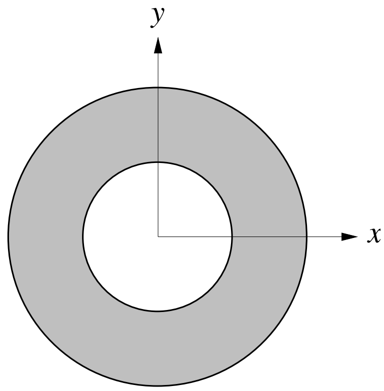
1. **ACE Core**
Consider the following 2D Poisson problem on the annular domain shown

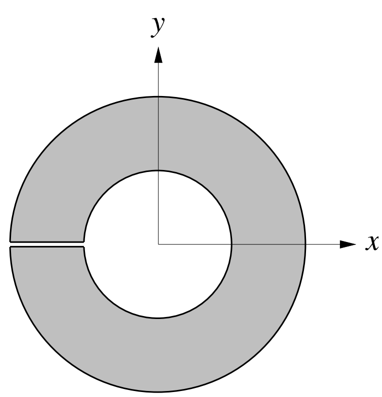


$$\nabla^2\phi = 0$$
$$\nabla\phi\cdot\mathbf{n} = \sigma(x,y) \qquad \text{(on outer boundary)}$$
$$\phi = 0 \qquad \text{(on inner boundary)}$$

The outer boundary flux $\sigma(x,y)$ is specified.

(a) What is the weak form of this problem? How would you solve this problem using a Galerkin finite element method?

(b) How would you solve this problem using a finite difference method?



In parts (c)-(e), we change the boundary condition on the inner boundary from Dirichlet to Neumann

$$\nabla\phi\cdot\mathbf{n} = 0 \qquad \text{(on inner boundary)}$$

In addition, the solution can have a constant jump across the branch cut (as shown in the second figure,) but derivatives of every order must be continuous across it.

(c) Under what condition does this problem have a solution? When it has a solution, does this problem have a unique solution? If not, how can it be modified into a well-posed problem?

Hint: $\arctan(y/x)$ can be an undetermined mode in the solution.

(d) The problem is solved numerically with a Galerkin finite element method. Describe how your modifications in the previous part would be implemented.

2. Numerical Linear Algebra

The eigenvalue and right-eigenvector pair $\lambda, \mathbf{v}$ of a matrix $\mathbf{A}$ satisfy

$$\mathbf{A}\mathbf{v} \;=\; \lambda\mathbf{v} \tag{1}$$

and are typically computed by any number of eigenvalue methods.

Some numerical applications in addition require the calculation of the left eigenvector $\mathbf{w}$ which satisfies

$$\mathbf{A}^T\mathbf{w} \;=\; \lambda\mathbf{w}$$

where λ is the same as in (1).

(a) Consider the iterative inverse power iteration method for calculation of $\mathbf{w}$ corresponding to a known λ,

$$\tilde{\mathbf{w}} \;=\; \left[\mathbf{A}^T - \lambda\mathbf{I}\right]^{-1}\mathbf{w}^n \tag{2}$$

$$\mathbf{w}^{n+1} \;=\; \frac{\tilde{\mathbf{w}}}{|\tilde{\mathbf{w}}|} \tag{3}$$

where n is the iteration index, and $\mathbf{w}^1$ is typically chosen as a random vector. What due you see as a potential problem with this method?

For parts (b) and (c), consider the following example matrix:

$$\mathbf{A} \;=\; \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

(b) Determine the exact eigenvalues $\lambda_{1,2}$, right-eigenvectors $\mathbf{v}_{1,2}$, and left-eigenvectors $\mathbf{w}_{1,2}$.

(c) Since all general eigenvalue methods are iterative, any eigenvalue λ extracted from a large $\mathbf{A}$ will in general have some uncertainty ϵ related to machine precision.

Determine the stability of the inverse power iteration method for a perturbed eigenvalue $\lambda = \lambda_1 + \epsilon$. Use the following initial guess.

$$\mathbf{w}_1 \;=\; \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\}$$

3. **Optimization**

You are to find the unconstrained minimum of a function $f(x_1, x_2, \ldots x_N)$.

(a) Derive the Newton method for finding the minimum location $x^* = (x_1^*, x_2^*, \ldots x_N^*)$. Specifically, given some current solution estimate x^k, determine the next estimate x^{k+1}, where k is the iteration index.

(b) When will an iteration step of this method fail? What feature of $f(x_1, x_2, \ldots)$ does this situation correspond to?

(c) If the method does converge to some x^*, is this point guaranteed to be a local minimum? If not, devise a test for whether it is a local minimum.

(d) Can the method guarantee that the resulting x^* is a global minimum?

(e) Write down the steepest-descent method to update x^k to x^{k+1}. In what special situation does this give nearly the same convergence behavior as the Newton method?