Algebraic models and arithmetic geometry of Teichmüller curves in genus two

Abhinav Kumar and Ronen E. Mukamel

March 17, 2014

Abstract

For each discriminant D of a real quadratic order, the Weierstrass curve W_D is a finite volume hyperbolic orbifold which is algebraically and isometrically immersed in the moduli space of genus two Riemann surfaces. The components of such an immersion are called Teichmüller curves. We describe a numerical method for computing algebraic models of Weierstrass curves and a rigorous certification procedure involving only arithmetic in function fields over number fields. We demonstrate our methods by giving an explicit model of W_D for the thirty fundamental discriminants $D < 100$. Our examples include the first explicit models of positive genus Teichmüller curves and give evidence that Teichmüller curves admit a rich arithmetic geometry.

Contents

1 Introduction

For each integer $D > 1$ with $D \equiv 0$ or $1 \bmod 4$, the *Weierstrass curve of discriminant D* is the moduli space W_D consisting of pairs $(X, [\omega])$ where: (1) X is a Riemann surface of genus two, (2) ω is a holomorphic one-form on X with a double zero and (3) the Jacobian $\mathrm{Jac}(X)$ admits real multiplication by the quadratic ring $\mathcal{O}_D$ of discriminant D and stabilizing the one-form up to scale $[\omega]$. The space W_D can be viewed as both an algebraic curve and a finite volume hyperbolic orbifold and emerges from the study of billiards in L-shaped polygons. Weierstrass curves are important in Teichmüller theory because the natural algebraic immersion into the moduli space of genus two Riemann surfaces

$$W_D \to \mathcal{M}_2$$

is locally isometric for Teichmüller metric on $\mathcal{M}_2$ [Mc1] (see also [Ca]). The irreducible components of such an immersion are called *Teichmüller curves* and the components of Weierstrass curves are the main examples of Teichmüller curves in $\mathcal{M}_2$ [Mc3].

Few explicit algebraic models of Teichmüller curves have appeared in the literature, and those that have appeared [BM1, BM2, Lo] all have genus zero and hyperbolic volume at most 3π. The primary goal of this paper is to describe methods for numerically computing and rigorously verifying algebraic models of Weierstrass curves and to demonstrate our methods by computing and verifying models of W_D for the thirty fundamental discriminants $1 < D < 100$.[1] Our examples give the first explicit models of Teichmüller curves with positive genus and include an example of genus eight and hyperbolic volume 60π. We will also present evidence drawn from our examples of a rich arithmetic geometry associated to Teichmüller curves.

Weierstrass curves in Hilbert modular surfaces. The starting point for our study of Weierstrass curves are the explicit algebraic models of Hilbert modular surfaces given in [EK]. The *Hilbert modular surface of discriminant* D is the complex orbifold $X_D = \mathbb{H} \times \mathbb{H}/\operatorname{PSL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$.[2] When viewed as an algebraic surface, X_D is a moduli space of principally polarized abelian surfaces with real multiplication by $\mathcal{O}_D$. In [EK], Hilbert modular surfaces for fundamental discriminants $D < 100$ are parametrized by studying elliptic fibrations of K3 surfaces yielding a birational model of X_D.

Theorem (Elkies-Kumar)**.** *For fundamental discriminants $1 < D < 100$, the Hilbert modular surface X_D is birational to the degree two cover of the (r, s)-plane branched along the curve $b_D(r, s) = 0$ where b_D is the polynomial in Table T.1.*

The period mapping sending a Riemann surface to its Jacobian lifts to an embedding of W_D into X_D. Our main theorems identify the locus corresponding to W_D in X_D in the models above. For discriminants $D \not\equiv 1 \bmod 8$, the curve W_D is irreducible with an explicit algebraic model given by the following theorem.

Theorem 1.1. *For fundamental discriminants $1 < D < 100$ with $D \not\equiv 1 \bmod 8$, the Weierstrass curve W_D is birational to the curve $w_D(r, s) = 0$ where w_D is the polynomial in Table T.2.*

For discriminants $D \equiv 1 \bmod 8$, the curve $W_D = W_D^0 \sqcup W_D^1$ is a disjoint union of two irreducible components distinguished by a spin invariant [Mc2]. For such discriminants, the components of W_D have Galois conjugate algebraic models defined over $\mathbb{Q}(\sqrt{D})$ [BM1]. Our next theorem identifies explicit models of these curves.

Theorem 1.2. *For fundamental discriminants $1 < D < 100$ with $D \equiv 1 \bmod 8$, the curve W_D^ϵ is birational to the curve $w_D^\epsilon(r, s) = 0$ where w_D^0 is the polynomial in Table T.3 and w_D^1 is the Galois conjugate of w_D^0.*

The first Weierstrass curve of positive genus is the curve W_{44} of genus one. The birational model $w_{44}(r, s) = 0$ of W_{44} is depicted in Figure 1 along with the curve $b_{44}(r, s) = 0$. Our proofs of Theorems 1.1 and 1.2 will yield an explicit birational model of the universal curve over W_D for fundamental discriminants $1 < D < 100$.

[1] Compare with Theorems 1.1 and 1.2. We have overwhelming numerical evidence that Theorems 1.1 and 1.2 are true as stated. The certification procedure we will describe involves only rigorous arithmetic in number fields (i.e. no floating point arithmetic) and, to date, we have certified our equations for discriminants $D \le 65$, $D = 73$ and $D = 88$. We will complete this certification process for the remaining eight discriminants shortly and remove this footnote.

[2] The surface X_D is isomorphic to $\mathbb{H} \times \overline{\mathbb{H}}/\operatorname{PSL}_2(\mathcal{O}_D)$ and is typically denoted $Y_-(D)$ in the algebraic geometry literature [vdG, HZ].

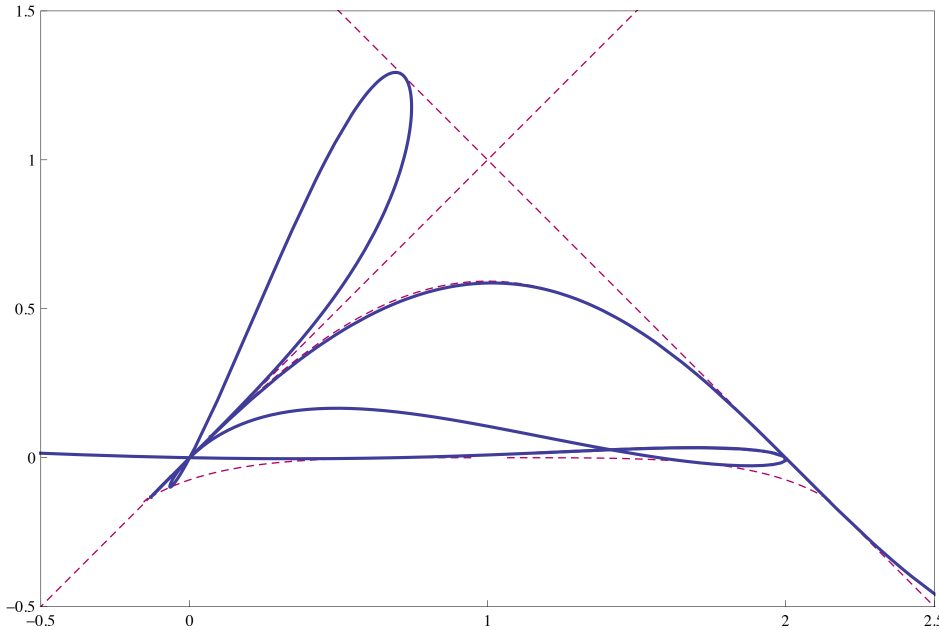

Figure 1: The Hilbert modular surface X_{44} is birational to degree two cover of the (r,s)-plane branched along the curve $b_{44}(r,s)=0$ (dashed). The Weierstrass curve W_{44} is birational to the curve $w_{44}(r,s)=0$ (solid).

Rational, hyperelliptic and plane quartic models. The polynomials w_D listed in Table T.2 are complicated in part because they reflect how W_D is embedded in X_D. The homeomorphism type of W_D is determined in [Ba, Mc2, Mu3] and in Table T.4 we list the homeomorphism type of W_D for the discriminants considered in this paper. For fundamental discriminants $D \leq 73$ with $D \neq 69$, the irreducible components of W_D have genus at most three and algebraic models simpler than those given by Theorems 1.1 and 1.2.

For discriminants $D \leq 41$, each irreducible component of W_D has genus zero. Our proof of Theorems 1.1 and 1.2 will give rational parametrizations of the irreducible components of $w_D(r,s)=0$ for such D and yield our next result.

Theorem 1.3. *For fundamental discriminants $D \leq 41$, each component of W_D is birational to $\mathbb{P}^1$ over $\mathbb{Q}(\sqrt{D})$. For $D \leq 41$ with $D \not\equiv 1 \bmod 8$ and $D \neq 21$, the curve W_D is also birational to $\mathbb{P}^1$ over $\mathbb{Q}$. The curve W_{21} has no rational points and is birational over $\mathbb{Q}$ to the conic $g_{21}(x,y)=0$ where:*

$$g_{21}(x,y) = 21\left(11x^2 - 182x - 229\right) + y^2.$$

The curve W_{44} of genus one and the curves W_{53} and W_{61} of genus two are hyperelliptic and the curves W_{56} and W_{60} of genus three are canonically embedded as smooth quartics in $\mathbb{P}^2$. Our next theorem identifies hyperelliptic and plane quartic models of these curves.

Theorem 1.4. *For $D \in \{44, 53, 56, 60, 61\}$, the curve W_D is birational to $g_D(x,y)=0$ where g_D is the polynomial listed in Table 1.1.*

Hyperelliptic and plane quartic models of Weierstrass curves
$g_{44}(x,y) = x^3 + x^2 + 160x + 3188 - y^2$
$g_{53}(x,y) = 7711875 + 3572389x + 777989x^2 + 100812x^3 + 8252x^4 + 401x^5 + 9x^6 - (1+x^2)y - y^2$
$g_{56}(x,y) = 35+10x-20x^2-2x^3+x^4-43y+15xy+5x^2y-x^3y+33y^2-xy^2-5x^2y^2-10y^3+4xy^3+4y^4$
$g^0_{57}(x,y) = x^3 + (1330\sqrt{57} - 4710)x^2 - (7130112\sqrt{57} - 40387584)x - y^2$
$g_{60}(x,y) = 4x^4 - 8x^3y - 4x^3 + 50x^2y^2 - 2x^2y - 44xy^3 - 56xy^2 + 10xy + 228y^4 - 32y^3 - 8y^2 + y$
$g_{61}(x,y) = 12717 - 527x - 6117x^2 + 1498x^3 - 604x^4 - 282x^5 + 324x^6 - \left(x^2+x+1\right)y - y^2$
$g^0_{65}(x,y) = x^3 + \left(27\sqrt{65} - 229\right)x^2 + \frac{1}{2}\left(11225\sqrt{65} - 90375\right)x - y^2$
$g^0_{73}(x,y) = x^3 + \frac{1}{2}\left(1+\sqrt{73}\right)x^2 - \frac{1}{2}\left(701 + 83\sqrt{73}\right)x + \frac{1}{8}\left(34553 + 4045\sqrt{73}\right) - y^2$

Table 1.1: For discriminants $44 \leq D \leq 73$ with $D \neq 69$, each irreducible component of W_D has either a hyperelliptic or plane quartic model defined above (cf. Theorems 1.4 and 1.5).

The irreducible components of W_{57}, W_{65} and W_{73} have genus one. We also identify hyperelliptic models of these curves.

Theorem 1.5. *For $D \in \{57, 65, 73\}$, the curve W^ϵ_D is birational to $g^\epsilon_D(x,y) = 0$ where g^0_D is the polynomial listed in Table 1.1 and g^1_D is the Galois conjugate of g^0_D.*

Arithmetic of Teichmüller curves. We hope that the models of Weierstrass curves in Tables 1.1, T.2 and T.3 will encourage the study of the arithmetic geometry of Teichmüller curves. To that end, we now list several striking facts about these examples that give evidence toward the theme:

Teichmüller curves are arithmetically interesting.

We will denote by $\overline{W}_D$ the smooth, projective curve birational to W_D. The curve $\overline{W}_D$ is obtained from W_D by filling in finitely many cusps on W_D (studied in [Mc2]) and smoothing finitely many orbifold points (studied in [Mu3]). Our rational, hyperelliptic and plane quartic birational models of low genus components of W_D extend to biregular models of components of $\overline{W}_D$. Throughout what follows, we identify $\overline{W}_D$ with these biregular models via the parametrizations in [KM2] as described in Section 7.

Singular primes. The first indication that the curves $\overline{W}_D$ have interesting arithmetic is the fact our low, positive genus examples are singular only at small primes. Our next two theorems suggest the following.

The primes of bad reduction for Teichmüller curves
have arithmetic significance.

To formulate a precise statement, we define

$$N(D) = 2 \cdot D \cdot \prod_e \frac{D - e^2}{4} \text{ where } e \text{ ranges in } \left\{e : e > 0, e \equiv D \bmod 2 \text{ and } e^2 < D\right\}. \tag{1.1}$$

The quantity $N(D)$ is closely related to the product locus $P_D \subset X_D$ parametrizing polarized products of elliptic curves with real multiplication. The curve P_D is a disjoint union of modular curves each of whose levels divide $N(D)$ ([Mc2], §2). In particular, the primes of bad reduction for P_D all divide $N(D)$. For many of our examples, we find that the same is true of the primes of bad reduction for $\overline{W}_D$.

Theorem 1.6. *For discriminants $D \in \{21, 44, 53, 56, 60, 61\}$, the curve $\overline{W}_D$ has bad reduction at the prime p only if p divides $N(D)$.*

For Weierstrass curves birational $\mathbb{P}^1$ over $\mathbb{Q}$, we define the *cuspidal polynomial* $c_D(t)$ to be the monic polynomial vanishing simply at the cusps of $\overline{W}_D$ in the affine t-line and nowhere else. Our explicit rational parametrizations of genus zero Weierstrass curves yield the following genus zero analogue of Theorem 1.6.

Theorem 1.7. *For $D \leq 41$ with $D \not\equiv 1 \bmod 8$ and $D \neq 21$, the cuspidal polynomial $c_D(t)$ is in $\mathbb{Z}[t]$ and a prime p divides the discriminant of $c_D(t)$ only if p divides $N(D)$.*

The primes of singular reduction for our models of low, positive genus Weierstrass curves are listed in Table 7.1 and the cuspidal polynomials for Weierstrass curves birational to $\mathbb{P}^1$ over $\mathbb{Q}$ are listed in Table 7.2.

Divisors supported at cusps. The divisors supported at cusps of $\overline{W}_D$ provide further evidence that Teichmüller curves are arithmetically interesting. The Fuchsian groups presenting Teichmüller curves as hyperbolic orbifolds are examples of Veech groups. Our next three theorems suggest that

Veech groups have a rich theory of modular forms.

The Veech groups uniformizing the components of W_D can be computed by the algorithm described in [Mu1]. For background on Veech groups see e.g. [MT, Zo].

By the Manin-Drinfeld theorem [Dr, Ma], the degree zero divisors supported at the cusps of the modular curve $X_0(m) = \overline{\mathbb{H}/\Gamma_0(m)}$ generate a finite subgroup of the Picard group $\mathrm{Pic}^0(X_0(m))$. The same is not quite true for divisors supported at cusps of Weierstrass curves.

Theorem 1.8. *The subgroup of $\mathrm{Pic}^0\left(\overline{W}_{44}\right)$ generated by divisors supported at the nine cusps of W_{44} is isomorphic to $\mathbb{Z}^2$.*

While the cuspidal subgroup of $\overline{W}_{44}$ is not finite, it is small in the sense that there are (many) principal divisors supported at cusps. In other words, there are non-constant holomorphic maps $W_{44} \to \mathbb{C}^*$. Several other Weierstrass curves also enjoy this property.

Theorem 1.9. *Each of the curves W_{44}, W_{53}, W_{57}, W_{60}, W_{65}, and W_{73} admits a non-constant holomorphic map to $\mathbb{C}^*$.*

For several of the genus two and three Weierstrass curves, we also find canonical divisors supported at cusps.

Theorem 1.10. *Each of the curves $\overline{W}_{53}$, $\overline{W}_{56}$ and $\overline{W}_{60}$ has a holomorphic one-form vanishing only at cusps. The curve $\overline{W}_{61}$ has no holomorphic one-form vanishing only at cusps.*

In Figure 2, the plane quartic model for $\overline{W}_{60}$ is shown with the locations of the cusps marked. The five dashed lines meet $\overline{W}_{60}$ only at cusps and each corresponds to a holomorphic one-form up to scale on $\overline{W}_{60}$ vanishing only at cusps. The ratio two such forms corresponds to a holomorphic map $W_{60} \to \mathbb{C}^*$.

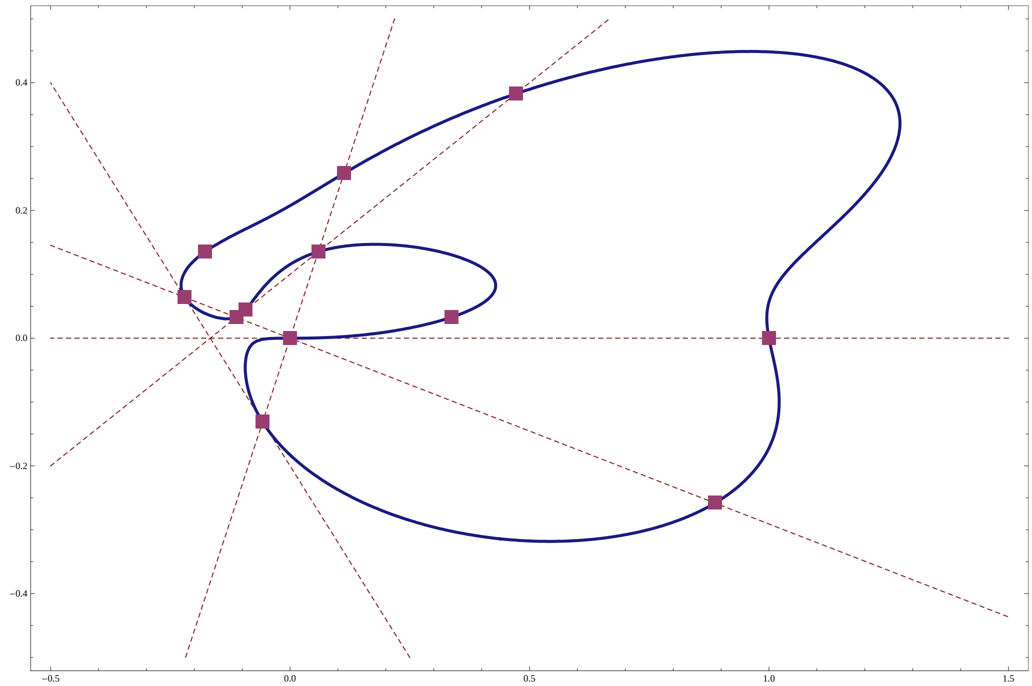

Figure 2: The curve $\overline{W}_{60}$ is biregular to the plane quartic $g_{60}(x, y) = 0$ (solid) and the five lines shown (dashed) meet $\overline{W}_{60}$ only at cusps (squares).

Numerical sampling and Hilbert modular forms. As we now describe, the equations in Table T.2 were obtained by numerically sampling the ratio of certain Hilbert modular forms. For $\tau = (\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H}$, define matrices

$$\Pi(\tau) = \begin{pmatrix} 1 & \frac{D+\sqrt{D}}{2} & \tau_1 \frac{1+\sqrt{D}}{2} & \tau_1 \frac{\sqrt{D}}{D} \\ 1 & \frac{D-\sqrt{D}}{2} & \tau_2 \frac{1-\sqrt{D}}{2} & \tau_2 \frac{-\sqrt{D}}{D} \end{pmatrix} \text{ and } M = \frac{1}{2} \begin{pmatrix} D+\sqrt{D} & 0 \\ 0 & D-\sqrt{D} \end{pmatrix}. \tag{1.2}$$

Since multiplication by M preserves the lattice $\Pi(\tau) \cdot \mathbb{Z}^4$, the abelian variety $B(\tau) = \mathbb{C}^2 / \left(\Pi(\tau) \cdot \mathbb{Z}^4\right)$ admits real multiplication by $\mathcal{O}_D$, and the forms dz_1 and dz_2 on $\mathbb{C}^2$ cover $\mathcal{O}_D$-eigenforms $\eta_1(\tau)$ and $\eta_2(\tau)$ on $B(\tau)$. There are meromorphic functions $a_k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ for $0 \le k \le 5$ so that, for most $\tau \in \mathbb{H} \times \mathbb{H}$, the Jacobian of the algebraic curve

$$Y(\tau) \in \mathcal{M}_2 \text{ with Weierstrass equation } z^2 = w^6 + a_5(\tau) w^5 + \cdots + a_1(\tau) w + a_0(\tau) \tag{1.3}$$

is isomorphic to $B(\tau)$ and the forms $\eta_1(\tau)$ and $\eta_2(\tau)$ pull back under the Abel-Jacobi map $Y(\tau) \to B(\tau)$ to the forms $\omega_1(\tau) = dw/z$ and $\omega_2(\tau) = w \cdot dw/z$. The functions a_k are modular for $\mathrm{PSL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ and the ratio of a_0 with the Igusa-Clebsch invariant of weight two

$$a_0/I_2 \text{ where } I_2 = -240 a_0 + 40 a_1 a_5 - 16 a_2 a_4 + 6 a_3^2$$

is $\mathrm{PSL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$-invariant. Since $a_0(\tau)$ is zero if and only if $\omega_2(\tau)$ has a double zero, a_0/I_2 covers an algebraic function on X_D which vanishes along W_D.

To obtain an explicit model for W_D, we numerically sample a_0/I_2 using the model for X_D in [EK] and the functions in MAGMA related to analytic Jacobians (cf. [vW3]). We then interpolate to find an exact rational function[3] $w_D(r,s)/I_2(r,s)$ which equals a_0/I_2 in these models and whose numerator appears in Table T.2. The function a_0/I_2 and its variants (e.g. a_0^5/I_{10}) have several other remarkable properties and will be studied along with the Hilbert modular forms a_k in [Mu2].

Eigenform certification. To prove Theorems 1.1 and 1.2, in Section 4 we develop an Eigenform Certification Algorithm (ECA, Figure 3). Recall that, for $Y \in \mathcal{M}_2$, there is a natural pairing between $T_Y\mathcal{M}_2$ and the space of holomorphic quadratic differentials $Q(Y)$ on Y. There is a well-known formula for this pairing which we recall in Section 3 in terms of a hyperelliptic model for Y.

Our certification algorithm is based on the following theorem, which is a consequence of Ahlfors' variational formula.

Theorem 1.11. *For τ in the domain of the meromorphic function $Y : \mathbb{H} \times \mathbb{H} \to \mathcal{M}_2$ defined by Equation 1.3, the line in $Q(Y(\tau))$ spanned by the quadratic differential*

$$q(\tau) = \omega_1(\tau) \cdot \omega_2(\tau)$$

annihilates the image of dY_τ.

Theorem 1.11 characterizes the eigenforms $\omega_1(\tau)$ and $\omega_2(\tau)$ on $Y(\tau)$ up to permutation and scale. Using the algebraic model for X_D given in [EK] and the formula in Section 3, we can use Theorem 1.11 to identify eigenforms for real multiplication by $\mathcal{O}_D$ and reduce Theorems 1.1 and 1.2 to linear algebra over function fields.

In [KM1], we will describe a second method of eigenform certification based on explicit algebraic correspondences and similar in spirit to [vW2, vW1]. This technique could be used to prove Theorems 1.1 and 1.2 and such a proof would, unlike the proofs in this paper, be logically independent of [EK]. We found this correspondence method practical for certifying single eigenforms and impractical for certifying positive dimensional families of eigenforms.

Computer files. Throughout this paper, we will refer to computer files in [KM2] and we provide a reader's guide in the file labeled README therein.

Outline. We conclude this Introduction by outlining the remaining sections of this paper.

1. We begin in Section 2 by studying families of marked Riemann surfaces whose Jacobians admit real multiplication. We prove that, for a Riemann surface Y whose Jacobian has real multiplication, there is a symplectic basis U for $H_1(Y,\mathbb{R})$ consisting of eigenvectors for real multiplication (Proposition 2.2) and that the period matrix for Y with respect to U is diagonal (Proposition 2.3). Using Ahlfors' variational formula, we deduce Proposition 2.5 which places a condition on eigenform products and generalizes Theorem 1.11.

2. We then study the pairing between the vector spaces $Q(Y)$ and $T_Y\mathcal{M}_2$ for a genus two algebraic curve Y with Weierstrass polynomial $f_a(w)$. There is a well known formula for this pairing in terms of the roots of $f_a(w)$. We recall this formula in Proposition 3.2 and deduce Proposition 3.1 which gives a formula in terms of the coefficients of $f_a(w)$. Proposition 3.1, while hard

[3]It turns out that a_0/I_2 is invariant under the involution $(\tau_1,\tau_2) \mapsto (\tau_2,\tau_1)$ which covers the deck transformation of the map X_D onto its image in $\mathcal{M}_2$. In the models in [EK], this involution corresponds to the deck transformation of the map from X_D to the (r,s)-plane.

to state (e.g. refers to [KM2]), is useful from a computational standpoint because the field generated by the coefficients of $f_a(w)$ is often simpler than the field generated by the roots of $f_a(w)$.

3. In Section 4, we combine the condition on eigenforms imposed by Proposition 2.5 with the pairing given in Section 3 to give an Eigenform Certification Algorithm. We demonstrate our algorithm by identifying the eigenforms for real multiplication by $\mathcal{O}_{12} = \mathbb{Z}\left[\sqrt{3}\right]$ on a particular genus two algebraic curve (Theorem 4.1).

4. In Section 5, we implement ECA over function fields to certify our models of irreducible W_D and prove Theorems 1.1 and 1.4.

5. We then turn to reducible Weierstrass curves in Section 6. Using the technique in Section 5, we can show that the curve $w_D^0(r,s) = 0$ gives a birational model for *an* irreducible component of W_D. In Section 6, we explain how to distinguish between the irreducible components of W_D by studying cusps, allowing us to prove Theorems 1.2 and 1.5.

6. In Section 7, we discuss the proofs of the remaining theorems stated in this introduction concerning the arithmetic geometry of Weierstrass curves.

Acknowledgments. The authors would like to thank Curt McMullen, Noam Elkies and Matt Emerton for helpful comments. A. Kumar was supported in part by National Science Foundation grant DMS-0952486 and by a grant from the MIT Solomon Buchsbaum Research Fund. R. E. Mukamel was supported in part by National Science Foundation grant DMS-1103654.

2 Jacobians with real multiplication

Throughout this section, we fix the following:

- a compact topological surface S of genus g,
- an order $\mathcal{O}$ in a totally real field K of degree g over $\mathbb{Q}$, and
- a proper, self-adjoint embedding of rings $\rho : \mathcal{O} \to \mathrm{End}(H_1(S,\mathbb{Z}))$.

Here, proper means that ρ does not extend to a larger subring of K and self-adjoint is with respect to the intersection symplectic form $E(S)$ on $H_1(S,\mathbb{Z})$, i.e. for each $x, y \in H_1(S,\mathbb{Z})$ and $\alpha \in \mathcal{O}$ we have $E(S)(\rho(\alpha)x, y) = E(S)(x, \rho(\alpha)y)$.

Our goal for this section is to define and study the Teichmüller space of the pair (S,ρ). The space $\mathrm{Teich}(S,\rho)$ consists of complex structures Y on S for which ρ extends to real multiplication by $\mathcal{O}$ on $\mathrm{Jac}(Y)$. In Proposition 2.2, we show that there is a basis U for $H_1(S,\mathbb{R})$ consisting of eigenvectors for ρ. In Proposition 2.3, we show that Y is in $\mathrm{Teich}(S,\rho)$ if and only if the period matrix for Y with respect to U is diagonal. In Proposition 2.5, we combine Ahlfors' variational formula with Proposition 2.3 to derive a condition satisfied by products of eigenforms for real multiplication on $Y \in \mathrm{Teich}(S,\rho)$. The condition in Theorem 1.11 follows easily from of Proposition 2.5.

The results in the section are, for the most part, well known. We include them as background and to fix notation. In Sections 4 and 5, we will use Proposition 2.5 to certify that certain algebraic one-forms are eigenforms for real multiplication and show that the equations in Table T.2 give algebraic models of Weierstrass curves. For additional background on abelian varieties, Jacobians and their endomorphisms see [BL], for background on Hilbert modular varieties see [Mc4, vdG] and for background on Teichmüller theory and moduli space of Riemann surfaces see [Hu, IT, HM].

Teichmüller space of S. Let $\mathrm{Teich}(S)$ be the Teichmüller space of S. The space $\mathrm{Teich}(S)$ is the fine moduli space representing the functor sending a complex manifold B to the set of holomorphic families over B whose fibers are marked by S up to isomorphism. In particular, a point $Y \in \mathrm{Teich}(S)$ corresponds to an isomorphism classes Riemann surface marked by S and there are canonical isomorphisms $H_1(Y, \mathbb{Z}) \cong H_1(S, \mathbb{Z})$, $\pi_1(Y) \cong \pi_1(S)$, etc. The space $\mathrm{Teich}(S)$ is a complex manifold homeomorphic to $\mathbb{R}^{6g-6}$ and is isomorphic to a bounded domain in $\mathbb{C}^{3g-3}$.

Moduli space. Let $\mathrm{Mod}(S)$ denote the *mapping class group* of S, i.e. the group of orientation preserving homeomorphisms from S to itself up to homotopy. The group $\mathrm{Mod}(S)$ acts properly discontinuously on $\mathrm{Teich}(S)$ and the quotient

$$\mathcal{M}_g = \mathrm{Teich}(S)/\,\mathrm{Mod}(S)$$

is a complex orbifold which coarsely solves the moduli problem for unmarked families of Riemann surfaces homeomorphic to S. We call $\mathcal{M}_g$ the *moduli space of genus g Riemann surfaces.*

Holomorphic one-forms and Jacobians. For each $Y \in \mathcal{M}_g$, let $\Omega(Y)$ be the vector space of holomorphic one-forms on Y and let $\Omega(Y)^*$ be the vector space dual to $\Omega(Y)$. By complex analysis, $\dim_{\mathbb{C}} \Omega(Y) = g$ and the map

$$f : H_1(Y, \mathbb{R}) \to \Omega(Y)^* \text{ given by } f(a)(\omega) = \int_a \omega \tag{2.1}$$

is an $\mathbb{R}$-linear isomorphism. In particular, $f(H_1(Y, \mathbb{Z}))$ is a lattice in $\Omega(Y)^*$ and the quotient

$$\mathrm{Jac}(Y) = \Omega(Y)^*/f(H_1(Y, \mathbb{Z})) \tag{2.2}$$

is a complex torus called the *Jacobian* of Y. The Hermitian form H^* on $\Omega(Y)^*$ dual to the form

$$H(\omega, \eta) = \int_Y \omega \wedge \overline{\eta} \text{ for each } \omega, \eta \in \Omega(Y) \tag{2.3}$$

defines a principal polarization on $\mathrm{Jac}(Y)$ since the pullback of $\mathrm{Im}(H^*)$ under f restricts to the intersection pairing $E(Y)$ on $H_1(Y, \mathbb{Z})$.

Jacobian endomorphisms. An *endomorphism* of $\mathrm{Jac}(Y)$ is a holomorphic homomorphism from $\mathrm{Jac}(Y)$ to itself. Since $\mathrm{Jac}(Y)$ is an abelian group, the collection $\mathrm{End}(\mathrm{Jac}(Y))$ of all endomorphisms of $\mathrm{Jac}(Y)$ forms a ring called the *endomorphism ring* of $\mathrm{Jac}(Y)$. Every endomorphism $R \in \mathrm{End}(\mathrm{Jac}(Y))$ arises from $\mathbb{C}$-linear map $\rho_a(R) : \Omega(Y)^* \to \Omega(Y)^*$ preserving the lattice $f(H_1(Y, \mathbb{Z}))$. The assignment

$$\rho_a : \mathrm{End}(\mathrm{Jac}(Y)) \to \mathrm{End}(\Omega(Y)^*) \text{ given by } R \mapsto \rho_a(R) \tag{2.4}$$

is an embedding of rings called the *analytic representation* of $\mathrm{End}(\mathrm{Jac}(Y))$. We will denote by ρ_a^* the representation of $\mathrm{End}(\mathrm{Jac}(Y))$ on $\Omega(Y)$ dual to ρ_a. The assignment

$$\rho_r : \mathrm{End}(\mathrm{Jac}(Y)) \to \mathrm{End}(H_1(Y, \mathbb{Z})) \text{ given by } \rho_r(R) = f^{-1} \circ \rho_a(R) \circ f \tag{2.5}$$

is also an embedding of rings and is called the *rational representation* of $\mathrm{End}(\mathrm{Jac}(Y))$.

For any endomorphism $R \in \mathrm{End}(\mathrm{Jac}(Y))$, there is another endomorphism $R^* \in \mathrm{End}(\mathrm{Jac}(Y))$ called the *adjoint* of R and characterized by the property that $\rho_r(R^*)$ is the $E(Y)$-adjoint of $\rho_r(R)$. The assignment $R \mapsto R^*$ defines an (anti-)involution on $\mathrm{End}(\mathrm{Jac}(Y))$ called the *Rosati involution.*

Real multiplication. Recall that K is a totally real number field of degree g over $\mathbb{Q}$ and $\mathcal{O}$ is an order in K, i.e. a subring of K which is also a lattice. We will say that $\mathrm{Jac}(Y)$ *admits real multiplication by* $\mathcal{O}$ if there is

$$\text{a proper, self-adjoint embedding } \iota : \mathcal{O} \to \mathrm{End}(\mathrm{Jac}(Y)). \tag{2.6}$$

Proper means that ι does not extend to a larger subring in K and self-adjoint means that $\iota(\alpha)^* = \iota(\alpha)$ for each $\alpha \in \mathcal{O}$. If $\mathcal{O}$ is maximal (i.e. $\mathcal{O}$ is not contained in a strictly larger order in K) then an embedding $\mathcal{O} \to \mathrm{End}(\mathrm{Jac}(Y))$ is automatically proper.

Teichmüller space of the pair (S, ρ). Recall that $\rho : \mathcal{O} \to \mathrm{End}(H_1(S, \mathbb{Z}))$ is a proper and self-adjoint embedding of rings. For $Y \in \mathrm{Teich}(S)$, we will say that ρ *extends to real multiplication by* $\mathcal{O}$ *on* $\mathrm{Jac}(Y)$ if there is

$$\text{an embedding } \iota : \mathcal{O} \to \mathrm{End}(\mathrm{Jac}(Y)) \text{ satisfying } \rho_r \circ \iota = \rho. \tag{2.7}$$

Equivalently, ρ extends to real multiplication if and only if the $\mathbb{R}$-linear extension of $f \circ \rho(\alpha) \circ f^{-1}$ to $\Omega(Y)^*$ is $\mathbb{C}$-linear for each $\alpha \in \mathcal{O}$. Since ρ is proper and self-adjoint, an ι as in Equation 2.7 is automatically proper and self-adjoint in the sense of the previous paragraph. In Equation 2.7, we have implicitly identified $H_1(Y, \mathbb{R})$ with $H_1(S, \mathbb{R})$ via the marking.

We define the Teichmüller space of the pair (S, ρ) to be the space

$$\mathrm{Teich}(S, \rho) = \{Y \in \mathrm{Teich}(S) : \rho \text{ extends to real multiplication by } \mathcal{O} \text{ on } \mathrm{Jac}(Y)\}. \tag{2.8}$$

If ρ_1 and ρ_2 are two proper, self-adjoint embeddings $\mathcal{O} \to \mathrm{End}(H_1(S, \mathbb{Z}))$ and $g \in \mathrm{Mod}(S)$ is a mapping class such that $g_* \in \mathrm{End}(H_1(S, \mathbb{Z}))$ conjugates $\rho_1(\alpha)$ to $\rho_2(\alpha)$ for each $\alpha \in \mathcal{O}$, then g gives a biholomorphic map between $\mathrm{Teich}(S, \rho_1)$ and $\mathrm{Teich}(S, \rho_2)$.

Symplectic K-modules and their eigenbases. The representation

$$\rho_K = \rho \otimes_{\mathbb{Z}} \mathbb{Q} : K \to \mathrm{End}(H_1(S, \mathbb{Q}))$$

turns $H_1(S, \mathbb{Q})$ into a K-module. We begin our study of $\mathrm{Teich}(S, \rho)$ by showing that there is a unique symplectic K-module that arises in this way.

Let $E(\mathrm{Tr})$ be the symplectic *trace form* on $K \oplus K$ defined by

$$E(\mathrm{Tr})\left((x_1, y_1), (x_2, y_2)\right) = \mathrm{Tr}^K_{\mathbb{Q}}(x_1 y_2 - y_1 x_2). \tag{2.9}$$

It is easy to check that multiplication by $k \in K$ is self-adjoint for $E(\mathrm{Tr})$.

Proposition 2.1. *Regarding $H_1(S, \mathbb{Q})$ as a K-module via $\rho_K = \rho \otimes_{\mathbb{Z}} \mathbb{Q}$, there is a K-linear isomorphism*

$$T : K \oplus K \to H_1(S, \mathbb{Q})$$

which is symplectic for the trace from $E(\mathrm{Tr})$ on $K \oplus K$ and the intersection form $E(S)$ on $H_1(S, \mathbb{Q})$.

Proof. Choose any $x \in H_1(S,\mathbb{Q})$ and set $L = \rho_K(K) \cdot x$. Since ρ_K is self-adjoint, L is isotropic. The non-degeneracy of the intersection form $E(S)$ ensures that there is a $y \in H_1(S,\mathbb{Q})$ such that $E(S)\,(\rho_K(k) \cdot x, y) = \mathrm{Tr}_{\mathbb{Q}}^{K}(k)$. Define a map $T : K \oplus K \to H_1(S,\mathbb{Q})$ by the formula

$$T(k_1,k_2) = \rho_K(k_1) \cdot x + \rho_K(k_2) \cdot y.$$

Clearly, the map T is K-linear. An easy computation shows that T satisfies $E(\mathrm{Tr})(v,w) = E(S)(T(v),T(w))$ for each $v,w \in K \oplus K$ which, together with the non-degeneracy of $E(\mathrm{Tr})$, implies that T is a symplectic vector space isomorphism. □

Now let $h_1,\ldots,h_g : K \to \mathbb{R}$ be the g places for K. Proposition 2.1 allows us to show that there is a symplectic basis for $H_1(S,\mathbb{R})$ adapted to ρ.

Proposition 2.2. *There is a symplectic basis $U = \langle a_1,\ldots,a_g,b_1,\ldots,b_g\rangle$ for $H_1(S,\mathbb{R})$ such that*

$$\rho(\alpha)a_i = h_i(\alpha) \cdot a_i \text{ and } \rho(\alpha)b_i = h_i(\alpha) \cdot b_i \text{ for each } \alpha \in \mathcal{O}. \tag{2.10}$$

Proof. Since the group $H_1(S,\mathbb{Q})$ is isomomorphic as a symplectic K-module to $K \oplus K$ with the trace pairing $E(\mathrm{Tr})$ (Proposition 2.1), it suffices to construct an analogous basis for $(K \oplus K) \otimes_{\mathbb{Q}} \mathbb{R}$. Let $\alpha_1,\ldots,\alpha_g$ be an arbitrary $\mathbb{Q}$-basis for K. Since $\mathrm{Tr} : K \times K \to \mathbb{Q}$ is non-degenerate, we can choose $\beta_1,\ldots,\beta_g \in K$ so that $\mathrm{Tr}_{\mathbb{Q}}^{K}(\alpha_i\beta_j) = \delta_{ij}$. Setting

$$a_i = \sum_{j=1}^{g}(\alpha_j,0) \otimes h_i(\beta_j) \text{ and } b_i = \sum_{j=1}^{g}(0,\beta_j) \otimes h_i(\alpha_j)$$

yields a basis with the desired properties. □

The period map. Now let $\mathcal{H}_g$ be the Siegel upper half-space consisting of $g \times g$ symmetric matrices with positive definite imaginary part. The space $\mathcal{H}_g$ is equal to an open, bounded and symmetric domain in the $(g^2+g)/2$-dimensional space of all symmetric matrices.

As we now describe, the basis U for $H_1(S,\mathbb{R})$ given by Proposition 2.2 allows us to define a holomorphic period map from $\mathrm{Teich}(S)$ to $\mathcal{H}_g$. For $Y \in \mathrm{Teich}(S)$, we can view U as a basis for $H_1(Y,\mathbb{R})$ via the marking by S. Let $\langle \omega_1(Y),\ldots,\omega_g(Y)\rangle$ be the basis for $\Omega(Y)$ dual to U, i.e. such that $\int_{a_j} \omega_k(Y) = \delta_{jk}$. The *period map* is defined by

$$P : \mathrm{Teich}(S) \to \mathcal{H}_g \text{ where } P_{jk}(Y) = \int_{b_j} \omega_k(Y). \tag{2.11}$$

Our next proposition characterizes the points in $\mathrm{Teich}(S,\rho)$.

Proposition 2.3. *For $Y \in \mathrm{Teich}(S)$, the homomorphism ρ extends to real multiplication by $\mathcal{O}$ on $\mathrm{Jac}(Y)$ if and only if the period matrix $P(Y)$ is diagonal.*

Proof. From $\int_{a_j} \omega_k(Y) = \delta_{jk}$ and $P_{jk}(Y) = P_{kj}(Y) = \int_{b_j} \omega_k(Y)$ we see that the map $f : H_1(Y,\mathbb{R}) \to \Omega(Y)^*$ of Equation 2.1 satisfies $f(b_j) = \sum_{k=1}^{g} P_{jk}(Y) \cdot f(a_k)$. In matrix–vector notation, we have

$$(f(b_1), f(b_2),\ldots,f(b_g)) = P(Y) \cdot (f(a_1), f(a_2),\ldots,f(a_g)). \tag{2.12}$$

For $\alpha \in \mathcal{O}$, let $h(\alpha)$ be the $g \times g$ diagonal matrix with diagonal entries $(h_1(\alpha), \ldots, h_g(\alpha))$. From Equation 2.10, the map $T(\alpha) = f \circ \rho(\alpha) \circ f^{-1}$ extends $\mathbb{C}$-linearly to $\Omega(Y)^*$ if and only if the matrix for $T(\alpha)$ is $h(\alpha)$ with respect to both the basis $\langle f(a_1), \ldots, f(a_g)\rangle$ and the basis $\langle f(b_1), \ldots, f(b_g)\rangle$. From Equation 2.12 this happens if and only if $P(Y)$ commutes with $h(\alpha)$. Since the embeddings $h_1, \ldots, h_g : \mathcal{O} \to \mathbb{R}$ are pairwise distinct, $P(Y)$ commutes with $h(\alpha)$ for every $\alpha \in \mathcal{O}$ if and only if $P(Y)$ is diagonal. □

Eigenforms for real multiplication. For $Y \in \mathrm{Teich}(S, \rho)$ and ι satisfying $\rho_r \circ \iota = \rho$, we saw in the proof of Proposition 2.3 that the matrix for $\rho_a(\iota(\alpha))$ with respect to the basis $\langle f(a_1), \ldots, f(a_g)\rangle$ for $\Omega(Y)^*$ is the diagonal matrix $h(\alpha)$. Since this basis is dual to the basis $\langle \omega_1(Y), \ldots, \omega_g(Y)\rangle$ for $\Omega(Y)$, we see that $\rho_a^*(\iota(\alpha)) \in \mathrm{End}(\Omega(Y))$ stabilizes $\omega_i(Y)$ up to scale. We record this fact in the following proposition.

Proposition 2.4. *For $Y \in \mathrm{Teich}(S, \rho)$ and $\alpha \in \mathcal{O}$, we have that $\rho_a^*(\alpha)\omega_i(Y) = h_i(\alpha)\omega_i(Y)$.*

In light of Proposition 2.4, we call the non-zero scalar multiples of $\omega_i(Y)$ the *h_i-eigenforms for $\mathcal{O}$*.

Moduli of abelian varieties. Now consider the homomorphism $M : \mathrm{PSp}(H_1(S, \mathbb{R})) \to \mathrm{PSp}_{2g}(\mathbb{R})$ sending a projective symplectic automorphism of $H_1(S, \mathbb{R})$ to its matrix with respect to U. There is an action of $\mathrm{PSp}_{2g}(\mathbb{R})$ on $\mathcal{H}_g$ by holomorphic automorphisms via generalized Möbius transformations such that, for $h \in \mathrm{Mod}(S)$ inducing $h_* \in \mathrm{End}(H_1(S, \mathbb{Z}))$, we have

$$M(h_*) \cdot P(Y) = P(h \cdot Y). \tag{2.13}$$

We conclude that the period map $P : \mathrm{Teich}(S) \to \mathcal{H}_g$ covers a holomorphic map

$$\mathrm{Jac} : \mathcal{M}_g \to A_g = \mathcal{H}_g/\Gamma_{\mathbb{Z}} \text{ where } \Gamma_{\mathbb{Z}} = M(\mathrm{PSp}(H_1(S, \mathbb{Z}))). \tag{2.14}$$

We also call this map the period map and denote it by Jac since the space A_g has a natural interpretation as a moduli space of principally polarized abelian varieties so that Jac is simply the map sending a Riemann surface to its Jacobian.

Hilbert modular varieties. Let Δ_g denote the collection of diagonal matrices in $\mathcal{H}_g$ and let $\mathrm{PSp}(H_1(S, \mathbb{Z}), \rho)$ denote the subgroup of $\mathrm{PSp}(H_1(S, \mathbb{Z}))$ represented by symplectic automorphisms commuting with $\rho(\alpha)$ for each $\alpha \in \mathcal{O}$. The group $\Gamma_\rho = M(\mathrm{PSp}(H_1(S, \mathbb{Z}), \rho))$ consists of matrices whose $g \times g$ blocks are diagonal. Consequently, Γ_ρ preserves Δ_g and, by Proposition 2.3, the map $\mathrm{Teich}(S, \rho) \to A_g$ covered by the period map P factors through the orbifold

$$X_\rho = \Delta_g/\Gamma_\rho. \tag{2.15}$$

The space X_ρ has a natural interpretation as a moduli space of abelian varieties with real multiplication. Each of the complex orbifolds $\mathcal{M}_g$, A_g and X_ρ can be given the structure of an algebraic variety so that the map in the period map Jac and the map $X_\rho \to A_g$ covered by the inclusion $\Delta_g \to \mathcal{H}_g$ are algebraic. The variety X_ρ is called a *Hilbert modular variety.*

Tangent and cotangent space to $\operatorname{Teich}(S)$**.** For $Y \in \operatorname{Teich}(S)$, let $B(Y)$ denote the vector space of L^∞-Beltrami differentials on Y. The measurable Riemann mapping theorem can be used to give a marked family over the unit ball $B^1(Y)$ in $B(Y)$ and construct a holomorphic surjection $\phi : B^1(Y) \to \operatorname{Teich}(S)$ with $\phi(0) = Y$. There is a pairing between $B(Y)$ and the space of holomorphic quadratic differentials $Q(Y)$ on Y given by

$$B(Y) \times Q(Y) \to \mathbb{C} \text{ where } (\mu, q) \mapsto \int_Y \mu \cdot q. \tag{2.16}$$

Now let $Q(Y)^\perp \subset B(Y)$ be the vector subspace consisting of Beltrami differentials annihilating every quadratic differential under the pairing in Equation 2.16. By Teichmüller theory, the space $Q(Y)^\perp$ is closed, has finite codimension and is equal to the kernel of $d\phi_0$. The tangent space $T_Y \operatorname{Teich}(S)$ is isomorphic to $B(Y)/Q(Y)^\perp$ and the pairing in Equation 2.16 covers a pairing between $T_Y \operatorname{Teich}(S)$ and $Q(Y)$ giving an isomorphism

$$T_Y^* \operatorname{Teich}(S) \cong Q(Y). \tag{2.17}$$

The pairing in Equation 2.16 and the isomorphism in Equation 2.17 are $\operatorname{Mod}(S)$-equivariant, and they give rise to a pairing between $Q(Y)$ and the orbifold tangent space $T_Y \mathcal{M}_g$.

Eigenform products. We can now establish the following proposition which places conditions on the eigenforms of $Y \in \operatorname{Teich}(S, \rho)$.

Proposition 2.5. *Suppose B is a smooth manifold and $g : B \to \operatorname{Teich}(S, \rho)$ is a smooth map. For each $j \neq k$ and $b \in B$, the quadratic differential*

$$q_{jk}(b) = \omega_j(g(b)) \cdot \omega_k(g(b)) \in Q(g(b))$$

annihilates the image of dg_b in $T_{g(b)} \operatorname{Teich}(S)$.

Proof. In light of Proposition 2.3, the image of $P \circ g$ is contained within the set Δ_g of diagonal matrices in $\mathcal{H}_g$. For $j \neq k$, the composition $P_{jk} \circ g : B \to \mathbb{C}$ is identically zero. The differential dP_{jk} annihilates the image of dg_b by the chain rule and is equal to $q_{jk}(b)$ by Ahlfors' variational formula. □

Theorem 1.11 is a special case of Proposition 2.5.

Proof of Theorem 1.11. Fix τ in the domain for the map $Y : \mathbb{H} \times \mathbb{H} \to \mathcal{M}_2$ defined in Equation 1.3 and let B be a neighborhood of τ on which Y lifts to a map $\widetilde{Y} : B \to \operatorname{Teich}(S)$ for a genus two surface S. Identify $H_1(S, \mathbb{Z})$ with $H_1(\widetilde{Y}(\tau), \mathbb{Z})$ via the marking and with the lattice $\Pi(\tau) \cdot \mathbb{Z}^4 = H_1(B(\tau), \mathbb{Z})$ (Equation 1.2) via the Abel-Jacobi map $\widetilde{Y}(\tau) \to B(\tau)$ and let $\rho : \mathcal{O}_D \to \operatorname{End}(H_1(S, \mathbb{Z}))$ be the proper, self-adjoint embedding with $\rho(\alpha)$ equal to multiplication by the diagonal matrix $\operatorname{Diag}(h_1(\alpha), h_2(\alpha))$ on $\Pi(\tau) \cdot \mathbb{Z}^4$. Clearly, the lift $\widetilde{Y}$ maps B into $\operatorname{Teich}(S, \rho)$ and Proposition 2.5 shows that the product $\omega_1(\tau) \cdot \omega_2(\tau)$ annihilates the image of $d\widetilde{Y}_\tau$. □

Genus two Jacobians with real multiplication. For typical pairs (S, ρ), we know little else about the space $\operatorname{Teich}(S, \rho)$ including whether or not $\operatorname{Teich}(S, \rho)$ is empty. For the remainder of this section, we impose the additional assumption that $g = 2$ so that we can say more.

Let D be the discriminant of $\mathcal{O}$. The first special feature when $g = 2$ is that the order $\mathcal{O}$ is determined by D and is isomorphic to $\mathcal{O}_D = \mathbb{Z}\left[\frac{D+\sqrt{D}}{2}\right]$. The discriminants of real quadratic orders are precisely the integers $D > 0$ and congruent to 0 or 1 mod 4, and the order of discriminant D is an order in a real quadratic field if and only if D is not a square.[4] The discriminant D is *fundamental* and the order $\mathcal{O}_D$ is *maximal* if $\mathcal{O}_D$ is not contained in a larger order in $\mathcal{O}_D \otimes \mathbb{Q}$.

The second special feature when $g = 2$ is that, for each real quadratic order $\mathcal{O}$, there is a unique proper, self-adjoint embedding $\rho : \mathcal{O} \to H_1(S, \mathbb{Z})$ up to conjugation by elements of $\mathrm{Sp}(H_1(S, \mathbb{Z}))$ ([Ru], Theorem 2). Since $\mathrm{Mod}(S) \to \mathrm{Sp}(H_1(S, \mathbb{Z}))$ is onto, the spaces $\mathrm{Teich}(S, \rho)$ and X_ρ and the maps to $\mathrm{Teich}(S, \rho) \to \mathcal{M}_2$ and $X_\rho \to A_2$ are determined by D up to isomorphism. The Hilbert modular variety X_ρ is isomorphic to the *Hilbert modular surface of discriminant D*

$$\begin{gathered} X_D = \mathbb{H} \times \mathbb{H} / \operatorname{PSL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee) \text{ where } \mathcal{O}_D^\vee = \frac{1}{\sqrt{D}} \cdot \mathcal{O}_D \text{ and} \\ \operatorname{PSL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{PSL}_2(K) : \begin{array}{c} ad - bc = 1, a, d \in \mathcal{O}_D \\ b \cdot \mathcal{O}_D^\vee \subset \mathcal{O}_D \text{ and } c \cdot \mathcal{O}_D \subset \mathcal{O}_D^\vee \end{array} \right\}. \end{gathered} \tag{2.18}$$

The two places $h_1, h_2 : \mathcal{O}_D \to \mathbb{R}$ give two homomorphisms $\operatorname{PSL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee) \to \operatorname{PSL}_2(\mathbb{R})$ which we also denote by h_1 and h_2. The action of $M \in \operatorname{PSL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ on $\mathbb{H} \times \mathbb{H}$ is the ordinary action of $h_1(M)$ by Möbius transformation the first coordinate and by $h_2(M)$ on the second.

The third special feature when $g = 2$ is that the algebraic map $\mathrm{Jac} : \mathcal{M}_2 \to A_2$ is birational with rational inverse Jac^{-1}. Composing Jac^{-1} with the natural map $X_D \to A_2$ gives a rational inverse period map

$$\mathrm{Jac}_D^{-1} : X_D \to \mathcal{M}_2 \tag{2.19}$$

whose image is covered by $\mathrm{Teich}(S, \rho)$.

3 Quadratic differentials and residues

To make use of Proposition 2.5 for a complex structure $Y \in \mathrm{Teich}(S, \rho)$ represented by an algebraic curve, we will need an algebraic formula for the pairing between $Q(Y)$ and $T_Y \,\mathrm{Teich}(S)$ (Equation 2.17). For genus two Riemann surfaces, the curve Y is biholomorphic to the hyperelliptic curve defined by $z^2 = f_a(w)$ with

$$f_a(w) = w^5 + a_4 w^4 + a_3 w^3 + a_2 w^2 + a_1 w + a_0 \tag{3.1}$$

for some $a = (a_0, \ldots, a_4) \in \mathbb{C}^5$. There is a well known formula for the pairing between $Q(Y)$ and $T_Y \,\mathrm{Teich}(S)$ for such curves (and for hyperelliptic curves more generally) involving residues and the roots of $f_a(w)$. We recall this formula in Proposition 3.2.

From Proposition 3.2, we deduce Proposition 3.1 which gives a formula in terms of the coefficients a_i of $f_a(w)$. Proposition 3.1 will be more useful than Proposition 3.2 from a computational standpoint since the field $\mathbb{Q}(a_0, \ldots, a_4)$ is typically simpler than the splitting field of $f_a(w)$.

Coefficients of Weierstrass polynomials. For $a = (a_0, \ldots, a_4) \in \mathbb{C}^5$, let $f_a(w)$ be the degree five polynomial defined in Equation 3.1 and let $V \subset \mathbb{C}^5$ be the open set corresponding to polynomials with non-zero discriminant

$$V = \left\{ a \in \mathbb{C}^5 : \mathrm{Disc}\left(f_a(w)\right) \neq 0 \right\}. \tag{3.2}$$

[4] Rings with square discriminants correspond to orders in $\mathbb{Z} \times \mathbb{Z}$ and can in principle be treated similarly to those we consider in this paper. Since equations for X_{d^2} do not appear in [EK], we will not consider such rings in this paper.

Consider the holomorphic and algebraic map

$$Y : V \to \mathcal{M}_2 \text{ where } Y(a) \text{ is birational to the curve defined by } z^2 = f_a(w). \tag{3.3}$$

While there is no universal curve over $\mathcal{M}_2$, it is easy to construct one over V. By Teichmüller theory, the map Y lifts locally to Teich(S) giving rise to a pairing

$$Q(Y(a)) \times T_a V \to \mathbb{C} \tag{3.4}$$

via the derivative dY_a. There is a natural identification of $T_a V$ with $\mathbb{C}^5$ since V is open in $\mathbb{C}^5$, and we can identify $Q(Y(a))$ with $\mathbb{C}^3$ by associating the vector $x = (x_0, x_1, x_2) \in \mathbb{C}^3$ with the quadratic differential

$$q_x = Q_x(w)dw^2/f_a(w) \text{ where } Q_x(w) = (x_0 + x_1 w + x_2 w^2). \tag{3.5}$$

Our main goal for this section is to establish the following proposition which gives an explicit formula for the pairing in Equation 3.4 in these coordinates.

Proposition 3.1. *Fix $a \in V$, $v \in T_a V$ and $x \in \mathbb{C}^3$ and let $M(a)$ be the matrix in [KM2]. The product of the quadratic differential q_x and the vector v is given by the formula*

$$q_x(v) = (-2\pi) \cdot x^T \cdot M(a) \cdot v.$$

We will prove Proposition 3.1 at the end of this section.

Roots of Weierstrass polynomials. Now consider the function $a : \mathbb{C}^5 \to \mathbb{C}^5$ whose value at $r = (r_1, r_2, r_3, r_4, r_5)$ are the coefficients of the polynomial with roots r_i, i.e. $f_{a(r)}(r_i) = 0$. In other words, the kth coordinate $a_k(r)$ of $a(r)$ is a symmetric polynomial in the coordinates of r up to a sign. Let $V^{rt} = a^{-1}(V)$ and consider the composition $Y \circ a : V^{rt} \to \mathcal{M}_2$.

The universal curve over V pulls back to a universal curve over V^{rt} and, as in the previous paragraph, Teichmüller theory gives a natural pairing between $T_r V^{rt}$ and $Q(Y(a(r)))$ which is related to pairing in Equation 3.4 with $a = a(r)$ by multiplication by da_r. The formula for this pairing in these coordinates is given by the following proposition.

Proposition 3.2. *Fix $r = (r_1, \ldots, r_5) \in V^{rt}$, $v = (v_1, \ldots, v_5) \in T_r V^{rt}$ and $x \in \mathbb{C}^3$. The pairing between $q_x = Q_x(w) \cdot dw^2/f_a(w)$ with v is given by*

$$q_x(v) = (-2\pi) \cdot \sum_{j=1}^{5} v_j \operatorname{Res}_{r_j} \left(\frac{Q_x(w)}{f_{a(r)}(w)} dw \right). \tag{3.6}$$

Proposition 3.2 is well-known.[5] We include a proof for completeness.

Proof. The Riemann surface $Y(a)$ is birational to the algebraic curve $z^2 = f_a(w)$. One can construct a smoothly varying family of diffeomorphisms

$$\Phi_t(w, z) = (W_t(w, z), Z_t(w, z)) : Y(a(r)) \to Y(a(r + tv)) \text{ for } t \text{ small}$$

[5] See e.g. [Pi] pg. 50 or [HS] Proposition 7.3. Equation 3.6 differs from the equations in [Pi] and [HS] by a constant factor arising from different definitions and the fact that our formula is in genus two and their formula is in genus zero.

such that $W_t(w,z) = w + tv_i$ for w in a neighborhood of $w^{-1}(r_j)$, $W_t(w,z) = w$ for w in a neighborhood of ∞ and $\Phi_0 = \mathrm{id}|_{Y(a(r))}$. Since Φ_t is holomorphic for large w, the computation below is unaffected by our identification of $Y(a(r+tv))$ with the affine plane curve $z^2 = f_{a(r+tv)}(w)$.

The family of Beltrami differentials $\mu(\Phi_t) = \overline{\partial}\Phi_t/\partial\Phi_t$ provides a lift of the local map from V^{rt} to $\mathrm{Teich}(S)$ to the unit ball $B^1(Y(a(r)))$ in the space of L^∞-Beltrami differentials on $Y(a(r))$. To compute $q_x(v)$, we compute $\mu(\Phi_t)$ to first order in t and evaluate the right hand side of

$$q_x(v) = \lim_{t\to 0} \frac{1}{t} \int_{Y(a(r))} \mu(\Phi_t) q_x. \tag{3.7}$$

Compare Equation 3.7 with Equation 2.16. Since Φ_t is holomorphic in a neighborhood of the zeros and poles of the meromorphic one-form dw on $Y(a(r))$, the Beltrami differential $\mu(\Phi_t)$ satisfies

$$\mu(\Phi_t) = \frac{\overline{\partial}\Phi_t}{\partial\Phi_t} = \frac{\partial W_t(w,z)/\partial\overline{w}\cdot d\overline{w}}{\partial W_t(w,z)/\partial w\cdot dw} = \frac{\partial W_t(w,z)/\partial\overline{w}}{1+O(t)}\cdot\frac{d\overline{w}}{dw} = \partial W_t(w,z)/\partial\overline{w}\cdot\frac{d\overline{w}}{dw} + O(t^2). \tag{3.8}$$

The product $q_x \cdot \mu(\Phi_t)$ is supported away from small disks about the Weierstrass points of $Y(a(r))$. From Equation 3.8 we see that the product $q_x\cdot\mu(\Phi_t)$ is nearly exact in such a neighborhood

$$q_x\cdot\mu(\Phi_t) = \frac{Q_x(w)\partial W_t/\partial\overline{w}}{f_{a(r)}(w)}|dw|^2 + O(t^2) = d\eta + O(t^2) \text{ where } \eta = \frac{i}{2}\cdot\frac{(W_t(w,z)-w)\cdot Q_x(w)}{f_{a(r)}(w)}dw. \tag{3.9}$$

The factor of $i/2$ arises from the equation $|dw|^2 = (i/2)\cdot dw\wedge d\overline{w}$. Stokes' theorem gives

$$\int_{Y(a(r))} q_x\cdot\mu(\Phi_t) = \int_{C_\infty}\eta + \sum_{j=1}^{5}\int_{C_j}\eta + O(t^2) \tag{3.10}$$

where C_j is a small loop around $w^{-1}(r_j)$ and C_∞ is a small loop around $w^{-1}(\infty)$. The formula in Equation 3.6 follows by observing that η is identically zero in a neighborhood of $w^{-1}(\infty)$ and is equal to the meromorphic one-form $(i/2)\cdot tv_j Q_x(w)dw/f_{a(r)}(w)$ in a neighborhood of $w^{-1}(r_j)$. Note the extra factor of two arising from the fact that w maps $C_j \subset Y(a(r))$ to a closed curve winding twice about r_j. □

Pairing matrices. Now let $N(r)$ be the 3×5-matrix whose jk-th entry is given by the formula

$$N(r)_{jk} = \mathrm{Res}_{r_j}\left(\frac{w^k}{f_r(w)}dw\right) = r_j^k\cdot\prod_{l\neq j}\frac{1}{r_j - r_l}. \tag{3.11}$$

Proposition 3.2 says the pairing between T_rV^{rt} and $Q(Y(a(r)))$ is given by

$$q_x(v) = (-2\pi)\cdot x^T\cdot N(r)\cdot v.$$

We are now ready to give the proof of Proposition 3.1.

Proof of Proposition 3.1. For $r\in V^{rt}$, the pairing of T_rV^{rt} with $Q(r) = Q(Y(a(r)))$ and the pairing of $T_{a(r)}V$ with $Q(r)$ correspond to linear maps $L_r : T_rV^{rt} \to (Q(r))^*$ and $L_a : T_{a(r)}V \to (Q(r))^*$ which are related to one another by composition with the derivative of $a : V^{rt}\to V$, i.e. $L_r = L_a\circ da_r$.

Most computer algebra systems will readily verify that the derivative da_r, the matrix $N(r)$ defined by Equation 3.11 and the matrix $M(a)$ found in [KM2] are related by

$$N(r) = M\left(a(r)\right) \cdot da_r. \tag{3.12}$$

Since $(-2\pi) \cdot N(r)$ is the matrix for L_r with respect to the obvious bases on $T_r V^{rt}$ and $Q(Y(a(r)))^*$, the matrix $(-2\pi) \cdot M(a)$ is the matrix for L_a. We have included code to verify Equation 3.12 in [KM2]. □

4 Eigenform certification

In this section, we develop a method of eigenform certification based on the condition on eigenform products imposed by Proposition 2.5 and the formula in Proposition 3.1. We demonstrate our method by proving the following theorem.

Theorem 4.1. *The Jacobian of the algebraic curve Y with Weierstrass model*

$$z^2 = w^5 - 2w^4 - 12w^3 - 8w^2 + 52w + 24 \tag{4.1}$$

admits real multiplication by $\mathcal{O}_{12} = \mathbb{Z}\left[\sqrt{3}\right]$ with eigenforms dw/z and $w \cdot dw/z$.

Since the one-form dw/z has a double zero, Theorem 4.1 immediately implies the following.

Corollary 4.2. *The one-form up to scale $(Y, [dw/z])$ defined by Equation 4.1 lies on W_{12}.*

We conclude this section by summarizing our method in the Eigenform Certification Algorithm (ECA, Figure 3).

Igusa-Clebsch invariants. For a genus two topological surface S, the Igusa-Clebsch invariants define a holomorphic map $IC : \mathrm{Teich}(S) \to \mathbb{P}(2,4,6,10)$ where $\mathbb{P}(2,4,6,10)$ is the weighted projective space

$$\mathbb{P}(2,4,6,10) = \mathbb{C}^4/\mathbb{C}^* \text{ where } \mathbb{C}^* \text{ acts by } \lambda \cdot (I_2, I_4, I_6, I_{10}) = (\lambda^2 I_2, \lambda^4 I_4, \lambda^6 I_6, \lambda^{10} I_{10}). \tag{4.2}$$

The coordinate I_k of IC is called the *Igusa-Clebsch invariant of weight k* (cf. [Ig]).

For $a \in V$ and $Y(a) \in \mathcal{M}_2$ as defined in Section 3, the invariants $I_k(Y(a))$ are polynomial[6] in a. The invariant I_{10} is the discriminant of the Weierstrass polynomial $f_a(w)$ and the curve Y defined by Equation 4.1 has

$$IC(Y) = (56 : -32 : -348 : -324).$$

Just as the j-invariant gives an algebraic bijection between $\mathcal{M}_1$ and $\mathbb{C}$, the Igusa-Clebsch invariants give an algebraic bijection between $\mathcal{M}_2$ and $\mathbb{C}^3$. The map $IC : \mathrm{Teich}(S) \to \mathbb{P}(2,4,6,10)$ is $\mathrm{Mod}(S)$ invariant and covers a bijection between $\mathcal{M}_2$ and the hyperplane complement

$$\{(I_2 : I_4 : I_6 : I_{10}) : I_{10} \neq 0\} \subset \mathbb{P}(2,4,6,10). \tag{4.3}$$

[6] We will not repeat the formula for I_k here. See [Ig], the function IgusaClebschInvariants() in MAGMA or the file IIa.magma in [KM2].

Real multiplication by $\mathcal{O}_{12}$. Recall from Section 2 that the Hilbert modular surface

$$X_{12} = \mathbb{H} \times \mathbb{H} / \operatorname{PSL}(\mathcal{O}_{12} \oplus \mathcal{O}_{12}^{\vee})$$

admits an inverse period map $\operatorname{Jac}_{12}^{-1} : X_{12} \to \mathcal{M}_2$ which is a rational map parametrizing the collection of genus two surfaces whose Jacobians admit real multiplication by $\mathcal{O}_{12}$ (cf. Equation 2.19). Set $H_{12} = \mathbb{C}^2$ (the (r,s)-plane) and consider the map $IC_{12} : H_{12} \to \mathbb{P}(2,4,6,10)$ defined by

$$\begin{aligned} IC_{12}(r,s) = \big(&-8(3+3r-3s-2s^2+2s^3) : -8(3+3r-3s-2s^2+2s^3) : \\ &-4(-1+s)^2(72r+90r^2+48rs+102r^2s-141rs^2-38rs^3+8s^4+67rs^4-8s^5-6s^6+6s^7) \\ &: -4r^3(-1+s)^6 \big). \end{aligned} \tag{4.4}$$

The map above is defined in [EK], where it is shown that the map $X_{12} \to \mathbb{P}(2,4,6,10)$ factors through IC_{12}.

Theorem 4.3 (Elkies-Kumar)**.** *The rational map $IC \circ \operatorname{Jac}_{12}^{-1} : X_{12} \to \mathbb{P}(2,4,6,10)$ is the composition of a degree two rational map $X_{12} \to H_{12}$ branched along the curve*

$$b_{12}(r,s) = 0 \text{ where } b_{12}(r,s) = (-1+s)(1+s)(16r+27r^2-18rs^2-s^4+s^6) \tag{4.5}$$

and the map $IC_{12} : H_{12} \to \mathbb{P}(2,4,6,10)$.

Compare Equation 4.5 with Table T.1. From Theorem 4.3 we see that $\operatorname{Jac}(Y)$ admits real multiplication by $\mathcal{O}_{12}$ if and only if $IC(Y)$ is in the closure of the image of IC_{12}.

Proposition 4.4. *The Jacobian of the genus two curve defined by Equation 4.1 admits real multiplication by $\mathcal{O}_{12}$.*

Proof. The curve Y defined by Equation 4.1 satisfies $IC(Y) = IC_{12}(b)$ where $b = (-3/8, -1/2)$. □

Deformations. Now set $a = (24, 52, -8, -12, -2) \in V$ so that the algebraic curve defined by Equation 4.1 is isomorphic to $Y(a)$ in the notation of Section 3. Also, set

$$v_r = (-36, 6, -76, -13, 9) \in T_aV \text{ and } v_s = (-80, 32, -112, 16, 12) \in T_aV. \tag{4.6}$$

Proposition 4.5. *There is an open neighborhood B of $b = (-3/8, -1/2)$ in $\mathbb{C}^2$ and a holomorphic map $g : B \to V$ such that*

$$dg_b((1,0)) = v_r, dg_b((0,1)) = v_s \text{ and } IC\left(Y(g(r,s))\right) = IC_{12}(r,s).$$

Proof. This proposition follows from the inverse function theorem and the following facts:

$$IC_{12}(b) = IC(Y(a)), IC \circ Y : V \to \mathbb{P}(2,4,6,10) \text{ is a submersion at } a,$$
$$d(IC \circ Y)_a(v_r) = d(IC_{12})_b((1,0)) \text{ and } d(IC \circ Y)_a(v_s) = d(IC_{12})_b((0,1)).$$

□

Using Proposition 3.1, we can compute the annihilator of the image of dg_b.

Proposition 4.6. *The annihilator of the image of dg_b is the line of quadratic differentials spanned by $w \cdot dw^2/f_a(w)$ in $Q(Y(a))$.*

Proof. Setting $a = (24, 52, -8, -12, -2)$, the matrix $M(a)$ cited in Proposition 3.1 is

$$M = \frac{1}{2^8 \cdot 3^6} \begin{pmatrix} -95 & 8 & -74 & -328 & -44 \\ 8 & -74 & -328 & -44 & -2752 \\ -74 & -328 & -44 & -2752 & -5000 \end{pmatrix}. \tag{4.7}$$

Let L be the matrix with columns v_r and v_s. The nullspace of $(M \cdot L)^T$ is spanned by $(0, 1, 0)$. By Proposition 3.1, the annihilator of dg_b is the line spanned by

$$(0 \cdot w^0 + 1 \cdot w^1 + 0 \cdot w^2)\frac{dw^2}{f_a(w)} = w\frac{dw^2}{f_a(w)}.$$

□

Eigenform certification. We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Possibly making the neighborhood B of b in Proposition 4.5 smaller, we can ensure that the map g constructed in Proposition 4.5 has a lift $\widetilde{g} : B \to \mathrm{Teich}(S)$ where S is a surface of genus two. By Theorem 4.3, we can choose a proper, self-adjoint embedding $\rho : \mathcal{O} \to \mathrm{End}(H_1(S, \mathbb{Z}))$ so that the image of $\widetilde{g}$ is contained in $\mathrm{Teich}(S, \rho)$.

Let $Y = \widetilde{g}(b)$ and, as in Section 2, let U be the symplectic basis for $H_1(S, \mathbb{R})$ adapted to ρ in the sense of Proposition 2.2 and let $\omega_1(Y)$ and $\omega_2(Y)$ be the eigenforms dual to the a-cycles in U. By Proposition 2.5, the product $\omega_1(Y) \cdot \omega_2(Y)$ annihilates the image of $d\widetilde{g}_b$. On the other hand, Y is biholomorphic to the algebraic curve defined by Equation 4.1, and in this model the annihilator of $d\widetilde{g}_b$ is spanned by $w \cdot dw^2/f_a(w) = w \cdot dw/z \cdot dw/z$ (Proposition 4.6). Since there is, up to scale and permutation, a unique pair of one-forms whose product is equal to $w \cdot dw^2/f_a(w)$, the forms dw/z and $w \cdot dw/z$ are eigenforms for real multiplication by $\mathcal{O}_{12}$ on $\mathrm{Jac}(Y)$. □

We summarize our method of eigenform certification in the Eigenform Certification Algorithm (ECA) in Figure 3. If **(ECA1)** and **(ECA2)** are true, we conclude as in the proof of Proposition 4.4 that $\mathrm{Jac}(Y(a))$ has real multiplication by $\mathcal{O}_D$ and that there is a neighborhood B of b in H_D and a lift $g : B \to V$ with $g(b) = a$ as in Proposition 4.5. If **(ECA4)** is true, we conclude that the annihilator of the image of dg_b is spanned by $w \cdot dw^2/f_a(w)$ as in the proof of Proposition 4.6. Using Proposition 2.5, a `True` eigenform certificate ensures that $w \cdot dw^2/f_a(w)$ is a product of eigenforms and that real multiplication by $\mathcal{O}_D$ on $\mathrm{Jac}(Y(a))$ stabilizes dw/z and $w \cdot dw/z$ up to scale.

5 Weierstrass curve certification

In this section, we discuss implementing the Eigenform Certification Algorithm over function fields to give birational models for irreducible Weierstrass curves. We demonstrate this process in detail for W_{12}, the first Weierstrass curve whose algebraic model has not previously appeared in the literature. We conclude with our proof of Theorem 1.1 giving birational models of irreducible W_D.

We start with the following theorem which is a straightforward application of ECA.

Theorem 5.1. *For generic $t \in \mathbb{C}$, the Jacobian of the algebraic curve $Y_{12}(t)$ defined by*

$$\begin{aligned} z^2 = w^5 + 2(3+2t)w^4 - 4(-1+t)(3+2t)w^3 - 8(3+t)(3+2t)^2w^2 \\ + 4(3+2t)^2(-27-18t+t^2)w + 8(3+2t)^3(21+14t+t^2) \end{aligned} \tag{5.1}$$

Eigenform Certification Algorithm (ECA)

Input: An algebraic map $IC_D : H_D \to \mathbb{P}(2,4,6,10)$ satisfying an analogue of Theorem 4.3 for X_D and points $a \in V$ and $b \in H_D$.
Output: Eigenform certificate. The eigenform certificate is **True** only if $\mathrm{Jac}(Y(a))$ has real multiplication by $\mathcal{O}_D$ with eigenforms dw/z and $w \cdot dw/z$.

If

(ECA1) $IC_D(b) = IC(Y(a))$, and

(ECA2) $d(IC \circ Y)_a$ is onto,

Then

(ECA3) Compute a matrix L so that

$$\mathrm{Range}\left(d(IC \circ Y)_a \cdot L\right) = \mathrm{Range}\left(d(IC_D)_b\right).$$

(ECA4) Return **True** if $(0,1,0)$ spans the nullspace of $(M(a) \cdot L)^T$.

Return **False**.

Figure 3: The Eigenform Certification Algorithm.

admits real multiplication by $\mathcal{O}_{12}$ with eigenforms dw/z and $w \cdot dw/z$.

Proof. Define $a(t) = (a_0(t), \dots, a_4(t))$ so that $a_k(t)$ is the coefficient of w^k on the right hand side of Equation 5.1 and $Y_{12}(t)$ is isomorphic to $Y(a(t))$. Also set $r(t) = -(13+10t+t^2)/t^3$, $s(t) = (t+3)/t$ and $b(t) = (r(t), s(t))$. Running our Eigenform Certification Algorithm with $IC_D = IC_{12}$, $b = b(t)$ and $a = a(t)$ verifies that dw/z and $w \cdot dw/z$ are eigenforms for real multiplication by $\mathcal{O}_{12}$ on $\mathrm{Jac}(Y_{12}(t))$ for generic $t \in \mathbb{C}$ (cf. cert12.magma in [KM2]). Each of the steps in ECA is linear algebra in the field $\mathbb{Q}(t)$. □

Since the form $dw/z \in \Omega(Y_{12}(t))$ has a double zero, the one-form up to scale $(Y_{12}(t), [dw/z])$ is in W_{12} for most t.

Corollary 5.2. *The map $t \mapsto (Y_{12}(t), [dw/z])$ defines a birational map $h_{12} : \mathbb{P}^1 \to W_{12}$.*

Proof. By Theorem 5.1, the pair $(Y_{12}(t), [dw/z])$ is in W_{12} for generic $t \in \mathbb{C}$ and $t \mapsto (Y_{12}(t), [dw/z])$ defines a rational map $h_{12} : \mathbb{P}^1 \to W_{12}$. To check that h_{12} is birational, we compute the composition $\mathbb{P}^1 \xrightarrow{h_{12}} W_{12} \to \mathcal{M}_2 \xrightarrow{IC} \mathbb{P}(2,4,6,10)$ and check that it is non-constant and birational onto its image. Since W_{12} is irreducible, we conclude that h_{12} birational. □

As a corollary, we can verify that the polynomial $w_{12}(r, s)$ in Table T.2 gives a birational model for W_{12} by checking that $b(t)$ defines a birational map from $\mathbb{P}^1$ to the curve $w_{12}(r, s) = 0$, yielding the following proposition.

Proposition 5.3. *The immersion $W_{12} \to \mathcal{M}_2 \xrightarrow{IC} \mathbb{P}(2,4,6,10)$ factors through the composition of a birational map*

$$W_{12} \to \{(r, s) \in H_{12} : w_{12}(r, s) = 0\} \text{ where } w_{12}(r, s) = 27r + (8 - 12s - 9s^2 + 13s^3) \tag{5.2}$$

and the map $IC_{12} : \mathbb{C}^2 \to \mathbb{P}(2,4,6,10)$ of Equation 4.4.

We can now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. For each fundamental discriminant D with $1 < D < 100$ and $D \not\equiv 1 \bmod 8$, we provide two files in [KM2]: ICDrs.magma and certD.magma. In ICDrs.magma we recall the parametrization

$$IC_D : H_D \to \mathbb{P}(2,4,6,10)$$

defined in [EK] and satisfying an analogue of Theorem 4.3 for X_D.[7] In all of our examples, $H_D = \mathbb{C}^2$. In certD.magma, we provide equations for an algebraic curve G_D over $\mathbb{Q}$ and define rational functions

$$a_D : G_D \to V \text{ and } b_D : G_D \to H_D$$

where b_D is birational onto the curve $w_D(r, s) = 0$ and $IC \circ Y \circ a_D$ is birational onto its image. We then call ECA.magma which carries out ECA with $a = a_D$ and $b = b_D$, certifying that

$$h_D(c) = (Y(a_D(c)), [dw/z]) \text{ defines a rational map } h_D : G_D \to W_D. \tag{5.3}$$

Each of the steps **(ECA1)–(ECA4)** is linear algebra in the field of algebraic functions on G_D. We conclude that the curves $w_D(r, s) = 0$, G_D and W_D are birational to one another. □

[7] For several discriminants, we change the coordinates given in [EK] by a product of Möbius transformations on $H_D = \mathbb{C}^2$ to simplify the equation for W_D.

Remark. An important ingredient in our proof of Theorem 1.1 is an explicit model of the universal curve over an open subset of G_D (i.e. the function $a_D : G_D \to V$), which is not easy to compute from IC_D and $w_D(r,s)$. The numerical sampling technique described in Section 1 that we used to compute $w_D(r,s)$ can also be used to sample the universal curve over G_D and was used to generate the equations in certD.magma.

Our proof of Theorem 1.1 also proves Theorem 1.4 giving Weierstrass and plane quartic models for W_D with $D \in \{44, 53, 56, 60, 61\}$.

Proof of Theorem 1.4. For $D \in \{44, 53, 56, 60, 61\}$, the curve G_D defined in certD.magma is the curve defined by the equation $g_D(x,y) = 0$ (cf. Table 1.1) and, by the proof of Theorem 1.1, is birational to W_D. □

Our proof of Theorem 1.1 also proves the second half of Theorem 1.3 concerning irreducible Weierstrass curves of genus zero.

Proposition 5.4. *For $D \leq 41$ with $D \not\equiv 1 \bmod 8$ and $D \neq 21$, the curve W_D is birational to $\mathbb{P}^1$ over $\mathbb{Q}$.*

Proof. For these discriminants, $G_D = \mathbb{P}^1$ and the maps a_D, b_D defined in certD.magma are defined over $\mathbb{Q}$. □

Proposition 5.5. *The curve W_{21} has no $\mathbb{Q}$-rational points and is birational over $\mathbb{Q}$ to the conic $g_{21}(x,y) = 0$ where:*

$$g_{21}(x,y) = 21\left(11x^2 - 182x - 229\right) + y^2. \tag{5.4}$$

Proof. The curve G_D defined in cert21.magma is the conic defined by $g_{21}(x,y) = 0$ and the maps a_D and b_D defined cert21.magma are defined over $\mathbb{Q}$. From the proof of Theorem 1.1, we see that W_{21} is birational over $\mathbb{Q}$ to the curve $g_{21}(x,y) = 0$. The closure of the conic $g_{21}(x,y) = 0$ in $\mathbb{P}^2$ has no integer points, as can be seen by homogenizing Equation 5.4 and reducing modulo 3. We conclude that W_{21} has no $\mathbb{Q}$-rational points. □

6 Cusps and spin components

Using the technique described in Section 5 for verifying our equations for irreducible W_D, we can also show that the curve $w_D^0(r,s) = 0$ parametrizes *an* irreducible component of reducible W_D. We now turn to distinguishing the components of W_D by spin.

Cusps on Weierstrass curves. Let $\overline{\mathcal{M}}_2$ be the Deligne-Mumford compactification of $\mathcal{M}_2$ by stable curves and let $\overline{W}_D$ be the smooth projective curve birational to W_D. The curve $\overline{W}_D$ is obtained from W_D by smoothing orbifold points and filling in finitely many cusps. Since $\overline{W}_D$ and $\overline{\mathcal{M}}_2$ are projective varieties, the map $W_D \to \mathcal{M}_2$ extends to an algebraic map from $\overline{W}_D$ to the coarse space associated to $\overline{\mathcal{M}}_2$. The cusps of $\overline{W}_D$ are sent into $\partial\overline{\mathcal{M}}_2$ under this map.

Locating cusps in birational models. The composition of $W_D \to \mathcal{M}_2 \xrightarrow{IC} \mathbb{P}(2,4,6,10)$ also extends to a map $\overline{W}_D \to \mathbb{P}(2,4,6,10)$ and this extension sends the cusps into the hyperplane $I_{10} = 0$. Given an explicit algebraic curve G_D and a rational map $a_D : G_D \to V$ giving rise to the birational map $h_D : G_D \to W_D$ (cf. the proof of Theorem 1.1), we can locate the smooth points in G_D corresponding to cusps of W_D by determining the poles of the algebraic function

$$c \mapsto (I_2(Y(a_D(c)))^5 / I_{10}(Y(a_D(c))). \tag{6.1}$$

Splitting prototypes. The cusps on W_D are enumerated in [Mc2]. A *splitting prototype of discriminant D* is a quadruple $(a,b,c,e) \in \mathbb{Z}^4$ satisfying

$$\begin{array}{lll} D = e^2 + 4bc, & 0 \le a < \gcd(b,c), & c + e < b, \\ 0 < b, & 0 < c, \text{ and} & \gcd(a,b,c,e) = 1. \end{array} \tag{6.2}$$

For example, the quadruple $(a,b,c,e) = (0,1,3,0)$ is a splitting prototype of discriminant 12.

Theorem 6.1 (McMullen)**.** *If D is not a square, then the cusps of W_D are in bijection with the set of splitting prototypes of discriminant D.*

Stable limits and Igusa-Clebsch invariants. algebraic models for the singular curves corresponding to cusps of W_D are described in [Ba] (see also [BM1], Proposition 3.2). From these models it is easy to prove the following.

Proposition 6.2. *Let $(Y_n, [\omega_n]) \in W_D$ be a sequence tending to the cusp with splitting prototype $p = (a,b,c,e)$. Then $\lim_{n\to\infty} IC(Y_n) = IC(p)$ where*

$$\begin{aligned} IC(p) = (&12b^4 - 8b^3c + 12b^2c^2 - 4b^2e^2 + 24bce^2 + 6e^4 + e(3e^2 + 3D - 4b^2)\sqrt{D} : b^4(e + \sqrt{D})^4 : \\ &b^4(e+\sqrt{D})^4(4b^4 - 4b^3c + 4b^2c^2 - 2b^2e^2 + 8bce^2 + 2e^4 + e(e^2 + D - 2b^2)\sqrt{D}) : 0). \end{aligned} \tag{6.3}$$

For instance, with $Y_{12}(t)$ the algebraic curve defined by Equation 5.1 we have

$$\lim_{t\to\infty} IC(Y_{12}(t)) = (96 : 289 : 8092 : 0) = IC\left((0,1,3,0)\right).$$

Spin invariant. Now suppose $D \equiv 1 \bmod 8$. For such discriminants, the curve W_D has two irreducible components W_D^ϵ distinguished by a spin invariant $\epsilon \in \mathbb{Z}/2\mathbb{Z}$.

Theorem 6.3 (McMullen)**.** *For a prototype $p = (a,b,c,e)$ of discriminant D with $D \equiv 1 \bmod 8$, the cusp corresponding to p lies on the spin $\epsilon(p)$-component of W_D where*

$$\epsilon(p) = \frac{e-f}{2} + (c+1)(a+b+ab) \bmod 2 \tag{6.4}$$

and f is the conductor[8] *of $\mathcal{O}_D$.*

We are now ready to prove Theorem 1.2:

[8] The conductor of $\mathcal{O}_D$ is the index of $\mathcal{O}_D$ in the maximal order of $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Q}$. Rings with fundamental discriminants have conductor $f = 1$.

Proof of Theorem 1.2. For each fundamental discriminant $1 < D < 100$ with $D \equiv 1 \bmod 8$, we provide files ICDrs.magma and certD.magma in [KM2]. In ICDrs.magma we recall the map IC_D in [EK] and in certD.magma we define an algebraic curve G_D^0 and rational functions

$$a_D^0 : G_D^0 \to V \text{ and } b_D^0 : G_D^0 \to H_D$$

so that $IC \circ Y \circ a_D^0$ is birational onto its image and b_D^0 is birational onto the curve $w_D^0(r,s) = 0$. As in the proof of Theorem 1.1, we then call ECA.magma which implements the Eigenform Certification Algorithm verifying that

$$h_D^0(c) = (Y(a_D^0(c)), [dw/z]) \text{ defines a rational map } h_D^0 : G_D^0 \to W_D. \tag{6.5}$$

We conclude that $w_D^0(r,s) = 0$ is birational to an irreducible component of W_D.

We then identify a smooth point $c \in G_D^0$ and call spin_check.magma which checks that c corresponds to a cusp of W_D (i.e. I_2^5/I_{10} has a pole at c), identifies the splitting prototypes p of discriminant D satisfying $IC(p) = IC_D(b_D(c))$ and verifies that they all have even spin using Equation 6.4. This shows that the curve $w_D^0(r,s) = 0$ is birational to W_D^0.

Applying the non-trivial field automorphism of $\mathbb{Q}(\sqrt{D})$ to all of the equations in certD.magma gives a curve G_D^1 and maps a_D^1, b_D^1 and h_D^1. Since the equations in ICDrs.magma have coefficients in $\mathbb{Q}$, ECA with $a = a_D^1$ and $b = b_D^1$ will return the same value as ECA with $a = a_D^0$ and $b = b_D^0$. We conclude that the Galois conjugate w_D^1 of w_D^0 defines a curve birational to another component of W_D. We verify that this component is W_D^1 by the method above applied to the point in G_D^1 Galois conjugate to $c \in G_D^0$. □

Remark. Some care has to be taken when choosing the point $c \in G_D^0$ in the proof of Theorem 1.2 since the stable limit $h_D^0(c)$ does not always uniquely identify the corresponding splitting prototype. For instance, the first coordinate a in the splitting prototype does not affect the stable limit, as reflected by the fact that a does not appear on the right hand side of Equation 6.3.

We can combine the parametrization h_D^0 and its Galois conjugate h_D^1 used in the proof of Theorem 1.2 into a birational map

$$h_D : G_D = G_D^0 \sqcup G_D^1 \to W_D = W_D^0 \sqcup W_D^1. \tag{6.6}$$

We will use h_D in the next section to give biregular models of reducible $\overline{W}_D$ for certain D in the next section.

Our proof of Theorem 1.2 also establishes Theorem 1.5 which gives Weierstrass models for the components of W_{57}, W_{65} and W_{73}.

Proof of Theorem 1.5. For $D \in \{57, 65, 73\}$, the curve G_D^0 defined in certD.magma is the curve defined by $g_D^0(x,y) = 0$. In the proof of Theorem 1.2, we saw that W_D^0 is birational to G_D^0 and W_D^1 is birational to the Galois conjugate of G_D^0. □

We can also complete the proof of Theorem 1.3 concerning Weierstrass curves of genus zero.

Proof of Theorem 1.3. For $D \leq 41$ with $D \equiv 1 \bmod 8$, the curve G_D^0 defined in certD.magma is $\mathbb{P}^1$ and the maps a_D and b_D are defined over $\mathbb{Q}(\sqrt{D})$. This shows that the components of W_D are birational to $\mathbb{P}^1$ over $\mathbb{Q}(\sqrt{D})$ for such discriminants. The remaining claims made in Theorem 1.3 are established in Propositions 5.4 and 5.5. □

7 Arithmetic geometry of Weierstrass curves

In this section, we study the arithmetic geometry of our examples of Weierstrass curves and prove the remaining Theorems stated in Section 1.

Biregular models for Weierstrass curves. For each fundamental discriminant $1 < D < 100$, we have now given a birational parametrization h_D of W_D by an explicit algbraic curve G_D (cf. proofs of Theorems 1.1 and 1.2). The curve G_D and parametrization $h_D : G_D \to W_D$ are defined [KM2].

Many of our birational models for small genus W_D easily extend to biregular models for the smooth, projective curve $\overline{W}_D$ birational to W_D. For $D \leq 41$ with $D \neq 21$, G_D is a union of $k = 1$ or 2 projective t-lines and is already smooth and projective, and h_D extends to a biregular map $h_D : \overline{G}_D = G_D \to \overline{W}_D$. For $D \in \{21, 44, 56, 57, 60, 65, 73\}$, each irreducible component of G_D is an affine plane curve with smooth closure in $\mathbb{P}^2$. The birational map h_D extends to a biregular map $h_D : \overline{G}_D \to \overline{W}_D$ where the irreducible components of $\overline{G}_D$ are disjoint and equal to the closures of irreducible components of G_D in $\mathbb{P}^2$. For $D \in \{53, 61\}$ the curve G_D is an irreducible affine curve of genus two and has singular closure in $\mathbb{P}^2$. The closure $\overline{G}_D$ of the algebraic set

$$\left\{\left((x : y : 1), (1/x : y/x^3 : 1)\right) : g_D(x, y) = 0, x \neq 0\right\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \tag{7.1}$$

is smooth, projective and birational to G_D in an obvious way, and the birational map h_D naturally extends to a biregular map $h_D : \overline{G}_D \to \overline{W}_D$.

For the remainder of this section, we will identify $\overline{W}_D$ for these discriminants ($D \leq 73$ with $D \neq 69$) with the biregular models described above via the biregular map h_D.

Singular primes and primes of bad reduction. Now that we have given smooth, projective models over $\mathbb{Z}$ for several Weierstrass curves, we can study their primes of singular and bad reduction. For general discussion of these notions we refer the reader to [Li] (in particular §10.1.2) and [Da]. For an affine plane curve C defined by $g \in \mathbb{Z}[x, y]$ and a prime $p \in \mathbb{Z}$, we say that p is a *prime of singular reduction for C* if the polynomial equations

$$g = 0, \partial g/\partial x = 0 \text{ and } \partial g/\partial y = 0$$

have a simultaneous solution in an algebraically closed field of characterstic p. For a projective curve C defined over $\mathbb{Z}$ and covered by plane curves $C_1, \ldots, C_n$ defined by polynomials $g_1, \ldots, g_n \in \mathbb{Z}[x, y]$, we will say that p is a *prime of singular reduction for C* if p is a prime of singular reduction for at least one of the curves C_k. For an affine or projective curve C defined over $\mathbb{Q}$ and a prime $p \in \mathbb{Z}$, we will call p a prime of *bad reduction* for C if p is a singular prime for every curve C' defined over $\mathbb{Z}$ and biregular to C over $\mathbb{Q}$. In particular, the primes of singular reduction for *any* integral model of C contain the primes of bad reduction of C.

Singular primes of low, positive genus Weierstrass curves. As we demonstrate in our next proposition, the primes of singular reduction for conic and hyperelliptic Weierstrass curves can be computed using discriminants and the primes of singular reduction for our genus three Weierstrass curves can be computed using elimination ideals.

Theorem 7.1. *For $D \in \{21, 44, 53, 56, 60, 61\}$, the primes of singular reduction for $\overline{W}_D$ are those listed in Table 7.1.*

D	Singular primes for $\overline{W}_D$
21	$\{2,3,5,7\}$
44	$\{2,5,11\}$
53	$\{2,11,13,53\}$
56	$\{2,5,7,13\}$
60	$\{2,3,5,7,11\}$
61	$\{2,3,5,13,61\}$

Table 7.1: For $D \in \{44,53,56,60,61\}$, the birational model $g_D(x,y)=0$ for the Weierstrass curve $\overline{W}_D$ has a singularity at the prime p for the primes listed above.

Proof. For a hyperelliptic curve or conic birational to the plane curve defined by a polynomial of the form $y^2+h(x)y+f(x) \in \mathbb{Z}[x,y]$, it is standard to show that the primes of singular reduction are precisely the primes dividing the discriminant of $h(x)^2-4f(x)$. From this we easily verify that the primes listed in Table 7.1 are the primes of singular reduction for $\overline{W}_{21}$, $\overline{W}_{44}$, $\overline{W}_{53}$ and $\overline{W}_{61}$.

Now set $D=56$ or 60 so that $\overline{W}_D$ is a smooth plane quartic and let $g_D^h \in \mathbb{Z}[X,Y,Z]$ be the homogeneous, degree four polynomial with $g_D^h(x,y,1)=g_D(x,y)$. Also set $g_1(x,y)=g_D^h(x,y,1)$, $g_2(x,y)=g_D^h(x,1,y)$ and $g_3(x,y)=g_D^h(1,x,y)$ so that $\overline{W}_D=C_1\cup C_2\cup C_3$ with C_k the plane curve defined by g_k. For each of the primes listed next to D in Table 7.1, we are able to find a simultaneous solution to $g_k=0$, $\partial g_k/\partial x=0$ and $\partial g_k/\partial y=0$ with coordinates in the finite field with p elements for some k. We conclude that each of these primes is a prime of singular reduction for $\overline{W}_D$. To show that there are no other primes of singular reduction for $\overline{W}_D$, we consider the elimination ideals

$$E_k = I_k \cap \mathbb{Z} \text{ where } I_k = (g_k, \partial g_k/\partial x, \partial g_k/\partial y)\,.$$

Elimination ideals can be computed using Gröbner bases and it is easy to compute E_k in MAGMA. Clearly, if p is a prime of singular reduction for the affine curve C_k, then the ideal generated by p divides E_k. The primes listed in Table 7.1 are precisely those dividing $E_1\cdot E_2\cdot E_3$ and contain all of the primes of singular reduction for $\overline{W}_D$. □

Theorem 1.6 about the primes of bad reduction for certain Weierstrass curves is a corollary of Theorem 7.1.

Proof of Theorem 1.6. The set primes of bad reduction for $\overline{W}_D$ is contained in the set of primes of singular reduction for our biregular model of $\overline{W}_D$. By inspecting Table 7.1, we see that each prime of singular reduction for our model of $\overline{W}_D$ divides the quantity $N(D)$ defined in Equation 1.1. □

Singular primes for genus zero Weierstrass curves. We now turn to the Weierstrass curves $\overline{W}_D$ biregular to the projective t-line $\mathbb{P}^1$ over $\mathbb{Q}$. In Section 1, we defined the *cuspidal polynomial* for these curves to be the monic polynomial $c_D(t)$ vanishing simply at the cusps of $\overline{W}_D$ in the affine t-line and non-zero elsewhere.

D	Cuspidal polynomial $c_D(t)$	Discriminant of $c_D(t)$
5	$t-4$	1
8	$t(t+1)$	1
12	$t^2+10t+13$	$2^4 \cdot 3$
13	$t(t^2-14t-3)$	$2^4 \cdot 3^2 \cdot 13$
24	$t(t-16)(t^2-6)(t^2-24t-72)$	$2^{36} \cdot 3^{14} \cdot 5^{14}$
28	$(t^2-24t-423)(t^2-63)(t^2+14t+21)$	$2^{30} \cdot 3^{38} \cdot 7^7$
29	$t(t^2-174t+145)(t^2+145t-3625)$	$2^{10} \cdot 5^{18} \cdot 7^8 \cdot 29^{10}$
37	$(t^2-2368)(t^2-1332)(t^2+74t+1221)$ $(t^3+51t^2-2220t-114108)$	$2^{60} \cdot 3^{23} \cdot 7^{32} \cdot 37^{28}$
40	$t(t+81)(t^2+110t+2025)$ $(t^2+270t-10935)(t^2+630t+18225)$ $(t^3+351t^2+10935t+164025)$	$2^{168} \cdot 3^{267} \cdot 5^{66}$

Table 7.2: For $\overline{W}_D$ birational to the projective t-line over $\mathbb{Q}$, the cuspidal polynomial $c_D(t)$ is the polynomial vanishing simply at cusps of $\overline{W}_D$ in the finite t-line and nowhere else zero.

As we described in Section 6, we can locate the cusps and compute $c_D(t)$ in each of these examples by determining the poles of the algebraic function I_2^5/I_{10} on $\overline{W}_D$. We list the polynomials $c_D(t)$ along with their discriminants in Table 7.2, allowing us to prove Theorem 1.7.

Proof of Theorem 1.7. The polynomial $c_D(t)$ listed in Table 7.2 is obviously in $\mathbb{Z}[t]$ and each of the primes dividing the discriminant of $c_D(t)$ divides the quantity $N(D)$ defined in Equation 1.1. □

A Weierstrass elliptic curve. The Weierstrass curve $\overline{W}_{44}$ is the only Weierstrass curve associated to a fundamental discriminant and birational to an elliptic curve over $\mathbb{Q}$. From our explicit Weierstrass model for $\overline{W}_{44}$, it is standard to compute various arithmetic invariants and easy to do so in MAGMA or Sage (also cf. [CS]). We collect these facts about $\overline{W}_{44}$ in the following theorem.

Proposition 7.2. *The Weierstrass curve $\overline{W}_{44}$ has j-invariant $j(\overline{W}_{44}) = 479^3/(11 \cdot 2^5 \cdot 5^5)$, conductor $N\left(\overline{W}_{44}\right) = 880$, endomorphism ring* $\mathrm{End}\left(\overline{W}_{44}\right)$ *isomorphic to $\mathbb{Z}$ and infinite cyclic Mordell-Weil group $\overline{W}_{44}(\mathbb{Q})$ generated by $(x, y) = (26, 160)$.*

Remark. We have numerical evidence, obtained using the functions related to analytic Jacobians in MAGMA, that the endomorphism rings of $\mathrm{Jac}\left(\overline{W}_{53}\right)$ and $\mathrm{Jac}\left(\overline{W}_{61}\right)$ are also isomorphic to $\mathbb{Z}$.

Our identification of $\overline{W}_{44}$ with an elliptic curve turns $\overline{W}_{44}$ into a group. We will call the subgroup of $\overline{W}_{44}$ generated by cusps the *cuspidal subgroup*. By the method described in Section 6, we can locate the cusps on $\overline{W}_{44}$ and prove the following proposition.

Proposition 7.3. *The cuspidal subgroup of $\overline{W}_{44}$ is freely generated by*

$$P_1 = \left(\frac{-38 - 48\sqrt{11}}{25}, \frac{-1584 + 1936\sqrt{11}}{125} \right) \text{ and } P_2 = \left(2 + 4\sqrt{11}, 44 + 16\sqrt{11}\right).$$

Prototype	(x,y)	(r,s)	Mordell-Weil
$(0,11,1,0)$	(∞,∞)	$(-1,0)$	$(0,0)$
$(0,7,1,4)$	$\left(\frac{1}{25}(-38-48\sqrt{11}),\frac{1}{125}(-1584+1936\sqrt{11})\right)$	$(-1,0)$	$(1,0)$
$(0,7,1,-4)$	$\left(\frac{1}{25}(-38+48\sqrt{11}),\frac{1}{125}(-1584-1936\sqrt{11})\right)$	$(-1,0)$	$(5,-6)$
$(0,5,2,-2)$	$\left(2+4\sqrt{11},44+16\sqrt{11}\right)$	$(1,0)$	$(0,1)$
$(0,5,2,2)$	$\left(2-4\sqrt{11},44-16\sqrt{11}\right)$	$(1,0)$	$(4,-5)$
$(0,2,1,-6)$	$(-9,10\sqrt{11})$	$\left(\frac{1}{15}(2-2\sqrt{11}),0\right)$	$(6,-9)$
$(0,1,2,-6)$	$(-9,-10\sqrt{11})$	$\left(\frac{1}{15}(2+2\sqrt{11}),0\right)$	$(-6,9)$
$(0,10,1,-2)$	$\left(66+20\sqrt{11},740+240\sqrt{11}\right)$	$(-1,0)$	$(-2,4)$
$(0,10,1,2)$	$\left(66-20\sqrt{11},740-240\sqrt{11}\right)$	$(-1,0)$	$(6,-8)$

Table 7.3: For each of the nine splitting prototypes of discriminant 44, we list the (x,y) coordinates in the Weierstrass model $g_{44}(x,y)=0$, the (r,s)-coordinates in the $w_{44}(r,s)=0$ model and the Mordell-Weil coordinates in the cuspidal subgroup of $\overline{W}_{44}$ for the corresponding cusp.

Proof. The second column of Table 7.3 identifies the locations of the cusps for $\overline{W}_{44}$ in our elliptic curve model $g_{44}(x,y)=0$ and the fourth column asserts relations among these points in the group law (e.g. the cusp at $Q=(-9,10\sqrt{11})$ is equal to $6P_1-9P_2$). It is standard to verify these relations and easy to do so in MAGMA. We conclude that the cuspidal subgroup is generated by P_1 and P_2.

To show that the cuspidal subgroup is freely generated by P_1 and P_2, we first check that P_1-P_2 is a $\mathbb{Q}$-rational point. By Proposition 7.2, the difference P_1-P_2 generates a free subgroup of $\overline{W}_{44}$. Next, we check that $n\cdot P_2$ is not $\mathbb{Q}$-rational for any $n\leq 18$. By Kamienny's bound on the torsion order of points on elliptic curves over quadratic fields [Ka], we conclude that P_1 and P_1-P_2 generate a free subgroup of $\overline{W}_{44}$ and the proposition follows. □

Theorem 1.8 concerning the subgroup of $\mathrm{Pic}^0\left(\overline{W}_{44}\right)$ generated by pairwise cusp differences is an immediate corollary.

Proof of Theorem 1.8. Since the identity $(x,y)=(\infty,\infty)$ in $\overline{W}_{44}$ is a cusp, the cuspidal subgroup is naturally isomorphic to the subgroup of $\mathrm{Pic}^0\left(\overline{W}_{44}\right)$ generated by pairwise cusp differences. By Proposition 7.3, the cuspidal group is isomorphic to $\mathbb{Z}^2$. □

Canonical divisors supported at cusps. A genus two curve with Weierstrass model given by $y^2+h(x)y+f(x)=0$ admits a hyperelliptic involution η given by the formula $\eta(x,y)=(x,-h(x)-y)$. The orbits of η are intersections with vertical lines $x=c$ and canonical divisors. By locating the cusps on $\overline{W}_{53}$ as described in Section 6, we find two canonical divisors supported at cusps.

Proposition 7.4. *The holomorphic one-forms on $\overline{W}_{53}$ given by*

$$\omega_1=\left(2x+7-2\sqrt{53}\right)dx/y \text{ and } \omega_2=\left(2x+7+2\sqrt{53}\right)dx/y$$

vanish only at cusps.

By contrast, after computing the cusp locations on $\overline{W}_{61}$, we find that there are no such forms on $\overline{W}_{61}$.

Proposition 7.5. *There are no holomorphic one-forms on $\overline{W}_{61}$ which vanish only at cusps.*

For both $\overline{W}_{53}$ and $\overline{W}_{61}$, the hyperelliptic involution η does not preserve the set of cusps, yielding our next proposition.

Proposition 7.6. *For $D \in \{53, 61\}$, the hyperelliptic involution on $\overline{W}_D$ does not restrict a hyperbolic isometry of W_D.*

Our smooth plane quartic Weierstrass curves—$\overline{W}_{56}$ and $\overline{W}_{60}$—are canonically embedded in $\mathbb{P}^2$. In particular, intersections with lines are canonical divisors. By computing the cusp locations on $\overline{W}_{56}$, we find a canonical divisor supported at cusps. In the following propositions, we let X, Y and Z be homogeneous coordinates on the projective closure of the (x, y)-plane, with $x = X/Z$ and $y = Y/Z$

Proposition 7.7. *The line $Y = 2Z$ meets $\overline{W}_{56}$ at a canonical divisor supported at cusps.*

On $\overline{W}_{60}$, we find five canonical divisors supported at cusps.

Proposition 7.8. *Each of the following five lines*

$$\begin{gathered} Y = 0, 4X + (6 - \sqrt{60})Y = 0, 4X + (6 + \sqrt{60})Y = 0, \\ -6X + 10X - Z = 0 \text{ and } 6X + 5X + Z = 0 \end{gathered} \tag{7.2}$$

meets $\overline{W}_{60}$ at a canonical divisor supported at cusps.

Combining the propositions of this paragraph, we can now prove Theorem 1.10.

Proof of Theorem 1.10. By Propositions 7.4, 7.7 and 7.8, each of the curves $\overline{W}_{53}$, $\overline{W}_{56}$ and $\overline{W}_{60}$ has a canonical divisor supported at cusps. By Proposition 7.5, the curve $\overline{W}_{61}$ has no canonical divisor supported at cusps. □

Principal divisors supported at cusps. We now prove the following theorem about principal divisors supported at cusps on Weierstrass curves.

Proposition 7.9. *Each of the curves $\overline{W}_{44}$, $\overline{W}_{53}$, $\overline{W}_{57}$, $\overline{W}_{60}$, $\overline{W}_{65}$ and $\overline{W}_{73}$ has a principal divisor supported at cusps.*

Proof. By Propositions 7.4 and 7.8, each of the curves $\overline{W}_{53}$ and $\overline{W}_{60}$ has a pair of holomorphic one-forms which are distinct up to scale and vanish only at cusps. The ratio of these two one-forms defines an algebraic function with zeros and poles only at cusps.

The irreducible components of the remaining curves all have genus one and our biregular models for these curves are elliptic curves. By the technique described in Section 6, we locate their cusps. We find that the identity $(x, y) = (\infty, \infty)$ is a cusp in each case and then search for (and find) relations among the cusps in the group law by computing small integer combinations among triples of cusps. We have already given many such relations for $\overline{W}_{44}$ in Table 7.3. For $D \in \{57, 65, 73\}$, we include a comment in certD.magma identifying the locations of several cusps and a relation among them. □

Theorem 1.9 is an immediate corollary of Proposition 7.9.

Proof of Theorem 1.9. For a projective curve C and a finite set $S \subset C$, the curve C has a principal divisor supported at S if and only if $C \setminus S$ admits a non-constant holomorphic map to $\mathbb{C}^*$. By Proposition 7.9, for each $D \in \{44, 53, 57, 60, 65, 73\}$, the curve $\overline{W}_D$ has a principal divisor supported at its cusps $S_D \subset \overline{W}_D$, so the curve $W_D = \overline{W}_D \setminus S_D$ admits a non-constant holomorphic map to $\mathbb{C}^*$. □

T Tables

In this Appendix, we provide tables listing birational models of the Hilbert modular surface X_D (Table T.1), the Weierstrass curve W_D (Tables T.2 and T.3) and the homeomorphism type of W_D (Table T.4) for fundamental discriminants $1 < D < 100$.

Algebraic models for Hilbert modular surfaces
$b_5(r,s) = 972r^5 + 324r^4 + 27r^3 + 4500r^2s + 1350rs - 6250s^2 + 108s$
$b_8(r,s) = 16r^3 + 32r^2s + 24r^2 + 16rs^2 - 40rs + 12r - s + 2$
$b_{12}(r,s) = 27r^2s^2 - 27r^2 - 18rs^4 + 34rs^2 - 16r + s^8 - 2s^6 + s^4$
$b_{13}(r,s) = 128r^3 + 27r^2s^2 - 656r^2s - 192r^2 - 108rs^3 + 468rs^2 - 568rs + 96r - 4s^2 + 16s - 16$
$b_{17}(r,s) = 4r^6 + 20r^5 - 48r^4s + 41r^4 + 236r^3s + 44r^3 + 192r^2s^2 + 346r^2s + 26r^2 + 464rs^2 + 144rs + 8r - 256s^3 + 185s^2 + 18s + 1$
$b_{21}(r,s) = 189r^6 - 594r^5s + 621r^4s^2 - 378r^4 - 216r^3s^3 + 1116r^3s - 954r^2s^2 + 205r^2 + 184rs^3 - 522rs + 16s^4 + 349s^2 - 16$
$b_{24}(r,s) = r^4 - 9r^3s^2 - 2r^3 + 24r^2s^4 - 25r^2s^2 - 16rs^6 + 36rs^4 - 22rs^2 + 2r - s^4 + 2s^2 - 1$
$b_{28}(r,s) = 100r^6 + 580r^5 - 192r^4s^2 + 1191r^4 - 728r^3s^2 + 1000r^3 + 84r^2s^4 - 1230r^2s^2 + 264r^2 + 148rs^4 - 1000rs^2 - 32r + 8s^6 + 39s^4 - 280s^2 - 16$
$b_{29}(r,s) = 1024r^5 + 27r^4s^2 - 288r^4s - 768r^4 - 18r^3s^2 + 200r^3s + 192r^3 + 5r^2s^2 - 280r^2s - 16r^2 - 6rs^3 + 102rs^2 + 8rs + s^4 - 11s^3 - s^2$
$b_{33}(r,s) = 8r^6 - 72r^5 - 25r^4s^2 + 280r^4 + 152r^3s^2 - 472r^3 + 26r^2s^4 - 400r^2s^2 + 336r^2 - 80rs^4 + 408rs^2 - 64r - 9s^6 + 104s^4 - 432s^2 - 16$
$b_{37}(r,s) = 108r^4s - 27r^4 - 126r^3s^2 - 176r^3s + 62r^3 + r^2s^4 + 28r^2s^3 + 142r^2s^2 + 102r^2s - 51r^2 - 2rs^3 - 44rs^2 - 54rs + 24r + s^2 + 20s - 8$
$b_{40}(r,s) = 9r^6 - 12r^5 - 26r^4s^2 - 2r^4 + 24r^3s^2 + 22r^3 + 25r^2s^4 + 2r^2s^2 - 5r^2 - 12rs^4 - 22rs^2 - 10r - 8s^6 + 6s^2 - 2$
$b_{41}(r,s) = 256r^4s^4 + 256r^4s^3 + 96r^4s^2 + 16r^4s + r^4 - 2048r^3s^4 - 11776r^3s^3 - 5248r^3s^2 - 544r^3s + 16r^3 + 4096r^2s^4 + 149504r^2s^3 + 63232r^2s^2 + 7936r^2s + 96r^2 - 688128rs^3 - 223232rs^2 - 20992rs + 256r + 1048576s^3 + 196608s^2 + 12288s + 256$
$b_{44}(r,s) = r^8 - 8r^7 - r^6s^2 + 2r^6s + 26r^6 + 6r^5s^2 - 12r^5s - 44r^5 - 14r^4s^2 + 46r^4s + 41r^4 + 16r^3s^2 - 104r^3s - 20r^3 - 18r^2s^3 + 54r^2s^2 + 92r^2s + 4r^2 + 36rs^3 - 124rs^2 - 8rs - 27s^4 + 52s^3 + 4s^2$
$b_{53}(r,s) = 27r^4s^4 + 26r^3s^5 + 18r^3s^4 + 18r^3s^3 + 11r^2s^6 + 138r^2s^5 + 383r^2s^4 + 506r^2s^3 + 353r^2s^2 + 120r^2s + 16r^2 + 104rs^6 + 406rs^5 + 630rs^4 + 496rs^3 + 210rs^2 + 46rs + 4r + 44s^7 + 252s^6 + 608s^5 + 799s^4 + 616s^3 + 278s^2 + 68s + 7$
$b_{56}(r,s) = 64r^6 + 384r^5 + 27r^4s^4 + 72r^4s^3 - 40r^4s^2 - 288r^4s + 752r^4 + 108r^3s^4 + 288r^3s^3 - 160r^3s^2 - 1152r^3s + 448r^3 - 52r^2s^5 - 20r^2s^4 + 416r^2s^3 + 224r^2s^2 - 1088r^2s - 64r^2 - 104rs^5 - 256rs^4 + 256rs^3 + 768rs^2 + 128rs - 4s^6 - 32s^5 - 96s^4 - 128s^3 - 64s^2$
$b_{57}(r,s) = 256r^6s^4 + 2176r^5s^3 - 3456r^5s - 176r^4s^4 + 4320r^4s^2 - 2160r^4 - 960r^3s^3 + 576r^3s + 40r^2s^4 - 688r^2s^2 - 648r^2 + 104rs^3 + 328rs - 3s^4 - 34s^2 + 361$
$b_{60}(r,s) = 9r^{10} - 44r^8s^2 - 156r^8 + 86r^6s^4 + 532r^6s^2 + 838r^6 - 84r^4s^6 - 668r^4s^4 - 1836r^4s^2 - 1188r^4 + 41r^2s^8 + 364r^2s^6 + 1382r^2s^4 + 2124r^2s^2 - 1095r^2 - 8s^{10} - 72s^8 - 368s^6 - 752s^4 - 648s^2 - 200$

Table T.1: The Hilbert modular surface X_D is birational to the degree two cover of the (r,s)-plane branched along the curve $b_D(r,s) = 0$.

$b_{61}(r,s) = r^4s^4-4r^4s^3+6r^4s^2-4r^4s+r^4-2r^3s^5+30r^3s^4+12r^3s^3+2r^3s^2-42r^3s+r^2s^6-46r^2s^5-19r^2s^4+42r^2s^3+39r^2s^2-44r^2s+20rs^6+10rs^5-26rs^4-2rs^3+24rs^2-8s^6+13s^4-16s^2$

$b_{65}(r,s) = 4r^4s^{10}+3r^4s^8+166r^4s^6-997r^4s^4+328r^4s^2-80r^4+32r^3s^{11}+56r^3s^9+1192r^3s^7-5016r^3s^5-5128r^3s^3+1184r^3s+96r^2s^{12}+264r^2s^{10}+3264r^2s^8-7184r^2s^6-21376r^2s^4-16184r^2s^2+3232r^2+128rs^{13}+480rs^{11}+4064rs^9-320rs^7-15808rs^5-40096rs^3-30368rs+64s^{14}+304s^{12}+1952s^{10}+3920s^8+7680s^6-6448s^4-37856s^2-35152$

$b_{69}(r,s) = r^6s^6-2r^6s^5+r^6s^4-2r^5s^6-24r^5s^5+100r^5s^4-118r^5s^3+44r^5s^2+r^4s^6+100r^4s^5-439r^4s^4+640r^4s^3-357r^4s^2+72r^4s-16r^4-118r^3s^5+640r^3s^4-1180r^3s^3+872r^3s^2-266r^3s+64r^3+44r^2s^5-357r^2s^4+872r^2s^3-830r^2s^2+314r^2s-83r^2+72rs^4-266rs^3+314rs^2-130rs+38r-16s^4+64s^3-83s^2+38s-11$

$b_{73}(r,s) = 16r^4s^2-64r^4s+64r^4+136r^3s^4-688r^3s^3+1120r^3s^2-640r^3s+128r^3+r^2s^6+56r^2s^5-384r^2s^4+448r^2s^3+432r^2s^2-512r^2s+64r^2+2rs^6-64rs^5+80rs^4+304rs^3-224rs^2+64rs+s^6+8s^5+24s^4+32s^3+16s^2$

$b_{76}(r,s) = 4r^6s^2-13r^4s^4-4r^4s^3-36r^4s^2-48r^4s-16r^4+32r^2s^6+80r^2s^5+202r^2s^4+224r^2s^3+100r^2s^2+16r^2s-288s^6-1104s^5-1853s^4-1628s^3-724s^2-128s$

$b_{77}(r,s) = r^6s^6-14r^6s^4-343r^6s^2-1372r^6+42r^5s^5-1124r^5s^3+7994r^5s-2r^4s^6+433r^4s^4-6268r^4s^2-4531r^4-68r^3s^5+2088r^3s^3+4892r^3s+r^2s^6-328r^2s^4-1763r^2s^2+362r^2+26rs^5+316rs^3-342rs-27s^4+54s^2-27$

$b_{85}(r,s) = -8r^8s^4+72r^8s^3-164r^8s^2+144r^8s-44r^8+16r^7s^4-44r^7s^3+236r^7s^2-388r^7s+180r^7-3r^6s^4+108r^6s^3+18r^6s^2-28r^6s-79r^6+6r^5s^4+44r^5s^3+120r^5s^2+388r^5s-398r^5-5r^4s^4+346r^4s^2+219r^4-6r^3s^4+44r^3s^3-120r^3s^2+388r^3s+398r^3-3r^2s^4-108r^2s^3+18r^2s^2+28r^2s-79r^2-16rs^4-44rs^3-236rs^2-388rs-180r-8s^4-72s^3-164s^2-144s-44$

$b_{88}(r,s) = 27r^6s^2+208r^5s^4-96r^5s^3+120r^5s^2+46r^5s-64r^4s^6-768r^4s^5+1376r^4s^4+112r^4s^3-64r^4s^2+120r^4s+27r^4-256r^3s^6-5376r^3s^5+640r^3s^4+2112r^3s^3+112r^3s^2-96r^3s-256r^2s^6-12032r^2s^5-8576r^2s^4+640r^2s^3+1376r^2s^2+208r^2s-8704rs^5-12032rs^4-5376rs^3-768rs^2-256s^4-256s^3-64s^2$

$b_{89}(r,s) = r^6s^4-4r^5s^5-6r^5s^4-4r^5s^3+6r^4s^6+16r^4s^5-49r^4s^4-26r^4s^3+6r^4s^2-4r^3s^7-12r^3s^6+100r^3s^5-52r^3s^4-146r^3s^3+70r^3s^2-4r^3s+r^2s^8-36r^2s^6+26r^2s^5+273r^2s^4-514r^2s^3+271r^2s^2-38r^2s+r^2+2rs^8-12rs^7+68rs^6-322rs^5+772rs^4-890rs^3+470rs^2-88rs+s^8-16s^7+102s^6-332s^5+593s^4-588s^3+304s^2-64s$

$b_{92}(r,s) = r^8s^4-4r^8s^3+6r^8s^2-4r^8s+r^8+2r^7s^5-20r^7s^4+44r^7s^3-32r^7s^2+2r^7s+4r^7+r^6s^6-56r^6s^5+114r^6s^4-20r^6s^3-77r^6s^2+32r^6s+6r^6-66r^5s^6+46r^5s^5+238r^5s^4-286r^5s^3+20r^5s^2+44r^5s+4r^5-26r^4s^7-98r^4s^6+388r^4s^5-168r^4s^4-238r^4s^3+114r^4s^2+20r^4s+r^4-96r^3s^7+202r^3s^6+144r^3s^5-388r^3s^4+46r^3s^3+56r^3s^2+2r^3s-27r^2s^8+30r^2s^7+169r^2s^6-202r^2s^5-98r^2s^4+66r^2s^3+r^2s^2+46rs^7-30rs^6-96rs^5+26rs^4-27s^6$

$b_{93}(r,s) = 16r^6s^6-32r^6s^4+16r^6s^2+168r^5s^6+8r^5s^4-392r^5s^2+216r^5-27r^4s^8+684r^4s^6+1246r^4s^4-1620r^4s^2-27r^4-216r^3s^8+872r^3s^6+3768r^3s^4-328r^3s^2-648r^2s^8-1080r^2s^6+2184r^2s^4+56r^2s^2-864rs^8-2496rs^6+288rs^4-432s^8-1312s^6-48s^4$

$b_{97}(r,s) = r^6s^4+14r^6s^3+r^6s^2+4r^5s^5+54r^5s^4-26r^5s^3+30r^5s^2+2r^5s+6r^4s^6+80r^4s^5-75r^4s^4+128r^4s^3-54r^4s^2+18r^4s+r^4+4r^3s^7+56r^3s^6-64r^3s^5+168r^3s^4-148r^3s^3+96r^3s^2-26r^3s+2r^3+r^2s^8+18r^2s^7-11r^2s^6+68r^2s^5-101r^2s^4+112r^2s^3-69r^2s^2+22r^2s+r^2+2rs^8+6rs^7-10rs^6+14rs^5+6rs^4-32rs^3+24rs^2+s^8-8s^7+24s^6-32s^5+16s^4$

Algebraic models for W_D
$w_5(r,s) = 15r + 2$
$w_8(r,s) = 4r + 4s + 1$
$w_{12}(r,s) = 27r + 13s^3 - 9s^2 - 12s + 8$
$w_{13}(r,s) = 26r + 108s^2 - 252s - 9$
$w_{17}(r,s) = 16r^4 + 136r^3 - 64r^2s + 196r^2 + 544rs + 102r - 1024s^2 + 288s + 18$
$w_{21}(r,s) = 108r^4 - 216r^3s - 513r^3 + 108r^2s^2 + 621r^2s - 925r^2 - 108rs^2 + 1650rs + 205r - 225s^2 + 795s + 500$
$w_{24}(r,s) = 125r^3 - 555r^2s^2 + 510r^2s + 45r^2 + 222rs^4 - 102rs^3 - 351rs^2 + 120rs + 111r - 8s^6 + 24s^5 + 9s^4 - 66s^3 + 24s^2 + 42s - 25$
$w_{28}(r,s) = 290r^5 + 378r^4s + 726r^4 - 272r^3s^2 + 342r^3s + 969r^3 - 432r^2s^3 - 366r^2s^2 - 549r^2s + 740r^2 + 90rs^4 - 846rs^3 - 114rs^2 - 630rs + 135r - 54s^5 + 144s^4 - 306s^3 - 128s^2 - 171s - 18$
$w_{29}(r,s) = 1856r^5 + 5488r^4s^2 + 15408r^4s - 400r^4 - 1388r^3s^2 - 12600r^3s - 645r^2s^3 + 8375r^2s^2 - 1800rs^3 + 125s^4$
$w_{33}(r,s) = 54r^6 - 108r^5s - 486r^5 - 54r^4s^2 + 675r^4s + 1989r^4 + 216r^3s^3 + 594r^3s^2 - 2088r^3s - 2790r^3 - 54r^2s^4 - 972r^2s^3 - 2034r^2s^2 + 1458r^2s + 2796r^2 - 108rs^5 - 108rs^4 + 2376rs^3 + 2790rs^2 - 4488rs - 168r + 54s^6 + 297s^5 - 243s^4 - 1458s^3 + 1548s^2 - 168s - 64$
$w_{37}(r,s) = 432r^6 - 72r^5s + 4300r^5 - 441r^4s^2 - 15152r^4s + 4252r^4 + 9898r^3s^2 + 5288r^3s - 49052r^3 - 3969r^2s^2 + 24192r^2s + 77652r^2 - 14256rs - 49248r + 11664$
$w_{40}(r,s) = 216r^8 - 432r^7s - 216r^7 - 432r^6s^2 + 576r^6s - 1089r^6 + 1296r^5s^3 - 72r^5s^2 + 1926r^5s + 2708r^5 - 1008r^4s^3 + 1116r^4s^2 - 4328r^4s - 799r^4 - 1296r^3s^5 + 792r^3s^4 - 3402r^3s^3 - 2176r^3s^2 - 256r^3s - 4168r^3 + 432r^2s^6 + 288r^2s^5 + 585r^2s^4 + 5416r^2s^3 + 2079r^2s^2 + 7428r^2s + 4401r^2 + 432rs^7 - 504rs^6 + 1476rs^5 - 532rs^4 - 194rs^3 - 2352rs^2 - 5238rs - 324r - 216s^8 + 144s^7 - 612s^6 - 1088s^5 - 830s^4 - 908s^3 + 837s^2 + 324s - 729$
$w_{44}(r,s) = 45r^9 + 426r^8s - 282r^8 - 140r^7s^2 - 3510r^7s + 637r^7 + 1000r^6s^3 + 3752r^6s^2 + 11379r^6s - 604r^6 + 40r^5s^3 - 17004r^5s^2 - 16928r^5s + 196r^5 + 348r^4s^3 + 37908r^4s^2 + 9604r^4s + 4640r^3s^4 - 22616r^3s^3 - 35476r^3s^2 + 2240r^2s^4 + 46844r^2s^3 - 26656rs^4 + 5488s^5$
$w_{53}(r,s) = 5488r^6s^4 + 17524r^5s^5 + 17236r^5s^4 - 17708r^5s^3 - 17420r^5s^2 + 83928r^4s^6 + 484792r^4s^5 + 1058759r^4s^4 + 1147886r^4s^3 + 674175r^4s^2 + 219336r^4s + 35152r^4 - 153196r^3s^7 - 837540r^3s^6 - 1910262r^3s^5 - 2374028r^3s^4 - 1782432r^3s^3 - 859764r^3s^2 - 269902r^3s - 44460r^3 - 39952r^2s^8 - 365904r^2s^7 - 1355673r^2s^6 - 2687366r^2s^5 - 3111735r^2s^4 - 2117412r^2s^3 - 785719r^2s^2 - 121638r^2s + 759r^2 + 65824rs^9 + 514008rs^8 + 1742400rs^7 + 3333792rs^6 + 3902976rs^5 + 2805264rs^4 + 1138368rs^3 + 174240rs^2 - 34848rs - 12584r + 21296s^{10} + 212960s^9 + 958320s^8 + 2555520s^7 + 4472160s^6 + 5366592s^5 + 4472160s^4 + 2555520s^3 + 958320s^2 + 212960s + 21296$
$w_{56}(r,s) = 4000r^8 + 3280r^7s^2 - 10400r^7s + 24000r^7 + 2197r^6s^4 + 4668r^6s^3 + 27980r^6s^2 - 80800r^6s + 48000r^6 + 428r^5s^4 + 9336r^5s^3 + 130560r^5s^2 - 198400r^5s + 32000r^5 + 412r^4s^5 - 27621r^4s^4 - 48172r^4s^3 + 312140r^4s^2 - 156800r^4s + 4154r^3s^5 - 43994r^3s^4 - 196144r^3s^3 + 273400r^3s^2 + 1339r^2s^6 + 13578r^2s^5 + 14068r^2s^4 - 199400r^2s^3 + 4778rs^6 + 15736rs^5 + 45800rs^4 + 50s^7 + 4600s^6 + 5000s^5$

Table T.2: For discriminants $1 < D < 100$ with $D \not\equiv 1 \bmod 8$, the Weierstrass curve is irreducible and birational to the plane curve $w_D(r,s) = 0$.

$$w_{60}(r,s)=54r^{12}-108r^{11}s-486r^{11}-216r^{10}s^2+648r^{10}s-513r^{10}+540r^9s^3+2106r^9s^2+3402r^9s+6156r^9+270r^8s^4-2916r^8s^3-1494r^8s^2-14796r^8s+22320r^8-1080r^7s^5-3564r^7s^4-12294r^7s^3-10206r^7s^2-55842r^7s-28134r^7+5184r^6s^5+9792r^6s^4+43020r^6s^3-26436r^6s^2+62244r^6s-311596r^6+1080r^5s^7+2916r^5s^6+16470r^5s^5-2826r^5s^4+150300r^5s^3+30360r^5s^2+493014r^5s+158670r^5-270r^4s^8-4536r^4s^7-15282r^4s^6-43776r^4s^5-35622r^4s^4-94872r^4s^3+327354r^4s^2-89904r^4s+975660r^4-540r^3s^9-1134r^3s^8-9666r^3s^7+11646r^3s^6-136602r^3s^5-35214r^3s^4-853398r^3s^3-155238r^3s^2-738930r^3s+454500r^3+216r^2s^{10}+1944r^2s^9+9729r^2s^8+17676r^2s^7+64800r^2s^6+66300r^2s^5+136074r^2s^4+169476r^2s^3-1279152r^2s^2-1682340r^2s-1989075r^2+108rs^{11}+162rs^{10}+2088rs^9-4770rs^8+42144rs^7+24204rs^6+356856rs^5-154284rs^4+1077492rs^3+726930rs^2+2220000rs+132750r-54s^{12}-324s^{11}-2232s^{10}-2124s^9-25062s^8-24888s^7-148304s^6+77400s^5-78930s^4+235388s^3-608760s^2-569100s-783250$$

$$w_{61}(r,s)=729r^6s^2-1386r^6s+225r^6+21708r^5s^3+1082r^5s^2-8244r^5s+450r^5-105786r^4s^4+13710r^4s^3-23921r^4s^2-15516r^4s+225r^4+191484r^3s^5-70050r^3s^4-30642r^3s^3-167542r^3s^2-68094r^3s-167751r^2s^6+137860r^2s^5+174078r^2s^4+155124r^2s^3-187315r^2s^2-59436r^2s+71280rs^7-117504rs^6-120676rs^5+122400rs^4+248124rs^3-81224rs^2-11664s^8+36288s^7+9180s^6-94916s^5-28092s^4+70356s^3-35152s^2$$

$$w_{69}(r,s)=225r^6s^8+3450r^6s^7-27425r^6s^6+79500r^6s^5-122625r^6s^4+110650r^6s^3-58575r^6s^2+16800r^6s-2000r^6-450r^5s^8-18264r^5s^7+112218r^5s^6-297960r^5s^5+477270r^5s^4-509400r^5s^3+357054r^5s^2-147048r^5s+26580r^5+225r^4s^8+34992r^4s^7-157485r^4s^6+230580r^4s^5-216225r^4s^4+471480r^4s^3-801747r^4s^2+599220r^4s-161040r^4-28542r^3s^7+77188r^3s^6+404760r^3s^5-1809192r^3s^4+2298318r^3s^3-645840r^3s^2-653640r^3s+356948r^3+8364r^2s^7-11751r^2s^6-472890r^2s^5+2333949r^2s^4-4103904r^2s^3+2730804r^2s^2-109284r^2s-375288r^2+86088rs^6-433512rs^5+657972rs^4-166392rs^3-98748rs^2-414864rs+369456r-78608s^6+549372s^5-1560288s^4+2309228s^3-1961736s^2+1014000s-293264$$

$$w_{76}(r,s)=250r^{11}s^3-120r^{10}s^4-2940r^{10}s^3-2070r^{10}s^2+7635r^9s^5+23784r^9s^4+27977r^9s^3+15192r^9s^2+3114r^9s+21584r^8s^6+72939r^8s^5+79416r^8s^4+77089r^8s^3+91836r^8s^2+59328r^8s+13770r^8+41904r^7s^7+287856r^7s^6+793188r^7s^5+1265556r^7s^4+1237083r^7s^3+702702r^7s^2+204147r^7s+21708r^7+88128r^6s^8+664656r^6s^7+1866688r^6s^6+2572188r^6s^5+1924440r^6s^4+735185r^6s^3+41652r^6s^2-74871r^6s-20250r^6+93312r^5s^9+527040r^5s^8+1024416r^5s^7+79936r^5s^6-2583282r^5s^5-4520772r^5s^4-3983499r^5s^3-2054862r^5s^2-586089r^5s-69984r^5-300672r^4s^9-2146944r^4s^8-7172352r^4s^7-13983216r^4s^6-16841850r^4s^5-12715176r^4s^4-5935041r^4s^3-1613844r^4s^2-215055r^4s-7290r^4+131328r^3s^9+1672320r^3s^8+7286448r^3s^7+17114576r^3s^6+24303996r^3s^5+21654252r^3s^4+12195711r^3s^3+4266162r^3s^2+869697r^3s+82620r^3+697088r^2s^9+3898176r^2s^8+9884304r^2s^7+14747584r^2s^6+14131428r^2s^5+9042912r^2s^4+3987063r^2s^3+1252746r^2s^2+272403r^2s+30618r^2-293760rs^9-2473536rs^8-8590656rs^7-16916736rs^6-20892033rs^5-16588260rs^4-8343378rs^3-2530602rs^2-417717rs-29160r-235136s^9-846336s^8-676896s^7+1628192s^6+4459599s^5+4712688s^4+2560788s^3+657072s^2+31347s-11664$$

$$w_{77}(r,s)=5537r^8s^8+50792r^8s^7-1132372r^8s^6+4186056r^8s^5+807926r^8s^4-14441672r^8s^3-20806772r^8s^2+48395928r^8s-53061071r^8+29106r^7s^8+196336r^7s^7-2517968r^7s^6-4872400r^7s^5+63345476r^7s^4-69890864r^7s^3-207472608r^7s^2+371313936r^7s-6148422r^7+56987r^6s^8+281736r^6s^7-100076r^6s^6-26503832r^6s^5+61709858r^6s^4+88812440r^6s^3-289022444r^6s^2+166556280r^6s-217764837r^6+40768r^5s^8+241360r^5s^7+2254632r^5s^6-9708400r^5s^5-57462264r^5s^4+107942512r^5s^3+13376888r^5s^2+198271536r^5s-110974440r^5-19845r^4s^8+297080r^4s^7-701564r^4s^6+18197144r^4s^5-51047854r^4s^4-33824920r^4s^3-38419740r^4s^2+53521800r^4s+16002251r^4-55958r^3s^8+382032r^3s^7-2638048r^3s^6+8796688r^3s^5+4907300r^3s^4+5250800r^3s^3-5794640r^3s^2-14429520r^3s+3581346r^3-39739r^2s^8+252056r^2s^7-828212r^2s^6-1849288r^2s^5-1031042r^2s^4+2942408r^2s^3+4705676r^2s^2-1345176r^2s-2806683r^2-12348rs^8+61936rs^7+144648rs^6-185808rs^5-359856rs^4+185808rs^3+335160rs^2-61936rs-107604r-1372s^8+5488s^6-8232s^4+5488s^2-1372$$

$w_{85}(r,s)$=$2916s^8r^{16}-23328s^7r^{16}+81648s^6r^{16}-163296s^5r^{16}+204120s^4r^{16}-163296s^3r^{16}+81648s^2r^{16}-23328sr^{16}+2916r^{16}+10368s^8r^{15}-33048s^7r^{15}-58968s^6r^{15}+467208s^5r^{15}-1020600s^4r^{15}+1165752s^3r^{15}-757512s^2r^{15}+266328sr^{15}-39528r^{15}+9369s^8r^{14}+62208s^7r^{14}-344061s^6r^{14}+233334s^5r^{14}+1161135s^4r^{14}-2798604s^3r^{14}+2687877s^2r^{14}-1237194sr^{14}+225936r^{14}-11569s^8r^{13}+254266s^7r^{13}-528136s^6r^{13}-163078s^5r^{13}+647950s^4r^{13}+1350254s^3r^{13}-3401296s^2r^{13}+2456926sr^{13}-605317r^{13}+36199s^8r^{12}-177352s^7r^{12}+1581385s^6r^{12}-3841010s^5r^{12}+2237285s^4r^{12}+2013436s^3r^{12}-2063873s^2r^{12}-193090sr^{12}+407020r^{12}-5622s^8r^{11}+491772s^7r^{11}-1193178s^6r^{11}+3760500s^5r^{11}-7097790s^4r^{11}+1876404s^3r^{11}+7080402s^2r^{11}-6439716sr^{11}+1527228r^{11}+135251s^8r^{10}-517820s^7r^{10}+2887313s^6r^{10}-3134246s^5r^{10}+407945s^4r^{10}-266960s^3r^{10}-3498081s^2r^{10}+7055346sr^{10}-3068748r^{10}-61145s^8r^9+1090970s^7r^9-2189150s^6r^9+5386010s^5r^9-127160s^4r^9-10970242s^3r^9+6679566s^2r^9+55758sr^9+135393r^9+97320s^8r^8-614280s^7r^8+2286570s^6r^8+975960s^5r^8-1262670s^4r^8+4089912s^3r^8-7646178s^2r^8-1405992sr^8+3474174r^8-76295s^8r^7+189050s^7r^7-189830s^6r^7-2442890s^5r^7+12026840s^4r^7-3497402s^3r^7-8930754s^2r^7+4713930sr^7-1835145r^7+62381s^8r^6-177668s^7r^6-2828407s^6r^6+5160314s^5r^6-5526335s^4r^6+5031032s^3r^6+8228511s^2r^6-9951438sr^6-127926r^6+43278s^8r^5-321348s^7r^5+1271682s^6r^5-8686752s^5r^5+2529090s^4r^5+9519900s^3r^5-4473138s^2r^5-599352sr^5+541536r^5+67049s^8r^4+705208s^7r^4-2606311s^6r^4+599114s^5r^4-2056175s^4r^4-6069436s^3r^4+6282543s^2r^4+4736634sr^4-1715586r^4+109981s^8r^3+95906s^7r^3+1980904s^6r^3-2002898s^5r^3-6529570s^4r^3+1912846s^3r^3+4903656s^2r^3+58914sr^3-400203r^3+77919s^8r^2+805008s^7r^2+479835s^6r^2-219930s^5r^2+518415s^4r^2-1404396s^3r^2-2173467s^2r^2+906870sr^2+1184850r^2+67868s^8r+295792s^7r+1214132s^6r+1618540s^5r-1197400s^4r-3420872s^3r-793468s^2r+1550572sr+752900r+27436s^8+245480s^7+525388s^6+82304s^5-842120s^4-739144s^3+164380s^2+419648s+133204$

$w_{88}(r,s)$=$2197r^{10}s^4+22896r^9s^6-90924r^9s^5-49284r^9s^4-5640r^9s^3+716976r^8s^8+863856r^8s^7+2086764r^8s^6+1241328r^8s^5+151980r^8s^4-21228r^8s^3-1482r^8s^2-56448r^7s^{10}-3568896r^7s^9+2550240r^7s^8+1878496r^7s^7+7068000r^7s^6+9706416r^7s^5+4668192r^7s^4+1000848r^7s^3+120084r^7s^2+9784r^7s+451584r^6s^{11}+4704768r^6s^{10}-30519936r^6s^9-14556288r^6s^8-8229408r^6s^7-12314688r^6s^6+4049616r^6s^5+10226280r^6s^4+4756800r^6s^3+1047180r^6s^2+136092r^6s+9261r^6-903168r^5s^{12}-7428096r^5s^{11}+32463360r^5s^{10}-100819968r^5s^9-135201792r^5s^8-42387840r^5s^7-29795328r^5s^6-33991200r^5s^5-13098672r^5s^4-1209744r^5s^3+191760r^5s^2+24420r^5s-700416r^4s^{12}-63608832r^4s^{11}+66746368r^4s^{10}-89135616r^4s^9-356568576r^4s^8-219396608r^4s^7-12925440r^4s^6+16489728r^4s^5-1341120r^4s^4-2341632r^4s^3-341268r^4s^2-4688r^4s+24551424r^3s^{12}-124084224r^3s^{11}+22806528r^3s^{10}+174600192r^3s^9-147545088r^3s^8-340414464r^3s^7-172284672r^3s^6-18544896r^3s^5+8804256r^3s^4+2455776r^3s^3+152592r^3s^2+91865088r^2s^{12}+61562880r^2s^{11}+90882048r^2s^{10}+339996672r^2s^9+354278400r^2s^8+102297600r^2s^7-41952000r^2s^6-33328896r^2s^5-6801024r^2s^4-198240r^2s^3+49968r^2s^2+122683392rs^{12}+380731392rs^{11}+539353088rs^{10}+456105984rs^9+235702272rs^8+55894016rs^7-11779584rs^6-12596736rs^5-3461248rs^4-342144rs^3+56623104s^{12}+273678336s^{11}+560185344s^{10}+639442944s^9+447197184s^8+196503552s^7+52835328s^6+7828992s^5+444672s^4-10368s^3$

$w_{92}(r,s)$=$1372r^4s^{14}+13328r^5s^{13}+12768r^4s^{13}-19760r^3s^{13}+54831r^6s^{12}+60260r^5s^{12}-88004r^4s^{12}-5216r^3s^{12}+117048r^2s^{12}+125881r^7s^{11}+78820r^6s^{11}-226380r^5s^{11}+87204r^4s^{11}+618720r^3s^{11}-22064r^2s^{11}-83504rs^{11}+177331r^8s^{10}-77287r^7s^{10}-476478r^6s^{10}+400662r^5s^{10}+1390158r^4s^{10}-548148r^3s^{10}-500148r^2s^{10}+99696rs^{10}+27436s^{10}+157437r^9s^9-353150r^8s^9-662128r^7s^9+822728r^6s^9+1582868r^5s^9-2131262r^4s^9-1217332r^3s^9+853684r^2s^9+296840rs^9+86093r^{10}s^8-457177r^9s^8-416740r^8s^8+1212602r^7s^8+921785r^6s^8-3571576r^5s^8-1053155r^4s^8+2754234r^3s^8+1100307r^2s^8+3896rs^8+26411r^{11}s^7-299600r^{10}s^7+79186r^9s^7+1277764r^8s^7+171499r^7s^7-3076594r^6s^7+505939r^5s^7+4473036r^4s^7+1856901r^3s^7+1066r^2s^7+3381r^{12}s^6-100345r^{11}s^6+262170r^{10}s^6+738484r^9s^6-348744r^8s^6-1570757r^7s^6+1421228r^6s^6+3772253r^5s^6+1510292r^4s^6-16613r^3s^6+17287r^2s^6-49r^{13}s^5-13142r^{12}s^5+129572r^{11}s^5+118872r^{10}s^5-513158r^9s^5-642614r^8s^5+793050r^7s^5+1523718r^6s^5+554266r^5s^5+29472r^4s^5+52625r^3s^5+245r^{13}s^4+19052r^{12}s^4-52862r^{11}s^4-236476r^{10}s^4-137882r^9s^4+234406r^8s^4+301584r^7s^4+136810r^6s^4+97340r^5s^4+54682r^4s^4+49r^3s^4-490r^{13}s^3-12408r^{12}s^3-12296r^{11}s^3+53898r^{10}s^3+115794r^9s^3+97736r^8s^3+75212r^7s^3+60258r^6s^3+21460r^5s^3+196r^4s^3+490r^{13}s^2+3617r^{12}s^2+8884r^{11}s^2+11223r^{10}s^2+11536r^9s^2+12439r^8s^2+9308r^7s^2+3233r^6s^2+294r^5s^2-245r^{13}s-794r^{12}s-99r^{11}s+2840r^{10}s+5045r^9s+3870r^8s+1411r^7s+196r^6s+49r^{13}+294r^{12}+735r^{11}+980r^{10}+735r^9+294r^8+49r^7$

$w_{93}(r,s)$=$37044r^4s^{16}+296352r^3s^{16}+889056r^2s^{16}+1185408rs^{16}+592704s^{16}-9801r^5s^{15}-639900r^4s^{15}-4744980r^3s^{15}-13620960r^2s^{15}-17236800rs^{15}-8024832s^{15}+18629r^6s^{14}+288876r^5s^{14}+4333743r^4s^{14}+25277260r^3s^{14}+64256268r^2s^{14}+74944800rs^{14}+35052224s^{14}+15933r^7s^{13}+206862r^6s^{13}+935346r^5s^{13}-4870338r^4s^{13}-48209232r^3s^{13}-126538944r^2s^{13}-130612608rs^{13}-48533760s^{13}-873r^8s^{12}-48456r^7s^{12}-592590r^6s^{12}-2144544r^5s^{12}+9277119r^4s^{12}+87549072r^3s^{12}+220122624r^2s^{12}+208914912rs^{12}+71156736s^{12}+4914r^8s^{11}+15306r^7s^{11}-154254r^6s^{11}-3964311r^5s^{11}-27184656r^4s^{11}-99324984r^3s^{11}-198318816r^2s^{11}-149760576rs^{11}-33225984s^{11}-9297r^8s^{10}+34728r^7s^{10}+908259r^6s^{10}+8318232r^5s^{10}+21709662r^4s^{10}+3090696r^3s^{10}+5486040r^2s^{10}+6351840rs^{10}+3181248s^{10}+2124r^8s^9+109107r^7s^9+911196r^6s^9-2860596r^5s^9-17723916r^4s^9+24390288r^3s^9+34519680r^2s^9+7126272rs^9+1016064s^9+16218r^8s^8-199104r^7s^8-1815556r^6s^8+3165504r^5s^8+51470058r^4s^8+50360048r^3s^8+19611072r^2s^8+1118880rs^8+592704s^8-20772r^8s^7-128100r^7s^7-660012r^6s^7-3242631r^5s^7-44396964r^4s^7-39882708r^3s^7-8446464r^2s^7-2032128rs^7+1638r^8s^6+374688r^7s^6+1457499r^6s^6-8217444r^5s^6+146547r^4s^6+1145484r^3s^6+2040444r^2s^6+14544r^8s^5-107133r^7s^5+164406r^6s^5+11551554r^5s^5+8544798r^4s^5+52704r^3s^5-10269r^8s^4-161592r^7s^4-358350r^6s^4-3769920r^5s^4-703197r^4s^4+162r^8s^3+102906r^7s^3-242406r^6s^3-50265r^5s^3+2475r^8s^2-264r^7s^2+156317r^6s^2-972r^8s-8019r^7s+108r^8$

Algebraic models for W_D^ϵ

$w^0_{17}(r,s)=(2+2\sqrt{17})r^2-(17-7\sqrt{17})r+64s-(9-3\sqrt{17})$

$w^0_{33}(r,s)=36r^3-36r^2s-(162+18\sqrt{33})r^2-36rs^2+(63+15\sqrt{33})rs+(447+63\sqrt{33})r+36s^3+(99+3\sqrt{33})s^2-(213+21\sqrt{33})s+(42+10\sqrt{33})$

$w^0_{41}(r,s)=16r^3s^2+8r^3s+1r^3+(864+160\sqrt{41})r^2s^2-(154-2\sqrt{41})r^2s-(-8\sqrt{41})r^2-(7680+1280\sqrt{41})rs^2+(2288+272\sqrt{41})rs+80r+(15872+2560\sqrt{41})s^2-(7200+1120\sqrt{41})s$

$w^0_{57}(r,s)=576r^3s^3+(864-96\sqrt{57})r^3s^2-(504+24\sqrt{57})r^3s-(792-72\sqrt{57})r^3-288r^2s^3+(2304+288\sqrt{57})r^2s^2-(2892+348\sqrt{57})r^2s+(228+148\sqrt{57})r^2-144rs^3-(828+60\sqrt{57})rs^2+(3294+486\sqrt{57})rs-(3078+302\sqrt{57})r+72s^3-(270+30\sqrt{57})s^2+(1083+159\sqrt{57})s-(1083+95\sqrt{57})$

$w^0_{65}(r,s)=50r^4s^8+(560-100\sqrt{65})r^4s^7+(7235-829\sqrt{65})r^4s^6+(3970-426\sqrt{65})r^4s^5-(9915-999\sqrt{65})r^4s^4+(1260-468\sqrt{65})r^4s^3+(4420-140\sqrt{65})r^4s^2+(2800-80\sqrt{65})r^4s-800r^4+400r^3s^9+(4480-800\sqrt{65})r^3s^8+(58280-6632\sqrt{65})r^3s^7+(36240-4208\sqrt{65})r^3s^6-(22540-1540\sqrt{65})r^3s^5+(37040-6000\sqrt{65})r^3s^4-(42900-7100\sqrt{65})r^3s^3+(5560-1928\sqrt{65})r^3s^2+(4480-960\sqrt{65})r^3s+(8800-160\sqrt{65})r^3+1200r^2s0+(13440-2400\sqrt{65})r^2s^9+(176040-19896\sqrt{65})r^2s^8+(122160-15024\sqrt{65})r^2s^7+(103920-14736\sqrt{65})r^2s^6+(205440-27168\sqrt{65})r^2s^5-(198040-27144\sqrt{65})r^2s^4+(22080-11360\sqrt{65})r^2s^3-(163360-18704\sqrt{65})r^2s^2-(36080-2064\sqrt{65})r^2s+(4080-1360\sqrt{65})r^2+1600rs1+(17920-3200\sqrt{65})rs0+(236320-26528\sqrt{65})rs^9+(180800-23232\sqrt{65})rs^8+(368880-45456\sqrt{65})rs^7+(417600-51648\sqrt{65})rs^6-(134320-18896\sqrt{65})rs^5+(129120-28064\sqrt{65})rs^4-(520880-61968\sqrt{65})rs^3-(222720-16768\sqrt{65})rs^2-(182000-22864\sqrt{65})rs-(113120-7456\sqrt{65})r+800s2+(8960-1600\sqrt{65})s1+(118960-13264\sqrt{65})s0+(99360-13216\sqrt{65})s^9+(300400-35632\sqrt{65})s^8+(289600-35136\sqrt{65})s^7+(109600-11744\sqrt{65})s^6+(169600-24832\sqrt{65})s^5-(338880-41440\sqrt{65})s^4-(221760-22400\sqrt{65})s^3-(279760-39344\sqrt{65})s^2-(243360-23712\sqrt{65})s-(13520-8528\sqrt{65})$

$w^0_{73}(r,s)=288r^4s^4-(2436-84\sqrt{73})r^4s^3+(7704-504\sqrt{73})r^4s^2-(10800-1008\sqrt{73})r^4s+(5664-672\sqrt{73})r^4+(51-15\sqrt{73})r^3s^5-(1934-230\sqrt{73})r^3s^4+(4044-444\sqrt{73})r^3s^3+(8232-840\sqrt{73})r^3s^2-(23648-2528\sqrt{73})r^3s+(11328-1344\sqrt{73})r^3-(730-34\sqrt{73})r^2s^5+(906+174\sqrt{73})r^2s^4+(14992-2128\sqrt{73})r^2s^3-(25592-3032\sqrt{73})r^2s^2-(7184-848\sqrt{73})r^2s+(5664-672\sqrt{73})r^2-(153-45\sqrt{73})rs^5+(2138-290\sqrt{73})rs^4+(5212-652\sqrt{73})rs^3-(30952-3592\sqrt{73})rs^2+(5664-672\sqrt{73})rs+(3186-378\sqrt{73})s^4-(12264-1416\sqrt{73})s^3+(1416-168\sqrt{73})s^2$

$w^0_{89}(r,s)=40r^6s^4-(345+35\sqrt{89})r^6s^3-120r^5s^5+(890+54\sqrt{89})r^5s^4-(744+56\sqrt{89})r^5s^3-(1246+138\sqrt{89})r^5s^2+120r^4s^6-(825+3\sqrt{89})r^4s^5+(1000-56\sqrt{89})r^4s^4+(4372+564\sqrt{89})r^4s^3-(7256+792\sqrt{89})r^4s^2+(759+77\sqrt{89})r^4s-40r^3s^7+(360-16\sqrt{89})r^3s^6-(736-128\sqrt{89})r^3s^5-(3670+698\sqrt{89})r^3s^4+(17184+2000\sqrt{89})r^3s^3-(19046+2042\sqrt{89})r^3s^2+(4728+488\sqrt{89})r^3s-80r^2s^7+(520-16\sqrt{89})r^2s^6-(1680-32\sqrt{89})r^2s^5-(6276+932\sqrt{89})r^2s^4+(33247+3653\sqrt{89})r^2s^3-(45640+4832\sqrt{89})r^2s^2+(23344+2456\sqrt{89})r^2s-(3740+396\sqrt{89})r^2-40rs^7-(1420+180\sqrt{89})rs^6+(13112+1464\sqrt{89})rs^5-(54648+5872\sqrt{89})rs^4+(115992+12320\sqrt{89})rs^3-(127036+13452\sqrt{89})rs^2+(67928+7192\sqrt{89})rs-(13888+1472\sqrt{89})r-(1700+180\sqrt{89})s^6$

$w_{97}(r,s)=288r^7s^4+(987-21\sqrt{97})r^6s^5-(330+42\sqrt{97})r^6s^4+(1239+39\sqrt{97})r^6s^3+(1233-63\sqrt{97})r^5s^6-(249+105\sqrt{97})r^5s^5+(4334+206\sqrt{97})r^5s^4-(2210+266\sqrt{97})r^5s^3+(1989+117\sqrt{97})r^5s^2+(657-63\sqrt{97})r^4s^7+(738-126\sqrt{97})r^4s^6+(4210+274\sqrt{97})r^4s^5-(2975+551\sqrt{97})r^4s^4+(6457+553\sqrt{97})r^4s^3-(3375+399\sqrt{97})r^4s^2+(1413+117\sqrt{97})r^4s+(123-21\sqrt{97})r^3s^8+(903-105\sqrt{97})r^3s^7+(992+128\sqrt{97})r^3s^6+(870-306\sqrt{97})r^3s^5+(4451+563\sqrt{97})r^3s^4-(4764+684\sqrt{97})r^3s^3+(3860+404\sqrt{97})r^3s^2-(1440+168\sqrt{97})r^3s+(375+39\sqrt{97})r^3+(246-42\sqrt{97})r^2s^8+(99-117\sqrt{97})r^2s^6+(4400+464\sqrt{97})r^2s^5-(5001+585\sqrt{97})r^2s^4+(3427+403\sqrt{97})r^2s^3-(2038+238\sqrt{97})r^2s^2+(750+78\sqrt{97})r^2s+(55+7\sqrt{97})r^2+(123-21\sqrt{97})rs^8-(1371+75\sqrt{97})rs^7+(3990+246\sqrt{97})rs^6-(4734+270\sqrt{97})rs^5+(5313+369\sqrt{97})rs^4-(5133+405\sqrt{97})rs^3+(1953+177\sqrt{97})rs^2-(1125+117\sqrt{97})s^7+(3375+351\sqrt{97})s^6-(3375+351\sqrt{97})s^5+(1062+126\sqrt{97})s^4$

Table T.3: For $1 < D < 100$ with $D \equiv 1 \bmod 8$, the curve W^0_D is birational the curve $w^0_D(r,s)=0$ and W^1_D is the Galois conjugate of W^0_D.

D	g	e_2	C	χ	D	g	e_2	C	χ
5^*	0	1	1	$-\frac{3}{10}$	56	3	2	10	-15
8^*	0	0	2	$-\frac{3}{4}$	57	$\{1,1\}$	$\{1,1\}$	$\{10,10\}$	$\{-\frac{21}{2},-\frac{21}{2}\}$
12	0	1	3	$-\frac{3}{2}$	60	3	4	12	-18
13	0	1	3	$-\frac{3}{2}$	61	2	3	13	$-\frac{33}{2}$
17	$\{0,0\}$	$\{1,1\}$	$\{3,3\}$	$\{-\frac{3}{2},-\frac{3}{2}\}$	65	$\{1,1\}$	$\{2,2\}$	$\{11,11\}$	$\{-12,-12\}$
21	0	2	4	-3	69	4	4	10	-18
24	0	1	6	$-\frac{9}{2}$	73	$\{1,1\}$	$\{1,1\}$	$\{16,16\}$	$\{-\frac{33}{2},-\frac{33}{2}\}$
28	0	2	7	-6	76	4	3	21	$-\frac{57}{2}$
29	0	3	5	$-\frac{9}{2}$	77	5	4	8	-18
33	$\{0,0\}$	$\{1,1\}$	$\{6,6\}$	$\{-\frac{9}{2},-\frac{9}{2}\}$	85	6	2	16	-27
37	0	1	9	$-\frac{15}{2}$	88	7	1	22	$-\frac{69}{2}$
40	0	1	12	$-\frac{21}{2}$	89	$\{3,3\}$	$\{3,3\}$	$\{14,14\}$	$\{-\frac{39}{2},-\frac{39}{2}\}$
41	$\{0,0\}$	$\{2,2\}$	$\{7,7\}$	$\{-6,-6\}$	92	8	6	13	-30
44	1	3	9	$-\frac{21}{2}$	93	8	2	12	-27
53	2	3	7	$-\frac{21}{2}$	97	$\{4,4\}$	$\{1,1\}$	$\{19,19\}$	$\{-\frac{51}{2},-\frac{51}{2}\}$

Table T.4: For discriminants $D > 8$, the homeomorphism type of each irreducible component of W_D is determined by its genus g, the number of cusps C and the number of points of orbifold order two e_2. For reducible W_D, we list the topological invariants of both spin components, with the invariants of W_D^0 appearing first. The curves W_5 and W_8 are isomorphic to the $(2,5,\infty)-$ and $(4,\infty,\infty)-$orbifolds respectively.

References

[Ba] M. Bainbridge. Euler characteristics of Teichmüller curves in genus two. *Geom. Topol.* **11**(2007), 1887–2073.

[BL] C. Birkenhake and H. Lange. *Complex Abelian Varieties.* Springer-Verlag, Berlin, second edition, 2004.

[BM1] I. Bouw and M. Möller. Differential equations associated with nonarithmetic Fuchsian groups. *J. Lond. Math. Soc.* **81**(2010), 65–90, arXiv:0710.5277.

[BM2] I. Bouw and M. Möller. Teichmüller curves, triangle groups, and Lyapunov exponents. *Ann. of Math.* **172**(2010), 139–185, arXiv:math/0511738.

[Ca] K. Calta. Veech surfaces and complete periodicity in genus two. *J. Amer. Math. Soc.* **17**(2004), 871–908.

[CS] J. Cremona and A. Sutherland. LMFDB – Elliptic Curves 880.i2 (Cremona label 880g1). `http://www.lmfdb.org/EllipticCurve/Q/880.i2`, Retrieved 2014.

[Da] C. S. Dalawat. Good reduction, bad reduction. *Preprint* (2006), arXiv:math/0605326.

[Dr] V. G. Drinfeld. Two theorems on modular curves. *Akademija Nauk SSSR. Funkcionalnyi Analiz i ego Priloenija* **7**(1973), 83–84.

[EK] N. Elkies and A. Kumar. K3 surfaces and equations for Hilbert modular surfaces. *Algebra Number Theory, to appear* (2013), arXiv:1209.3527.

[HM] J. Harris and I. Morrison. *Moduli of Curves.* Springer, New York, NY, 1998.

[HZ] F. Hirzebruch and D. Zagier. Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. *Inventiones mathematicae* **36**(1976), 57–113.

[Hu] J. H. Hubbard. *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics*, volume 1. Matrix Editions, Ithaca, NY, 2006.

[HS] J. H. Hubbard and D. Schleicher. The spider algorithm. In *Complex Dynamical Systems*, volume 49 of *Proc. Sympos. Appl. Math.*, pages 155–180. Amer. Math. Soc., Providence, RI, 1994.

[Ig] J. Igusa. Arithmetic variety of moduli for genus two. *Ann. of Math.* **72**(1960), 612–649.

[IT] Y. Imayoshi and M. Taniguchi. *An Introduction to Teichmüller Spaces.* Springer-Verlag, Tokyo, 1992.

[Ka] S. Kamienny. Torsion points on elliptic curves. *Bull. Amer. Math. Soc.* **23**(1990), 371–373.

[KM1] A. Kumar and R. E. Mukamel. Eigenform certification via algebraic correspondences. *In preparation.*

[KM2] A. Kumar and R. E. Mukamel. Algebraic models for Weierstrass curves [Computer File]. `http://math.uchicago.edu/~ronen/papers/wDcertify.tar.gz`, Retrieved 2014.

[Li] Q. Liu. *Algebraic Geometry and Arithmetic Curves.* Oxford University Press, Oxford, 2002. Translated by R. Erné.

[Lo] P. Lochak. On arithmetic curves in the moduli space of curves. *J. Inst. Math. Jussieu* **4**(2005), 443–508.

[Ma] J. I. Manin. Parabolic points and zeta functions on modular curves. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* **36**(1972), 19–66.

[MT] H. Masur and S. Tabachnikov. Rational billiards and flat structures. In *Handbook of dynamical systems, Vol. 1A*, pages 1015–1089. North-Holland, Amsterdam, 2002.

[Mc1] C. McMullen. Billiards and Teichmüller curves on Hilbert modular surfaces. *J. Amer. Math. Soc.* **16**(2003), 857–885.

[Mc2] C. McMullen. Teichmüller curves in genus two: Discriminant and spin. *Math. Ann.* **333**(2005), 87–130.

[Mc3] C. McMullen. Teichmüller curves in genus two: Torsion divisors and ratios of sines. *Invent. Math.* **165**(2006), 651–672.

[Mc4] C. T. McMullen. Foliations of Hilbert modular surfaces. *Amer. J. Math.* **129**(2007), 183–215.

[Mu1] R. E. Mukamel. Fundamental domains and generators for lattice Veech groups. *Preprint* (2013).

[Mu2] R. E. Mukamel. Hilbert and Veech modular forms and principal divisors Teichmüller curves. *In preparation.*

[Mu3] R. E. Mukamel. Orbifold points on Teichmüller curves and Jacobians with complex multiplication. *Geom. Topol., to appear.*

[Pi] K. Pilgrim. Riemann surfaces, dynamics, groups, and geometry (course notes). http://mypage.iu.edu/~pilgrim/Teaching/M731F2008.pdf, Retrieved 2014.

[Ru] B. Runge. Endomorphism rings of abelian surfaces and projective models of their moduli spaces. *Tohoku Math. J.* **51**(1999), 283–304.

[vdG] G. van der Geer. *Hilbert Modular Surfaces.* Springer-Verlag, Berlin, 1988.

[vW1] P. B. van Wamelen. Examples of genus two CM curves defined over the rationals. *Math. Comp.* **68**(1999), 307–320.

[vW2] P. B. van Wamelen. Proving that a genus 2 curve has complex multiplication. *Math. Comp.* **68**(1999), 1663–1677.

[vW3] P. B. van Wamelen. Computing with the analytic Jacobian of a genus 2 curve. In *Discovering mathematics with Magma*, pages 117–135. Springer, 2006.

[Zo] A. Zorich. Flat surfaces. In *Frontiers in Number Theory, Physics and Geometry. Volume 1: On random matrices, zeta functions and dynamical systems*, pages 439–586. Springer-Verlag, Berlin, 2006.