Decision Making with Multiple Observations

We observe $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \ldots, \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, which are the realizations of random vectors $\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}, \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix}, \ldots, \begin{bmatrix} X_n \\ Y_n \end{bmatrix}$, respectively. The random vectors are generated by one of the following hypotheses:

$$H_1 \quad : \quad \begin{bmatrix} X_i \\ Y_i \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{E_\text{s}}{2}} \\ \sqrt{\frac{E_\text{s}}{2}} \end{bmatrix} + \begin{bmatrix} Q_i \\ R_i \end{bmatrix}, \qquad \text{for each } i = 1, 2, \ldots, n$$

$$H_0 \quad : \quad \begin{bmatrix} X_i \\ Y_i \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{E_\text{s}}{2}} \\ -\sqrt{\frac{E_\text{s}}{2}} \end{bmatrix} + \begin{bmatrix} Q_i \\ R_i \end{bmatrix}, \qquad \text{for each } i = 1, 2, \ldots, n,$$

where $Q_1, Q_2, \ldots, Q_n, R_1, R_2, \ldots, R_n$ are independent and identically distributed normal $\mathcal{N}\left(0, \frac{N_0}{2}\right)$ random variables, and $N_0 > 0$ and $E_\text{s} > 0$ are known constants.

1. Design a likelihood ratio test with a threshold η based on the observations $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$, i.e.,

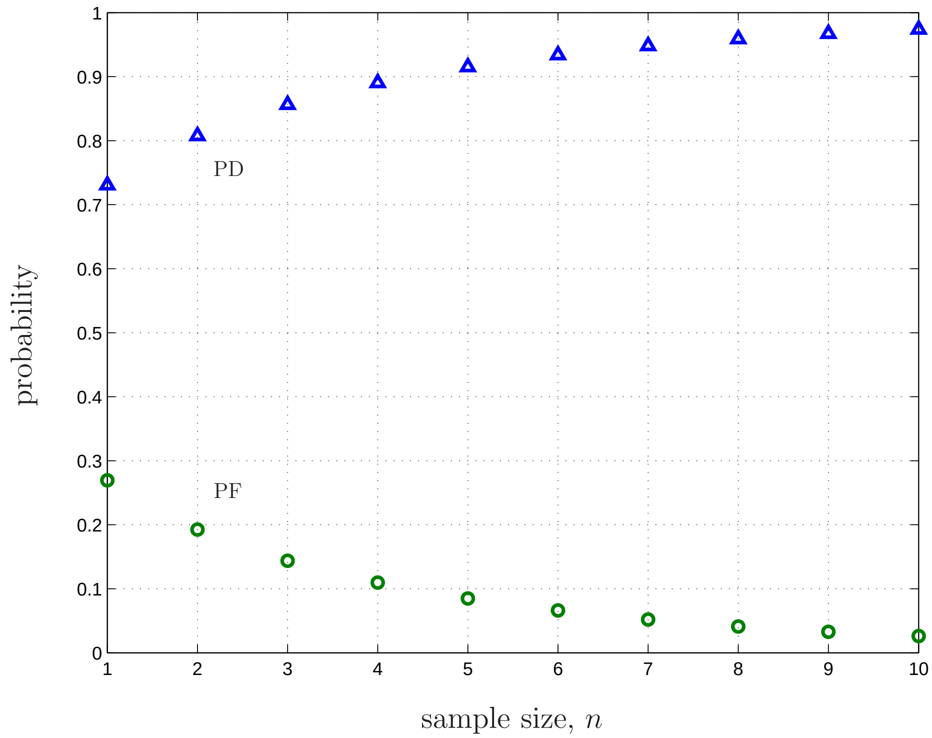
   $$\Lambda(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \quad \underset{H_0}{\overset{H_1}{\gtrless}} \quad \eta.$$

   [*Hint*: $\Lambda(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ is a function of $\sum_{i=1}^{n}(x_i + y_i)$.]

2. Suppose that the two hypotheses are equally likely to occur and suppose that $\eta = 1$. Write the expression of the probability of error in terms of the Q-function or the cumulative distribution function $\Phi(\cdot)$ of the standard normal random variable.

3. The probability of detection (PD) and the probability of false alarm (PF), as functions of the sample size n, are shown in the figure for $\eta = 1$ and for some specific values of the parameters.

   What is the minimum sample size that gives us $P_\text{D} \geq 0.8$ *and* $P_\text{F} \leq 0.1$?



4. Suppose that $n = 1$ and $\eta = 1$. Draw the decision regions $\mathcal{R}_0$ and $\mathcal{R}_1$, where

$$\mathcal{R}_0 = \left\{(x_1, y_1) \in \mathbb{R}^2 \,:\, \widehat{H} = H_0\right\}$$
$$\mathcal{R}_1 = \left\{(x_1, y_1) \in \mathbb{R}^2 \,:\, \widehat{H} = H_1\right\},$$

and $\widehat{H}$ is the decision from the likelihood ratio test in part (1).