Department of Aeronautics and Astronautics
Field Oral Exam
Aerospace Computational Engineering
January 2009

Each student will attempt **two questions:** the core question and the one of the two questions corresponding to his/her electives. The choice of which of the two electives to answer is up to the student.

1. **Core Content Question (required for every student)**

   We consider the problem:

$$
\begin{array}{rcllll}
-\nabla^2 u & = & f & \quad\text{in} & \Omega \\
\nabla u \cdot \boldsymbol{n} & = & g & \quad\text{on} & \partial\Omega
\end{array}
\tag{1}
$$

   together with the additional condition:

$$
\int_\Omega u \, dV = 0 \tag{2}
$$

   Here, Ω is a spatial domain, $\partial\Omega$ its boundary and $\boldsymbol{n}$ the outward unit normal to the boundary, u is the scalar unknown function and f and g are given functions.

   If we use the standard finite element Galerkin method to discretize the above problem, we obtain a system of equations of the form

$$
K\boldsymbol{u}_h = \boldsymbol{F} \qquad \text{and} \qquad \boldsymbol{m}^T \boldsymbol{u}_h = 0 \tag{3}
$$

   where K is the stiffness matrix, $\boldsymbol{u}_h$ is the vector of nodal unknowns and $\boldsymbol{F}$ and $\boldsymbol{m}$ are vectors of dimension equal to the number of degrees of freedom.

   Questions:

   (a) Is condition (2) necessary to uniquely define u? Are there any conditions on f and g for this system problem to have a solution?

   (b) Write a variational formulation for (1)

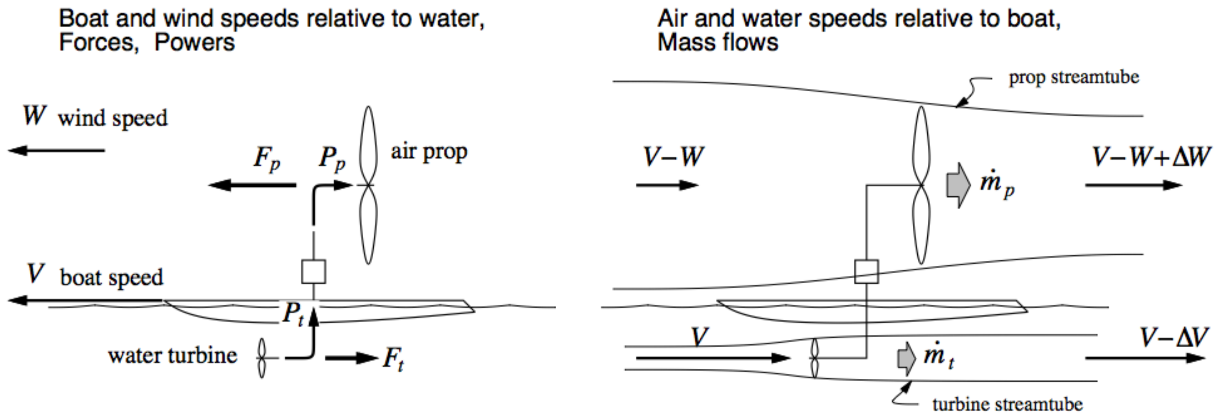
   (c) Is the matrix K singular? If so, what is its null space?

   (d) Is the matrix K symmetric and/or positive semi-definite? Show it.

   (e) How would you go about solving the system of equations (3) for the unknown vector $\boldsymbol{u}_h$?

   (f) Assuming that the exact solution is sufficiently smooth and that your approximation space is made up of piecewise linear polynomials, how would you expect the finite element solution u_h to converge to the exact solution as the mesh element size h tends to zero?

2. **Fluid Mechanics Option**



The left figure above shows a boat equipped with a water turbine and an air propeller, linked by a mechanical transmission. The boat is moving through the water with speed V, in the same direction as a slower wind moving relative to the water with speed W. As shown in the right figure, the water turbine therefore sees a water velocity of V, while the air prop sees an air velocity of $V - W$, both opposite the boat motion. The prop and turbine streamtube mass flows are $\dot{m}_p$ and $\dot{m}_t$. The downstream velocity changes in the prop and turbine streamtubes are ΔW and ΔV.

1. Determine the turbine and propeller forces F_t and F_p based on the velocities and mass flows. First define control volumes to which your analysis applies.

2. Determine the mechanical powers P_t and P_p based on the velocities and mass flows. If possible, use your previous force results to eliminate the mass flows.

3. The vehicle is in steady motion. What must be its overall drag D?

3. **Solid Mechanics Option**

The elastic deformations of a rubber balloon may be described within the framework of hyperelasticity as follows. Assume a spherical reference configuration for the balloon with internal radius R and wall thickness H. For this configuration the internal (Cauchy) pressure is equal to zero ($p = 0$). Assume also that $H << R$ and that the deformation is uniform and radially symmetric. The rubber material may be assumed incompressible with an elastic response adequately described by the neo-Hookean strain energy density (per unit undeformed volume):

$$W = \frac{\mu}{2}\left[tr(\mathbf{C}) - 3\right]$$

The objectives of this question are to:

- derive an expression for the pressure as a function of the radius change $\lambda = \frac{r}{R}$, where r is the deformed radius of the balloon, and to
- find the value of $\lambda = \lambda_{\max}$ for which the pressure reaches a maximum $p = p_{\max}$, and the value of $p_{\max}$

The following may serve as a guide for the derivation:

(a) Express the membranal stretches λ_1, λ_2 as a function of the deformed radius r by considering the extension of a diametral circumference of the sphere in two orthogonal directions

(b) Use the incompressibility condition to derive the through-thickness stretch λ_3 in terms of λ

(c) Express the strain energy density in terms of the membranal stretch λ

(d) Write the expression of the total potential energy of the balloon: $\Phi = \Pi_{int} + \Pi_{ext}$

(e) Find the expression for the pressure sought by applying the principle of minimum potential energy

(f) Show that in the (physical) range $0 < \lambda < \infty$, $p(\lambda)$ has local maximum. Find the value of λ at which the maximum is attained and the value of the maximum itself. Comment on its physical significance.

(g) Sketch the plot of $p(\lambda)$ for $0 \leq \lambda \leq 5\lambda_{max}$

4. **Numerical Linear Algebra Option**

(a) Let A and X be m-by-m matrices with X nonsingular and consider $B = X^{-1}AX$.

i. Show that A and B have the same characteristic polynomial, and therefore the same eigenvalues.

ii. If $\boldsymbol{x}$ is an eigenvector of A with eigenvalue λ, what is the corresponding eigenvector of B?

(b) Construct a projection matrix P ($P^2 = P$) that projects any vector (x, y) *onto* the space $\{(2, 1)\}$, *along* the space $\{(-1, 1)\}$. Show that this matrix satisfies $P^2 = P$.

(c) In many algorithms, we use Householder reflectors

$$F = I - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}$$

i. What are the eigenvalues of F? *(Hint: Consider possible eigenvectors parallel and perpendicular to $\boldsymbol{v}$)*

ii. Give a geometric interpretation of F. Show that if you choose $\boldsymbol{v} = ||\boldsymbol{x}||\boldsymbol{e}_1 - \boldsymbol{x}$ then $F\boldsymbol{x} = ||\boldsymbol{x}||\boldsymbol{e}_1$. (*Here,* $\boldsymbol{e}_1 = (1, 0, 0, \ldots, 0)^T$)

5. **Optimization Methods Option**

(a) You are given access to software that can solve an LP in standard form

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}$$

Indicate how would you write in standard form the following problem:

$$\begin{array}{llrl} \text{minimize} & 6z_1 + 4z_2 + z_3 & & \\ \text{subject to} & 4z_1 - 6z_3 & \geq & 6 \\ & |z_3| & \leq & 2 \\ & z_1, z_2 & \geq & 0 \end{array}$$

How would you have handled the constraint on the absolute value if it had been replaced by $|z_3| \geq 2$?

(b) Consider the following problem with a quadratic objective and linear constraints

$$\begin{array}{ll} \text{minimize} & \boldsymbol{x}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \end{array}$$

where the vector $\boldsymbol{x}$ and $\boldsymbol{b}$ are vectors of dimension n and m, respectively, and A is a matrix of size m-by-n.

i. Introduce Lagrange multipliers to deal with the constraints and write the equations that must be satisfied by $\boldsymbol{x}$ and the lagrange multipliers at the optimum point.

ii. Derive the dual problem.