Department of Aeronautics and Astronautics
Field Oral Exam
Aerospace Computational Engineering
January 2012

Each student will attempt **two questions:** the core question and the one question corresponding to his/her chosen option

Make sure to manage your time carefully and devote adequate time to each question.

1. **Core Content Question (required for every student)**

We consider the integration of an ordinary differential equation (ODE) of the form,

$$u_t = f(u), \qquad 0 < t < T \tag{1}$$

with an initial condition $u(0) = u_0$.

We will apply a discontinuous Galerkin (DG) finite element discretization to this ODE. That is, we will divide the time range into N elements, each of size $\Delta t = T/N$. Denote the n-th element as the element spanning from $t^n < t < t^{n+1}$ where $t^n = n\Delta t$. The DG solution space will be polynomials of order p on each element (i.e. $p = 0$ is a constant function, $p = 1$ is a linear function, etc). The basis will be chosen so that it is non-zero in only a single element and zero for all other elements. In particular, note that the DG method uses a solution space that is discontinuous from element-to-element.

The DG discretization can be found by weighting the ODE by a function, $v^n(t)$ in the solution space which is non-zero in element n and integrating (by parts) over element n to produce the following,

$$\begin{aligned} \int_{t^n}^{t^{n+1}} v^n u_t \, dt &= \int_{t^n}^{t^{n+1}} v^n f(u) \, dt, \\ v^n \hat{u}\big|_{t^n}^{t^{n+1}} - \int_{t^n}^{t^{n+1}} v_t^n u \, dt &= \int_{t^n}^{t^{n+1}} v^n f(u) \, dt, \end{aligned}$$

where $\hat{u}(t^n)$ is the solution at $t = t^n$. However, since the solution is discontinuous between elements, $\hat{u}(t^n)$ could be taken (in principle) from either element $n-1$ or n. Since the problem is integrated forward in time, then a reasonable choice is $\hat{u}(t^n) = u^{n-1}(t^n)$ where $u^{n-1}(t^n)$ is the solution in element $n-1$ evaluated at $t = t^n$. Thus, the final form of the DG time discretization is,

$$v^n(t^{n+1})u^n(t^{n+1}) - v^n(t^n)u^{n-1}(t^n) - \int_{t^n}^{t^{n+1}} v_t^n u^n \, dt = \int_{t^n}^{t^{n+1}} v^n f(u^n) \, dt. \tag{2}$$

Questions:

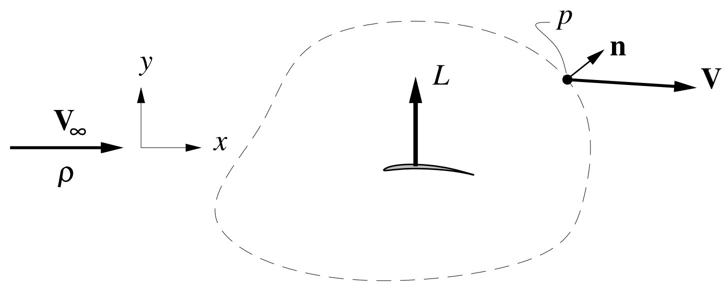
(a) Consider the $p = 0$ DG discretization. What common integration method is this $p = 0$ DG discretization equivalent to?

(b) Next, consider the $p = 1$ DG discretization. Use a nodal basis in which the two unknowns from element n correspond to the value of the DG solution in element n at times t^n and t^{n+1}. Sketch the two basis functions corresponding to the unknowns in element n.

(c) Still considering the $p = 1$ DG discretization, describe how you would implement this discretization including calculation of any integrals and solving the resulting discrete equations.

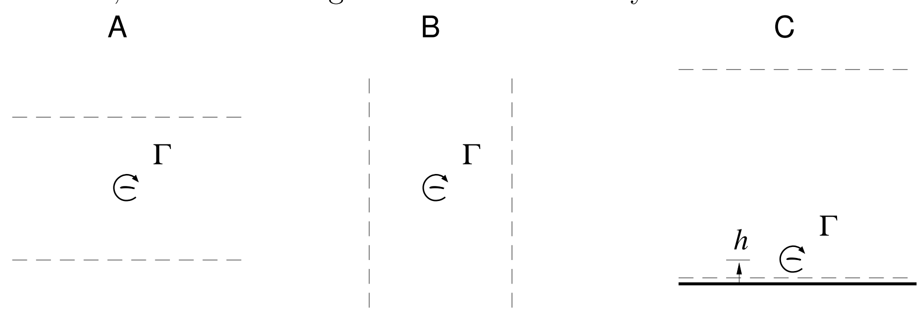
(d) Consider the linear problem for which $f = \lambda u$ and λ is a negative real number. Let $T \to \infty$ while keeping Δt fixed such that $N \to \infty$. Is the $p = 0$ discretization asymptotically stable (i.e. does the solution remain bounded for $N \to \infty$) for $\lambda\Delta t \to -\infty$? Is the $p = 1$ discretization asymptotically stable for $\lambda\Delta t \to -\infty$?

2. **Fluid Mechanics Option** A 2D airfoil is operating in an incompressible inviscid flow with freestream velocity $\mathbf{V}_\infty$ and density ρ.



(a) Write an integral for the airfoil lift L in terms of flow quantities around some control volume or contour surrounding the airfoil, and the unit vector $\mathbf{n}$ normal to the contour, as shown above. It may be more convenient to work with the cartesian perturbation velocities u, v, where $\mathbf{V} = (V_\infty + u)\mathbf{i} + v\mathbf{j}$.

(b) Now assume that the contour is very far from the airfoil relative to its chord c. The airfoil also has some known circulation Γ. Evaluate the lift integral from the first question above for the rectangular contour A, where the left and right boundaries are at $x = \pm\infty$, thus confirming the Kutta-Joukowsky theorem for this case.



(c) Now evaluate the lift integral for contour B, where the top and bottom boundaries are at $y = \pm\infty$.

(d) In case C, the airfoil is at some height $h \gg c$ above an infinite ground plane. Determine the contributions of the lift integral on the top and bottom boundaries of the contour C.

3. **Numerical Linear Algebra Option**

Consider a large, $n \times n$ matrix A, with distinct, real eigenvalues. For a much smaller $m << n$, the following iterative algorithm aims to compute A's right eigenvectors corresponding to the largest m eigenvalues (in magnitude):

1. Start at iteration number $k = 0$ with m distinct vectors $y_1^0, \ldots, y_m^0$.
2. For $i = 1, \ldots, m$, compute $y_i^{k+\frac{1}{2}} = A\, y_i^k$.
3. Perform the following orthonormalization

$$\begin{aligned} y_1^{k+1} &= b_{11}\, y_1^{k+\frac{1}{2}}, \\ y_2^{k+1} &= b_{12}\, y_1^{k+\frac{1}{2}} + b_{22}\, y_2^{k+\frac{1}{2}}, \\ y_3^{k+1} &= b_{13}\, y_1^{k+\frac{1}{2}} + b_{23}\, y_2^{k+\frac{1}{2}} + b_{33}\, y_3^{k+\frac{1}{2}}, \\ &\cdots \end{aligned}$$

   so that the new vectors $y_1^{k+1}, \ldots, y_m^{k+1}$ are orthonormal.
4. Increment iteration number k, go to step 2. Iterate until convergence.

Denote the converged vectors as $y_1, \ldots, y_m$.

(a) What are the possible algorithms for performing the orthonormalization in Step 3? Discuss the number floating point operations and stability of each algorithm.

(b) Consider a matrix D with distinct, real eigenvalues. Suppose that the left eigenvectors (as row vectors) of D form an upper triangular matrix, show that the right eigenvectors (as column vectors) of D also form an upper triangular matrix; in addition, show that D itself is an upper triangular matrix.

(c) Continue from part (a): the obtained vectors $y_1, \ldots, y_m$ are not actually the desired eigenvectors. Instead, they are linear combinations of them. Denote the desired eigenvector corresponding to the m largest eigenvalues as $x_1, \ldots, x_m$. The vectors $y_1, \ldots, y_m$ obtained from the algorithm are in fact

$$\begin{aligned} y_1 &= c_{11}\, x_1, \\ y_2 &= c_{12}\, x_1 + c_{22}\, x_2, \\ y_3 &= c_{13}\, x_1 + c_{23}\, x_2 + c_{33}\, x_3, \\ &\cdots \end{aligned}$$

for an array of unknown coefficients c_{ij}. (Take this as a given fact. You are not required to prove this.) Please figure out how to recover the desired eigenvectors $x_1, \ldots, x_m$ from the vectors $y_1, \ldots, y_m$. Discuss the number of floating point operations of the algorithm you come up with. (Hint: Multiply A with the y_i's, use the relation between y_i and x_i, and use the result of Question (b).)

4. **Optimization Methods Option**

Consider a set of n points $\{(x_1, y_1), ..., (x_n, y_n)\}$ in the plane. We want to find a point (x, y) such that the sum of the Euclidean distances from this point to all the other points is minimized.

(a) Give a nonlinear optimization formulation of this problem.

(b) Is the objective function differentiable? Is this a convex optimization problem?

(c) Derive the conditions for stationarity of the objective function.

(d) Are the stationarity conditions necessary for optimality? Are they sufficient for optimality? State clearly your assumptions.

(e) Provide a semidefinite programming formulation of this problem.