1 Background theory

1.1 The PV flux

The flux form of the PV equation has been discussed at length in Haynes and McIntyre (1987), Marshall and Nurser (1992), Rhines (1993), Schar (1993) and Bretherton and Schar (1993). Here we very briefly outline the approach, drawing out issues that are particularly pertinent to the ocean.

First let's clarify some issues related to the flux-form of the PV equation. Following Bretherton and Schar (1993) we can write, making use of the definition of PV and that $\nabla \cdot \omega = 0$:

$$\frac{\partial}{\partial t}(\rho Q) = \frac{\partial}{\partial t}(-\omega \cdot \nabla\sigma) = -\frac{\partial}{\partial t}\nabla \cdot (\omega\sigma) = -\nabla \cdot \mathbf{J} \tag{1}$$

with $\mathbf{J} = \frac{\partial}{\partial t}(\omega\sigma)$.

This shows that one can always write a conservation law in flux form from the definition of PV. This is true for any scalar field σ and non-divergent vector ω. If one adds any non-divergent vector to $\mathbf{J}$, the flux-form equation will still be satisfied. The problem is then to set the gauge and this choice has to be made on physical grounds.

Expanding the partial derivative in $\mathbf{J}$, one can write:

$$\mathbf{J} = \omega\frac{\partial\sigma}{\partial t} + \sigma\frac{\partial\omega}{\partial t} \tag{2}$$

Now, using the definition $\omega = 2\mathbf{\Omega} + \nabla \times \mathbf{u}$ and some vector identity, one can write:

$$\mathbf{J} = \omega\frac{\partial\sigma}{\partial t} + \frac{\partial\mathbf{u}}{\partial t} \times \nabla\sigma + \nabla \times \left(\sigma\frac{\partial\mathbf{u}}{\partial t}\right) \tag{3}$$

Note that the third term above is non-divergent and hence does not contribute to the PV flux-form equation. The problem is then to determine the 'gauge' $\mathbf{X}$ such that:

$$\mathbf{J} = \omega\frac{\partial\sigma}{\partial t} + \frac{\partial\mathbf{u}}{\partial t} \times \nabla\sigma + \mathbf{X} \tag{4}$$

has physical significance.

$\mathbf{X} = 0$ is not a satisfying choice because that would imply, as noted by BC, that $\mathbf{J} = 0$ in the steady state. Instead, we require that $\mathbf{J}$ reduces to the advective flux $\rho Q\mathbf{u}$ in the absence of diabatic and frictional forcings. This choice will uniquely determine $\mathbf{X}$ and hence $\mathbf{J}$. Using the previous expression for $\mathbf{J}$, $\mathbf{X}$ can be written:

$$\mathbf{X} = \rho Q\mathbf{u} - \omega\frac{\partial \sigma}{\partial t} - \frac{\partial \mathbf{u}}{\partial t} \times \nabla\sigma \tag{5}$$

Now we use the definitions of PV and the thermodynamic and momentum equations:

$$\rho Q = -\omega \cdot \nabla\sigma \tag{6}$$

$$\frac{\partial \sigma}{\partial t} = -\mathbf{u} \cdot \nabla\sigma \tag{7}$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\omega \times \mathbf{u} - \nabla\pi + \frac{\Phi}{\rho_o}\nabla\rho' \tag{8}$$

In (8), the density ρ of the Boussinesq fluid is given by $\rho = \rho_o + \rho'$ where ρ_o is a constant reference density. Φ is the geopotential and π is the Bernoulli function written in the Boussinesq approximation:

$$\pi = \frac{|\mathbf{u}|^2}{2} + \frac{p}{\rho_o} + \frac{\rho'}{\rho_o}\Phi \tag{9}$$

In (9), p is the deviation of the pressure from that of a resting, hydrostatically balanced ocean. Replacing in (5), we have:

$$\mathbf{X} = -\left(\omega \cdot \nabla\sigma\right)\mathbf{u} + \omega\left(\mathbf{u} \cdot \nabla\sigma\right) + \left(\omega \times \mathbf{u}\right) \times \nabla\sigma + \nabla\pi \times \nabla\sigma \tag{10}$$

(omitting the term $\frac{\Phi}{\rho_o}\nabla\rho' \times \nabla\sigma$). The first three terms on the right hand side cancel leaving:

$$\mathbf{X} = \nabla\pi \times \nabla\sigma \tag{11}$$

$\mathbf{X}$ sets the gauge for $\mathbf{J}$. Replacing in (4), we find:

$$\mathbf{J} = \omega\frac{\partial \sigma}{\partial t} + \left(\frac{\partial \mathbf{u}}{\partial t} + \nabla\pi\right) \times \nabla\sigma \tag{12}$$

expression written by Schar (1993) in the atmospheric context.

Note that Eq.(12):

1. as noted by Haynes and McIntyre (1987), reveals the 'impermeability theorem' in a transparent way - the first term on the lhs, when projected in the direction normal to the σ surface, is equal to $v_\sigma \rho Q$ where $v_\sigma = -|\nabla\sigma|^{-1} \frac{\partial\sigma}{\partial t}$ is the velocity of the σ surface normal to itself. The remaining terms represent a flux that is always parallel to the σ surface.

2. shows that $\mathbf{J}$ can be evaluated without explicit reference to frictional ($\mathbf{F}$) and buoyancy ($\frac{D\sigma}{Dt}$) sources, if the rates of change of σ and $\mathbf{u}$ are known. Of course, as seen above, part of this result originates from the definition of PV. Note however the crucial role played by the 'gauge' term $\nabla\pi \times \nabla\sigma$.

3. shows that in the steady state π is the streamfunction for the $\mathbf{J}$ vector on σ surfaces, a very general result first noted by Schar (1993) and Bretherton and Schar (1993).

To obtain an expression for $\mathbf{J}$ that includes explicitely the frictional and buoyancy sources, we follow the same procedure as above, replacing in (12) the rates of change of σ and $\mathbf{u}$ from the thermodynamic and momentum equations but retaining the terms $\frac{D\sigma}{Dt}$ and $\mathbf{F}$, respectively. We find:

$$\mathbf{J} = \rho Q \mathbf{u} + \omega \frac{D\sigma}{Dt} + \mathbf{F} \times \nabla\sigma + \frac{\Phi}{\rho_o} \nabla\rho' \times \nabla\sigma \qquad (13)$$

which is the PV flux written down in Marshall and Nurser (MN, Eqs.1b and 1c), but modified by the non-advective thermobaric term $\frac{\Phi}{\rho_o}\nabla\rho' \times \nabla\sigma$. The latter term (neglected in MN) is zero if $\sigma = \sigma(\rho')$ - for example if we ignore the pressure dependence of ρ' on p. However, if $\nabla\rho' \times \nabla\sigma \neq 0$, Lagrangian conservation of PV no longer pertains even in adiabatic, frictionless flow - see McDougall (1988). But more importantly in the present context, the flux form of the PV equation can never be compromised.

In the remaining of this paper, unless noted otherwise, we will make use of the form (12) for $\mathbf{J}$.