Appendix to “Long-Run Risk, the Wealth-Consumption Ratio, and the Temporal Pricing of Risk”

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In this appendix, we first derive four risk premia: the expected excess returns on a consumption claim, on equity and on real and nominal bonds. We then obtain the Alvarez and Jermann (2005) decomposition of the SDF in the long-run risk model. We present the parameter values used in our calibration and our simulation results. Finally, we report robustness checks and empirical variance ratios.

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I. Wealth-Consumption Ratio and Consumption Risk Premium

We start from the aggregate budget constraint:

\[ W_{t+1} = R_{t+1}^c (W_t - C_t). \]

The beginning-of-period (or cum-dividend) total wealth \( W_t \) that is not spent on aggregate consumption \( C_t \) earns a gross return \( R_{t+1}^c \) and leads to beginning-of-next-period total wealth \( W_{t+1} \). The return on a claim to aggregate consumption, the total wealth return, can be written as

\[ R_{t+1}^c = \frac{W_{t+1}}{W_t} = \frac{C_{t+1} W_{t+1}}{C_t W_t} - 1. \]

We use the Campbell (1991) approximation of the log total wealth return \( r_t^c = \log(R_t^c) \) around the long-run average log wealth-consumption ratio \( \mu_{wc} \) to

\[ r_{t+1}^c = \kappa_0^c + \Delta c_{t+1} + wc_{t+1} - \kappa_1^c wc_t, \]

where the linearization constants \( \kappa_0^c \) and \( \kappa_1^c \) are non-linear functions of the unconditional mean log wealth-consumption ratio \( \mu_{wc} \):

\[ \kappa_1^c = \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1} > 1 \quad \text{and} \quad \kappa_0^c = -\log(e^{\mu_{wc}} - 1) + \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1} \mu_{wc}. \]

Throughout the paper, we use lower letters to denote logs.

The Euler equation for any asset \( i \) with lognormal return \( R_i \) implies:

\[ 0 = E_t [sdf_{t+1}] + E_t [r_{t+1}^i] + \frac{1}{2} \text{Var}_t [sdf_{t+1}] + \frac{1}{2} \text{Var}_t [r_{t+1}^i] + \text{Cov}_t [sdf_{t+1}, r_{t+1}^i] \]

We conjecture that the wealth-consumption ratio is linear in the state variables \( x_t, \sigma_{g,t}^2 \) and \( \sigma_{w,t}^2 \):

\[ wc_t = \mu_{wc} + W_x x_t + W_g (\sigma_{gt}^2 - \sigma_g^2) + W_{w,t} (\sigma_{wt}^2 - \sigma_w^2) \]

We first compute the different components of equation \( 2 \)

\[ r_{t+1}^c = r_0^c + [1 + W_x (\rho - \kappa_1^c)] x_t + W_{g,t} (\nu_g - \kappa_1^c) (\sigma_{gt}^2 - \sigma_g^2) + W_{w,t} (\nu_x - \kappa_1^c) (\sigma_{xt}^2 - \sigma_x^2) \]

\[ + \sigma_{gt} \eta_{t+1} + W_{x,t} \sigma_{x,t+1} + W_{gs} \sigma_{g,w,t+1} + W_{xs} \sigma_{x,w,t+1} \]

\[ E_t [r_{t+1}^c] = r_0 + [1 + W_x (\rho - \kappa_1^c)] x_t + W_{g,t} (\nu_g - \kappa_1^c) (\sigma_{gt}^2 - \sigma_g^2) + W_{w,t} (\nu_x - \kappa_1^c) (\sigma_{xt}^2 - \sigma_x^2) \]

\[ + \sigma_{gt} \eta_{t+1} + W_{x,t} \sigma_{x,t+1} + W_{gs} \sigma_{g,w,t+1} + W_{xs} \sigma_{x,w,t+1} \]

\[ r_{t+1}^c - E_t [r_{t+1}^c] = \sigma_{gt} \eta_{t+1} + W_{x,t} \sigma_{x,t+1} + W_{gs} \sigma_{g,w,t+1} + W_{xs} \sigma_{x,w,t+1} \]

\[ V_t [r_{t+1}^c] = \sigma_{g,t}^2 + W_x^2 \sigma_{x,t}^2 + W_{g,t}^2 \sigma_{g,t}^2 + W_{x,t}^2 \sigma_{x,t}^2 \]

\[ r_0^c = \kappa_0^c + \mu_g + (1 - \kappa_1^c) \mu_{wc} \]
Epstein and Zin (1989) show that the log real stochastic discount factor is

\[ sdt_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{t+1}^c \]
\[ = \mu_s + \left\{ -\frac{\theta}{\psi} + (\theta - 1) [1 + W_x (\rho - \kappa_1^c)] \right\} x_t \]
\[ + \{W_{gs} (\nu_g - \kappa_1^c) (\theta - 1)\} (\sigma^2_{gt} - \sigma^2_g) + \{W_{xs} (\nu_x - \kappa_1^c) (\theta - 1)\} (\sigma^2_{xt} - \sigma^2_x) \]
\[ + \left\{ \theta \left( 1 - \frac{1}{\psi} \right) - 1 \right\} \sigma_{gt} \eta_{t+1} + (\theta - 1) \{W_x \sigma_{xt} \epsilon_{t+1} + W_{gs} \sigma_{gw} w_{g,t+1} + W_{xs} \sigma_{xw} w_{x,t+1} \} \]

\[ sdt_{t+1} - E_t [sdt_{t+1}] = \left\{ \theta \left( 1 - \frac{1}{\psi} \right) - 1 \right\} \sigma_{gt} \eta_{t+1} + (\theta - 1) \{W_x \sigma_{xt} \epsilon_{t+1} + W_{gs} \sigma_{gw} w_{g,t+1} + W_{xs} \sigma_{xw} w_{x,t+1} \} \]
\[ E_t [sdt_{t+1}] = \mu_s + \left\{ -\frac{\theta}{\psi} + (\theta - 1) [1 + W_x (\rho - \kappa_1^c)] \right\} x_t \]
\[ + \{W_{gs} (\nu_g - \kappa_1^c) (\theta - 1)\} (\sigma^2_{gt} - \sigma^2_g) + \{W_{xs} (\nu_x - \kappa_1^c) (\theta - 1)\} (\sigma^2_{xt} - \sigma^2_x) \]
\[ \forall_t [sdt_{t+1}] = \left\{ \theta \left( 1 - \frac{1}{\psi} \right) - 1 \right\} \sigma^2_{gt} + (\theta - 1)^2 \{W_x^2 \sigma^2_{xt} + W_{gs}^2 \sigma^2_{gw} + W_{xs}^2 \sigma^2_{xw} \} \]
\[ \mu_s = \theta \log \delta - \frac{\theta}{\psi} \mu_g + (\theta - 1) r_0^c \]

\[ \text{Cov}_t [r_{t+1}^c, sdt_{t+1}] = E_t [(r_{t+1}^c - E_t [r_{t+1}^c]) (sdt_{t+1} - E_t [sdt_{t+1}])] \]
\[ = \left\{ \theta \left( 1 - \frac{1}{\psi} \right) - 1 \right\} \sigma^2_{gt} + W_x^2 (\theta - 1) \sigma^2_{x} + W_{gs}^2 (\theta - 1) \sigma^2_{gw} + W_{xs}^2 (\theta - 1) \sigma^2_{xw} \]

Plugging these different components into equation (2) evaluated at \( i = c \) yields:

\[ 0 = r_0^c + \mu_s + \frac{\theta^2}{2} \left\{ \left( 1 - \frac{1}{\psi} \right)^2 \sigma^2_g + W_x^2 \sigma^2_x + W_{gs}^2 \sigma^2_{gw} + W_{xs}^2 \sigma^2_{xw} \right\} \]
\[ + \theta \left\{ -\frac{1}{\psi} + [1 + W_x (\rho - \kappa_1^c)] \right\} x_t \]
\[ + \frac{\theta}{2} \left\{ 2W_{gs} (\nu_g - \kappa_1^c) + \theta \left( 1 - \frac{1}{\psi} \right)^2 \right\} (\sigma^2_{gt} - \sigma^2_g) \]
\[ + \frac{\theta}{2} \left\{ 2W_{xs} (\nu_x - \kappa_1^c) + \theta W_x^2 \right\} (\sigma^2_{xt} - \sigma^2_x) \]
Then setting all coefficients equal to zero we obtain:

\[ W_x = \frac{1 - \frac{\psi}{\kappa_1}}{\kappa_1 - \rho} \]

\[ W_{gs} = \frac{\theta \left( 1 - \frac{1}{\psi} \right)^2}{2 (\kappa_1^g - \nu_g)} \]

\[ W_{xs} = \frac{\theta}{2 (\kappa_1^x - \nu_x)} \left( 1 - \frac{1}{\psi} \right)^2 \]

If the IES exceeds 1, then \( W_x > 0 \), \( W_{gs} < 0 \), and \( W_{xs} < 0 \).

Plugging these coefficients back into equation (3) implicitly defines a nonlinear equation in one unknown \( \mu_{wc} \), which can be solved for numerically, characterizing the average wealth-consumption ratio.

According to (2), the risk premium (expected excess real return corrected for a Jensen term) on the consumption claim is given by:

\[
\mathbb{E}_t [r_{t+1}^{c,e}] = -\text{Cov}_t [r_{t+1}^{c,e}, sdf_{t+1}] \\
= \left\{ 1 - \theta \left( 1 - \frac{1}{\psi} \right) \right\} \sigma_{gt}^2 + W_x^2 (1 - \theta) \sigma_{xt}^2 + W_{gs}^2 (1 - \theta) \sigma_{gw}^2 + W_{xs}^2 (1 - \theta) \sigma_{xw}^2 \\
= \lambda_\eta \sigma_{gt}^2 + W_x \lambda_c \sigma_{xt}^2 + W_{gs} \lambda_{gw} \sigma_{gw}^2 + W_{xs} \lambda_{xw} \sigma_{xw}^2
\]

with the market price of risk vector \( \Lambda = [\lambda_\eta, \lambda_c, \lambda_{gw}, \lambda_{xw}] \) given by:

\[ \lambda_\eta = -\left\{ \theta \left( 1 - \frac{1}{\psi} \right) - 1 \right\} = \gamma > 0 \]

\[ \lambda_c = (1 - \theta) W_x = \frac{\gamma - \frac{1}{\psi}}{\kappa_1^1 - \rho} \]

\[ \lambda_{gw} = (1 - \theta) W_{gs} = -\frac{(\gamma - 1) (\gamma - \frac{1}{\psi})}{2 (\kappa_1^g - \nu_g)} \]

\[ \lambda_{xw} = (1 - \theta) W_{xs} = -\frac{(\gamma - 1) (\gamma - \frac{1}{\psi})}{2 (\kappa_1^x - \nu_x)} (\kappa_1^x - \rho)^2 \]

If the IES is sufficiently large \( (\gamma > 1/\psi) \), then \( \lambda_c > 0 \), \( \lambda_{gw} < 0 \), and \( \lambda_{xw} < 0 \).

### II. Equity Risk Premium

We log-linearize return on portfolio:

\[ r_{t+1} = \kappa_0 + \Delta d_{t+1} + pd_{t+1} - \kappa_1 pd_t \]

and conjecture that the price-dividend ratio is linear in the state variables:

\[ pd_t = \mu_{pd} + D_x x_t + D_{gs} (\sigma_{gt}^2 - \sigma_{g}^2) + D_{xs} (\sigma_{xt}^2 - \sigma_{x}^2) \]

As we did for the return on the consumption claim, we compute innovations in the

\[ y_t(1) = -\text{E}_t [sdf_{t+1}] - \frac{1}{2} \text{Var}_t [sdf_{t+1}] \]
Plugging these different components into equation (2):

\[ r_{t+1} = r_0 + \{ \phi_x + D_x (\rho - \kappa_1) \} x_t + D_{gs} (\nu_g - \kappa_1) \left( \sigma_{gt}^2 - \sigma_g^2 \right) + D_{xs} (\nu_x - \kappa_1) \left( \sigma_{xt}^2 - \sigma_x^2 \right) \]
\[ + \varphi_d \sigma_{gt} \eta_{d,t+1} + D_{xt} \sigma_{xt} \epsilon_{t+1} + D_{gx} \sigma_{gw} \eta_{g,t+1} + D_{xt} \sigma_{xw} \eta_{x,t+1} \]
\[ r_{t+1} - \mathbb{E}_t [r_{t+1}] = \varphi_d \sigma_{gt} \eta_{d,t+1} + D_{xt} \sigma_{xt} \epsilon_{t+1} + D_{gx} \sigma_{gw} \eta_{g,t+1} + D_{xt} \sigma_{xw} \eta_{x,t+1} \]
\[ \mathbb{E}_t [r_{t+1}] = r_0 + \{ \phi_x + D_x (\rho - \kappa_1) \} x_t + D_{gs} (\nu_g - \kappa_1) \left( \sigma_{gt}^2 - \sigma_g^2 \right) \]
\[ + D_{xs} (\nu_x - \kappa_1) \left( \sigma_{xt}^2 - \sigma_x^2 \right) \]
\[ \text{Var}_t [r_{t+1}] = \varphi_d^2 \sigma_{gt}^2 + D_{xt}^2 \sigma_{xt}^2 + D_{gx}^2 \sigma_{gw}^2 + D_{xs}^2 \sigma_{xw}^2 \]
\[ r_0 = \kappa_0 + \mu_{pd} (1 - \kappa_1) + \mu_d \]

\[ \text{Cov}_t [r_{t+1}, s df_{t+1}] = (\theta - 1) [W_{gs} D_{gs} \sigma_{gw}^2 + W_{xs} D_{xs} \sigma_{xw}^2] - \gamma \varphi_d \tau_g d \sigma_{gt}^2 + (\theta - 1) W_x D_x \sigma_{zt}^2 \]

Plug these different components into equation (2):

\[
0 = \mu_x + r_0 + \frac{1}{2} \left[ \gamma^2 - 2 \gamma \varphi_d \tau_g d + \varphi_d^2 \right] \sigma_g^2 + \frac{1}{2} \left[ W_x (\theta - 1) + D_x \right] \sigma_x^2 + \frac{1}{2} \left[ W_{gs} (\theta - 1) + D_{gs} \right] \sigma_{gw}^2 \\
+ \frac{1}{2} [W_{xt} (\theta - 1) + D_{xt}] \sigma_{xw}^2 \\
+ \left\{ \frac{1}{\eta} \left[ \phi_x + D_x (\rho - \kappa_1) \right] x_t \right\} \\
+ \left\{ \phi_d \left[ \gamma^2 - 2 \gamma \varphi_d \tau_g d + \varphi_d^2 \right] + W_{gs} (\kappa_1 - \nu_g) (1 - \theta) + D_{gs} (\nu_g - \kappa_1) \right\} \left( \sigma_{gt}^2 - \sigma_g^2 \right) \\
+ \left\{ \frac{1}{2} \left[ W_x (\theta - 1) + D_x \right]^2 + W_{xs} (\kappa_1 - \nu_x) (1 - \theta) + D_{xs} (\nu_x - \kappa_1) \right\} \left( \sigma_{xt}^2 - \sigma_x^2 \right)
\]

Then setting all coefficients equal to zero we get:

\[ \phi_x = \frac{\frac{1}{\eta}}{\kappa_1 - \rho} \]
\[ \frac{1}{2} \left[ \phi_d - \frac{1}{\kappa_1 - \rho} \right] \gamma + \frac{1}{2} \left( \frac{1}{\kappa_1 - \rho} - \frac{\gamma}{\kappa_1 - \rho} \right) (\gamma - 1) \]
\[ \frac{1}{2} \left[ \phi_d - \frac{1}{\kappa_1 - \rho} \right] \gamma + \frac{1}{2} \left( \frac{1}{\kappa_1 - \rho} - \frac{\gamma}{\kappa_1 - \rho} \right) (\gamma - 1) \]
\[ \frac{1}{2} \left[ \phi_d - \frac{1}{\kappa_1 - \rho} \right] \gamma + \frac{1}{2} \left( \frac{1}{\kappa_1 - \rho} - \frac{\gamma}{\kappa_1 - \rho} \right) (\gamma - 1) \]

Plugging these into \( \phi_1 \) implicitly defines a nonlinear equation in one unknown (i.e., \( \mu_{pd} \)), which can be solved for numerically, characterizing the mean price-dividend ratio.

The \( D \) coefficients are the betas of the equity market portfolio with respect to the four fundamental consumption growth shocks.
The equity risk premium is equal to:

\[ E_t [r_{t+1}^e] = -\text{Cov}_t[r_{t+1}, sdf_{t+1}] \]

\[ = \left[ (\varphi_d \gamma_d) \lambda_0 \sigma_{gt}^2 + D_x \lambda_v \sigma_{xt}^2 + D_g \lambda_w \sigma_{gw}^2 + D_x \lambda_x \sigma_{xw}^2 \right] \]

\[ = \left[ G_0 + G_g \sigma_{gw}^2 + G_x \sigma_{xw}^2 \right] + G_g \left( \sigma_{gt}^2 - \gamma_d^2 \right) + G_x \left( \sigma_{xt}^2 - \gamma_x^2 \right) \]

\[ G_0 = D_g \lambda_w \sigma_{gw}^2 + D_x \lambda_x \sigma_{xw}^2 \]

\[ G_g = \varphi_d \gamma_d \]

\[ G_x = D_x \lambda_v \]

### III. Real Bond Returns and Risk Premium

We start off the expression for the real stochastic discount factor derived in the first sub-section above. Let define the following three parameters: \( s_x \equiv -\frac{1}{\psi}, s_{gs} \equiv -\frac{1}{2} (\gamma - 1)(\gamma - \frac{1}{\psi}), \) and \( s_{xs} \equiv -\frac{1}{2} (\gamma - 1)(\gamma - \frac{1}{\psi}) \frac{1}{(\kappa_1^c - \rho)^2} \). Using notation defined above and in the previous sub-sections, the real stochastic discount factor is:

\[ sdf_{t+1} = \mu_s + s_{x} x_t + s_{gs} \left( \sigma_{gt}^2 - s_g^2 \right) + s_{xs} \left( \sigma_{xt}^2 - s_x^2 \right) \]

\[ -s_x \gamma \sigma_{gt} \eta_{t+1} - \lambda_e \sigma_{xt} e_{t+1} - \lambda_w \sigma_{gw} w_{g,t+1} - \lambda_x \sigma_{xw} w_{x,t+1} \]

Let \( p_t^b(n) = \log (P_t^b(n)) \) be the log price and \( y_t^b(n) = -\frac{1}{n} p_t^b(n) \) the yield of an \( n \)-period real bond.

We conjecture that the log prices of real bonds are linear in the state variables: \( p_t(n) = -B_0(n) - B_x(n)x_t - B_{gs}(n) (\gamma - \frac{1}{\psi}) - B_{xs}(\gamma - \frac{1}{\psi}) = \)

The coefficients are initialized at zero and satisfy the following recursions:

\[ B_0(n) = B_0(n-1) - \mu_s - \frac{1}{2} \left( \lambda_{gw} + B_{gs}(n-1) \right)^2 \sigma_{gw}^2 \]

\[ -\frac{1}{2} \left\{ \left( \lambda_{xw} + B_{xs}(n-1) \right)^2 \sigma_{xw}^2 + \lambda_e^2 \right\} \]

\[ -\frac{1}{2} \left\{ \left( \lambda_{e} + B_x(n-1) \right)^2 \sigma_{x}^2 \right\} \]

\[ B_x(n) = \rho B_x(n-1) + \frac{1}{\psi} \]

\[ B_{gs}(n) = \nu_g B_{gs}(n-1) + \frac{1}{2} (\gamma - 1)(\gamma - \frac{1}{\psi}) - \frac{1}{2} \gamma^2 \]

\[ B_{xs}(n) = \nu_x B_{xs}(n-1) + \frac{1}{2} (\gamma - 1)(\gamma - \frac{1}{\psi}) \frac{(\gamma - \frac{1}{\psi})}{(\kappa_1^c - \rho)^2} - \frac{1}{2} \left[ \frac{\gamma - \frac{1}{\psi}}{\kappa_1^c - \rho} + B_x(n-1) \right]^2 \].
These recursions imply the following limit values:

\[ B_x(\infty) = \frac{1}{\psi(1 - \rho)} \]
\[ B_{gs}(\infty) = \frac{\frac{1}{2}(\gamma - 1)(\gamma - \frac{1}{\psi}) - \frac{1}{2}\gamma^2}{1 - \nu_g} \]
\[ B_{xs}(\infty) = \frac{\frac{1}{2}(\gamma - 1)(\gamma - \frac{1}{\psi}) - 1}{1 - \nu_x} \left[ \frac{\gamma - \frac{1}{\psi}}{\alpha_t - \rho} + B_x(\infty) \right]^2. \]

We define \( B(\infty) \equiv [B_x(\infty), B_{gs}(\infty), B_{xs}(\infty)]' \).
The real bond risk premium on monthly holding period returns is equal to:

\[
\begin{align*}
    r_{t+1}^b(n) & \equiv n_B(n) - (n-1)g_{t+1}(n-1) \\
    r_{t+1}^b(n) - E_t[r_{t+1}^b(n)] & = -B_x(n-1)\sigma_x \gamma_{t+1} - B_{gs}(n-1)\gamma_{gs}w_{gs,t+1} \\
    -B_{xs}(n-1)\sigma_{xs}w_{xs,t+1} \\
    E_t[r_{t+1}^{b,c}(n)] & = -Cov_t[r_{t+1}^b, sdf_t^{b}]
\end{align*}
\]
\[
\begin{align*}
    & = \left[ F_0(n) + F_{gs}(n)\sigma_g^2 + F_{xs}(n)\sigma_x^2 \right] + F_{gs}(n) \left( \sigma_{gs}^2 - \sigma_g^2 \right) + F_{xs}(n) \left( \sigma_{xs}^2 - \sigma_x^2 \right) \\
    F_0(n) & = -B_{gs}(n-1)\lambda gw \sigma_{gw} - B_{xs}(n-1)\lambda_{gw} \sigma_{gw}, \\
    F_{gs}(n) & = 0, \\
    F_{xs}(n) & = -B_x(n-1)\lambda_{gw}.
\end{align*}
\]

We now define some vectors and matrices to present results in a more compact way.
Let the vector \( X_t \) summarize all real state variables: \( X_t \equiv [x_t, \sigma_{gt}^2, \sigma_{zt}^2 - \sigma_{zt}^2]' \).
Let \( \varepsilon_{t+1} \) denote the corresponding gaussian, i.i.d shocks: \( \varepsilon_{t+1} \equiv [\varepsilon_{t+1}, w_{gs,t+1}, w_{xs,t+1}]' \).
We define \( \Sigma_t \equiv diag(\sigma_{xt}^2, \sigma_{gw}^2, \sigma_{gw}^2) \). The law of motion of the state vector \( X_t \) is \( X_{t+1} = \Gamma X_t + \Sigma_{\varepsilon}^{\frac{1}{2}} \varepsilon_{t+1} \), where \( \Gamma \) is a 3 by 3 diagonal matrix with \( \rho, \varphi_{zg}, \) and \( \varphi_{zx} \) on the diagonal.
Let \( B(n) \) denote all the \( n \)-period real bond parameters: \( B(n) \equiv [B_x(n), B_{gs}(n), B_{xs}(n)]' \).
Using this notation, we can rewrite the real bond risk premium as:

\[ E_t[r_{t+1}^{b,c}(n)] = -B(n-1)'\Sigma_t \Lambda. \]

**IV. Nominal Bond Returns and Risk Premium**

We start off the expression for the real stochastic discount factor derived above. We use a \(^8\) superscript to denote nominal variables. The nominal stochastic discount factor is then:

\[
\begin{align*}
    sdf_t^8(n) & \equiv sdf_t^{b+1} - \pi_t^{b+1} \\
    & = \mu - \mu_{g} + s_x \gamma_{t+1} + s_{gs} (\sigma_{gt}^2 - \sigma_g^2) + s_{xs} (\sigma_{zt}^2 - \sigma_x^2) - (\bar{\pi}_t - \mu_{\pi}) \\
    & - (\lambda_{g} + \varphi_{g}) \gamma_{t+1} - (\lambda_{x} + \varphi_{x}) \gamma_{zt} \gamma_{t+1} - \lambda_{gw} \sigma_{gw} w_{gs,t+1} - \lambda_{gw} \sigma_{gw} w_{xs,t+1} - \sigma_{\pi} \xi_{t+1}
\end{align*}
\]

Let \( y_t^8(n) = \log \{ P_t^8(n) \} \) be the log price and \( y_t^8(n) = -\frac{1}{n} p_t^8(n) \) the yield of an \( n \)-period
nominal bond. We conjecture that the log prices of nominal bonds are linear in the state variables: 

$$p_t^B = -B_0^B(n) - B_{-1}^B(n)x_t - B_{gs}^B(n)(\sigma_{gt}^2 - \sigma_{g}^2) - B_{xs}^B(\sigma_{zt}^2 - \sigma_{z}^2) - B_{\pi}^B(n)(\pi_t - \mu_{\pi})$$

The coefficients are initialized at zero and satisfy the following recursions:
Define the following vector and matrix objects:

\[ \hat{\Lambda}^8 \equiv [\lambda_n + \varphi_{xg}, \lambda_e + \varphi_{xx}, \lambda_{gw}, \lambda_{xw}, \sigma], \]
\[ \hat{B}^8(n) \equiv [\rho^g(n)\varphi_{xg}, \rho^x(n)\varphi_{xx}, \rho_{gs}(n), \rho_{xs}(n), \sigma^g, \sigma^x, \sigma_{gw}, \sigma_{xw}], \]
\[ \hat{\Sigma}_t \equiv \text{diag}[\sigma^2_{gt}, \sigma^2_{xt}, \sigma^2_{gw}, \sigma^2_{xw}, 1], \]
\[ \hat{\varepsilon}_{t+1} \equiv [\eta_{t+1}, e_{t+1}, w_{g,t+1}, w_{x,t+1}, \xi_{t+1}] \]

Then we can write the nominal bond risk premium compactly as:

\[ \mathbb{E}_t \left[ r_{t+1}^{h, e}(n) \right] = -\hat{B}^8(n-1)\hat{\Sigma}_t\hat{\Lambda}^8. \]

V. Decomposition of the Real SDF

The following proposition shows how to decompose the SDF of the long-run risk model into a martingale component and the dominant pricing component.

**Proposition 1.** The stochastic discount factor of the long-run risk model can be decomposed into a martingale component and the dominant pricing component:

\[
\frac{M_t^T}{M_t^P} = \beta \exp \left( -B'_\infty (I - \Gamma) X_t + B'_\infty \Sigma^\frac{1}{2}_t \varepsilon_{t+1} \right),
\]
\[
\frac{M_{t+1}^P}{M_t^P} = \beta^{-1} \exp \left( \mu_s + [S' + B'_\infty (I - \Gamma)] X_t - (\Lambda' + B'_\infty) \Sigma^\frac{1}{2}_t \varepsilon_{t+1} - \frac{1}{2} \lambda \eta \sigma_{gt} \eta_{t+1} \right).
\]

To show this, we start from the definition of the dominant pricing component of the pricing kernel:

\[ M_t^P = \lim_{n \to \infty} \frac{\beta^{t+n}}{\mathbb{P}_t^b(n)}. \]

Recall that log real bond prices are affine in the state vector:

\[ p_t^b(n) = -B_0(n) - B_x(n)x_t - B_{gs}(n) (\sigma^2_{gt} - \sigma^2_g) - B_{xs}(n) (\sigma^2_{xt} - \sigma^2_x) \]
\[ = -B_0(n) - B(n)'X_t. \]

We can then write the dominant pricing component of the SDF as:

\[ M_t^T = \lim_{n \to \infty} \beta^{t+n} \exp (B_0(n) + B(n)'X_t). \]

The constant \( \beta \) is chosen in order to satisfy Assumption 1 in Alvarez and Jermann (2005):

\[ 0 < \lim_{n \to \infty} \frac{\mathbb{P}_t^b(n)}{\beta^n} < \infty. \]

Recall that \( B_0(n) \) is defined recursively:

\[ B_0(n) = B_0(n-1) - \mu_s - \frac{1}{2} \left\{ (\lambda_{gw} + B_{gs}(n-1))^2 \sigma^2_{gw} \right\} \]
\[ - \frac{1}{2} \left\{ (\lambda_{xw} + B_{xs}(n-1))^2 \sigma^2_{xw} + \lambda^2_{gw} \sigma^2_g + [\lambda_e + B_x(n-1)]^2 \sigma^2_x \right\} \]
Because of the affine term structure of the model and the stationarity of the state vector \( X \), the limit \( \lim_{n \to \infty} B(n) = B(\infty) \) is finite. Taking limits on both sides of the equation above leads to:

\[
\lim_{n \to \infty} B_0(n) - B_0(n - 1) = -\mu_s - \frac{1}{2} \left\{ [\lambda_{gw} + B_{gs}(\infty)]^2 \sigma_{gw}^2 \right\} - \frac{1}{2} \left\{ [\lambda_{xw} + B_{xs}(\infty)]^2 \sigma_{xw}^2 + \lambda^2 \sigma^2 + [\lambda_e + B_x(\infty)]^2 \sigma_{x}^2 \right\}
\]

The limit of \( B_0(n) - B_0(n - 1) \) is finite, so that \( B_0(n) \) grows at a linear rate in the limit. We choose the constant \( \beta \) to offset the growth in \( B_0(n) \) as \( n \) becomes very large. Setting

\[
\beta = \exp \left( \mu_s + \frac{1}{2} \left\{ [\lambda_{gw} + B_{gs}(\infty)]^2 \sigma_{gw}^2 + [\lambda_{xw} + B_{xs}(\infty)]^2 \sigma_{xw}^2 + \lambda^2 \sigma^2 + [\lambda_e + B_x(\infty)]^2 \sigma_{x}^2 \right\} \right)
\]

guarantees that Assumption 1 in Alvarez and Jermann (2005) is satisfied.

We can now write the dominant pricing component of the SDF as:

\[
\frac{M_{t+1}^T}{M_t} = \beta \exp \left( -B_\infty' (I - \Gamma) X_t + B_\infty' \Sigma_t \frac{1}{2} \epsilon_t + \lambda \sigma_{gt} \eta_{t+1} \right),
\]

To derive the martingale component of the SDF, let us go back to the SDF itself. Let \( S \) and \( \Lambda \) denote the parameters of the real SDF: \( S \equiv [s_x, s_{gs}, s_{xs}]; \Lambda \equiv [\lambda_e, \lambda_{gw}, \lambda_{xw}] \). Then the real SDF is:

\[
SDF_{t+1} = \frac{M_{t+1}}{M_t} = \exp \left( \mu_s + S' X_t - \Lambda' \Sigma_t \frac{1}{2} \epsilon_t + \lambda \sigma_{gt} \eta_{t+1} \right).
\]

As a result, the martingale component of the SDF is:

\[
\frac{M_{t+1}^P}{M_t^P} = \frac{M_{t+1}}{M_t} \left( \frac{M_{t+1}^T}{M_t^T} \right)^{-1} = \beta^{-1} \exp \left( \mu_s + [S' + B_\infty' (I - \Gamma)] X_t - (A' + B_\infty') \Sigma_t \frac{1}{2} \epsilon_t + \lambda \sigma_{gt} \eta_{t+1} \right).
\]

We need to verify that the martingale component is a martingale, i.e that \( E_t[M_{t+1}^P / M_t^P] = 1 \).

To do this, recall that the bond parameters evolve as:

\[
B_x(n) = \rho B_x(n - 1) - s_x,
\]

\[
B_{gs}(n) = \nu_g B_{gs}(n - 1) - s_{gs} - \frac{1}{2} \lambda^2 \eta_n,
\]

\[
B_{xs}(n) = \nu_x B_{xs}(n - 1) - s_{xs} - \frac{1}{2} [\lambda_e + B_x(n - 1)]^2.
\]

Taking limits as \( n \to \infty \) leads to:

\[
B(\infty)'(I - \Gamma) = -S' + [0, -\frac{1}{2} \lambda^2 \eta, -\frac{1}{2} [\lambda_e + B_x(\infty)]^2 \nu].
\]
To check the martingale condition, plug the definition of $\beta$ in the following expression:

$$E_t \left[ \frac{M_{t+1}^P}{M_t^P} \right] = \beta^{-1} \exp \left( \mu_s + [S' + B'_{\infty} (I - \Gamma)] X_t + \frac{1}{2} (\Lambda' + B'_\infty) \Sigma_t (\Lambda + B_\infty) + \frac{1}{2} \lambda_{gt}^2 \sigma_{gt}^2 \right).$$

The term in front of $X_t$ is equal to $[0, - \frac{1}{2} \lambda_{gt}^2, - \frac{1}{2} [\lambda_e + B_x(\infty)]^2]$. Terms in $\sigma_{gt}^2$ and $\sigma_{zt}^2$ cancel out. We next plug in the expression for $\beta$ and check that $E_t \left[ \frac{M_{t+1}^P}{M_t^P} \right] = 1$.

We now turn to the conditional variances of the log SDF and its dominant pricing and martingale components, $\text{Var}_t[\text{sdf}_{t+1}], \text{Var}_t[\text{sdf}_{t+1}^T]$ and $\text{Var}_t[\text{sdf}_{t+1}^P]$.

$$\text{Var}_t[\text{sdf}_{t+1}] = \Lambda' \Sigma_t \Lambda + \lambda_{gt}^2 \sigma_{gt}^2$$

$$\text{Var}_t[\text{sdf}_{t+1}^T] = B'_\infty \Sigma_t B_\infty$$

$$\text{Var}_t[\text{sdf}_{t+1}^P] = (\Lambda' + B'_\infty) \Sigma_t (\Lambda + B_\infty) + \lambda_{gt}^2 \sigma_{gt}^2.$$

The conditional variance ratio $\text{Var}_t[\text{sdf}_{t+1}^P]/\text{Var}_t[\text{sdf}_{t+1}]$ equals

$$\frac{\text{Var}_t[\text{sdf}_{t+1}^P]}{\text{Var}_t[\text{sdf}_{t+1}]} = 1 - \frac{B'_\infty \Sigma_t \Lambda - \frac{1}{2} B'_\infty \Sigma_t B_\infty}{\frac{1}{2} \Lambda' \Sigma_t \Lambda + \frac{1}{2} \lambda_{gt}^2 \sigma_{gt}^2}$$

The first term in the numerator corresponds to the bond risk premium ($-B'_\infty \Sigma_t \Lambda$). It includes the Jensen term ($\frac{1}{2} B'_\infty \Sigma_t B_\infty$). As a result, the numerator corresponds to the bond risk premium without the Jensen term. The denominator corresponds to the maximum risk premium (also without the Jensen term).

Note that the maximal Sharpe ratio in the model is:

$$\text{MaxSR}_t = \sigma_t (\log SD F_{t+1})$$

$$= \sqrt{\lambda_{gt}^2 \sigma_{gt}^2 + \lambda_{gw}^2 \sigma_{gw}^2 + \lambda_{xw}^2 \sigma_{xw}^2 + \lambda_{gt}^2 \sigma_{gt}^2}$$

$$= (\Lambda' \Sigma_t \Lambda + \lambda_{gt}^2 \sigma_{gt}^2)^{\frac{1}{2}}$$

VI. Decomposition of the Nominal SDF

The following proposition shows how to decompose the nominal SDF of the long-run risk model into a martingale and a dominant pricing component. To avoid confusion, we use $MN$ to denote the nominal pricing kernel.

**Proposition 2.** The stochastic discount factor of the long-run risk model can be decomposed into a martingale component and the dominant pricing component:

$$\frac{MN_{t+1}^P}{MN_t^P} = \tilde{\beta} \exp \left( -B_{\infty}^g (I - \hat{\Gamma}) \hat{X}_t + B_{\infty}^g \hat{\Sigma}_t \hat{\epsilon}_{t+1} \right),$$

$$\frac{MN_{t+1}^M}{MN_t^M} = \tilde{\beta}^{-1} \exp \left( \mu_s - \mu_x + [\hat{S}' + B_{\infty}^g (I - \hat{\Gamma})] \hat{X}_t - (\hat{\Lambda}^g + B_{\infty}^g \hat{\Sigma}_t \hat{\epsilon}_{t+1}) \right).$$

To show this, we start from the definition of the dominant pricing component of the
pricing kernel:

\[ MN_t^T = \lim_{n \to \infty} \frac{\tilde{\beta}^{t+n}}{P_t^{sb}(n)}, \]

Recall that log real bond prices are affine in the state vector:

\[ p_t^{sb}(n) = -B_0^8(n) - B_2^8(n)x_t - B_2^8(n) (\sigma^2_{zt} - \sigma^2_g) - B_2^8 (\sigma^2_{xt} - \sigma^2_x) - B_2^8 (\tilde{\pi}_t - \mu_x) \]

where we define \( \tilde{X}_t = [x_t, \sigma^2_{zt} - \sigma^2_g, \sigma^2_{xt} - \sigma^2_x, \tilde{\pi}_t - \mu_x] \).

We can then write the dominant pricing component of the SDF as:

\[ MN_t^T = \lim_{n \to \infty} \tilde{\beta}^{t+n} \exp \left( B_0^8(n) + B_2^8(n)' \tilde{X}_t \right). \]

The constant \( \tilde{\beta} \) is chosen in order to satisfy Assumption 1 in Alvarez and Jermann (2005):

\[ 0 < \lim_{n \to \infty} \frac{P_t^{sb}(n)}{\beta_n} < \infty. \]

Recall that \( B_0^8(n) \) is defined recursively:

\[
B_0^8(n) = B_0^8(n-1) - \mu_x + \mu_\pi - \frac{1}{2} \left\{ \left[ \sigma_\pi + B_2^8(n-1) \sigma_x \right]^2 + \left[ \lambda_{gw} + B_{gs}^8(n-1) \right]^2 \sigma^2_{gw} \right\} \\
- \frac{1}{2} \left\{ \left[ \lambda_{xw} + B_{xs}^8(n-1) \right]^2 \sigma^2_{xw} + \left[ \varphi_{\pi g} + \lambda_{\eta} + \varphi_{z g} B_{z\pi}^8(n-1) \right]^2 \sigma^2_g \right\} \\
- \frac{1}{2} \left\{ \varphi_{xz} + \lambda_c + B_{z\pi}^8(n-1) + \varphi_{xx} B_{z\pi}^8(n-1) \right\}^2 \sigma^2_x
\]

Because of the affine term structure of the model and the stationarity of the state vector \( \tilde{X} \), the limit \( \lim_{n \to \infty} B_0^8(n) = B^8(\infty) \) is finite. Taking limits on both sides of the equation above leads to:

\[
\lim_{n \to \infty} B_0^8(n) - B_0^8(n-1) = -\mu_x + \mu_\pi - \frac{1}{2} \left\{ \left[ \sigma_\pi + B_2^8(\infty) \sigma_x \right]^2 + \left[ \lambda_{gw} + B_{gs}^8(\infty) \right]^2 \sigma^2_{gw} \right\} \\
- \frac{1}{2} \left\{ \left[ \lambda_{xw} + B_{xs}^8(\infty) \right]^2 \sigma^2_{xw} + \left[ \varphi_{\pi g} + \lambda_{\eta} + \varphi_{z g} B_{z\pi}^8(\infty) \right]^2 \sigma^2_g \right\} \\
- \frac{1}{2} \left\{ \varphi_{xz} + \lambda_c + B_{z\pi}^8(\infty) + \varphi_{xx} B_{z\pi}^8(\infty) \right\}^2 \sigma^2_x
\]

The limit of \( B_0^8(n) - B_0^8(n-1) \) is finite, so that \( B_0^8(n) \) grows at a linear rate in the limit. We choose the constant \( \tilde{\beta} \) to offset the growth in \( B_0^8(n) \) as \( n \) becomes very large.
where
\[ S \equiv \text{guarantees that Assumption 1 in Alvarez and Jermann (2005) is satisfied.} \]

Setting
\[
\hat{\beta} = \exp \left( \mu_s - \mu_\pi + \frac{1}{2} \left( \sigma_\pi + B^8_s(\infty) \sigma_\pi \right)^2 + \left[ \lambda_{gw} + B^8_{gw}(\infty) \right]^2 \sigma_{gw}^2 \right) 
\]
\[
+ \frac{1}{2} \left( \lambda_{xw} + B^8_{xw}(\infty) \right)^2 \sigma_{xw}^2 + \left[ \varphi_{\pi x} + \lambda_\pi + \varphi_{z \pi} B^8_\pi(\infty) \right] \left[ \sigma_\pi + \lambda_\pi + \varphi_{z \pi} B^8_\pi(\infty) \right]^2 \sigma_\pi^2 
\]
\[
+ \frac{1}{2} \left[ \varphi_{\pi x} + \lambda_\pi + B^8_\pi(\infty) + \varphi_{z x} B^8_\pi(\infty) \right] \sigma_\pi^2 
\]
guarantees that Assumption 1 in Alvarez and Jermann (2005) is satisfied.

We can now write the dominant pricing component of the SDF as:
\[
\frac{MN_{t+1}}{MN_t} = \hat{\beta} \exp \left( -B^8_\infty (I - \tilde{\Gamma}) \tilde{X}_t + B^8_\pi(\infty) \varphi_{z \pi} \sigma_{zt} \xi_{t+1} + B^8_\pi(\infty) \sigma_{z \pi} \xi_{t+1} \right) 
\]
\[
+ \left[ B^8_\pi(\infty) + B^8_\pi(\infty) \varphi_{z \pi} \sigma_{zt} \xi_{t+1} + B^8_{gw}(\infty) \sigma_{gw} \nu_{g,t+1} + B^8_{xw}(\infty) \sigma_{xw} \nu_{x,t+1} \right] 
\]
\[
= \hat{\beta} \exp \left( -B^8_\infty (I - \tilde{\Gamma}) \tilde{X}_t + \tilde{B}^8_\infty \hat{\Gamma}_t^5 \tilde{\xi}_{t+1} \right), 
\]
where
\[
\tilde{\Gamma} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \nu_\pi & 0 & 0 \\ 0 & 0 & \nu_\pi & 0 \\ \alpha_\xi & 0 & 0 & \alpha_\pi \end{bmatrix} 
\]

To derive the martingale component of the SDF, let us go back to the SDF itself. Let \( \tilde{S} \equiv [s_x, s_{gw}, s_{zx}, -1]^T \). Then the nominal SDF is:
\[
\frac{MN_{t+1}}{MN_t} = \exp \left( \mu_s - \mu_\pi + \tilde{S}^T \tilde{X}_t - (\lambda_\pi + \varphi_{z \pi}) \sigma_{zt} \xi_{t+1} \right) 
\]
\[
- (\lambda_\pi + \varphi_{z \pi}) \sigma_{zt} \xi_{t+1} - \lambda_{gw} \sigma_{gw} \nu_{g,t+1} - \lambda_{xw} \sigma_{xw} \nu_{x,t+1} - \sigma_\pi \xi_{t+1} \right) 
\]
\[
= \exp \left( \mu_s - \mu_\pi + \tilde{S}^T \tilde{X}_t - \hat{\Lambda}^{5} \Sigma^{5} \tilde{\xi}_{t+1} \right) 
\]

As a result, the martingale component of the SDF is:
\[
\frac{MN^P_{t+1}}{MN^P_t} = \frac{MN_{t+1}}{MN_t} \left( \frac{MN^T_{t+1}}{MN^T_t} \right)^{-1} 
\]
\[
= \hat{\beta}^{-1} \exp \left( \mu_s - \mu_\pi + \tilde{S}^T \tilde{X}_t - (\lambda_\pi + \varphi_{z \pi} + B^8_\pi(\infty) \varphi_{z \pi}) \sigma_{zt} \xi_{t+1} \right) 
\]
\[
- \left[ \lambda_\pi + \varphi_{z \pi} + B^8_\pi(\infty) \varphi_{z \pi} \sigma_{zt} \xi_{t+1} \right] 
\]
\[
- \left[ \lambda_{gw} + B^8_{gw}(\infty) \sigma_{gw} \nu_{g,t+1} - (\lambda_{xw} + B^8_{xw}(\infty)) \sigma_{xw} \nu_{x,t+1} - \sigma_\pi \xi_{t+1} \right] 
\]
\[
= \hat{\beta}^{-1} \exp \left( \mu_s - \mu_\pi + \tilde{S}^T + B^8_\pi(\infty) \right) \tilde{X}_t - \left( \hat{\Lambda}^5 + \tilde{B}^8_\infty \hat{\Sigma}^5 \tilde{\xi}_{t+1} \right). 
\]

We need to verify that the martingale component is a martingale, i.e.
\[
E_{t}[M^P_{t+1}/M^P_t] = 
\]
The first term in the numerator corresponds to the nominal bond risk premium of an infinite horizon bond, which includes a Jensen term. The second term in the numerator is that Jensen term. As a result, the numerator corresponds to the nominal bond risk premium without the Jensen term. The denominator corresponds to the maximum nominal risk premium, also without the Jensen term.
VII. Calibration

Table 1 reports the model parameter values we use; they are the ones proposed in Bansal and Shaliastovich (2007). Table 2 reports the model loadings on state variables. The model is simulated for 60,000 months and aggregated up to quarterly frequency for comparison with our quarterly data. In the simulation, negative values for $\sigma_{g,t+1}^2$ and $\sigma_{x,t+1}^2$ are replaced by very small positive values in simulation.

Table 4 reports the mean, standard deviation and autocorrelation of the stochastic discount factor (SDF), its martingale ($SDFP$) and dominant pricing ($SDFT$) components, the conditional variance ratio $\omega$, the maximum risk premium without Jensen adjustment ($Max\ RP$) and the risk premium of an infinite maturity bond without Jensen adjustment ($BRP(\infty)$). Table 3 reports the mean and standard deviations of the real and nominal yields and bond risk premia in the model and compare them to the same moments in the actual nominal data. Table 5 reports moments of quarterly inflation in the model and in the data. Quarterly inflation is obtained as the sum of three consecutive monthly inflation rates.

The Bansal and Shaliastovich (2007) calibration generates an annual consumption growth rate of 2.12 percent with a standard deviation of 3.52 percent. It generates an annual inflation rate of 3.52 percent with a standard deviation of 2.49 percent.

VIII. Robustness Checks

As robustness checks, we considered both changes on the real and on the nominal side of the economy.

On the real side, we conduct two experiments. First, we find that a slight decrease in the persistence of the long-run component in consumption growth $\rho_x$ could decrease the long-horizon consumption variance ratios and the real variance ratio significantly, and increase the long term real yield from negative to positive values. As a result, the model would need to rely less on a large inflation risk premium in order to match the nominal yield curve, thus lowering the variation of $M^T_t$ in the nominal pricing kernel. However, if all the other parameters are maintained at their previous values, the model would then imply too much volatility of the wealth-consumption ratio and an equity risk premium that is much too low. Second, we shut down the heteroscedasticity in consumption growth by calibrating $\sigma_{xw}$ and $\sigma_{gw}$ to very low values. We keep all the other parameters at their previous values. In this case, the real and nominal conditional variance ratios are respectively 1.20 and 0.63 (see Table 6). They are closer to 1, but equity and bond risk premia are constant.

On the nominal side, we first check the robustness of our results to a slightly different calibration of the inflation dynamics. First, we vary each inflation parameter independently in either direction. We report in Table 7 the mean maximum risk premium ($MRP$), the mean bond risk premium $BRP_f$ (including the Jensen term) and the mean variance ratio $\omega$ for different values of the inflation parameters. We simulate the model for a low and a high value of each parameter (25 percent above and below the benchmark value reported in Table 1). The only exception is the parameter $\alpha_{\pi}$, which we cannot increase by 25 percent without running into stationarity issues. The high value is a 10 percent increase for that parameter. We find that $\omega$ only changes noticeably with $\alpha_x$ and $\alpha_{\pi}$.

To further investigate the sensitivity to these two parameters, Figure 1 in the appendix plots $\omega_t$ (left axis) and the five-year nominal bond risk premium (right axis) against $\alpha_x$ (horizontal axis). As we vary $\alpha_x$ away from its benchmark value of -0.35, we simultaneously vary $\alpha_{\pi}$ to match the observed persistence of quarterly inflation. We also choose $\mu_{\pi}$ and $\sigma_{\pi}$ to keep the mean and volatility of inflation at their benchmark values. The
figure shows that $\omega_t$ is essentially unchanged over a wide range of values for $\alpha_x$ and never comes close to the desired value of one.

Next, we consider a calibration that matches the observed mean, variance, and persistence of inflation, the 5-1-year yield spread, and the persistence of the 5-year nominal bond risk premium. This calibration delivers a nominal variance ratio $\omega_t$ that is much too high.

Finally, we ask whether we can find inflation parameters that deliver a nominal variance ratio of 1. We find that we can, while matching the mean inflation, the slope of the nominal term structure, and the persistence of the nominal BRP, but inflation ends up being 2.5 times too volatile and not persistent enough.

![Figure 1. Variance Ratio and Nominal Bond Risk Premium: Sensitivity Analysis](image)

The figure plots the conditional variance ratio $\omega_t$ (against the left axis) and the five-year nominal bond risk premium (against the right axis) for different values of the parameter $\alpha_x$ (on the horizontal axis). As we vary $\alpha_x$ away from its benchmark value of -0.35, we simultaneously vary $\alpha_\pi$ to match the observed persistence of quarterly inflation. We also choose $\mu_\pi$ and $\sigma_\pi$ to keep the mean and volatility of inflation at their benchmark values.

### IX. Empirical Variance Ratios

Alvarez and Jermann (2005) show that – assuming that the process $X_t$ satisfies the same regularity conditions as above and that $X_{t+1}/X_t$ is strictly stationary and $\lim_{k \to \infty} \frac{1}{k} \text{Var}(E_{t+k}[X_t]) = 0$ – then

$$\text{Var} \left( \frac{X_{t+1}^F}{X_t^F} \right) = \lim_{k \to \infty} \frac{1}{k} \text{Var} \left( \frac{X_{t+k}}{X_t} \right),$$

Note that the entropy measure used by Alvarez and Jermann (2005) collapses to the half-variance since all variables are conditionally normal. This result implies that long-horizon variance ratios are informative about the variance of the martingale component. We now turn to the empirical variance ratios of the two components of the SDF, e.g. consumption growth and the wealth consumption ratio.

If changes in log consumption or changes in the log wealth-consumption ratio are i.i.d., then the variance of long-horizon changes in each variable should grow with the horizon. We compute variance ratios at horizon $h$ as $VR(h) = \text{Var}(\sum_{j=0}^{h} \Delta x_{t+j})/[h \text{Var}(\Delta x_t)]$, for $x = c$ and $x = wc$. We simulate the model at monthly frequency. Table [1] in the appendix reports the model parameters. We start from the parameter values in Bansal and Shaliastovich (2007).
Figure 2 reports these variance ratios for consumption growth, the change in the wealth-consumption ratio, and inflation. The left panel corresponds to actual data; the right panel uses simulated series. Let us first focus on actual data. The variance ratio of the wealth-consumption ratio clearly decreases with the horizon. It is below 0.6 within five years. Consumption growth exhibits a very different pattern: its variance ratio first increases for horizons up to 5 years; it then decreases, but even after 15 years, the variance ratio is still above one. As a result, there is strong evidence of persistence and mean-reversion in the wealth-consumption ratio, but not in consumption growth.

Let us now turn to simulated data. The variance ratios of the wealth-consumption ratio are in line with the data. They decrease linearly with the horizon, from 1 to approximately 0.5 at the 30-year horizon. In the data, the variance ratio decreases from 1 to 0.6. Consumption growth, however, exhibits a very different pattern. At long horizons, it displays more persistence in the model than in the data. The bottom panel shows that the inflation persistence is similar in model and data, with a slight divergence maybe at longer horizons.

The variance ratio of $\Delta x_t$ is equal to $VR(h) = \frac{\text{Var}\left(\sum_{j=0}^{h} \Delta x_{t-j}\right)}{h \text{Var}(\Delta x_t)}$. The left panel corresponds to actual data. The right panel corresponds to simulated data. Data are quarterly. Actual data come from Lustig et al. (2009). The sample is 1952:II-2008:IV.
REFERENCES


<table>
<thead>
<tr>
<th>Parameter</th>
<th>BS(2007)</th>
</tr>
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<tr>
<td><strong>Preference Parameters:</strong></td>
<td></td>
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<tr>
<td>Subjective discount factor</td>
<td>$\delta$</td>
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<tr>
<td>Intertemporal elasticity of substitution</td>
<td>$\psi$</td>
</tr>
<tr>
<td>Risk aversion coefficient</td>
<td>$\gamma$</td>
</tr>
</tbody>
</table>

| **Consumption Growth Parameters:**              |           |
| Mean of consumption growth                     | $\mu_g$   | 0.0016    |
| Long-run risk persistence                      | $\rho$    | 0.991     |
| News volatility level                          | $\sigma_g$| 0.004     |
| News volatility persistence                     | $\nu_g$   | 0.85      |
| News volatility of volatility                  | $\sigma_{gw}$ | 1.15e−6 |
| Long run-risk volatility level                 | $\sigma_x$| 0.004$\sigma_g$ |
| Long run-risk volatility persistence            | $\nu_x$   | 0.996     |
| Long run-risk volatility of volatility         | $\sigma_{xw}$ | 0.06$^2\sigma_{gw}$ |

| **Dividend Growth Parameters:**                 |           |
| Mean of dividend growth                         | $\mu_d$   | 0.0015    |
| Dividend leverage                               | $\phi_x$  | 1.5       |
| Dividend loading on news volatility             | $\phi_{gs}$ | 0       |
| Dividend loading on long-run risk volatility    | $\phi_{xx}$ | 0       |
| Volatility loading of dividend growth           | $\varphi_d$ | 6.0    |
| Correlation of consumption and dividend news    | $\tau_{gd}$ | 0.1    |

| **Inflation Parameters:**                       |           |
| Mean of inflation rate                          | $\mu_\pi$ | 0.0032    |
| Inflation leverage on news                      | $\varphi_{\pi g}$ | 0    |
| Inflation leverage on long-run news             | $\varphi_{\pi x}$ | −2.0 |
| Inflation shock volatility                      | $\sigma_\pi$ | 0.0035 |
| Expected inflation AR coefficient              | $\alpha_\pi$ | 0.83    |
| Expected inflation loading on long-run risk     | $\alpha_x$  | −0.35    |
| Expected inflation leverage on news             | $\varphi_{\pi g}$ | 0    |
| Expected inflation leverage on long-run news    | $\varphi_{\pi x}$ | −1.0  |
| Expected inflation shock volatility             | $\sigma_z$  | 4.0e−6    |

This table reports the calibrated parameters values for our simulation. We take them from Table IV and Table C.I in Bansal and Shaliastovich (2007).
This table reports the model loadings on a constant and the state variables. We consider the log wealth-consumption ratio \((wc)\), the log price-dividend ratio \((pd)\), the equity risk premium \((ERP)\), the real and nominal bond risk premia \((BRP)\) at the \(n\)-year horizon.

<table>
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<tr>
<th>State Variables</th>
<th>(x)</th>
<th>(\sigma^2_{\omega t} - \sigma^2_\omega)</th>
<th>(\sigma^2_{\omega t} - \sigma^2_\omega)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(wc)</td>
<td>(\mu_{wc})</td>
<td>(W_x = \frac{1}{\kappa_1 - \rho})</td>
<td>(W_{gs} = \frac{\theta (\frac{1}{\kappa_1} - 1)^2}{2(\kappa_1 - \rho)})</td>
</tr>
<tr>
<td></td>
<td>6.4</td>
<td>31</td>
<td>-7.7</td>
</tr>
<tr>
<td>(pd)</td>
<td>(\mu_{pd})</td>
<td>(D_x = \frac{\phi w - \frac{1}{\kappa_1 - \rho}}{\kappa_1 - \rho})</td>
<td>(D_{gs} = \frac{1}{\kappa_1 - \rho}\left[\frac{\gamma}{\kappa_1 - \rho} + \frac{\gamma}{\kappa_1 - \rho} + \frac{\gamma}{\kappa_1 - \rho}\right] \frac{2}{(\kappa_1 - \rho)^2})</td>
</tr>
<tr>
<td></td>
<td>5.6</td>
<td>66</td>
<td>1.3 \times 10^2</td>
</tr>
<tr>
<td>(ERP)</td>
<td>0.003</td>
<td>0</td>
<td>4.8</td>
</tr>
<tr>
<td>(BRP) (Real)</td>
<td>(-0.0014)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(BRP) (Nominal)</td>
<td>(-0.0014)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

This table reports the model loadings on a constant and the state variables. We consider the log wealth-consumption ratio \((wc)\), the log price-dividend ratio \((pd)\), the equity risk premium \((ERP)\), the real and nominal bond risk premia \((BRP)\) at the \(n\)-year horizon.
### Table 3—Real and Nominal Yield Curves

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>30</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Nominal Bonds - Data</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean Yields</td>
<td>5.33</td>
<td>5.52</td>
<td>5.69</td>
<td>5.80</td>
<td>5.89</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Std</td>
<td>2.81</td>
<td>2.77</td>
<td>2.70</td>
<td>2.69</td>
<td>2.65</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Nominal Bonds - Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean Yields</td>
<td>5.19</td>
<td>5.46</td>
<td>5.75</td>
<td>6.06</td>
<td>6.38</td>
<td>12.82</td>
<td>20.02</td>
</tr>
<tr>
<td>Std</td>
<td>2.92</td>
<td>2.79</td>
<td>2.65</td>
<td>2.53</td>
<td>2.43</td>
<td>1.60</td>
<td>0.36</td>
</tr>
<tr>
<td>Mean BRP</td>
<td>0.33</td>
<td>0.93</td>
<td>1.59</td>
<td>2.27</td>
<td>2.97</td>
<td>16.81</td>
<td>24.43</td>
</tr>
<tr>
<td>Std</td>
<td>0.07</td>
<td>0.18</td>
<td>0.28</td>
<td>0.38</td>
<td>0.46</td>
<td>1.13</td>
<td>1.18</td>
</tr>
<tr>
<td><strong>Real Bonds - Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean Yields</td>
<td>1.26</td>
<td>1.05</td>
<td>0.83</td>
<td>0.61</td>
<td>0.39</td>
<td>−4.71</td>
<td>−13.63</td>
</tr>
<tr>
<td>Std</td>
<td>1.39</td>
<td>1.35</td>
<td>1.32</td>
<td>1.30</td>
<td>1.29</td>
<td>1.10</td>
<td>0.25</td>
</tr>
<tr>
<td>Mean BRP</td>
<td>−0.39</td>
<td>−0.83</td>
<td>−1.28</td>
<td>−1.73</td>
<td>−2.19</td>
<td>−11.14</td>
<td>−16.21</td>
</tr>
<tr>
<td>Std</td>
<td>0.05</td>
<td>0.10</td>
<td>0.15</td>
<td>0.19</td>
<td>0.23</td>
<td>0.52</td>
<td>0.55</td>
</tr>
</tbody>
</table>

The top panel reports the mean and standard deviation of nominal bond yields in the Fama-Bliss data. The data are for 1952 until 2008, and only bond yields of maturities one through five years are available. The maturity is in years. The yields and returns are annualized and reported in percentage points. The middle panel does the same for nominal bond yields for a 60,000 month simulation of the LRR model. It also reports the mean and standard deviation of the nominal bond risk premia. The bottom panel reports the same model-implied moments for real bonds.

### Table 4—Conditional Variance Ratio

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std</th>
<th>AR(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Nominal SDF</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SDF$^{g}$</td>
<td>0.99</td>
<td>0.23</td>
<td>−0.01</td>
</tr>
<tr>
<td>SDF$^{g,p}$</td>
<td>1.00</td>
<td>0.14</td>
<td>−0.01</td>
</tr>
<tr>
<td>SDF$^{g,t}$</td>
<td>0.98</td>
<td>0.10</td>
<td>−0.01</td>
</tr>
<tr>
<td>$\omega_t^g$</td>
<td>0.37</td>
<td>0.06</td>
<td>0.98</td>
</tr>
<tr>
<td>Max RP</td>
<td>30.62</td>
<td>2.52</td>
<td>0.99</td>
</tr>
<tr>
<td>BRP($\infty$)</td>
<td>18.72</td>
<td>1.04</td>
<td>0.99</td>
</tr>
<tr>
<td><strong>Real SDF</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SDF$^{p}$</td>
<td>1.00</td>
<td>0.30</td>
<td>−0.01</td>
</tr>
<tr>
<td>SDF$^{e}$</td>
<td>1.02</td>
<td>0.07</td>
<td>−0.01</td>
</tr>
<tr>
<td>$\omega_t$</td>
<td>1.65</td>
<td>0.11</td>
<td>0.98</td>
</tr>
<tr>
<td>Max RP</td>
<td>30.69</td>
<td>2.54</td>
<td>0.99</td>
</tr>
<tr>
<td>BRP($\infty$)</td>
<td>−19.05</td>
<td>0.58</td>
<td>0.99</td>
</tr>
</tbody>
</table>

This table reports the mean, standard deviation and autocorrelation of the stochastic discount factor ($SDF$), its martingale ($SDF^{p}$) and dominant pricing ($SDF^{e}$) components, the conditional variance ratio $\omega$, the maximum risk premium without Jensen adjustment (Max RP) and the risk premium of an infinite maturity bond without Jensen adjustment (BRP($\infty$)). The table reports the autocorrelation of each monthly variable in logs. The top (bottom) panel focuses on the nominal (real) stochastic discount factor. The numbers are computed from a 60,000 month simulation.
### Table 5—Inflation: Model vs Data

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th></th>
<th></th>
<th>Model</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
<td>AR(1)</td>
<td>Mean</td>
<td>Std</td>
<td>AR(1)</td>
</tr>
<tr>
<td>$\pi_t$</td>
<td>0.85</td>
<td>0.62</td>
<td>0.86</td>
<td>0.88</td>
<td>1.25</td>
<td>0.76</td>
</tr>
</tbody>
</table>

This table reports the mean, standard deviation and autocorrelation of the quarterly inflation rate. The left panel corresponds to actual data, from Lustig, Van Nieuwerburgh and Verdelhan (2009). The right panel corresponds to simulated data, from the model. The mean and standard deviation are in percentage.

### Table 6—Conditional Variance Ratio: No Heteroscedasticity

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std</th>
<th>AR(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nominal SDF</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$SDF^g$</td>
<td>1.00</td>
<td>0.12</td>
<td>-0.01</td>
</tr>
<tr>
<td>$SDF^g,P$</td>
<td>1.00</td>
<td>0.13</td>
<td>-0.01</td>
</tr>
<tr>
<td>$SDF^g,T$</td>
<td>1.00</td>
<td>0.01</td>
<td>-0.01</td>
</tr>
<tr>
<td>$\omega_t^g$</td>
<td>1.20</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Max RP</td>
<td>8.74</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$BRP(\infty)$</td>
<td>-1.74</td>
<td>0.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Real SDF</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$SDF$</td>
<td>0.99</td>
<td>0.12</td>
<td>-0.01</td>
</tr>
<tr>
<td>$SDF^p$</td>
<td>1.00</td>
<td>0.10</td>
<td>-0.01</td>
</tr>
<tr>
<td>$SDF^t$</td>
<td>0.99</td>
<td>0.03</td>
<td>-0.01</td>
</tr>
<tr>
<td>$\omega_t$</td>
<td>0.63</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Max RP</td>
<td>8.70</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$BRP(\infty)$</td>
<td>3.18</td>
<td>0.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

This table reports the mean, standard deviation and autocorrelation of the stochastic discount factor ($SDF$), its martingale ($SDF^P$) and dominant pricing ($SDF^T$) components, the conditional variance ratio $\omega$, the maximum risk premium without Jensen adjustment (Max RP) and the risk premium of an infinite maturity bond without Jensen adjustment ($BRP(\infty)$). The table reports the autocorrelation of each monthly variable in logs. The top (bottom) panel focuses on the nominal (real) stochastic discount factor. The numbers are computed from a 60,000 month simulation.
TABLE 7—SENSITIVITY TO INFLATION SPECIFICATION

<table>
<thead>
<tr>
<th></th>
<th>$\mu_p$</th>
<th>$\varphi_{\pi g}$</th>
<th>$\varphi_{\pi x}$</th>
<th>$\sigma_{\pi}$</th>
<th>$\alpha_{\pi}$</th>
<th>$\alpha_x$</th>
<th>$\varphi_{xg}$</th>
<th>$\varphi_{xx}$</th>
<th>$\sigma_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean RP</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>30.62</td>
<td>30.62</td>
<td>18.72</td>
<td>18.72</td>
<td>0.37</td>
<td>0.37</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High</td>
<td>30.62</td>
<td>30.62</td>
<td>18.72</td>
<td>18.72</td>
<td>0.37</td>
<td>0.37</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This table reports the mean maximum risk premium (Max RP), the mean bond risk premium $BRP(\infty)$ (including the Jensen term) and the mean variance ratio $\omega$. We vary one parameter at a time, and simulate the model for a low and a high value of each parameter (25 percent above and below the benchmark value reported in the first column of Table 1). The only exception is the parameter $\alpha_{\pi}$, which we cannot increase by 25 percent without running into stationarity issues. The high value is a 10 percent increase for that parameter.