Department of Aeronautics and Astronautics
Field Oral Exam
Aerospace Computational Engineering
January 2011

Each student will attempt **two questions:** the core question and the question corresponding to his/her chosen option

Make sure to manage your time carefully and devote adequate time to each question.

1. **Core question**

Consider the following scalar conservation law in 1D,

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0, \qquad \text{for} \quad -\infty < x < \infty, \quad t > 0$$

where u is the conserved quantity and $F(u)$ is the flux function.

The initial condition is

$$u(x,0) = \begin{cases} u_L = 2, & x < 0, \\ u_R = 1, & x > 0. \end{cases}$$

(a) Determine the solution for $t > 0$ when $F(u) = \frac{1}{2}u^2$.

(b) Determine the solution for $t > 0$ when $F(u) = -\frac{1}{2}u^2$.

(c) On a uniform discretization in which the interval size is h and the time step is Δt, we consider schemes of the form

$$u_i^{n+1} - u_i^n = -\frac{\Delta t}{h}\left[F_{i+1/2}^n - F_{i-1/2}^n\right]$$

with the particular choice for the numerical flux function

$$F_{i+1/2}^n = \frac{1}{2}\left(F(u_{i+1}^n) + F(u_i^n)\right).$$

Comment on the suitability of this scheme to solve the above equation. In particular, determine whether the scheme is conservative and stable.

(d) Suggest a different numerical flux function that would lead to a stable and conservative scheme.

2. **Linear Algebra Option**

Finite element discretization of an integral-differential equation yields the following matrix equations

$$D\,x + w\,w^{\mathrm{T}}x = b \qquad (1)$$

where D is an $n \times n$ diagonal matrix with positive diagonal entries; w, x and b are column vectors of length n.

(a) Discuss the options for solving Equation (1) for x. About how many floating point multiplications (and divisions) are involved in each solution method?

(b) Now consider

$$D\,x + W\,W^{\mathrm{T}}x = b. \qquad (2)$$

The difference is that W is now an $n \times n$ matrix. How would you solve this system? What is the rough number of floating point multiplications involved?

(c) Describe the concept of numerical stability of an algorithm for solving a linear system. In the context of Problem (1), describe a situation when numerical stability is a serious concern. Among the methods you proposed to solve the previous problems, which one is more numerically stable? Explain why.

3. **Optimization Option.**

Consider the nonlinear optimization problem

$$
\text{(P1)} \qquad \begin{aligned} \min_{\mathbf{x}_0} \quad & \sum_{i=0}^{n} f(\mathbf{x}_i) \\ \text{s.t.} \quad & \mathbf{x}_1 = g(\mathbf{x}_0) \ , \\ & \mathbf{x}_2 = g(\mathbf{x}_1) \ , \\ & \ldots \\ & \mathbf{x}_n = g(\mathbf{x}_{n-1}) \ , \end{aligned}
$$

where each $\mathbf{x}_i, i = 0, 1, \ldots, n$ is a length m vector. f is a function with m-dimensional input and scalar output; g is a function with m-dimensional input and m-dimensional output.

(a) Write out the Karusch-Kuhn-Tucker (KKT) optimality conditions for this system.

(b) Which of the following case(s) are convex optimization problems?

1. $f(\mathbf{x}) = \|\mathbf{x}\|_1, \ g(\mathbf{x}) = B\mathbf{x} + c$; where B is an $m \times m$ matrix; c is a length m vector.
2. $f(\mathbf{x}) = e^{-\frac{1}{2}\mathbf{x}^\top\mathbf{x}}, \ g(\mathbf{x}) = a\mathbf{x} + c$; where a is a scalar in $(0, 1)$, c is a length m vector.
3. $f(\mathbf{x}) = (\mathbf{x} - A\mathbf{x})^\top (\mathbf{x} - A\mathbf{x}) \ , \ g(\mathbf{x}) = -a\mathbf{x}$; where A is an $m \times m$ matrix; a is a scalar in $(0, 1)$.
4. $f(\mathbf{x}) = \dfrac{1}{2}\mathbf{x}^\top\mathbf{x} - c^\top\mathbf{x}, \ g(\mathbf{x}) = \dfrac{B\mathbf{x}}{\|\mathbf{x}\|_2}$; where B is a $m \times m$ positive-definite matrix, c is a length m vector.

(c) Consider the special case $n = 1$,

$$
g(\mathbf{x}) = B\,\mathbf{x} + c \ ; \quad f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{x}
$$

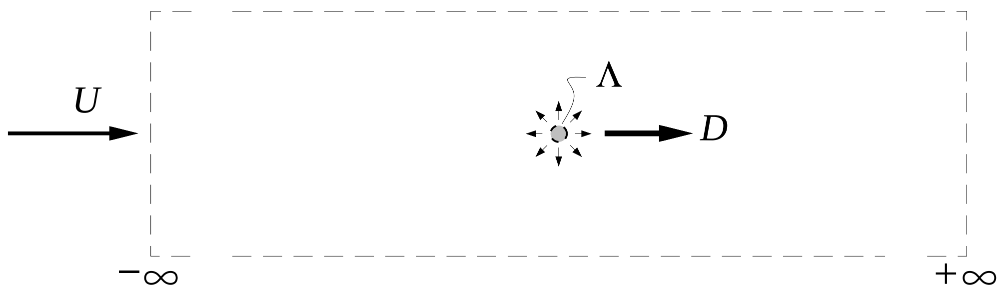
where B is an $m \times m$ matrix. Derive the dual problem of (P1).

4. **Fluid Mechanics Option**

A 2D cylindrical object is placed in an incompressible flow of freestream velocity U and density ρ. You are to examine the drag force D on the object using control volume analysis. As shown in the figure, the recommended control volume has some finite height of your choice, and extends from $-\infty$ to $+\infty$ along the freestream.

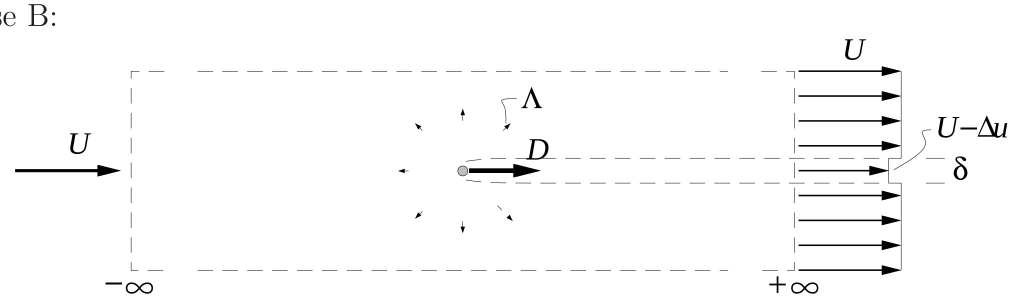
(a) In this case A, the body is issuing fluid from its porous surface, at an overall volume flow rate of Λ. The fluid issues from the body in such a way that the flow everywhere is irrotational. Determine the drag force D for this case.

Case A:



(b) In case B, the cylinder has no net fluid issuing from it. The fluid is viscous at a sufficiently high Reynolds number, so that there is now a thin viscous wake. Here it will be idealized as a small constant wake velocity defect of magnitude Δu and height δ, as shown in the Figure.

Case B:



Apply mass conservation to the control volume for case B, and explain why the flow outside of the wake, which is irrotational, must still have the same source-like form as the previous blowing case. Determine the apparent source strength Λ.

(c) Determine the drag force D for case B, and compare to the previous case A.