Aerospace Computational Engineering (ACE) Field Exam 2018

During the oral portion of the field exam, students will respond to questions on the ACE core content and their chosen ACE elective topic.

1. **ACE Core**

Consider the scalar nonlinear hyperbolic conservation law in one spatial dimension,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = g(x, t) \tag{1}$$

with a given source term $g(x, t)$ in the spatial domain $x \in [0, 1]$.

(a) Explain the characteristic curve of the equation by answering the following two questions: What mathematical equation does the characteristic curve satisfy? What is the significance of the characteristic curve?

(b) Under what condition is it appropriate to specify a boundary condition for the hyperbolic conservation law at $x = 0$ but not at $x = 1$? Under what condition is it appropriate to specify a boundary condition at $x = 1$ but not at $x = 0$? Under what condition is it appropriate to specify boundary conditions at both or neither boundaries?

(c) We want to solve a steady state version of this conservation law,

$$\frac{\mathrm{d}f(u)}{\mathrm{d}x} = g(x)$$

using **finite elements**. Assuming that the solution is continuous and differentiable, and that zero Dirichlet boundary conditions are applied for both $x = 0$ and $x = 1$, write down the weak form of the partial differential equation. Use a weak form that does not involve the derivative of the flux f.

(d) Now discretize the domain $x \in [0, 1]$ into N elements with nodes $0 = x_0 < x_1 < \ldots < x_N = 1$. Using a nodal basis of continuous, piecewise linear functions and a Galerkin discretization, write down the discretized equations. Please use a one-point Gauss quadrature to approximate integrals over each element.

(e) What are the similarities between this finite element scheme and a finite volume scheme? What are the practical and conceptual differences?

2. **ACE Optimization**

Professor X bikes from campus to home. The route goes up a hill and then down a hill. The first part of the route has length L_1 and a positive slope of s_1; the second part has length L_2 and a negative slope of s_2. The energy it takes Professor X to bike a unit length of slope s is

$$a\,v^2 + b\,s\,v$$

where $a > 0$ and $b > 0$.

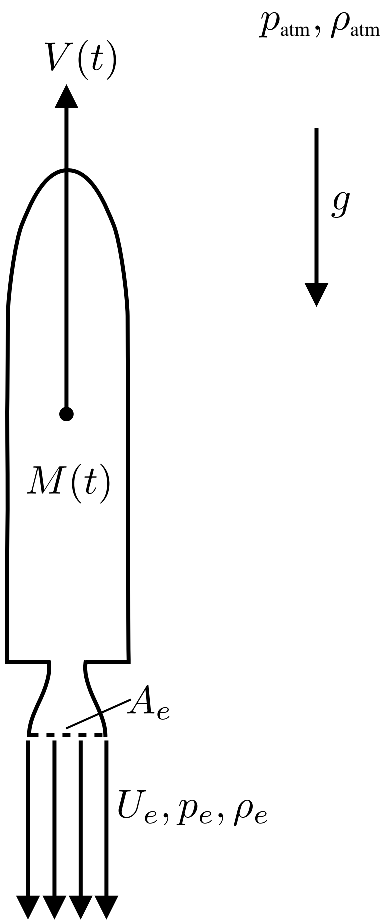
(a) Suppose that Professor X wants to get home for dinner as fast as possible. But he only has $E > 0$ amount of energy left in his body for biking. How fast should Professor X bike over the first and second parts of the route? Formulate the question as an optimization problem, assuming Professor X always bikes at a constant speed in the first part of the route and another constant speed in the second part of the route.

(b) Now suppose that Professor X needs to get home in $T > 0$ amount of time to submit a proposal. He wants to spend as little energy biking as possible. How fast should Professor X bike over the first and second parts of the route? Formulate the question as an optimization problem, still assuming Professor X always bikes at a constant speed in the first part of the route and another constant speed in the second part of the route.

(c) Do Part (a) and Part (b) always have a solution? Are their solutions unique? Why or why not?

(d) When you plot the solution to Problem (a) in the (v_1, v_2) space for different E, as well as the solution to Problem (b) for different T, will the plots lie on the same curve or different curves? Why or why not?

3. **ACE Fluid Mechanics**



Consider the motion of a rocket as shown in the figure where $V(t)$ is the rocket's velocity as measured in a stationary reference frame. During this motion, assume that the flow conditions leaving the rocket nozzle do not vary with time. Thus, the velocity U_e, pressure p_e and density ρ_e are constant. Note that U_e is the velocity of the nozzle exit flow in the rocket's reference frame (i.e. relative to the rocket). The exit plane of the nozzle has area A_e. The gravity g may be assumed constant during this motion (i.e. you do not need to account for variations in the gravity due to altitude).

The total mass of the rocket (i.e. including propellant, structure, etc) is $M(t)$. The initial velocity and mass of the rocket during this motion is V_0 and M_0. Let this initial condition correspond to the time $t = 0$.

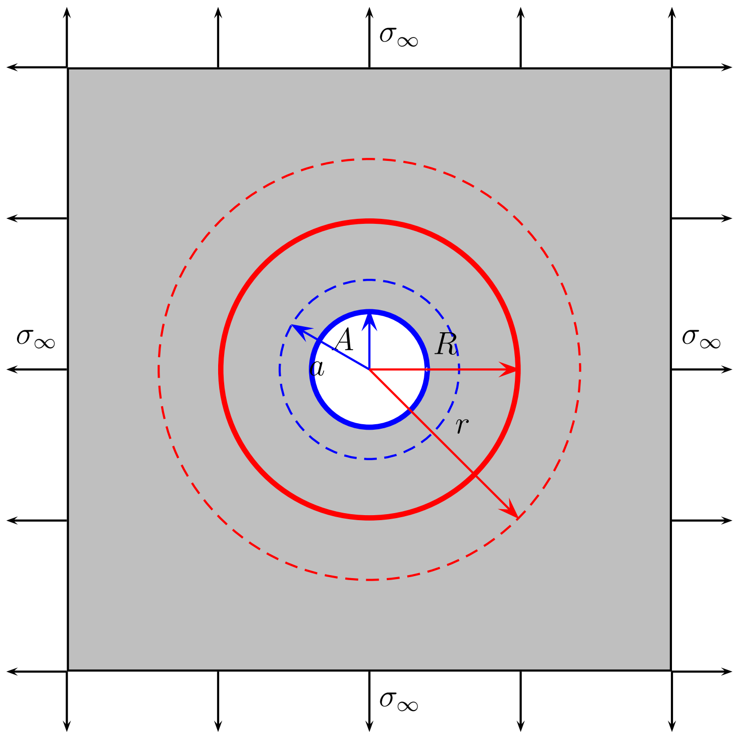
(a) Applying the integral form of conservation of mass, determine $M(t)$ in terms of (at most) M_0, A_e, and the nozzle exit flow conditions (U_e, p_e, ρ_e).

(b) Suppose that the rocket is outside the atmosphere so that $p_{\text{atm}} = 0$ and $\rho_{\text{atm}} = 0$. Applying the integral form of conservation of momentum, determine the acceleration $\mathrm{d}V/\mathrm{d}t$ of the rocket $V(t)$. Note: although $p_{\text{atm}} = 0$, the nozzle exit pressure is finite (i.e. $p_e > 0$) and cannot in general be ignored.

(c) Suppose that the motion was occurring inside the atmosphere (i.e. p_{atm} and ρ_{atm} are no longer zero). How would your result for the acceleration change?

4. **ACE Solid Mechanics**

Cavitation in nonlinear elasticity



We consider the expansion of a spherical cavity of initial radius A in an infinite elastic medium, subject to a remote hydrostatic tenstion σ_∞. The linear elastic solution corresponds to the well-known Lamé problem, which predicts a stress concentration for the hoop stress at the surface of the cavity of 1.5 times the remote stress. The linearity also implies that the cavity radius will increase linearly with the remote stress σ_∞. For ductile solids capable of experiencing large elastic deformations (e.g. elastomers) nonlinear effects may result in instabilities in which the cavity grows unboundedly when the remote stress reaches a certain threshold (cavitation).

The purpose of this problem is to determine the relation between the applied remote stress σ_∞ and the deformed radius of the cavity a and explore the possible appearance of cavitation.

The material may be assumed incompressible with an elastic response adequately described in principal Cauchy stresses σ_i, by the neo-Hookean model:

$$\sigma_i = \mu\left(\lambda_i^2 - q\right) \tag{2}$$

where λ_i are the principal stretches, μ is the shear modulus and q is the reaction pressure required to accommodate the incompressibility constraint.

The elastic deformations of the infinite medium may be described within the framework of hyperelasticity as follows (follow the guide and answer the questions, PLEASE BE

EFFICIENT WITH TIME, IF IN ANY OF THE PARTS BELOW YOU GET STUCK IN THE ALGEBRA AND CANNOT OBTAIN THE RESULT, MAKE SURE YOU EXPLAIN THE PROCESS AND MOVE ON TO THE NEXT PART, YOU CAN USE THE PROVIDED RESULTS EVEN IF YOU WERE NOT ABLE TO OBTAIN THEM):

(a) *Kinematics of deformation:* A material point at a distance R from the cavity center in the unstressed configuration (in solid red in the figure) is displaced to a distance $r(R)$ (in dashed red), which specializes to $a = r(A)$ on the cavity surface (in solid and dashed blue in the figure) .

Using the definition of principal stretch, obtain expressions for the radial λ_r and hoop $\lambda_\theta, \lambda_\phi$ stretches in terms of the deformed $r(R)$ and the undeformed R coordinates

(b) Relate the principal stretches using the incompressibility constraint to show that the function $r(R)$ can be expressed in terms of the undeformed distance R, the undeformed cavity radius A, and the deformed cavity radius a, as:

$$r(R) = (R^3 - A^3 + a^3)^{1/3} \tag{3}$$

Note that this implies that it suffices to know the deformed radius of the cavity to determine the complete deformation field $r(R)$.

(c) To complete the kinematic description, express the radial and hoop stretches in terms of the deformed or spatial coordinate r. (The reason is that we are seeking a spatial formulation of the boundary value problem)

(d) The only relevant equilibrium equation for radial symmetry can be expressed in spherical coordinates as:

$$\frac{d\sigma_{rr}}{dr} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{12}$$

Use the constitutive law, (2), and the expression for the stretches $\lambda_r(r), \lambda_\theta(r)$ to obtain either one of the following expressions for the equilibrium equation for σ_{rr} with either one of the stretches as the independent variable.

$$\frac{d\sigma_{rr}}{d\lambda_\theta} + 2\mu(\lambda_\theta^{-5} + \lambda_\theta^2) = 0 \tag{13}$$

$$\frac{d\sigma_{rr}}{d\lambda_r} - \mu(\lambda_r + \lambda_r^{-1/2}) = 0 \tag{14}$$

(e) Determine expressions for the boundary conditions for the equilibrium equation (13) in terms of $\lambda_\theta(a(A)) \equiv \lambda_a$ and $\lambda_\theta(r(R \to \infty)) \equiv \lambda_\infty$

(f) Integrate the ODE (13) and apply the boundary conditions to show that the deformed radius of the cavity a depends on the remote applied stress σ_∞ according to the expression:

$$\frac{\sigma_\infty}{\mu} = \frac{5}{2} - \left(\frac{a}{A}\right)^{-1} - \frac{1}{2}\left(\frac{a}{A}\right)^{-4} \tag{21}$$

(g) From (21), show that the cavitation limit for this material is $\sigma_\infty = \frac{5}{2}\mu$