Aerospace Computational Engineering (ACE) Field Exam 2017

During the oral portion of the field exam, students will respond to questions on the ACE core content and their chosen ACE elective topic.

1. **ACE Core**

Consider the Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + f(x) = 0 \tag{1}$$

with a given forcing function $f(x)$. The spatial domain is **periodic** in $[0, 2\pi]$. It is discretized by a **non-uniform grid** $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 2\pi$ using finite element method. In the finite element method, the solution is represented as a linear combination of N basis functions,

$$u(x) = \sum_{i=1}^{N} a_i \phi_i(x)$$

(a) Denote X_h as the space of all continuous, piece-wise linear functions in this periodic domain. What is the dimension N of the space?

(b) Derive a Galerkin finite element approximation to the differential equation. Derive a matrix equation for solving the finite element approximation. In this part of the question, we do not assume any specific form of $\phi_i(x)$.

(c) We would like to choose a set of basis functions $\phi_i(x)$ such that $a_i = u(x_i)$. Write down the mathematical form of a set of $\phi_i(x)$ that satisfy this property. With these ϕ_i, derive entries of the matrix you obtained in the last question in terms of the grid points $x_i, i = 0, \ldots, n-1$. Show that the matrix has a zero eigenvalue corresponding to a constant vector $[1, \ldots, 1]$. Does the system have a solution? If so, is it unique?

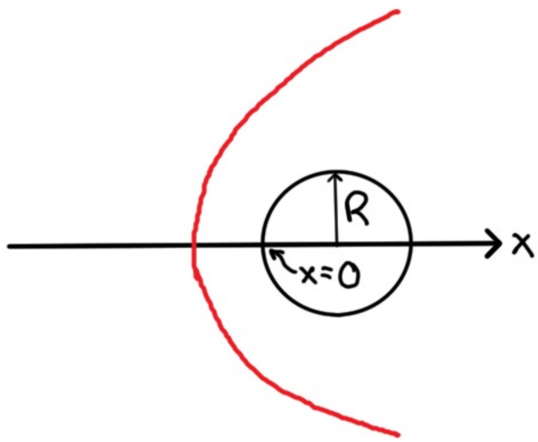
(d) We modify the original equation (1) by adding an unknown constant λ and one more equation constraining the integral of u to be 0. The modified system becomes:

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + f(x) = \lambda, \\ &\int_0^{2\pi} u \, dx = 0 \end{aligned} \tag{2}$$

How can you modify the finite element method to solve this modified problem? (Hint: the easiest approach is likely using the same basis functions as in the previous question)

2. **ACE Fluid Mechanics**

Consider the supersonic two-dimensional flow of air over a cylinder. The freestream velocity is $V_\infty = 1200\,\text{m/s}$ and the speed of sound $a_\infty = 300\,\text{m/s}$. Assume that the air may be modeled as a perfect gas with specific heat ratio of $\gamma = 1.4$. A bow shock will occur upstream of the cylinder, sketched qualitatively in the figure. In particular, we will consider the behavior of the flow along the x-axis upstream of the cylinder.



(a) First, assume that the flow is inviscid. In the vicinity of the shock, the flow along the x-axis can be well-approximated as one-dimensional. Recall that the Mach number across a normal shock satisfies the following jump condition:

$$M_2^2 = \frac{1 + \frac{\gamma-1}{2} M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}}$$

where M_1 is the Mach number just upstream of the shock and M_2 is the Mach number just downstream of the shock. For this cylinder flow, determine M_1 and M_2 for the shock at the x-axis.

(b) Still assuming inviscid flow, sketch $M(x)$, $T_0(x)/T_\infty$ (the total temperature), and $T(x)/T_\infty$ (the temperature) along the x-axis from the freestream condition through the shock and to the surface of the cylinder (i.e. for $x < 0$). Note: T_∞ is the freestream temperature.

(c) Next, consider the effect of viscosity on the flow where ν_∞ is the freestream kinematic viscosity. The shock will now have a finite thickness δ_s. How would you expect δ_s to depend on the freestream conditions? Assume that ν_∞ is small, but finite, compared to $V_\infty R$. Specifically, perform an order of magnitude analysis on the momentum equation to determine the expected scaling of δ_s/R with respect to $V_\infty R/\nu_\infty$.

(d) The viscous flow will also have a boundary layer on the cylinder surface. Let δ be the thickness of the boundary layer at some point along the cylinder surface. How would you expect δ/R to scale with respect to $V_\infty R/\nu_\infty$? You may assume the flow is laminar. Specifically, show how this scaling can be determined from an order of magnitude analysis of the appropriate governing equations.

3. **ACE Mechanics of Solid Materials**

Inflation of a thin rubber cylinder

The elastic deformations of a thin rubber cylinder may be described within the framework of hyperelasticity as follows. Assume a reference configuration for the cylinder with radius R and wall thickness T and deformed values r and t, respectively. In the reference configuration the internal (Cauchy) pressure is equal to zero ($p = 0$). Assume also that $T << R$ and that the state of deformation is uniform and corresponds to plane strain conditions (stretching or contraction not allowed in the axial direction of the cylinder). The rubber material may be assumed incompressible with an elastic response adequately described in principal Cauchy stresses by the expression:

$$\sigma_i = \mu\left(\lambda_i^2 - q\right)$$

where μ is the shear modulus and q is the reaction pressure required to accommodate the incompressibility constraint. We suggested the following axes convention:

$$\sigma_1 = \sigma_{rr},\ \sigma_2 = \sigma_{\theta\theta},\ \sigma_3 = \sigma_{zz}$$

The objective of this question is to derive an expression for the pressure as a function of the radius change $\frac{r}{R}$.

The following may serve as a guide for the derivation:

(a) From the uniformity of the strain field, use a geometric argument to infer a kinematic assumption for this problem and with this information express the radial λ_1 and circumferential λ_2 principal stretches as a function of the deformed and reference values of the cylinder radius and thickness.

(b) Relate the principal stretches using the incompressibility constraint to show that the deformed thickness of the cylinder t is inversely proportional to the deformed radius r, $t \sim \frac{1}{r}$

(c) From elementary equilibrium considerations in the deformed configuration, find expressions for the principal Cauchy stresses in terms of the deformed thickness and radius.

(d) Combine the equilibrium equation for σ_2 and the kinematic equations to express σ_1 in terms of the principal stretches and undeformed dimensions

(e) Considering the state of stress obtained from equilibrium and using the incompressibility constraint, simplify the constitutive equations and obtain expressions for σ_2 and σ_3 in terms of μ, λ_2 alone

(f) Combine the equilibrium and constitutive results to obtain an expression for the pressure as a function of the hoop principal stretch λ_2

(g) Express the pressure p, the hoop stress σ_2 and the axial stress σ_3 as a function of the deformed radius r and sketch these functions. Based on the shape of these functions, are the equilibrium solutions you obtained unique? Why or why not?

4. **ACE Probability**

(a) Warm-up: if a real-valued random variable Z has cumulative distribution function (CDF) F_Z, what is the distribution of $F_Z(Z)$?

Now for the main question. *Copulas* are a useful tool for modeling dependence among random variables. In general, a copula is the CDF of a random vector $\boldsymbol{U} \coloneqq (U_1, \ldots, U_n)$, where each component has a uniform marginal, $U_i \sim \mathcal{U}(0, 1)$, and the components may be dependent. In this problem, we will restrict our attention to the two-dimensional ($n = 2$) case.

We construct a particular family of copulas as follows. Consider a bivariate Gaussian distribution with mean zero and positive definite covariance matrix $\Sigma \in \mathbb{R}^{2\times 2}$. Let $\Phi_{12} : \mathbb{R}^2 \to [0, 1]$ be the joint CDF of this Gaussian distribution, and let $\Phi_1, \Phi_2 : \mathbb{R} \to [0, 1]$ be the CDFs of its marginals. Then the **Gaussian copula** C_Σ is defined as:

$$\Phi_{12}(x_1, x_2) = C_\Sigma \left(\Phi_1(x_1), \Phi_2(x_2)\right).$$

(b) Verify that C_Σ satisfies the definition of a copula given above.

(c) Now consider two arbitrary (univariate) CDFs on $\mathbb{R}$, F_1 and F_2. Consider the joint CDF defined by these marginals and the Gaussian copula:

$$H(x_1, x_2) = C_\Sigma \left(F_1(x_1), F_2(x_2)\right).$$

Describe a procedure for generating samples from a distribution whose CDF is H.

(d) The *mutual information* $I(X; Y)$ is a measure of dependence between two random variables X and Y. Let X and Y be real-valued random variables with joint probability density f_{XY} and marginal densities f_X and f_Y. Then

$$I(X; Y) \coloneqq \int f_{XY}(x, y) \log \frac{f_{XY}(x, y)}{f_X(x) f_Y(y)} \, dx \, dy.$$

Now suppose that X and Y have the joint distribution $F_{XY}(x, y) = C\left(F_X(x), F_Y(y)\right)$, where C is an arbitrary (not necessarily Gaussian) smooth copula and F_X, F_Y are the marginal distributions of X, Y. Derive an expression linking the mutual information to the copula C.