1. **ACE Core**

Consider the unsteady partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{1}$$

with a given forcing function $f(x,t)$ and initial condition $u(x,t=0) = u_0(x)$, The spatial domain is **periodic** in $[0, 2\pi]$. It is discretized by a **non-uniform grid** $0 = x_0 < x_1 < \ldots < x_{K-1} < 2\pi$.

In this question, we focus on spatial discretization. A forward Euler time integration scheme is assumed throughout the problem.

(a) Derive a consistent finite difference scheme for the PDE, using forward Euler for time discretization. Please only use a stencil of three grid points x_{i-1}, x_i and x_{i+1}. Show the order of accuracy of your spatial approximation.

(b) Consider the L_∞-norm of the numerical solution when $f = 0$. What **sufficient** conditions can you impose to ensure L_∞-stability?

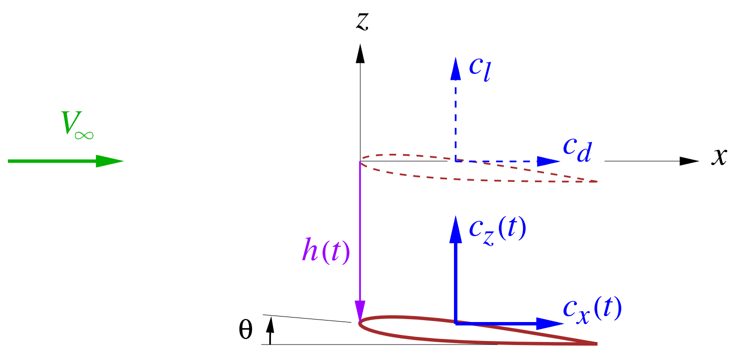
(c) Consider the numerical solution at a fixed time t_f. Does the error in the numerical solution go to zero as N (the number of time steps) goes to infinity, Δt goes to zero (keeping $N\Delta t = t_f$), and K (the number of spatial grid points) goes to infinity? Why or why not? If not, what sufficient conditions can you impose to ensure that the numerical solution goes to zero?

2. **Fluid Mechanics**

A 2D airfoil of chord c is placed in an incompressible flow which has freestream speed V_∞ along the horizontal x axis. The airfoil maintains a small angle θ above the x axis, while undergoing an unsteady sinusoidal vertical displacement

$$h(t) \;=\; h_1 \sin\omega t$$

where by convention h is positive down, and h_1 is the half-amplitude of the displacement.



(a) What are the physical boundary condition on the surface of the airfoil for i) viscous flow, and ii) inviscid flow? Assume that the normal vector $\hat{\mathbf{n}} = n_x\hat{\mathbf{x}} + n_z\hat{\mathbf{z}}$ is known everywhere on the airfoil surface.

(b) If $h_1 = 0$, then this situation reduces to the usual steady 2D airfoil case (shown dashed), with angle of attack $\alpha = \theta$. For $\alpha \ll 1$, the drag and lift coefficients (always defined parallel and normal to the airfoil's onset flow), are then given by:

$$\frac{D'}{\frac{1}{2}\rho V_\infty^2 c} \equiv c_d \;=\; c_{d_0} \tag{8}$$

$$\frac{L'}{\frac{1}{2}\rho V_\infty^2 c} \equiv c_\ell \;=\; 2\pi\alpha \tag{9}$$

For the more general case $h_1 \neq 0$, what are the requirements on the problem parameters $\rho, V_\infty, c, h_1, \omega$ such that the flow is quasi-steady, and equations (8) and (9) are valid?

(c) Assuming the requirements above hold, determine the instantaneous force coefficients

$$c_x(t) \;\equiv\; \frac{F_x(t)}{\frac{1}{2}\rho V_\infty^2 c}$$

$$c_z(t) \;\equiv\; \frac{F_z(t)}{\frac{1}{2}\rho V_\infty^2 c}$$

along the horizontal and vertical axes x, z. Also determine the corresponding time-averaged force coefficients $\bar{c}_x$ and $\bar{c}_z$ defined over any integer number of oscillation periods. You may use small-angle approximations.

(d) Consider the case where the condition(s) for the quasi-steady assumption in part (a) do not quite hold. In which direction will the resulting $c_d(t)$ and $c_\ell(t)$ forces change from their quasi-steady values you obtained for part (b)?

3. **Numerical Linear Algebra**

An iterative method is used to solve the following linear system,

$$Ax = b \tag{10}$$

where $A \in R^{N \times N}$, and $x,\ b \in R^N$. You may assume that A is a full rank matrix. The iterative method is of the following form,

$$Mx^{n+1} = Nx^n + b \tag{11}$$

where $A = M - N$. You may assume that M is also a full rank matrix.

(a) Define the error $e^n \equiv x - x^n$ (i.e. x being the exact solution). Determine G which is the matrix relating e^n and e^{n+1}, specifically,

$$e^{n+1} = Ge^n \tag{12}$$

Please express your answer for G in terms of M and A (not N).

(b) Where in the complex plane must the eigenvalues of $M^{-1}A$ be such that $||e^n|| \to 0$ as $n \to \infty$ for any initial guess x^0?

(c) The residual at iteration n is defined as $r^n \equiv b - Ax^n$. H is the matrix relating r^n and r^{n+1}, specifically,

$$r^{n+1} = Hr^n \tag{20}$$

How is H related to G? How are the eigenvalues of H related to the eigenvalues of G?

(d) A difficult aspect of iterative methods is to determine when to stop iterating based on a desired error level. For example, suppose that the desired error tolerance is $\tau_e > 0$ such that $||e^n||/||x|| \le \tau_e$ and $||\cdot||$ is a vector norm. Since the exact solution x and therefore the error e^n are unknown, this criteria cannot generally be enforced. Alternatively, suppose that a residual criteria is used as a stopping criteria. Specifically, the iterative method is terminated when $||r^n||/||b|| \le \tau_r$ for some positive τ_r. Determine lower and upper bounds on $||e^n||/||x||$ in terms of $||r^n||/||b||$ and $\kappa(A) = ||A||\,||A^{-1}||$ (i.e. the condition number of A where the matrix norms are those induced by the vector norms).

4. **Optimization Methods**

Consider the least squares problem

$$\underset{\theta}{\text{minimize}} ||X\theta - y||_2^2, \tag{30}$$

corresponding to the over-constrained linear relationship

$$\begin{bmatrix} \\ X \\ \\ \end{bmatrix} \begin{bmatrix} \theta \end{bmatrix} \approx \begin{bmatrix} \\ y \\ \\ \end{bmatrix}.$$

The analytical solution θ^* to this problem satisfies

$$(X^\mathsf{T} X)\theta^* = X^\mathsf{T} y. \tag{31}$$

For intuition, you may assume the problem size $X \in \mathbb{R}^{10{,}000 \times 100}$, $y \in \mathbb{R}^{10{,}000}$. In many settings, it is desireable that θ not be too large. One way to achieve this is *ridge regression*, which penalizes $||\theta||_2^2$:

$$\underset{\theta}{\text{minimize}} ||X\theta - y||_2^2 + \lambda ||\theta||_2^2. \tag{32}$$

For now, assume $\lambda > 0$ is a given parameter.

(a) Determine the analytical optimum θ^* to (32).

(b) Assume you have available a reliable and efficient black box for solving (30). The inputs to this program are X and y, and the output is θ^*. You want to use this program (unmodified) to solve (32), given a value for λ. How will you do it? Give expressions for X_{aug} and y_{aug}, the inputs you will provide to the solver.

In typical ridge regression solutions, many entries in θ^* are close to, but not equal to, 0. Another regularizer, the *Lasso*, has sparsity-inducing effects: it drives some of the entries in θ^* all the way to 0. The Lasso formulation is

$$\begin{aligned} \underset{\theta}{\text{minimize}} \quad & ||X\theta - y||_2^2 \\ \text{subject to} \quad & ||\theta||_1 \leq t, \end{aligned} \tag{33}$$

for some given $t > 0$, where $||\theta||_1$ is the sum of the absolute values of the entries in θ.

(c) Formulate Lasso as a quadratic program (QP), an optimization problem with the form

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & x^\mathsf{T} Q x + c^\mathsf{T} x \\ \text{subject to} \quad & Ax \leq b. \end{aligned}$$

Give expressions for x, Q, c, A, and b.

(d) Sketch, on a plot, how the optimal cost $||X\theta - y||_2^2$ of (33) changes as t is varied. Be sure to identify any salient features and values.