singular points which appear when we quotient X by a Nikulin involution. We have a map $\pi_* : H^2(X, Z) \to H^2(Y, Z)$ which takes $(\sum \mathbb{Z}[G_i] \oplus \sum \mathbb{Z}[H_i])^\perp$ bijectively to $(\sum [F_i] \oplus \sum [D_i])^\perp$. So in order to make the markings commute with the maps on cohomology, we are forced to define the map ρ on that sublattice as $\rho(z) = \phi(\pi_*^{-1}(z))$. Finally, we need to define $\rho : \sum \mathbb{Z}[D_i] \to \Upsilon$ so that ρ, defined a priori as a map from $\sum [F_i] \oplus \sum [D_i] \oplus (\sum [F_i] \oplus \sum [D_i])^\perp$ to $\Gamma \oplus \Upsilon \oplus \Theta$, will extend to a map $H^2(Y, \mathbb{Z}) \to L$. Let $K = (\sum \mathbb{Z}[F_i] \oplus \sum \mathbb{Z}[D_i])^\perp$. Then we have $\rho_{partial} : \sum [F_i] \oplus K \to L$, with the image being $\Gamma \oplus \Theta$.

The only choice remaining is for the images of the $[C_i]$. Each D_i will map to a root of the lattice $\Upsilon \overset{\beta}{\cong} N$. Now, N has exactly 16 roots. Let us label the corresponding members of Υ by $d_1, \dots, d_8, -d_1, \dots, -d_8$. Now we know that Y is a K3 surface (in fact, a Kummer surface), and so there exists a marking $\rho' : H^2(Y, \mathbb{Z}) \to L$. We have the following diagram:

$$\begin{array}{ccccc} H^2(Y,\mathbb{Z}) & \xrightarrow{\rho'} & L & & \\ \uparrow & \nearrow & \vdots\ \exists ?g & & \\ K \oplus \sum [F_i] & \xrightarrow[\rho_{partial}]{} & L & \longleftarrow & \Gamma \oplus \Theta \end{array}$$

Now, ρ' and $\rho_{partial}$ define two primitive embeddings of $K \oplus \sum [F_i] \cong E_8(-1) \oplus U(2)^3$ into L. We know that up to automorphisms of L, there is only one primitive embedding which has orthogonal complement isomorphic to N. Therefore there exists an automorphism of g which makes the above diagram commute. We modify the marking ρ' and set $\rho = g \circ \rho'$:

$H^2(Y,\mathbb{Z}) \to L$. The new marking extends $\rho_{partial}$ to all of $H^2(X,\mathbb{Z})$.

So now we have a marking $\rho : H^2(Y,\mathbb{Z}) \to L$. Since $\sum[D_j]$ is the orthogonal complement of $\sum[F_i] \oplus K$, we see that ρ maps $\sum[D_j]$ to the orthogonal complement of $\Gamma \oplus \Theta$ in L, which is $\Upsilon \cong N$. Each $[D_j]$ maps to some d_j or $-d_j$. If it maps to $-d_j$, we can use the automorphism of L given by the Weyl reflection corresponding to d_j (which fixes the sublattices Γ and Θ pointwise, as well as fixing the other roots of Υ pointwise, while negating d_j) to change the marking so that D_i goes to d_j. Doing this for every D_i, and after relabeling the D_i, we can assume that $\rho(D_i) = d_i$.

Step 4: Polarization

Now via the marking, let us identify L with $H^2(X,\mathbb{Z})$.

Lemma 4.12. *The class v is the cohomology class of an ample line bundle on Y.*

Proof. First, note that $v = k\pi_*(u) - \frac{1}{2}\sum d_i = k\pi_*(u) - \frac{1}{2}\sum[D_i]$, it is clear that v is algebraic. We will show that for $\ell \geq 27$, the class $v_0 = \ell\pi_*(u) - \sum[D_i]$ is ample. Applying this to $\ell = 2k$, we get that $2k\pi_*(u) - \sum d_i$ and therefore v is ample. First, let us assume $\ell = mn$, with $n \geq 3, m \geq 9$.

A simple calculation shows that $v_0^2 > 0$. We will show directly that v_0 is an ample class. Let $\sigma : \tilde{X} \to X$ be the blow-up of X at the eight fixed points $z_1, \ldots, z_8$ of the Nikulin involution, and let $Z_1, \ldots, Z_8$ be the eight exceptional curves on $\tilde{X}$. Now, since u is the class of an ample line bundle on X, we know by the theorem quoted below that $nu = w$ is the class of a very ample line bundle on X for every $n \geq 3$. We claim that $w = m\sigma^*(w) - \sum[Z_i]$ is the class of an ample line bundle on $\tilde{X}$, for $m \geq 9$. It is easily verified that $x^2 > 0$. It is clear that $x.[Z_i] > 0$ for all i. Now let $\tilde{C}$ be an irreducible curve on $\tilde{X}$ distinct from the Z_i's. Letting $C = \sigma(\tilde{C})$, we have $[\tilde{C}] = [\sigma^*(C)] - \sum \mu_{z_i}(C)[Z_i]$. Here $\mu_{z_i}(C) \geq 0$ is the multiplicity of C at the point z_i. We compute

$$
\begin{aligned}
x.[\tilde{C}] &= (m\sigma^*(w) - \sum[Z_i]) \cdot (\sigma^*(C) - \sum \mu_{z_i}(C)[Z_i]) \\
&= m(w.[C]) - \sum \mu_{z_i}(C) \\
&\geq m(w.[C]) - 8(w.[C]) > 0.
\end{aligned}
$$

Here we have used the lemma below to note that the multiplicity of the curve C at any point inside the projective embedding corresponding to the very ample divisor W (whose class is w) is at most its degree $W.C = w.[C]$.

Therefore x is ample on $\tilde{X}$, and its push-forward to Y, namely v_0, is ample as well. This proves our claim for the special case when $\ell = mn$ as above. In general, if $\ell \geq 27$, we just write $3\ell = 3(9 + 9 + (\ell - 18)) = \ell_1 + \ell_2 + \ell_3$. Now from the above, we have that $(3 \cdot 9)\pi_*(u) - \sum[D_i]$ is ample, and so is $(3 \cdot (\ell - 18))\pi_*(u) - \sum[D_i]$. Therefore, twice the first plus the second, or $3\ell\pi_*(u) - 3\sum[D_i]$ is also ample. Therefore, $\ell\pi_*(u) - \sum[D_i]$ is ample. ☐

Thus we see that v is the class of an ample divisor on Y, and so the marked surface (Y, ρ) lies in $\mathcal{M}_\nu$. This concludes the proof of the proposition. $\square$

Theorem 4.13. (Saint-Donat [SD]) *Let D be an ample divisor on a K3 surface X. Then $3D$ is very ample.*

Lemma 4.14. *Let C be a curve of degree d in some $\mathbb{P}^n$. Then the multiplicity of C at any point is at most d.*

Proof. Let P be a point of multiplicity m on C. Intersect C with a hyperplane H passing through P and not containing C. Then we have $d = C.H$ is the sum of the points of intersection counted with multiplicity, and P contributes at least m. Hence $d \geq m$. $\square$

Step 5: Injectivity

We would like to show that the map $\eta_{\nu,\mu}$ of moduli spaces is finite. First of all, given $(X, \phi) \in \mathcal{M}_\mu$ a marked K3 surface with a symmetric polarization, the quotient K3 surface Y is certainly determined as an algebraic surface, because the Nikulin involution on X is determined from the lattice theoretic data contained in μ. Therefore we only need to consider the case (Y, ρ) and (Y, ρ') lying over (X, ϕ) and show they are isomorphic as marked K3 surfaces.

Consider the map $\rho^{-1} \circ \rho' : H^2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$. We claim that it is an effective Hodge isometry. It is a Hodge isometry since the global $(2,0)$-form on X is induced by the form on Y (the period points are the same in $L \otimes \mathbb{C}$). It preserves the Kähler cone since we prescribed that the class of an ample divisor is fixed: namely $v = (v_1, v_2, v_3) \in L$. Therefore we have an automorphism f of Y such that $\rho' = \rho \circ f^*$, and the two marked surfaces (Y, ρ) and (Y, ρ') are isomorphic.

Another quick way to put this argument is that the period map is an isomorphism from $\mathcal{M}_\mu$ to an open subset of Ω_μ, and also an isomorphism from $\mathcal{N}_\nu$ to an open subset Ω_ν, whereas the induced map on period spaces $\Omega_\nu \to \Omega_\mu$ is a linear isomorphism.

Step 6: Holomorphicity

Lemma 4.15. *The map $\eta_{\nu,\mu}$ is a holomorphic map of complex manifolds.*

Proof. We will deduce holomorphicity by using the period mapping. Let Ω_μ be the space of periods for the surfaces in $\mathcal{M}_\mu$.

$$\Omega_\mu = \{\omega \in \mathbb{P}(L \otimes \mathbb{C}) \,|\, \langle \omega, \omega \rangle = 0, \langle \omega, \overline{\omega} \rangle > 0, \langle \omega, u \rangle = 0, \langle \omega, \Gamma_1 \oplus \Gamma_2 \rangle = 0\}$$

Similarly, we can define a space of periods Ω_ν for the surfaces in $\mathcal{M}_\nu$.

$$\Omega_\nu = \{\omega \in \mathbb{P}(L \otimes \mathbb{C}) \,|\, \langle \omega, \omega \rangle = 0, \langle \omega, \overline{\omega} \rangle > 0, \langle \omega, v \rangle = 0, \langle \omega, \Gamma \oplus \Upsilon \rangle = 0\}$$

The period map for a K3 surface in $\mathcal{M}_\mu$ lands in Ω_μ, is a holomorphic map, and is locally an isomorphism of complex 3-manifolds. A similar assertion holds for $\mathcal{M}_\nu$ and Ω_ν. Therefore, to prove that the map $\eta_{\nu,\mu}$ is holomorphic, it suffices to show that the induced map on periods is holomorphic. But that assertion is obvious, since the $(2,0)$ form on X is pulled pack from that on Y, and the transcendental lattices are related by π_* (or π^*), which has been prescribed lattice-theoretically. So in fact the map on the period spaces is linear and hence holomorphic. □

This concludes the proof of the theorem. □

Now, we would like to get rid of the marking. To do this, we want to identify (X,ϕ) and $(X, g\circ\phi)$ where $g \in O(L)$ is an element which fixes the sublattice $(\Gamma_1 \oplus \Gamma_2) + \mathbb{Z}v$. That is, let

$$G_\mu = \{g \in O(L) \,|\, gx = x \text{ for } x \in (\Gamma_1 \oplus \Gamma_2) + \mathbb{Z}u\}.$$

The group G_μ acts on $\mathcal{M}_\mu$ by $g \cdot (X,\phi) = (X, g\circ\phi)$. Note that this action doesn't affect the embedding of $(\Gamma_1 \oplus \Gamma_2) + \mathbb{Z}u = M_\mu$ inside L. Using arguments along the lines of [D1], we will show that $\mathcal{M}_\mu/G_\mu$ has the structure of a quasi-projective variety. Similarly, let

$$G_\nu = \{g \in O(L) \,|\, gx = x \text{ for } x \in (\Gamma \oplus \Upsilon) + \mathbb{Z}v\}.$$

Also recall that

$$G_0 = \{g \in O(L) \,|\, gx = x \text{ for } x \in \Gamma + \mathbb{Z}v, g(\Upsilon) = \Upsilon\}.$$

Then $\mathcal{M}_\nu/G_\nu$ is also a quasi-projective variety. We will show that G_ν is a normal subgroup of finite index in G_0 and that $\mathcal{M}_\nu/G_0$ is also a quasi-projective variety.

Theorem 4.16. *The map $\eta_{\nu,\mu}$ gives rise to an isomorphism of quasi-projective varieties $\mathcal{M}_\nu/G_0 \to \mathcal{M}_\mu/G_\mu$.*

Proof. We start with a lemma relating the actions of the groups involved.

Lemma 4.17. *There is a homomorphism $\psi : G_0 \to G_\mu$ such that for $(Y,\rho) \in \mathcal{M}_\nu$ and $(X,\phi) = \eta_{\nu,\mu}((Y,\rho)) \in \mathcal{M}_\mu$, we have $\psi(g)\cdot(X,\phi) = \eta_{\nu,\mu}(g\cdot(Y,\rho))$*

Proof. Let $g \in G_0$. Then g fixes Γ pointwise and Υ as a whole, and therefore acts on the orthogonal complement $\Theta = U(2)^3$. Therefore we get an action on U^3. Now just define $\psi(g)$ to fix Γ_1, Γ_2 pointwise and act on the orthogonal complement $\Xi = U^3$ by this action. This defines $\psi(g)$ on all of L. It's clear that $\psi(g)$ fixes u, since we have that $u = u_1 + u_2 + u_3$ is completely determined by $v = v_1 + v_2 + v_3$ by $u_1 = u_2 = \frac{1}{2k}v_1$ and $u_3 = \frac{1}{k}v_3(\frac{1}{2})$, and we know that the components v_i are fixed by g because v is fixed, and the lattices Γ, Υ and Θ are fixed as a whole.

Now we show compatibility with the actions on the moduli spaces. Starting from (Y, ρ) and $(Y, g \circ \rho)$ we first note that the X constructed is the same, since it is the double cover of Y branched on the same divisor. Now, we need to show that the markings ϕ and ϕ' on X are related by $\psi(g)$. But this is clear: the images of $[G_i]$ and $[H_i]$ are fixed to be g_i and h_i. So $\phi' \circ \phi^{-1}$ fixes $\Gamma_1 \oplus \Gamma_2$. It also fixes u, which is determined by v, Γ and Υ. Finally we have that $\phi' \circ \phi^{-1}$ acts on U^3 as $g(\frac{1}{2})$ as described above, by the following picture.

$$\begin{array}{ccccc} U^3 & \xleftarrow{\phi'} & K_1 & \xrightarrow{\phi} & U^3 \\ \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle (2)} \\ U(2)^3 & \xleftarrow{g \circ \rho} & K_2 & \xrightarrow{\rho} & U(2)^3 \end{array}$$

□

Step 1: Map of spaces

It follows immediately from the above lemma that $\mathcal{M}_\nu / G_0 \to \mathcal{M}_\mu / G_\mu$ is well-defined.

Step 2: Bijection

Now, we want to show that the map is bijective. Since $\eta_{\nu,\mu}$ is already surjective, so is the map on quotient spaces. Therefore we just have to show injectivity.

Lemma 4.18. *The map $\psi : G_0 \to G_\mu$ is surjective.*

Proof. Let $h \in G_\mu$. Then h fixes $\Gamma_1 \oplus \Gamma_2$, so basically h is an automorphism of $\Xi = U^3$ which fixes u_3. Therefore we get an automorphism of $U(2)^3 = \Theta$ which fixes v_3. Now, since $U(2)^3$ has a unique primitive embedding into $E_8(-1) \oplus U^3$ such that the orthogonal complement is isomorphic to N, we see that there is a lift g_1 of this automorphism of $U(2)^3$ to all of L and fixing Γ. Now g_1 takes the roots $d_1, \ldots, d_8$ of Υ to eight orthogonal roots of Υ. Recall that Υ has only 16 roots, namely $\pm d_1, \ldots, \pm d_8$. Then, by using reflections in the d_i, we can assume that we have modified g_1 to get some g such that $d_1, \ldots, d_8$ go to some $d_{\tau(1)}, \ldots, d_{\tau(8)}$ for some permutation $\tau \in S_8$, the symmetric group on eight elements. Then $\sum d_i \mapsto \sum d_i$ under g, and so we have that g fixes each of v_1, v_2, v_3. Hence $g \in G_0$ and $\psi(g) = h$. □

Lemma 4.19. *G_ν is a normal subgroup of finite index in G_0.*

Proof. We claim that any element $g \in G_0$ takes $d_1, \ldots, d_8$ to $d_{\tau(1)}, \ldots, d_{\tau(8)}$ for some permutation $\tau \in S_8$. This is because g must fix v and $d_1, \ldots, d_8$ are exactly the positive roots of Υ. Hence we have a homomorphism $G_0 \mapsto S_8$, and the kernel is G_ν. Since S_8 is finite, we are done. □

It follows easily that the map $\mathcal{M}_\nu/G_0 \to \mathcal{M}_\mu/G_\mu$ is injective and therefore bijective. For if we have two $\overline{(Y,\rho)}$ and $\overline{(Y',\rho')}$ in $\mathcal{M}_\nu/G_0$ mapping to the same $\overline{(X,\phi)}$ in $\mathcal{M}_\mu/G_\mu$, then we have $\eta_{\nu,\mu}(Y,\rho) = h \circ \eta_{\nu,\mu}(Y',\rho')$ for some $h \in G_\mu$. Then let $h = \psi(g)$ for $g \in G_0$. We have $\eta_{\nu,\mu}(Y,\rho) = \psi(g) \circ \eta_{\nu,\mu}(Y',\rho') = \eta_{\nu,\mu}(Y', g \circ \rho')$. Since $\eta_{\nu,\mu}$ is an isomorphism, we get that $\overline{(Y,\rho)}$ and $\overline{(Y',\rho')}$ are equal in $\mathcal{M}_\nu/G_0$.

Step 3: Quasi-projective varieties

To show that the moduli spaces $\mathcal{M}_\nu/G_0$ and $\mathcal{M}_\mu/G_\mu$ are quasi-projective varieties, we use the period map. We recall the necessary facts from [D1]. Let $M \subset L$ be a primitive sublattice of signature $(1,t)$, and suppose we have chosen a polarization of M, i.e. the following data. We fix one of the two connected components of $V(M) = \{x \in M \otimes \mathbb{R} | x^2 > 0\}$ and call it $V(M)^+$. Let $\Delta(M) = \{\delta \in M | \delta^2 = -2\}$. We have a choice of the positive and negative roots, i.e. a partition $\Delta(M) = \Delta(M)^+ \sqcup \Delta(M)^-$ satisfying the usual properties, i.e.

- $\Delta(M)^- = \{-\delta | \delta \in \Delta(M)^+\}$.
- If $\delta_1, \ldots, \delta_r \in \Delta(M)^+$ and $\delta = \sum n_i \delta_i$ with $n_i \geq 0$ then $\delta \in \Delta(M)^+$.

This choice defines the "ample classes in M", namely

$$C(M)^+ = \{x \in V(M)^+ \cap M \,|\, \langle x, \delta \rangle > 0 \text{ for all } \delta \in \Delta(M)^+\}$$

Then we define an ample marked M-polarized K3 surface to be a marked M-polarized K3 surface (X,ϕ) such that $\alpha^{-1}(V(M)) \subset V(X)^+$ and $\alpha^{-1}(\Delta(M)^+) = \alpha^{-1}(M) \cap \Delta(X)^+$, that is, the polarization we chose on M comes from that on X. Note that this condition is equivalent to saying $\alpha^{-1}(C(M)^+) \cap C(X) \cap NS(X) \neq \phi$. In the terminology of Chapter 3, an ample marked M-polarized K3 surface for us will be an element of $\mathcal{M}_M^{+P}$, where P is the specific partition chosen above. If $m \in M$ is a vector with $m^2 \geq -2$, then m represents an effective class on any ample marked M-polarized K3 surface if and only if m has positive inner product with any and all classes in $C(M)^+$.

Now, for a marked M-polarized K3 surface (X,ϕ), recall that the period point is $\phi(H^{2,0}(X)) \in \mathbb{P}(L_\mathbb{C})$. Since $\phi(H^{2,0}(X))$ is orthogonal to $NS(X) \supset M$, we see that we may naturally consider the period point as an element of $\mathbb{P}(N_\mathbb{C}) \subset \mathbb{P}(L_\mathbb{C})$, where $N = M^\perp$ is the orthogonal complement of M in L. Let Q be the quadric hypersurface in $\mathbb{P}(N_\mathbb{C})$ corresponding to the quadratic form on $N_\mathbb{C}$ defined by the lattice N. Since for $\omega \in H^{2,0}(X)$ we have $\langle \omega, \omega \rangle = 0$ and $\langle \omega, \bar{\omega} \rangle > 0$, we see that the image of the period map lies in an open subset D_M of the quadric Q. We can assign to $H^{2,0}$ the positive definite 2-plane $P_X \subset N_\mathbb{R} = \phi((H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X,\mathbb{R}))$ together with an orientation defined by the choice of the isotropic line $\phi(H^{2,0}(X)) \subset P_X \otimes \mathbb{C}$ (namely, ω gives the oriented basis $(\Re\omega, \Im\omega)$). Thus we can identify D_M with the symmetric homogenous space

$O(2, 19-t)/SO(2) \times O(19-t)$ of oriented positive definite planes in N_R. The space consists of two connected components, each isomorphic to a bounded Hermitian domain of type IV_{19-t} (see [H]). The involution interchanging the two components is induced by complex conjugation $Q \to Q$.

Now, our surfaces have an ample divisor that comes from M, so that we cannot have any roots in $NS(X)$ which come from N (because if δ is such a root, then either δ or $-\delta$ comes from a (possibly reducible) rational curve, and then the ample divisor would have strictly positive intersection with it). This basically forces the period point to lie in D^0_M, which is defined by

$$D^0_M = D_M \setminus \Big(\bigcup_{\delta \in \Delta(N)} H_\delta \cap D_M \Big).$$

where for $\delta \in \Delta(N) = \{x \in N | x^2 = -2\}$ we define

$$H_\delta = \{z \in N_{\mathbb{C}} \,|\, (z, \delta) = 0\}.$$

The period map induces a bijective map from the moduli space of ample marked M-polarized K3 surfaces to D^0_M. We let $O(L)$ be the orthogonal group of the lattice L (i.e. the isometries of L which fix the origin), and similarly for $O(N)$. We also let $\Gamma(M) = \{g \in O(L) \,|\, g(m) = m$, for all $m \in M\}$. Then $\Gamma(M)$ is a normal subgroup of finite index in $O(N)$. We get the moduli space of K3 surfaces with an ample M-polarized structure (i.e. get rid of the marking) by considering $D^0_M/\Gamma(M)$. The group $O(N)$ is an arithmetic subgroup of $O(2, 19-t)$ and so is $\Gamma(M)$, since it has finite index in $O(N)$. Therefore $D_M/\Gamma(M)$ is a quasi-projective variety. Now $O(N)$ has only finitely many orbits in the set of primitive vectors in N with given value of the quadratic form (this follows from Prop 1.15.1 of [N1]) and therefore $\Gamma(M)$, a finite index subgroup of $O(N)$, has finitely many orbits in its action on $\Delta(N)$. Therefore, $D^0_M/\Gamma(M)$ is $D_M/\Gamma(M)$ minus finitely many hypersurfaces, and is therefore a quasi-projective algebraic variety.

Now, we need to put in the further condition that a specific subset T of roots of M come from irreducible rational curves on the K3 surface. This means that we should not have the equation

$$t = u + v$$

for any effective divisor classes u and v,, and any $t \in T$.

Lemma 4.20. *Let t be the class of a divisor T on a K3 surface X, such that $t^2 = -2$. Let c be an ample class on X. Then t is an irreducible class if and only if there is no decomposition $t = u + v$ in $NS(X)$ such that $u^2 \geq -2$, $t \cdot u < 0$, and $0 < c \cdot u < c \cdot t$.*

Proof. Suppose T is reducible. Then $T = \sum \mu_i T_i$ with T_i irreducible effective divisors. Then we have $T_i^2 \geq -2$ for each i, by the genus formula. Furthermore, $-2 = T \cdot T =$

$T \cdot (\sum \mu_i T_i) = \sum \mu_i (T \cdot T_i)$. So we must have $T \cdot T_j < 0$ for some T_j. Letting the t_i denote the class of T_i, we also have $c \cdot t = \sum \mu_i (c \cdot t_i)$, a sum of (more than one) positive terms. So we see that $c \cdot t > c \cdot t_i > 0$ for each term. Now let u be the t_j chosen above, then one half of the lemma is proved.

Conversely, if $t = u + v$ with $u^2 \geq -2$, $t \cdot u < 0$, and $0 < c \cdot u < c \cdot t$, then first we have that u is the class of an effective divisor (by Riemann-Roch, and the fact that $u \cdot c > 0$. Also, $u \cdot t \leq -1$ so that $t - u$ also satisfies $(t-u)^2 \geq -2$ and $c \cdot (t-u) > 0$. Therefore $t - u$ is also effective. Therefore t is reducible. □

Now, let us assume that t is not reducible within M, i.e. we do not have an equation $t = u + v$ as above with $u, v \in M$ effective. Otherwise, t can never represent a smooth rational curve on an ample marked M-polarized surface.

If t is reducible in $NS(X)$, we assume as above that u satisfies the conditions above. In particular, $u^2 \geq -2$. For every such u we can write $u = u_M + u_N$ with u_M and u_N being the projections of u to $M \otimes \mathbb{Q}$ and $N \otimes \mathbb{Q}$, and $u_N \neq 0$. Then it is clear that $u_M \in M^*$, the dual of M considered as a subspace of $M \otimes Q$ and $u_N \in N^*$. Let d be the minimal positive integer such that du_N is in N and du_M is in M (d certainly exists because $disc(M) = [M^* : M]$ works). Note that du_N is a primitive vector in N. To further constrain u_M and u_N, we will need the following lemma, which is a variant of the Hodge index theorem.

Lemma 4.21. *Let D_1, D_2 be divisors on a surface X with $D_1^2 > 0$. Then*

$$(D_1^2)(D_2^2) \leq (D_1 \cdot D_2)^2$$

Proof. We let $m = D_1 \cdot D_2$ and $n = -D_1^2$ so that $D_1 \cdot (mD_1 + nD_2) = 0$. Then the Hodge index theorem implies $(mD_1 + nD_2)^2 \leq 0$, which gives the desired inequality. □

Let c be a fixed ample class in $C(M)^+$. Then for any ample marked M-polarized surface X, c represents the class of an ample divisor. Therefore we have $c \cdot t > 0, c \cdot u > 0, c \cdot v > 0$, and therefore bounds $0 < c \cdot u < c \cdot t$. Furthermore, $c \in M$ implies $c \cdot (du_N) = 0$ and $c \cdot (du_M) = c \cdot du = d(c \cdot u)$. The lemma above implies

$$\begin{aligned} (du_M)^2 &\leq (c \cdot du_M)^2/c^2 = (c \cdot du)^2/c^2 < d^2(c \cdot t)^2/c^2 \\ (du_N)^2 &\leq (c \cdot du_N)^2/c^2 = 0 \end{aligned}$$

Since $(du_M)^2 + (du_N)^2 = d^2u^2 \geq -2d^2$ is bounded below, we see that the norms of du_M and du_N are both bounded above as well as below.

Now, note that for an ample marked M-polarized K3 surface X, $u \in NS(X)$ iff $du \in NS(X)$ (because $NS(X)$ is a primitive lattice) iff $du_N \in NS(X)$ since $du_M \in M \subset NS(X)$ already. So in the period space D_M^0, we need to avoid ω such that $\omega^\perp \in L \otimes \mathbb{C}$ contains such vectors u_N.

Notice that the equation $t = u + v$ transforms under $g \in \Gamma(M)$ to $t = g(u) + g(v)$, and similarly $du = du_M + du_N$ transforms to $dg(u) = du_M + g(du_N)$. Also, note that $g(c) = c$. So if $t = u + v$ is a decomposition showing that t is not an irreducible nodal class, then so is $t = g(u) + g(v)$. Therefore, avoiding $du_N \in NS(X)$ necessarily entails avoiding the whole orbit of du_N under $\Gamma(M)$. The period space which bijects with the moduli space of ample marked M-polarized surfaces such that elements of T are classes of irreducible (smooth) rational curves, is therefore the subset of D_M given by

$$D^0_{M,T} = D_M \setminus \left(\Big(\bigcup_{\delta \in \Delta(N)} H_\delta \cap D_M \Big) \bigcup \Big(\bigcup_{w \in N'} H_w \cap D_M \Big) \right)$$

Here the notation $w \in N'$ means w ranges over all primitive $w = w_N \in N$ such that for some $t \in T$, and some u as above depending on t, we have $v \in L$ and for some positive integer d we can write $t = u + v$, $du = w_M + w_N$. The argument above shows that the norms of such w_N are bounded above and below. As before, $O(N)$ and therefore $\Gamma(M)$ has finitely many orbits in its action on N'. It follows that $D^0_{M,T}/\Gamma(M)$ is the same as $D_M/\Gamma(M)$ minus finitely many hypersurfaces, and is therefore a quasi-projective algebraic variety. We summarize the results in the proposition below.

Proposition 4.22. *Let $M \subset L$ be a fixed saturated sublattice of signature $(1, t)$, and choose a polarization of M. Suppose T is a subset of M consisting of roots which are positive and simple for the chosen polarization. Then the moduli space $\mathcal{M}^a_{M,T}$ of ample marked M-polarized K3 surfaces with every $t \in T$ represented by a smooth rational curve is a quasi-projective algebraic variety.*

We apply the above proposition to the space $\mathcal{M}_\mu$ with the polarization given by $u = (u_1, u_2, u_3)$, with $M = (\Gamma_1 \oplus \Gamma_2) + \mathbb{Z}u$, and with $T = \{g_i, h_i | 1 \le i \le 8\}$. Note that M is a saturated lattice since u_3 is primitive in U^3. We know from the hypothesis that the curves g_i and h_i are irreducible in the lattice. This proves that $\mathcal{M}_\mu/G_\mu$ is a quasi-projective algebraic variety. Similarly, $\mathcal{M}_\nu/G_\nu$ and $\mathcal{M}_\nu/G_0$ are quasi-projective algebraic varieties. Note that while [D1] and the proposition above deal with primitive sublattices, the sublattice M_ν is not primitive. However, we can simply replace it by its saturation, the smallest primitive lattice in L containing M_ν.

Step 4: Morphism

Since the map on period domains is linear and hence holomorphic, we see from the definition of the algebraic structure on the quotient varieties that the map $\eta_{\nu,\mu} : \mathcal{M}_\nu/G_0 \to \mathcal{M}_\mu/G_\mu$ is an algebraic morphism. It has a bijection, and by Zariski's main theorem we deduce that it is an isomorphism.

This completes the proof of the theorem. □

Chapter 5

Explicit construction of isogenies

In this chapter, we consider a family of elliptic K3 surfaces of Picard number 17 for which we can explicitly write down the isogeny to a Kummer surface.

5.1 Basic theory of elliptic surfaces

We recall here a few facts about elliptic surfaces needed in the sequel. References for these are [S] and [Si2].

Definition 5.1. *An elliptic surface is a smooth projective algebraic surface X with a proper morphism $\pi : X \to C$ to a smooth projective algebraic curve C, such that*

1. *There exists a section $\sigma : C \to X$.*
2. *The generic fiber E is an elliptic curve.*
3. *π is relatively minimal.*

Concretely, we will be considering the case $C = \mathbb{P}^1$, and then we will choose a Weierstrass equation for the generic fiber, which is an elliptic curve over the function field $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(t)$, namely

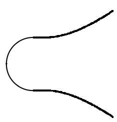
$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

where a_i are rational functions of t. In fact, by multiplying x and y by suitable polynomials, we can make $a_i(t)$ polynomials in t. This can be done in such a way that the degree of the discriminant is minimal. We can read out some properties of the surface directly from the Weierstrass equation. For instance, if $p_a(X)$ is the arithmetic genus of X, then

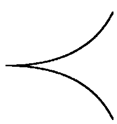
$p_a(X)+1 = \chi(\mathcal{O}_X)$ is the minimal n such that $\deg a_i \leq ni$ for $i = 1,2,3,4,6$. In particular, for a K3 surface X, we need to have degree $a_i \leq 2i$.

All but finitely many of the fibers of the elliptic surface are nonsingular and hence elliptic curves. Tate's algorithm [T] allows us to compute the description of the singular fibers.

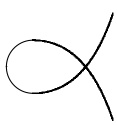
Each fiber is one of the types shown in the figure below. We note that the reducible fibers are unions of nonsingular rational curves, and they occur in configurations as shown below. The dual graph of the components is an extended Dynkin diagram of type A, D or E. The lattices labeled below are the ones spanned by the non-identity components of the fiber in $NS(X)$. The subscript in the root lattices indicated below is the rank of the lattice, which is also the number of non-identity components. Thus, the I_n fiber has n components, whereas the I_n^* fiber has $n+5$ components.



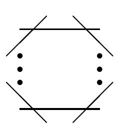
I_0, a nonsingular fiber (elliptic curve)



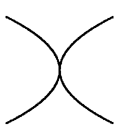
II, a cusp



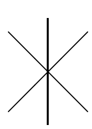
I_1, a node



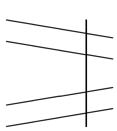
$I_n, n \geq 2$ (A_{n-1})



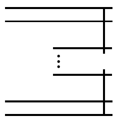
III (A_1)



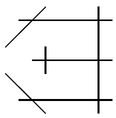
IV (A_2)



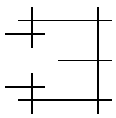
I_0^* (D_4)



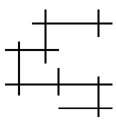
$I_n^*, n \geq 1$ (D_{n+4})



IV^* (E_6)



III^* (E_7)



II^* (E_8)

The Néron-Severi lattice of X is generated by the classes of all the sections of π (i.e. the Mordell-Weil group of X) considered as curves on the surface X, together with the class F of a fiber, and all the non-identity components of the reducible fibers. Let $R = \{v \in C(\mathbb{C}) \,|\, F_v \text{ is reducible}\}$, and for each $v \in R$, let $F_v = \pi^{-1}(v) = \Theta_{v,0} + \sum_{i=1}^{m_v-1} \mu_{v,i}\Theta_{v,i}$, where $\Theta_{v,0}$ is the component which intersects the identity, and the other $\Theta_{v,i}$ are the non-identity components. The intersection pairing satisfies:

- for any section P, $P^2 = O^2 = -\chi$,
- $P \cdot F = O \cdot F = 1$,
- $F^2 = 0$,
- $O \cdot \Theta_{v,i} = 0$ for $i \geq 1$,
- $\Theta_{v,i} \cdot \Theta_{w,j} = 0$ for $v \neq w$.

The intersection pairing for $\Theta_{v,i}$ and $\Theta_{v,j}$ is -2 if $i = j$, and $0, 1, 2$ if $i \neq j$ according to the figures above (2 occurs only for types I_2 and III). For a general section P, the intersection pairing with each $\Theta_{v,i}$ can be computed locally. In particular, for each v exactly one of the the intersection numbers is 1, for some i such that $\mu_{v,i} = 1$, and the others vanish. The rank of the Néron-Severi group is given by the formula

$$\rho = r + 2 + \sum_v m_v - 1.$$

The discriminant of the sublattice T spanned by the non-identity components of all the fibers is $\prod_v m_v^{(1)}$, where $m_v^{(1)}$ is the number of multiplicity one components of F_v.

5.2 Elliptic K3 surface with E_8 and E_7 fibers

Let X be an elliptic K3 surface with bad fibers of type E_8 and E_7 at ∞ and 0 respectively. A generic such K3 surface has a Néron-Severi lattice $NS(X) \cong U \oplus E_8(-1) \oplus E_7(-1)$ by Shioda's explicit description of the Néron-Severi of an elliptic surface. This lattice has rank 17, signature $(1, 16)$ and discriminant 2. The transcendental lattice T_X has rank 5, signature $(2, 3)$ and discriminant 2. We deduce that $T_X \cong U^2 \oplus \langle -2 \rangle$.

The transcendental lattice of a generic principally polarized abelian surface, that is, the Jacobian $J(C)$ for C a generic curve of genus 2, also satisfies the same property, since the Néron-Severi of $J(C)$ is spanned by the theta divisor, which has self-intersection 2, by the genus formula on the abelian surface

$$2 = 2g - 2 = C.(C + K) = C^2.$$

Therefore the orthogonal complement in $H^2(J(C), \mathbb{Z}) \cong U^3$ is exactly $U^2 \oplus \langle -2 \rangle$. We expect that the elliptic K3 surface X has a Shioda-Inose structure such that the quotient by the Nikulin involution gives the Kummer surface of a principally polarized abelian surface $\mathrm{Km}(J(C))$. Dolgachev [D2] proves that, in fact, X corresponds to a unique C up to isomorphism. However, an explicit identification of the quotient as a Kummer surface was not known. Below, we give an explicit construction of the correspondence.

We begin with the K3 surface X given by the equation

$$y^2 = x^3 + t^3(at + a')x + t^5(b''t^2 + bt + b').$$

In the next section we will show how to obtain this Weierstrass equation. The surface X has an II^* or E_8 fiber at $t = \infty$ and a III^* or E_7 fiber at $t = 0$. We can write it by scaling x, y as

$$y^2 = x^3 + (a + a'/t)x + (b''t + b + b'/t).$$

Now replacing y by y/t gives the equation

$$y^2 = b''t^3 + (x^3 + ax + b)t^2 + (a'x + b')t$$

and again replacing (y, t) by $(y/b'', t/b'')$ gives finally

$$y^2 = t^3 + (x^3 + ax + b)t^2 + b''(a'x + b')t$$

which is an elliptic surface over the x-line with an I_{10}^* or D_{14} fiber at $x = \infty$ and an I_2 or A_1 fiber at $x = -b'/a'$ and a 2-torsion section $(y, t) = (0, 0)$. The translation by the 2-torsion section is a Nikulin involution. We write down the isogenous elliptic surface Y as

$$y^2 = t^3 - 2(x^3 + ax + b)t^2 + ((x^3 + ax + b)^2 - 4b''(a'x + b'))t.$$

This is an elliptic surface over the x-line with an I_5^* or D_9 fiber at $x = \infty$ and I_2 or A_1 fibers at the roots of the sextic $(x^3 + ax + b)^2 - 4b''(a'x + b')$, and with a 2-torsion section $(t, y) = (0, 0)$. The Néron-Severi lattice of a generic such surface has signature $(1, 16)$ and discriminant $4 \cdot 2^6/2^2 = 2^6$. In fact, we will identify it with the Néron-Severi lattice of a generic Kummer surface (which we call the $(16, 6)$ lattice) in a later section. This will lead to the identification of the Kummer surface of $J(C)$ as an elliptic K3 surface with bad fibers of type I_5^* fiber at ∞, I_2 fibers at the roots of a sextic derived from C, and with a 2-torsion section. First we need some preliminaries.

5.3 Parametrization

In this section, we derive the family of elliptic surfaces described in the last section, using Tate's algorithm [T].

We consider an elliptic surface over $\mathbb{P}^1$ with bad fibers of type E_8 at $t = \infty$ and E_7 at $t = 0$. Its Weierstrass equation can be put in the form

$$y^2 = x^3 + r(t)x + s(t)$$

with degree$(r) \leq 8$, degree$(s) \leq 12$. Now for an E_7 fiber at $t = 0$, we need to have $t^5|s(t)$ and $t^3||r(t)$. To figure out the reduction at $t = \infty$, we change coordinates by replacing $t = 1/u, x = x/u^4, y = y/u^6$ to get

$$y^2 = x^3 + \tilde{r}(u)x + \tilde{s}(u)$$

where $\tilde{r}(u) = u^8 r(\frac{1}{u})$ and $\tilde{s}(u) = t^{12} s(\frac{1}{u})$. To have an E_8 fiber we need $u^4|\tilde{r}(u)$ and $u^5||\tilde{s}(u)$. Therefore, r has degree at most 4 and s has degree exactly 7. Combining all the information, we get that

$$r(t) = t^3(at + a')$$

$$s(t) = t^5(b''t^2 + bt + b')$$

with $a' \neq 0$ and $b'' \neq 0$. There is a further condition to ensure that there are no other reducible fibers: we compute the discriminant

$$\Delta = t^9(27b''^2t^5 + 54bb''t^4 + 54b'b''t^3 + 27b^2t^3 + 4a^3t^3 + 54bb't^2 + 12a^2a't^2 + 27b'^2t + 12aa'^2t + 4a'^3),$$

divide by t^9 and compute the discriminant of Δ with respect to t, and require it to be nonzero, which eliminates any double roots.

All the components of the E_8 and E_7 fibers are automatically rational, because there are no nontrivial automorphisms of the Dynkin diagram which fix the zero section.

5.4 Curves of genus two

Here we describe the basic geometry and moduli of curves of genus 2. For more background we refer the reader to [CF], [Cl], [I], [Me]. Let C be such a curve defined over a field k of characteristic zero. Then the canonical K_C bundle of C has degree 2 and $h^0(C, K_C) = 2$. That is, the corresponding complete linear system is a g^1_2 (and it is unique). We therefore have a map

$$x : C \to \mathbb{P}^1$$

which is ramified at 6 points by the Riemann-Hurwitz formula, and the function field of C is a quadratic extension of $k(x)$. Therefore, we may write the equation of C as

$$y^2 = f(x) = \sum_{i=0}^{6} f_i x^i.$$

The roots of the sextic are the six ramification points as of the map $C \to \mathbb{P}^1$. Their pre-images on C are the six Weierstrass points. Now, the isomorphism class of C over $\bar{k}$, the algebraic closure of k, is determined by the isomorphism class of the sextic $f(x)$, where two sextics are equivalent if there is a transformation in $PGL_2(\bar{k})$ which takes the set of roots (considered inside $\mathbb{P}^1$) to the roots of the other. Clebsch was the first to determine the invariants of binary sextics. He defined invariants of I_2, I_4, I_6, I_{10} of weights $2, 4, 6, 10$ respectively, and Clebsch and Bolza showed that they determined the sextic up to $\bar{k}$-equivalence. Therefore, the point $(I_2(f) : I_4(f) : I_6(f) : I_{10}(f))$ in weighted projective space $\mathbb{P}^3$ determines the isomorphism class of C. In fact, C and C' are isomorphic over k iff there is an $r \in k^*$ such that $I_d(f') = r^d I_d(f)$. Igusa generalized Clebsch's theory to hold in all characteristics by defining choosing a different algebraic equation for the curve C (through an embedding as a quartic in $\mathbb{P}^2$ with one node) and defining invariants J_2, J_4, J_6, J_8 and J_{10}. He thus obtained a moduli space of genus two curves defined over $\operatorname{Spec}\mathbb{Z}$. The invariants $I_2, \ldots, I_{10}$ are called the Igusa-Clebsch invariants.

However, if the Igusa-Clebsch invariants of a curve C lie in a field k, it does not necessarily mean that C can be defined over k: there is usually an obstruction in $Br_2(k)$. Therefore, C can always be defined over a quadratic extension of k.

5.5 Kummer surface of $J(C)$

Let C be a curve of genus 2, which we can write as

$$y^2 = f(x) = \sum_{i=0}^{6} f_i x^i$$

Let $\theta_i, i = 1, \ldots, 6$ be the roots of the of the sextic, so that

$$f(x) = f_6 \prod_{i=1}^{6} (x - \theta_i)$$

We shall concern ourselves with the embedding of the singular Kummer surface as a quartic in $\mathbb{P}^3$, which comes from the complete linear system 2Θ, twice the theta divisor which defines the principal polarization. We shall use the formulas from [CF]. The quartic is given by the equation

$$K(z_1, z_2, z_3, z_4) = K_2 z_4^2 + K_1 z_4 + K_0 = 0$$

where

$$\begin{aligned}
K_2 &= z_2^2 - 4z_1z_3, \\
K_1 &= -4z_1^3 f_0 - 2z_1^2 z_2 f_1 - 4z_1^2 z_3 f_2 - 2z_1z_2z_3 f_3 - 4z_1 z_3^2 f_4 - 2z_2 z_3^2 f_5 - 4z_3^3 f_6, \\
K_0 &= -4z_1^4 f_0 f_2 + z_1^4 f_1^2 - 4z_1^3 z_2 f_0 f_3 - 2z_1^3 z_3 f_1 f_3 - 4z_1^2 z_2^2 f_0 f_4 \\
&\quad +4z_1^2 z_2 z_3 f_0 f_5 - 4z_1^2 z_2 z_3 f_1 f_4 - 4z_1^2 z_3^2 f_0 f_6 + 2z_1^2 z_3^2 f_1 f_5 \\
&\quad -4z_1^2 z_3^2 f_2 f_4 + z_1^2 z_3^2 f_3^2 - 4z_1 z_2^3 f_0 f_5 + 8z_1 z_2^2 z_3 f_0 f_6 \\
&\quad -4z_1 z_2^2 z_3 f_1 f_5 + 4z_1 z_2 z_3^2 f_1 f_6 - 4z_1 z_2 z_3^2 f_2 f_5 - 2z_1 z_3^3 f_3 f_5 \\
&\quad -4z_2^4 f_0 f_6 - 4z_2^3 z_3 f_1 f_6 - 4z_2^2 z_3^2 f_2 f_6 - 4z_2 z_3^3 f_3 f_6 - 4z_3^4 f_4 f_6 + z_3^4 f_5^2.
\end{aligned}$$

The 16 singular points define ordinary double points on the quartic, which are called nodes. These are given explicitly by the coordinates

$$\begin{aligned}
p_0 &= (0 : 0 : 0 : 1) \\
p_{ij} &= (1 : \theta_i + \theta_j : \theta_i\theta_j : \beta_0(i, j))
\end{aligned}$$

for $1 \leq i < j \leq 6$.

Here $\beta_0(i, j)$ is defined as follows. Let

$$f(x) = (x - \theta_i)(x - \theta_j)h(x) \text{ with } h(x) = \sum_{n=0}^{4} h_n x^n.$$

Then

$$\beta_0(i, j) = -h_0 - h_2(\theta_i\theta_j) - h_4(\theta_i\theta_j)^2.$$

The singular point p_0 comes from the 0 point of the Jacobian, whereas the p_{ij} come from the 2-torsion point which is the difference of divisors $[(\theta_i, 0)] - [(\theta_j, 0)]$ corresponding to two distinct Weierstrass points on C. The sixteen singular points are called **nodes**.

There are also sixteen hyperplanes in $\mathbb{P}^3$ which are tangent to the Kummer quartic. These are called **tropes**. Each trope intersects the quartic in a conic with multiplicity 2, and contains 6 nodes. Conversely, each node is contained in exactly 6 nodes. This beautiful configuration is called the $(16, 6)$ Kummer configuration.

The explicit formulae for the tropes are as follows. Six of the tropes are given by

$$\theta^2 z_1 - \theta_i z_2 + z_3 = 0.$$

We call this trope T_i. It contains the nodes p_0 and p_{ij}. The remaining ten tropes are labeled T_{ijk} and corresponds to a partition of $\{1,2,3,4,5,6\}$ into two sets of three, say $\{i,j,k\}$ and its complement $\{l,m,n\}$. Set

$$G(X) = (x-\theta_i)(x-\theta_j)(x-\theta_k) = \sum_{r=0}^{3} g_r x^r,$$

$$H(X) = (x-\theta_l)(x-\theta_m)(x-\theta_n) = \sum_{r=0}^{3} h_r x^r.$$

Then the equation of T_{jk} is

$$f_6(g_2h_0 + g_0h_2)z_1 + f_6(g_0 + h_0)z_2 + f_6(g_1 + h_1)z_3 + z_4 = 0.$$

The Néron-Severi lattice of the nonsingular Kummer contains classes of rational curves E_0 and E_{ij} coming from the nodes, and C_i and C_{ijk} coming from the tropes. We will denote the lattice generated by these as $\Lambda_{(16,6)}$. It has signature $(1, 16)$ and discriminant 2^6 and is the Néron-Severi lattice of the Kummer surface of a generic principally polarized abelian surface.

Let L be the class of a hyperplane section, so that we have the following intersection numbers and relations in the Néron-Severi lattice.

$$\begin{aligned}
L^2 &= 4,\\
E_0^2 &= -2,\\
E_{ij}^2 &= -2,\\
E_0 \cdot E_{ij} &= 0,\\
E_{ij} \cdot E_{kl} &= 0, \{i,j\} \neq \{k,l\},\\
C_i &= (L - E_0 - \sum_{j \neq i} E_{ij})/2,\\
C_{ijk} &= (L - E_{ij} - E_{jk} - E_{ik} - E_{lm} - E_{mn} - E_{ln})/2.
\end{aligned}$$

We consider the following construction outlined in [Na]. Projection to a hyperplane from p_0 defines a 2 to 1 map of the Kummer to $\mathbb{P}^2$, and thus identifies the Kummer as a double cover of $\mathbb{P}^2$, ramified along the union of six lines, which are the projections of the conics C_i (or the tropes T_i). The exchange of sheets gives an involution of the sheets, which acts by

$$\begin{array}{rcl} E_0 & \mapsto & 2L - 3E_0, \\ E_{ij} & \mapsto & E_{ij}, \\ L & \mapsto & 3L - 4E_0. \end{array}$$

We can explicitly write down the projection to $\mathbb{P}^2$ as $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$. The involution which is the exchange of sheets is $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, -x_4)$. Let q_0, q_{ij} be the projections of the p_0, p_{ij}.

5.6 The elliptic fibration on the Kummer

We would like to identify the Kummer quartic as an elliptic surface with bad fibers of type I_5^* at ∞ and six I_2 fibers, with a 2-torsion section. We first identify the Néron-Severi lattices involved (namely $\Lambda_{(16,6)}$ and $(D_9 \oplus A_1^6 \oplus U)^+$). Next, we use the identification to find out the rational functions x, y, t on the Kummer which make satisfy the Weierstrass equation for an ellptic surface with the requisite bad fibers and 2-torsion section. The details are given in section 5.7.

We find that the Kummer surface is

$$y^2 = x^3 - 2(t^3 + at + b)x^2 + ((t^3 + at + b)^2 - 4b''(a't + b'))x$$

with

$$\begin{array}{rcl} a & = & -I_4/12, \\ a' & = & -1, \\ b & = & (I_2 I_4 - 3I_6)/108, \\ b' & = & I_2/24, \\ b'' & = & I_{10}/4, \end{array}$$

where I_2, I_4, I_6, I_{10} are the Igusa-Clebsch invariants of degrees $2, 4, 6, 10$ respectively of the genus 2 curve $C : y^2 = f(x)$. This elliptic fibration has a I_5^* fiber at $t = \infty$ and I_2 fibers at the roots of the sextic $(t^3 + at + b)^2 - 4b''(a't + b')$.

Theorem 5.2. *Let C be a curve of genus two, and $Y = \mathrm{Km}(J(C))$ the Kummer surface of its Jacobian. Let I_2, I_4, I_6, I_10 be the Igusa-Clebsch invariants of Y. Then there is an elliptic fibration on Y for which the Weierstrass equation may be written*

$$y^2 = x^3 - 2\left(t^3 - \frac{I_4}{12}t + \frac{I_2 I_4 - 3I_6}{108}\right)x^2 + \left(\left(t^3 - \frac{I_4}{12}t + \frac{I_2 I_4 - 3I_6}{108}\right)^2 + I_{10}\left(t - \frac{I_2}{24}\right)\right)x.$$

There is an elliptic K3 surface X given by

$$y^2 = x^3 - t^3\left(\frac{I_4}{12}t + 1\right)x + t^5\left(\frac{I_{10}}{4}t^2 + \frac{I_2I_4 - 3I_6}{108}t + \frac{I_2}{24}\right)$$

with fibers of type E_8 and E_7 at $t = \infty$ and $t = 0$ respectively, and a Nikulin involution on X, such that the quotient K3 surface is Y.

Remark 5.3. The Nikulin involution on X may be written as follows:

$$(x, y, t) \mapsto \left(\frac{16\,x\,(-x + I_2t^2/24)^2}{I_{10}^2\;t^8}, \frac{64\,y\,(-x + I_2t^2/24)^3}{I_{10}^3\;t^{12}}, \frac{4\,(-x + I_2t^2/24)}{I_{10}\;t^3}\right)$$

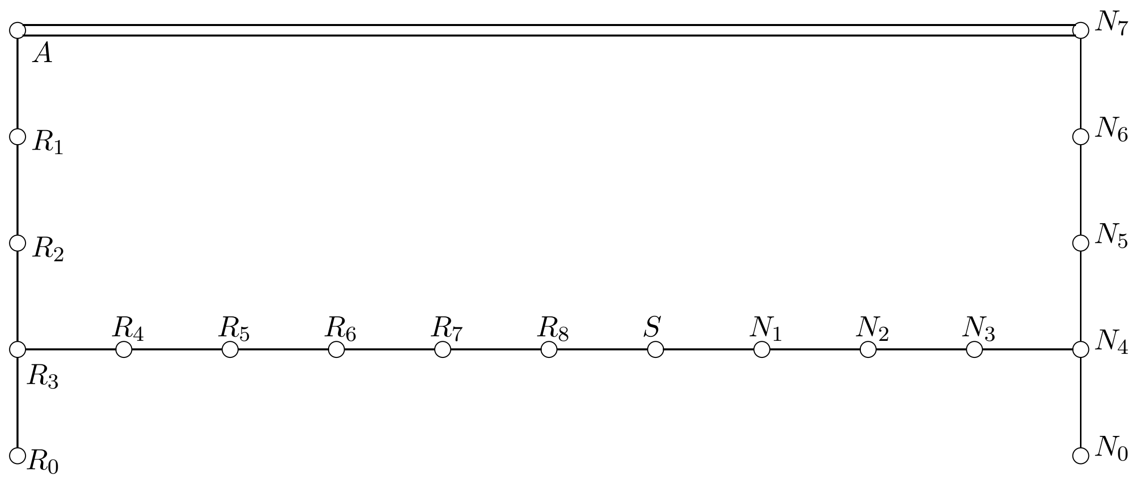
Remark 5.4. Notice that in addition to the correspondence of elliptic K3 surfaces having E_8 and E_7 bad fibers with Kummer surfaces of principally polarized abelian surfaces, we get a correspondence of curves of genus 2 between C and W, the curve given by

$$y^2 = \left(x^3 - \frac{I_4}{12}x + \frac{I_2I_4 - 3I_6}{108}\right)^2 + I_{10}\left(x - \frac{I_2}{24}\right).$$

5.7 Finding the isogeny via the Néron-Severi group

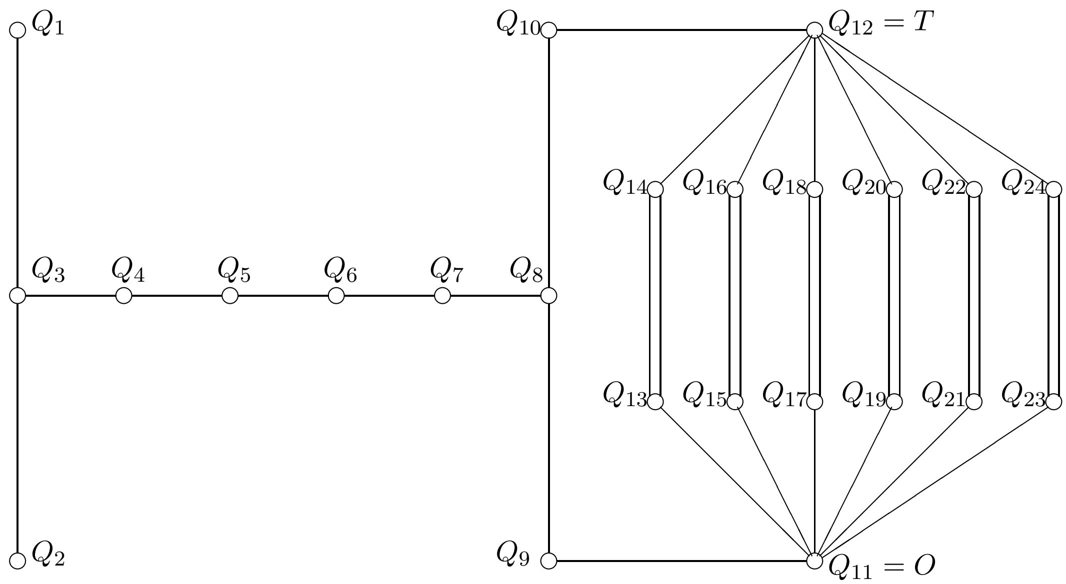
In this section, we give the details of how to put an elliptic fibration on the Kummer surface of a Jacobian of a curve of genus 2, with a 2-torsion section, a I_5^* fiber and six I_2 fibers. We use the construction of [Na], which gives an embedding of the lattice $N \oplus E_8(-1)$ inside $\Lambda_{(16,6)}$.

First, we start with the Néron-Severi lattice of the K3 surface X which has E_8 and E_7 fibers. The roots of the $NS(X)$ which correspond to the smooth rational curves on X are drawn below (we use the notation from [D2]).



There is an elliptic fibration on X which has $R_8+2R_7+3R_6+4R_5+5R_4+6R_3 4R_2+2R_1+3R_0$ as an II^* or E_8 fiber, $N_7+2N_6+3N_5+4N_4+N_3+2N_2+N_1+2N_0$ as a III^* or E_7 fiber, and S as the zero section. This is the fibration over $\mathbb{P}^1_t$. The fibration over $\mathbb{P}^1_x$ has the I^*_{10} or D_{14} fiber given by $R_0+R_2+2(R_3+R_4+R_5+R_6+R_7+R_8+S+N_1+N_2+N_3+N_4)+N_0+N_5$, an I_2 or A_1 fiber $A+N_7$, a 2-torsion section (say R_1) and a zero section N_6.

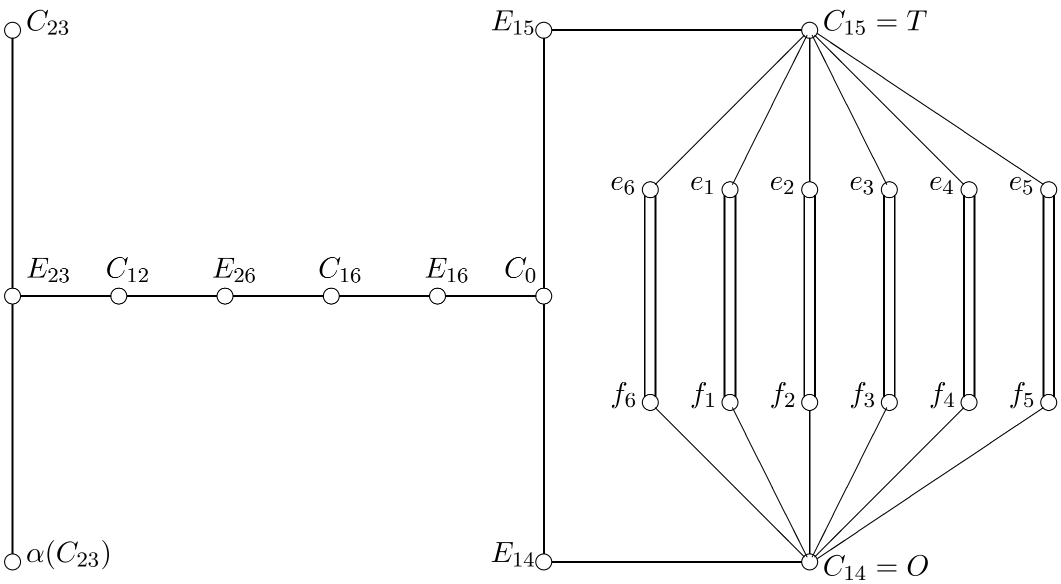
The Nikulin involution σ is translation by the 2-torsion section. It reflects the above picture about its vertical axis of symmetry. There are two obvious copies of $E_8(-1)$ switched by σ, namely the sublattices of $NS(X)$ spanned by the roots $\{S, N_1, N_2, N_3, N_4, N_0, N_5, N_6\}$ and $\{R_7, R_6, R_5, R_4, R_3, R_0, R_2, R_1\}$. Next, we write down some roots on $NS(Y)$, where Y is the quotient K3 surface of X by the involution. As we have described, Y has six I_2 or A_1 fibers $Q_{13}+Q_{14}, \ldots, Q_{23}+Q_{24}$, a I^*_5 or D_9 fiber, namely $Q_1+Q_2+2(Q_3+Q_4+Q_5+Q_6+Q_7+Q_8)+Q_9+Q_{10}$, a 2-torsion section $T=Q_{12}$ and its zero section is $O=Q_{11}$.



It is easily checked that the rational components of the E_8 describe above map as $N_6 \mapsto O$ (recall that N_6 is the zero section of the D_{14} fibration, on which the quotient map is an isogeny of elliptic surfaces), $N_5 \mapsto Q_9, N_4 \mapsto Q_8, N_0 \mapsto Q_{10}, N_3 \mapsto Q_7, N_2 \mapsto Q_6, N_1 \mapsto Q_5, S \mapsto Q_4$. Hence, we see a natural copy of E_8 within the Néron-Severi of Y. On the other hand, we can also see eight roots orthogonal to all the generators of E_8 as well as to each other, namely $Q_{14}, Q_{16}, Q_{18}, Q_{20}, Q_{22}, Q_{24}, Q_1$ and Q_2.

Now, we use the construction of Naruki [Na] which gives an explicit embedding of $N \oplus E_8(-1)$ inside the Néron-Severi lattice of a Kummer surface of a generic principally polarized abelian surface, or $\Lambda_{(16,6)}$. We extend this construction to get an identification of $\Lambda_{(16,6)}$ with $NS(Y)$, i.e. the lattice generated by the roots in the diagram above.

The identification is as follows:



Here $\alpha(C_{23}) = C_{23} + L - 2E_0$.

The class of the fiber is

$$\begin{aligned} F &= C_{23} + \alpha(C_{23}) + 2(E_{23} + C_{12} + E_{26} + C_{16} + E_{16} + C_0) + E_{15} + E_{14} \\ &= 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{36} + E_{45}) \end{aligned}$$

and the $e_1, \ldots, e_6, f_1, \ldots, f_6$ are given by

$$\begin{aligned} e_1 &= (L - E_0) - (E_{12} + E_{46}) \\ e_2 &= 2(L - E_0) - (E_{12} + E_{13} + E_{24} + E_{46} + E_{56}) \\ e_3 &= 3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{36} + E_{45} + E_{46} + E_{56}) \\ e_4 &= 4(L - E_0) - 2(E_{12} + E_{13} + E_{46}) - (E_{24} + E_{25} + E_{36} + E_{45} + E_{56}) \\ e_5 &= 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{34} + E_{36} + E_{45}) \\ e_6 &= E_{35} \\ (f_i &= F - e_i \text{ for all } i) \\ f_4 &= (L - E_0) - (E_{12} + E_{56}) \\ f_3 &= 2(L - E_0) - (E_{12} + E_{13} + E_{25} + E_{46} + E_{56}) \\ f_2 &= 3(L - E_0) - 2E_{12} - (E_{13} + E_{25} + E_{36} + E_{45} + E_{46} + E_{56}) \\ f_1 &= 4(L - E_0) - 2(E_{12} + E_{13} + E_{56}) - (E_{24} + E_{25} + E_{36} + E_{45} + E_{46}) \\ f_6 &= 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{34} + E_{36} + E_{45}) \\ f_5 &= E_{34} \end{aligned}$$

Notice that under the switch of indices $4 \leftrightarrow 5$ we have the permutation of fibers $\tau = (14)(23)(56)$ and in fact $e_i \mapsto f_{\tau(i)}$, $f_i \mapsto e_{\tau(i)}$.

Next, we describe how one may use all this information from the Néron-Severi group to construct x, y and t in the Weierstrass equation for $Y = \mathrm{Km}(J(C))$

$$y^2 = x^3 + a(t)x^2 + b(t)x$$

Consider the class of the fiber $F \in NS(\mathrm{Km}(J(C)))$.

$$F = 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{36} + E_{45}).$$

We can write down the parameter on the base by computing explicitly the sections of $H^0(Y, \mathcal{O}_Y(F))$. This linear system consists of (the pullback of) quintics passing through the points q_0 and q_{ij} which pass through $q_{24}, q_{25}, q_{36}, q_{45}$, having a double point at q_{13}, q_{46}, q_{56} and a triple point at q_{12}. This linear system is 2-dimensional, and taking the ratio of two linearly independent sections gives us the parameter t on the base, $\mathbb{P}^1$, for the elliptic fibration. Now, t is only determined up to the action of PGL_2, but the first restriction we make is to put the I_5^* fiber at $t = \infty$, which fixes t up to affine linear transformations. Any elliptic K3 surface with a 2-torsion section can be written in the form

$$y^2 = x^3 - 2q(t)x^2 + p(t)x$$

with $p(t)$ of degree at most 8 and $q(t)$ of degree at most 4. The 2-torsion section is $(x, y) = (0, 0)$. The discriminant of this elliptic surface is a multiple of $p^2(q^2 - p)$. In fact, we see that p must have degree exactly 6, and the positions $t_1, \ldots, t_6$ of the I_2 fibers are the roots of the polynomial $p(t) = p_0 \prod_{i=1}^{6}(x - t_i)$. Now t is determined up to transformations of the form $t \mapsto at + b$. To have exactly a I_5^* fiber at ∞, we must have $p(t) = q(t)^2 + r(t)$ where $q(t)$ is a monic cubic polynomial and $r(t)$ is a *linear* polynomial in t. We can further fix t up to scalings $t \mapsto at$ by translating t so that the quadratic term of $q(t)$ vanishes. We notice that the top coefficient p_0 of $p(t)$ is a square, and so by scaling t, x, y appropriately, we may assume $p_0 = 1$, i.e. that $p(t)$ and $q(t)$ are monic.

Now we describe how to obtain x. It is a Weil function, so that the horizontal component of its divisor equals $2T - 2O$, and the vertical component is uniquely determined by that fact that (x) is linearly (and hence numerically) equivalently to zero. So we deduce that the divisor of x is $2T - 2O + Q_{10} - Q_9 + Q_{14} + Q_{16} + Q_{18} + Q_{20} + Q_{22} + Q_{24} - 3F_0$, where

$$F_0 = Q_1 + Q_2 + 2(Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8) + Q_9 + Q_{10}$$

is the D_9 fiber.

To convert this to formulas, we figure out the functions which cut out $Q_{16}, \ldots, Q_{24}, F_0, T$ and O. There is a quintic s_1 which cuts out $O = C_{14}$. Now, notice that the D_9 fiber contains C_{12}, C_{16} and C_0. Therefore s_1 is divisible by T_2, T_6 and T_1. We write

$$s_1 = q_1 T_1 T_2 T_6$$

with a quadratic q_1. Next, we know that T_4 cuts out $C_{14} = O$ and T_5 cuts out $C_{15} = T$. To find, for instance, the function which cuts out e_2, we find the quadratic (unique up to constants) which passes through $q_{12}, q_{13}, q_{24}, q_{46}, q_{56}$. Call this function e_1, by abuse of notation. Similarly, we find $e_2, \dots, e_5$. We also note that the factor of T_5 in the numerator of x, which gives a zero along T, also gives a zero along $e_6 = E_{35}$ owing to the fact that $T = C_{15}$ intersects E_{35} (recall that we are working with the singular Kummer, on which the image of the curve E_{35} is just a single point).

Putting everything together, we can write x up to scaling as a quotient of two homogeneous polynomials of degree 16 as follows:

$$x = \frac{e_1 e_2 e_3 e_4 e_5 T_5}{s_1^3 T_4} = \frac{e_1 e_2 e_3 e_4 e_5 T_5}{(T_1 T_2 T_6 q_1)^3 T_4}$$

Finally, we have to scale x and t so that $x^3 + a(t)x^2 + b(t)x$ becomes a square of a function y on the Kummer.

We note that in the equation of the Kummer

$$K_2 z_4^2 + K_1 z_4 + K_0 = 0$$

we can complete the square for z_4 to obtain

$$(K_2 z_4 + K_1/2)^2 = K_1^2/4 - K_0 K_2 = 4T_1 T_2 T_3 T_4 T_5 T_6$$

We let y be a constant multiple of

$$\frac{e_1 e_2 e_3 e_4 e_5 (K_2 z_4 + K_1/2)}{T_1^5 T_2^3 T_4^2 T_6^4 q_1^2},$$

a quotient of two homogenous polynomials of degree 18, and verify that this makes the Weierstrass equation hold. The computation is carried out in a Maxima program which is listed in the appendix and available at

`http://math.harvard.edu/~abhinav/k3maxima.txt`

We noted earlier that the permutation (45) on the A_1 fibers by $\tau = (14)(23)(56)$, and takes e_i to $f_{\tau(i)}$. That is, it switches the components intersecting the identity and 2-torsion sections as well. In addition, it switches the zero section C_{14} and the 2-torsion section C_{15}, and on the D_9 fibers it switches the two near leaves E_{15} and E_{14}, namely, again the components intersecting T and O. On the other hand, consider the action on $NS(Y)$ induced by the translation by T. Under this map, T and O get swapped, the 2-torsion and identity components of the D_9 and A_1 fibers all get switched, and the far leaves of the D_9 fiber also get switched (this can be seen, for instance, from the fact that the group of simple components of the special D_9 fiber is $\mathbb{Z}/4\mathbb{Z}$). There is no permutation of the A_1 fibers themselves. Therefore the effect of the permutation 45 is the same as translation by 2-torsion composed with a pure involution (14)(23)(56) of the A_1 fibers and a switch of the far leaves of the D_9 fiber. Since the far leaves of the D_9 fiber are switched by the Galois involution that multiplies the square root of $b'' = I_{10}/4$ by -1, this tells us that we have the correct twist, since I_{10} is within a square factor of the discriminant of the sextic.

5.8 The correspondence of sextics

The construction above gave us a correspondence of sextics

$$f(x) = \sum f_i x^i = \prod (x - x_i)$$

and

$$g(x) = \left(x^3 - \frac{I_4}{12}x + \frac{I_2 I_4 - 3I_6}{108}\right)^2 + I_{10}\left(x - \frac{I_2}{24}\right).$$

Therefore, over an algebraically closed field, we get a birational map from the moduli space of 6 points in $\mathbb{P}^1$ (i.e. the quotient of $(\mathbb{P}^1)^6$ under the action of PGL_2 and S_6) with the space of roots up to scaling of

$$(x^3 + ax + b)^2 + (a'x + b')$$

as a, b, a', b' vary (we suppressed b'' since it just scales a' and b'). This latter space is cut out inside $\mathbb{P}^5 = \{(X_1 : X_2 : X_3 : X_4 : X_5 : X_6)\}$ by the hyperplane $\sigma_1(X) = X_1 + \ldots + X_6 = 0$ and by the quartic hypersurface $\sigma_2(X)^2 = 4\sigma_4(X)$, where σ_2 and σ_4 are the second and fourth elementary symmetric functions of the X_i. Thus, we get a model as a quartic threefold in $\mathbb{P}^4$, which is known as the Igusa quartic.

There is no simple one-one correspondence between the roots of $f(x)$ and $g(x)$, since the two actions of S_6 acting by the permuation representation on the six roots of $f(x)$ on the six roots of $g(x)$ are related by an outer automorphism. To see this, we recall from the last section that the permutation (45) on the roots of $f(x)$ (the Weierstrass points) acts on the roots of $g(x)$ (which are the locations of the A_1 fibers) by the permutation (14)(23)(56). By symmetry, all the transpositions of S_6 act by a product of three transpositions on the roots of $g(x)$. Thus we get a homomorphism $S_6(f) \to S_6(g)$ which is an outer automorphism.

5.9 Verifying the isogeny via the Grothendieck-Lefschetz trace formula

We describe a method suggested by Elkies which provides a necessary criterion for two surfaces to be related by a rational isogeny. This method can be used as a check that the correspondence between the original K3 surface X with E_8 and E_7 fibers and the Kummer surface Y obtained as a quotient by a Nikulin involution is defined over $\mathbb{Q}$. Recall that $Y = \mathrm{Km}(J(C))$, where C is the genus 2 curve whose equation is given by

$$y^2 = f(x) = \sum_{i=0}^{6} f_i x^i.$$

Here $f(x)$ is a sextic polynomial whose Igusa-Clebsch invariants are I_2, I_4, I_6, I_{10}. On the other hand, the K3 surface X with E_8 and E_7 fibers at ∞ and 0 is given by

$$y^2 = x^3 - \frac{1}{12}t^3(I_4 t + 12)x + \frac{1}{216}t^5(54I_{10}t^2 + 2(I_2I_4 - 3I_6)t + 9I_2).$$

Now, suppose $f_0, \ldots, f_6 \in \mathbb{Z}$. We would like to count the number of points on X modulo p for some prime $p > 3$. We assume the reduction of $f_0, \ldots, f_6$ modulo p are generic, so that the sextic $f(x)$ has distinct roots, and the K3 surface X mod p has no other reducible fibers, and no worse reduction than E_8 and E_7 at $t = \infty$ and 0. Then, by the Grothendieck-Lefschetz trace formula for X, we have

$$\#X_p(\mathbb{F}_p) = \#Y_p(\mathbb{F}_p) = \sum_{i=0}^{4} \mathrm{Tr}(Fr_p^*|H^i(Y_p, \mathbb{Q}_l))$$

where $l \neq p$ is some prime. Now, recall that since Y_p is a Kummer and therefore a K3 surface, $h^1(Y_p, \mathbb{Q}_l) = 0$ and $h^3(Y_p, \mathbb{Q}_l) = 0$ by standard comparison theorems. Therefore only H^0, H^2, H^4 contribute to the trace. In addition, we already know that the trace of Frobenius on H^0 is 1 and that on H^4 is p^2 by standard yoga of weights. The only term remaining to examine is $H^2(Y_p, \mathbb{Q}_l)$. Recall that for the K3 surface Y, $H^2(Y, \mathbb{Q}) = H^2(Y, \mathbb{Z}) \otimes \mathbb{Q}$ decomposes as $(NS(Y) \otimes \mathbb{Q}) \oplus (T_Y \otimes \mathbb{Q})$. Within the first subspace lies the span of the sixteen rational curves on the Kummer formed by the desingularizing the image of the sixteen 2-torsion points on A. The action of Fr_p on that subspace contributes $16p$ to the trace, and the trace on the complementary subspace is exactly equal to the trace of Fr_p on the six-dimensional space $H^2(J(C)_p, \mathbb{Q}_l)$. That trace may be computed as follows: $H^1(C_p, \mathbb{Q}_l) = H^1(J(C)_p, \mathbb{Q}_l)$ by definition. Now suppose the eigenvalues of Fr_p on $H^1(C_p, \mathbb{Q}_l)$ are $\alpha_1, \ldots, \alpha_4$. Then the trace of Fr_p on $H^2(J(C)_p, \mathbb{Q}_l)$ is $\sum_{i<j} \alpha_i\alpha_j$. On the other hand, we have

$$C_p(\mathbb{F}_p) = 1 + p + \sum_i \alpha_i \text{ and } C_p(\mathbb{F}_{p^2}) = 1 + p^2 + \sum_i \alpha_i^2$$

Therefore we may read out $\sum_i \alpha_i$ and $\sum_i \alpha_i^2$ and therefore $\sum_{i<j} \alpha_i\alpha_j = \frac{1}{2}((\sum \alpha_i)^2 - (\sum \alpha_i^2))$ by counting the number of $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$-valued points of C.

On the other hand, because the irreducible components of the E_8 and E_7 fibers of X are defined rationally, we have that

$$\begin{aligned} X(\mathbb{F}_p) &= (9p+1) + (8p+1) + \sum_{t \in \mathbb{F}_p - \{0\}} \#X_{p,t} \\ &= 17p + 2 + \sum_{t \in \mathbb{F}_p - \{0\}} \left(p + 1 + \sum_{x \in \mathbb{F}_p} \left(\frac{R(x,t)}{p} \right) \right) \\ &= p^2 - 1 + 17p + 2 + \sum_{t \in \mathbb{F}_p - \{0\}, x \in \mathbb{F}_p} \left(\frac{R(x,t)}{p} \right) \end{aligned}$$

where

$$R(x,t) = x^3 - \frac{1}{12}t^3(I_4t + 12)x + \frac{1}{216}t^5(54I_{10}t^2 + 2(I_2I_4 - 3I_6)t + 9I_2)$$

is the expression on the right hand side of the equation of the K3 surface X. We would want the above point count to equal

$$1 + p^2 + 16p + \mathrm{Tr}(F_p^*|H^2(J(C)_p, \mathbb{Q}_l))$$

We can use a computer program to calculate both sides and check that they are equal. This provides experimental verification of an isogeny between these surfaces. In the specific case above, the point counts are indeed always equal.

Chapter 6

Appendix

In this appendix we give the Maxima code which verifies the assertions of Section 5.7. We may assume, without loss of generality, that the curve C is given by

$$y^2 = x(x-1)(x-2)(x-t_4)(x-t_5)(x-t_6)$$

since we may move three of the Weierstrass points to $0, 1, 2$ by an element of PGL_2. The choice of $0, 1, 2$ instead of $0, 1, \infty$ is made to be able to use the formulas of [CF], which assumes that the Weierstrass points are finite.

```
t:[0,1,2,t4,t5,t6];

F:expand(product(x-t[i],i,1,6));
f0:coeff(F,x,0);
f1:coeff(F,x,1);
f2:coeff(F,x,2);
f3:coeff(F,x,3);
f4:coeff(F,x,4);
f5:coeff(F,x,5);
f6:coeff(F,x,6);

K2:z2^2-4*z1*z3;

K1:-4*z1^3*f0-2*z1^2*z2*f1-4*z1^2*z3*f2-2*z1*z2*z3*f3-4*z1*z3^2*f4-2*z2*
z3^2*f5-4*z3^3*f6;

K0:-4*z1^4*f0*f2+z1^4*f1^2-4*z1^3*z2*f0*f3-2*z1^3*z3*f1*f3-4*z1^2*z2^2
*f0*f4+4*z1^2*z2*z3*f0*f5-4*z1^2*z2*z3*f1*f4-4*z1^2*z3^2*f0*f6+2*z1^2*z3^
2*f1*f5-4*z1^2*z3^2*f2*f4+z1^2*z3^2*f3^2-4*z1*z2^3*f0*f5+8*z1*z2^2*z3*f0
```

```
*f6-4*z1*z2^2*z3*f1*f5+4*z1*z2*z3^2*f1*f6-4*z1*z2*z3^2*f2*f5-2*z1*z3^3*f3*
f5-4*z2^4*f0*f6-4*z2^3*z3*f1*f6-4*z2^2*z3^2*f2*f6-4*z2*z3^3*f3*f6-4*z3^4*
f4*f6+z3^4*f5^2;

K:K2*z4^2 + K1*z4 + K0;

/* Kummer surface is K2*z4^2 + K1*z4 + K0 = 0 */

p0:[z1=0,z2=0,z3=0,z4=1];
p12:[z1=1, z2=1, z3=0, z4=-2*t6*t5*t4];
p13:[z1=1, z2=2, z3=0, z4=-t6*t5*t4];
p14:[z1=1, z2=t4, z3=0, z4=-2*t6*t5];
p15:[z1=1, z2=t5, z3=0, z4=-2*t6*t4];
p16:[z1=1, z2=t6, z3=0, z4=-2*t5*t4];
p23:[z1=1, z2=3, z3=2, z4=(-2*t5-2*t6)*t4+(-2*t6*t5-4)];
p24:[z1=1, z2=t4 + 1, z3=t4, z4=-t4^2+((-t6-2)*t5-2*t6)*t4];
p25:[z1=1, z2=t5 + 1, z3=t5, z4=(-t6-2)*t5*t4+(-t5^2-2*t6*t5)];
p26:[z1=1, z2=t6 + 1, z3=t6, z4=(-t6*t5-2*t6)*t4+(-2*t6*t5-t6^2)];
p34:[z1=1, z2=t4 + 2, z3=2*t4, z4=-4*t4^2+((-2*t6-2)*t5-2*t6)*t4];
p35:[z1=1, z2=t5 + 2, z3=2*t5, z4=(-2*t6-2)*t5*t4+(-4*t5^2-2*t6*t5)];
p36:[z1=1, z2=t6 + 2, z3=2*t6, z4=(-2*t6*t5-2*t6)*t4+(-2*t6*t5-4*t6^2)];
p45:[z1=1, z2=t4 + t5, z3=t5*t4, z4=-t5^2*t4^2+(-3*t6-2)*t5*t4];
p46:[z1=1, z2=t4 + t6, z3=t6*t4, z4=-t6^2*t4^2+(-3*t6*t5-2*t6)*t4];
p56:[z1=1, z2=t5 + t6, z3=t6*t5, z4=-3*t6*t5*t4+(-t6^2*t5^2-2*t6*t5)];

/* some tropes */

T1:t[1]^2*z1 -t[1]*z2 + z3;
T2:t[2]^2*z1 -t[2]*z2 + z3;
T3:t[3]^2*z1 -t[3]*z2 + z3;
T4:t[4]^2*z1 -t[4]*z2 + z3;
T5:t[5]^2*z1 -t[5]*z2 + z3;
T6:t[6]^2*z1 -t[6]*z2 + z3;

/* pencil of quintics */

G:c1*z3^5+c2*z2*z3^4+c7*z1*z3^4+c3*z2^2*z3^3+c8*z1*z2*z3^3+c12*z1^2*z3^3
+c4*z2^3*z3^2+c9*z1*z2^2*z3^2+c13*z1^2*z2*z3^2+c16*z1^3*z3^2+c5*z2^4*z3
+c10*z1*z2^3*z3+c14*z1^2*z2^2*z3+c17*z1^3*z2*z3+c19*z1^4*z3+c6*z2^5+
c11*z1*z2^4+c15*z1^2*z2^3+c18*z1^3*z2^2+c20*z1^4*z2+c21*z1^5;

G1: diff(G,z1);
G2: diff(G,z2);
G3: diff(G,z3);
```

```
G11: diff(G1,z1);
G12: diff(G1,z2);
G13: diff(G1,z3);
G22: diff(G2,z2);
G23: diff(G2,z3);
G33: diff(G3,z3);

v: solve([ev(G,p24)=0,ev(G,p25)=0,ev(G,p36)=0,ev(G,p45)=0,
ev(G1,p13)=0,ev(G2,p13)=0,ev(G3,p13)=0,
ev(G1,p46)=0,ev(G2,p46)=0,ev(G3,p46)=0,ev(G1,p56)=0,ev(G2,p56)=0,
ev(G3,p56)=0,ev(G11,p12)=0,ev(G12,p12)=0,ev(G13,p12)=0,ev(G22,p12)=0,
ev(G23,p12)=0,ev(G33,p12)=0],[c1,c2,c3,c4,c5,c6,c7,c8,c9,
c10,c11,c12,c13,c14,c15,c16,c17,c18,c19,c20,c21]);

s1:ev(G,ev(v, %r1=1,%r2=0));
s1:num(rat(s1));
s1: factor(s1);

s2:ev(G,ev(v, %r1=0,%r2=1));
s2:num(rat(s2));

/* will modify s2 later */

/* computing e4 */

G: c1*z3^4+c2*z2*z3^3+c6*z1*z3^3+c3*z2^2*z3^2+c7*z1*z2*z3^2+c10*z1^2*z3^2+
c4*z2^3*z3+c8*z1*z2^2*z3+c11*z1^2*z2*z3+c13*z1^3*z3+c5*z2^4+c9*z1*z2^3+
c12*z1^2*z2^2+c14*z1^3*z2+c15*z1^4;


G1: diff(G,z1);
G2: diff(G,z2);
G3: diff(G,z3);

v: solve([ev(G,p24)=0,ev(G,p25)=0,ev(G,p36)=0,ev(G,p45)=0,ev(G,p56)=0,
ev(G1,p13)=0,ev(G2,p13)=0,ev(G3,p13)=0,
ev(G1,p46)=0,ev(G2,p46)=0,ev(G3,p46)=0,
ev(G1,p12)=0,ev(G2,p12)=0,ev(G3,p12)=0],
[c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,c11,c12,c13,c14,c15]);

e4: ev(G,ev(v, %r3=1));
e4: num(rat(e4));

/* computing e3 */
```

```
G: c1*z3^3+c2*z2*z3^2+c5*z1*z3^2+c3*z2^2*z3+c6*z1*z2*z3+c8*z1^2*z3+c4*z2^3+
c7*z1*z2^2+c9*z1^2*z2+c10*z1^3;

G1: diff(G,z1);
G2: diff(G,z2);
G3: diff(G,z3);

v: solve([ev(G,p13)=0,ev(G,p24)=0,ev(G,p46)=0,ev(G,p36)=0,ev(G,p45)=0,
ev(G,p56)=0,ev(G1,p12)=0,ev(G2,p12)=0,ev(G3,p12)=0],
[c1,c2,c3,c4,c5,c6,c7,c8,c9,c10]);

e3: ev(G,ev(v, %r4=1));
e3: num(rat(e3));

/* computing e2 */

G: c1*z3^2+c2*z2*z3+c4*z1*z3+c3*z2^2+c5*z1*z2+c6*z1^2;

v: solve([ev(G,p13)=0,ev(G,p24)=0,ev(G,p46)=0,ev(G,p12)=0,ev(G,p56)=0],
[c1,c2,c3,c4,c5,c6]);

e2: ev(G,ev(v, %r5=1));
e2: num(rat(e2));

/* computing e1 */

G: z3*c1 + z2*c2 + z1*c3;

v: solve([ev(G,p46)=0,ev(G,p12)=0],
[c1,c2,c3]);

e1: ev(G,ev(v, %r6=1));
e1: -num(rat(e1));

/* computing e5 */

G: c1*z3^5+c2*z2*z3^4+c7*z1*z3^4+c3*z2^2*z3^3+c8*z1*z2*z3^3+c12*z1^2*z3^3+
c4*z2^3*z3^2+c9*z1*z2^2*z3^2+c13*z1^2*z2*z3^2+c16*z1^3*z3^2+c5*z2^4*z3+
c10*z1*z2^3*z3+c14*z1^2*z2^2*z3+c17*z1^3*z2*z3+c19*z1^4*z3+c6*z2^5+
c11*z1*z2^4+c15*z1^2*z2^3+c18*z1^3*z2^2+c20*z1^4*z2+c21*z1^5;

G1: diff(G,z1);
G2: diff(G,z2);
```

```
G3: diff(G,z3);
G11: diff(G1,z1);
G12: diff(G1,z2);
G13: diff(G1,z3);
G22: diff(G2,z2);
G23: diff(G2,z3);
G33: diff(G3,z3);

v: solve([ev(G,p24)=0,ev(G,p25)=0,ev(G,p36)=0,ev(G,p45)=0,ev(G,p34)=0,
ev(G1,p13)=0,ev(G2,p13)=0,ev(G3,p13)=0,ev(G1,p46)=0,ev(G2,p46)=0,
ev(G3,p46)=0,ev(G1,p56)=0,ev(G2,p56)=0,ev(G3,p56)=0,ev(G11,p12)=0,ev
(G12,p12)=0,ev(G13,p12)=0,ev(G22,p12)=0,ev(G23,p12)=0,ev(G33,p12)=0],
[c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,c11,c12,c13,c14,c15,c16,c17,c18,c19,c20,c21]);

e5: ev(G,ev(v, %r7=1));
e5: num(rat(e5));

/* computing d1 */

G: c1*z3^4+c2*z2*z3^3+c6*z1*z3^3+c3*z2^2*z3^2+c7*z1*z2*z3^2+c10*z1^2*z3^2+
c4*z2^3*z3+c8*z1*z2^2*z3+c11*z1^2*z2*z3+c13*z1^3*z3+c5*z2^4+c9*z1*z2^3+
c12*z1^2*z2^2+c14*z1^3*z2+c15*z1^4;

G1: diff(G,z1);
G2: diff(G,z2);
G3: diff(G,z3);

v: solve([ev(G,p24)=0,ev(G,p25)=0,ev(G,p36)=0,ev(G,p45)=0,ev(G,p46)=0,
ev(G1,p13)=0,ev(G2,p13)=0,ev(G3,p13)=0,
ev(G1,p56)=0,ev(G2,p56)=0,ev(G3,p56)=0,
ev(G1,p12)=0,ev(G2,p12)=0,ev(G3,p12)=0],
[c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,c11,c12,c13,c14,c15]);

d1: ev(G,ev(v, %r8=1));
d1: -num(rat(d1));

/* computing d2 */

G: c1*z3^3+c2*z2*z3^2+c5*z1*z3^2+c3*z2^2*z3+c6*z1*z2*z3+c8*z1^2*z3+c4*z2^3
+c7*z1*z2^2+c9*z1^2*z2+c10*z1^3;

G1: diff(G,z1);
G2: diff(G,z2);
G3: diff(G,z3);
```

```
v: solve([ev(G,p13)=0,ev(G,p25)=0,ev(G,p46)=0,ev(G,p36)=0,
ev(G,p45)=0,ev(G,p56)=0,ev(G1,p12)=0,ev(G2,p12)=0,ev(G3,p12)=0],
[c1,c2,c3,c4,c5,c6,c7,c8,c9,c10]);

d2: ev(G,ev(v, %r9=1));
d2: -num(rat(d2));

/* computing d3 */

G: c1*z3^2+c2*z2*z3+c4*z1*z3+c3*z2^2+c5*z1*z2+c6*z1^2;

v: solve([ev(G,p13)=0,ev(G,p25)=0,ev(G,p46)=0,ev(G,p12)=0,ev(G,p56)=0],
[c1,c2,c3,c4,c5,c6]);

d3: ev(G,ev(v, %r10=1));
d3: -num(rat(d3));

/* computing d4 */

G: z3*c1 + z2*c2 + z1*c3;

v: solve([ev(G,p56)=0,ev(G,p12)=0],
[c1,c2,c3]);

d4: ev(G,ev(v, %r11=1));
d4: num(rat(d4));

/* computing d6 */

G: c1*z3^5+c2*z2*z3^4+c7*z1*z3^4+c3*z2^2*z3^3+c8*z1*z2*z3^3+c12*z1^2*z3^3
+c4*z2^3*z3^2+c9*z1*z2^2*z3^2+c13*z1^2*z2*z3^2+c16*z1^3*z3^2+c5*z2^4*z3+
c10*z1*z2^3*z3+c14*z1^2*z2^2*z3+c17*z1^3*z2*z3+c19*z1^4*z3+c6*z2^5+
c11*z1*z2^4+c15*z1^2*z2^3+c18*z1^3*z2^2+c20*z1^4*z2+c21*z1^5;

G1: diff(G,z1);
G2: diff(G,z2);
G3: diff(G,z3);
G11: diff(G1,z1);
G12: diff(G1,z2);
G13: diff(G1,z3);
G22: diff(G2,z2);
G23: diff(G2,z3);
G33: diff(G3,z3);
```

```
v: solve([ev(G,p24)=0,ev(G,p25)=0,ev(G,p36)=0,ev(G,p45)=0,ev(G,p35)=0,
ev(G1,p13)=0,ev(G2,p13)=0,ev(G3,p13)=0,ev(G1,p46)=0,ev(G2,p46)=0,
ev(G3,p46)=0,ev(G1,p56)=0,ev(G2,p56)=0,ev(G3,p56)=0,ev(G11,p12)=0,
ev(G12,p12)=0,ev(G13,p12)=0,ev(G22,p12)=0,ev(G23,p12)=0,ev(G33,p12)=0],
[c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,c11,c12,c13,c14,c15,c16,c17,c18,c19,c20,c21]);

d6: ev(G,ev(v, %r12=1));
d6: num(rat(d6));

u1: (s2-d1*e1)/s1;
u2: (s2-d2*e2)/s1;
u3: (s2-d3*e3)/s1;
u4: (s2-d4*e4)/s1;
u5: (s2-e5)/s1;
u6: (s2-d6)/s1;

u0: (u1+u2+u3+u4+u5+u6)/6;

u1: u1-u0;
u2: u2-u0;
u3: u3-u0;
u4: u4-u0;
u5: u5-u0;
u6: u6-u0;

s2: s2-u0*s1;

u1+u2+u3+u4+u5+u6;

rat((s2-s1*u6)/(d6));
rat((s2-s1*u5)/(e5));
rat((s2-s1*u4)/(d4*e4));
rat((s2-s1*u3)/(d3*e3));
rat((s2-s1*u2)/(d2*e2));
rat((s2-s1*u1)/(d1*e1));

q1: s1/(T1*T2*T6);

M:factor(-K2*K+(K2*z4+K1/2)^2);

M/(4*T1*T2*T3*T4*T5*T6);

/*
```

```
y: e1*e2*e3*e4*e5*(
t4*t5*(t5-t4)*t6*(t6-t4)*(t6-t5))*(c2*z4+c1/2)/(T1^5*T2^3*T4^2*T6^4*q1^2);

x: e1*e2*e3*e4*e5*T5/((T1*T2*T6*q1)^3*d5);

u:s2/s1;
*/

p: (u-u1)*(u-u2)*(u-u3)*(u-u4)*(u-u5)*(u-u6);
coeff(p,5,0);
w: u^3 + coeff(p,u,4)/2*u + coeff(p,u,3)/2;
q: p-w^2;

/* write down numerators and denominators of everything in sight */

wnum: s2^3 + coeff(p,u,4)/2*s2*s1^2 + coeff(p,u,3)/2*s1^3;
wden: s1^3;
pnum: (s2-u1*s1)*(s2-u2*s1)*(s2-u3*s1)*(s2-u4*s1)*(s2-u5*s1)*(s2-u6*s1);

Note that
pnum = (e1*e2*e3*e4*d1*d2*d3*d4*e5*d6);

pden: s1^6;

xnum: e1*e2*e3*e4*e5*T5;
xden: ((T1*T2*T6*q1)^3*d5);

The equation of the surface is

y^2 = x*(x^2 -2*w*x + p)

The identity boils down to checking the following:

4*(t4*t5*(t5-t4)*t6*(t6-t4)*(t6-t5))^2*T2^4*T3*T6^2*q1^5
=
(e1*e2*e3*e4*e5*T5^2) - 2*wnum*T4*T5 + T4^2*d1*d2*d3*d4*d6
*/

((e1*e2*e3*e4*e5*T5^2) - 2*wnum*T4*T5 + T4^2*d1*d2*d3*d4*d6)/(
4*(t4*t5*(t5-t4)*t6*(t6-t4)*(t6-t5))^2*T2^4*T3*T6^2*q1^5);
```

It remains to check that the Igusa-Clebsch invariants of the sextic

$$y^2 = x(x-1)(x-2)(x-t_4)(x-t_5)(x-t_6)$$

are related to the the coefficients of the polynomials w and q, as asserted. But this is easily verified.

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