Department of Aeronautics and Astronautics
Field Oral Exam
Aerospace Computational Engineering
January 2010

Each student will attempt **two questions:** the core question and the one of the questions of his/her choice from the two chosen options

Make sure to manage your time carefully and devote adequate time to each question.

1. **Core Content Question (required for every student)**

We consider a linear convection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1}$$

for $a > 0$ constant, $0 < x < 1$ and $t > 0$. The initial condition is of the form $u(x, 0) = u_0(x)$ and the boundary condition at $x = 0$ is of the form $u(0, t) = g(t)$ for some given smooth functions of $u_0(x)$ and $g(t)$.

Questions:

(a) Give the general form of the solution $u(x, t)$ in terms of $u_0(x)$ and $g(t)$. Explain what will happen if $u_0(0) \neq g(0)$.

(b) In order to solve equation (1), we consider the Lax-Wendroff finite difference method on a uniform grid consisting of $N + 1$ points. i.e. $i = 0, 1, \dots N$ and $h = 1/N$. One way to derive the Lax-Wendroff scheme is to first perform the time discretization using a Taylor series expansion and write $u(x, t^{n+1}) \approx u(x, t^n) + \Delta t(\partial u/\partial t)_{t=t^n} + (\Delta t^2/2)(\partial^2 u/\partial t^2)_{t=t^n}$ and then use the original equation (1) to replace time derivatives by spatial derivatives to obtain

$$u(x, t^{n+1}) \approx u(x, t^n) - a\Delta t \left(\frac{\partial u}{\partial x}\right)_{t=t^n} + \frac{a^2\Delta t^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_{t=t^n} \tag{2}$$

By approximating the first and second order spatial derivatives using using a three point central difference approximation one obtains the Lax-Wendroff scheme. For interior points $i = 1, 2, \dots N - 1$, we get:

$$u_i^{n+1} = u_i^n - \frac{C}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{C^2}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

where $C = ah/\Delta t$. Provide a suitable expression for calculating u_0^{n+1} and u_N^{n+1}. Is the resulting scheme conservative? What is the expression for the numerical flux? How do you expect this scheme to perform for both smooth and discontinuous solutions?

(c) Now, if instead of using finite differences to approximate the spatial derivatives in (2), we use a Galerkin finite element method with *continuous linear shape functions* we obtain the so-called Taylor-Galerkin scheme. You are asked to derive the Taylor-Galerkin scheme on a uniform mesh of N elements in a step-by-step manner by i) writing the variational statement identifying the space of shape and weighting functions, ii) deriving the discrete equation obtained by considering an interior shape function for a typical node i, iii) doing the same for the boundary node N.

What are the differences and similarities between the Lax-Wendroff and Taylor-Galerkin schemes?

2. **Numerical Linear Algebra Option.**

(a) A is a $m \times n$ matrix ($m > n$). Explain how you would solve the minimization problem

$$\min \left\| A\,x - b \right\|_2$$

using the normal equation and QR decomposition.

(b) Which method is more numerically stable? why?

(c) Construct an orthonormal matrix of the form $P = I + uv^{\mathrm{T}}$, where both u and v are column vectors of length n, so that $P\,c$ is some multiple of $e_1 = (1, 0, \ldots, 0)^{\mathrm{T}}$.

(d) Consider the minimization problem with a single linear equality constraint

$$\min \left\| A\,x - b \right\|_2 \quad \text{s.t.} \quad c^{\mathrm{T}} x = d \tag{3}$$

where c is a column vector of length n, and d is a scalar. Construct an algorithm, using the orthonormal matrix constructed in (c), to solve (3).

(e) Construct a method to solve the least squares problem with multiple linear equality constraints:

$$\min \left\| A\,x - b \right\|_2 \quad \text{s.t.} \quad C^{\mathrm{T}} x = d$$

where C is an $n \times l$ matrix ($l << n$), d is a column vector of length l.

3. **Optimization Option.**

Consider the nonlinear optimization problem

$$
\text{(P1)} \qquad \begin{array}{rrcl} \min\limits_{\mathbf{x}} & f(\mathbf{x}) & & \\ \text{s.t.} & g_1(\mathbf{x}) & \leq & 0 \\ & g_2(\mathbf{x}) & \leq & 0, \end{array}
$$

where f is the scalar objective function, $\mathbf{x} \in \mathbb{R}^n$ is the vector of n design variables, and there are two nonlinear constraint functions g_1 and g_2.

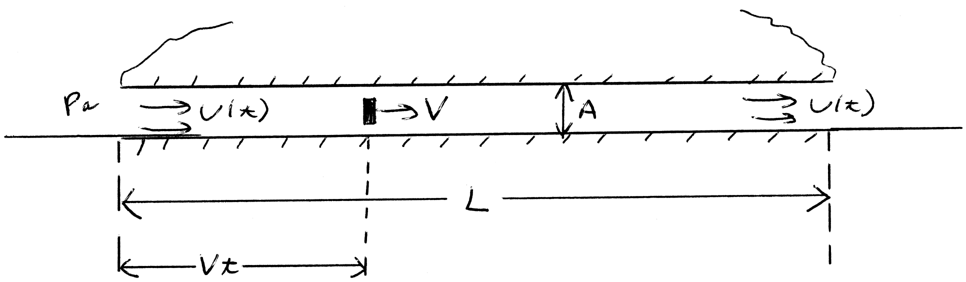
(a) Write out the Karusch-Kuhn-Tucker (KKT) optimality conditions for this system.

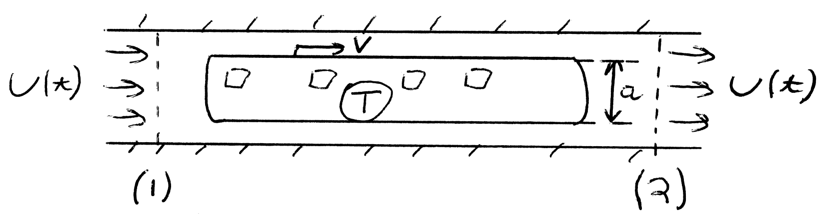
(b) Linearize the optimization problem (P1) about the point $\mathbf{x} = \mathbf{x}_0$. Write your linearized optimization problem in standard primal form.

(c) What is the dual of the linearized optimization problem? Explain how this dual problem relates to the KKT conditions of P1.

4. **Fluid Mechanics Option.**
An MBTA subway train enters a tunnel at $t = 0$ and moves through it with constant speed V. The tunnel length L is many kilometers. The length of the train is much shorter, such that the train appears to occupy only a point on the scale of the tunnel as a whole. (See upper figure.) The cross-sectional area of the tunnel is A and the cross-sectional area of the train is a. The pressure at the entrance of the tunnel is equal to the pressure at the exit.

We would like to predict the wind speed $U(t)$ in the unoccupied parts of the tunnel. You may assume that the air is incompressible and that shear forces on the tunnel wall are negligible. The wind speed U then depends only on time and is equal to zero at $t = 0$.





(a) Obtain an expression for the pressure $p_1(t)$ at a point (1) that is just behind the train on the scale of the tunnel, but far enough behind the train for the velocity to have regained a uniform distribution. (See lower figure.)

(b) Obtain an expression for the pressure $p_2(t)$ at a similar point (2) just ahead of the train.

(c) Now suppose that the drag coefficient C_D of the train in the tunnel is specified. Obtain a differential equation for $U(t)$ in terms of the given quantities. You do not need to solve this equation.

(d) What if the shear stress on the tunnel walls were not negligible, but rather specified by a nonzero skin friction coefficient $c_f = \tau_w / \frac{1}{2}\rho U^2$. How will your answers to parts (a) or (b) change?

5. **Solid Mechanics Option**

According to material frame-indifference requirements, the most general strain energy density (per unit undeformed volume) for a hyperelastic material is:

$$W = W(\mathbf{C})$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy-Green deformation tensor, $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$ is the deformation gradients tensor; and $\mathbf{x}$ and $\mathbf{X}$ define, respectively, the deformed and reference material point positions.

(a) how does this general dependence specialize for an isotropic hyperelastic material? Provide as much information as possible about the new argument(s) of the strain energy density, including their mathematical expressions in terms of the original tensor(s)

(b) how does the expression of the strain energy further specialize for the case of an incompressible material? Why? (provide as much detail as possible)

(c) Consider the new-Hookean strain energy density function for incompressible materials

$$W = k\big[tr(\mathbf{C}) - 3\big]$$

where k is a material constant. Determine the expression for the Cauchy stress in terms of the stretch for the case of uniaxial deformation. Determine the relation between k and the Young's modulus of the material.