Communications and Networks: Communications Option

Probability: Sum of Independent Gaussian Variables

Let $Z = X + Y$ where X and Y are two independent Gaussian r.v.'s with means μ_X and μ_Y, and variances σ_X^2 and σ_Y^2, respectively, i.e., $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$.

1. Determine the distribution of Z.
2. Determine the mean μ_Z and the variance σ_Z^2 of Z.

If you want to use characteristic functions, then the characteristic function of a Gaussian r.v. θ with mean μ_θ and variance σ_θ^2 is $\phi_\theta(w) = \mathbb{E}\left\{e^{jw\theta}\right\} = \exp\left(+jw\mu_\theta - \frac{1}{2}w^2\sigma_\theta^2\right)$.

Communications: Photon Bucket Array Reception

Consider a deep-space optical communications link employing array of N telescopes. The photodetector outputs $X_1, X_2, \cdots, X_N$ can be characterized by independent, identically distributed (i.i.d.) random variables (r.v.'s) with Poisson probability mass function (pmf), i.e.,

$$p_X(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots .$$

where $\lambda > 0$. Recall that the mean and variance of X_1 are, respectively,

$$\mathbb{E}\{X_1\} = \lambda$$
$$\mathbb{V}\{X_1\} = \lambda.$$

In many subsystems, it is desirable to use the estimator $\widehat{\lambda}(\mathbf{X})$ of λ based on the observation $\mathbf{X}$.

1. Find a sufficient statistic for λ based on observation $\mathbf{X} = [X_1\ X_2\ \cdots\ X_N]$.
2. Find a minimal sufficient statistic for λ based on observation $\mathbf{X}$.
3. Derive the maximum likelihood estimator $\widehat{\lambda}_{\mathrm{ML}}(\mathbf{X})$ of λ based on the observation $\mathbf{X}$.

4. Assume that your answer in part (b) is also a complete sufficient statistic for λ. Find a uniformly minimum variance unbiased (UMVU) estimator $\widehat{\lambda}_{\mathrm{UMVU}}(\mathbf{X})$ of λ based on the observation $\mathbf{X}$.

Bonus Questions (attempt only if you completed all the above questions)

5. Prove that your answer in part (b) is also a complete sufficient statistic for λ based on observation $\mathbf{X}$.

6. Derive the Fisher information $I_{\mathbf{X}}(\lambda)$ in $\mathbf{X}$ about λ. Does $\widehat{\lambda}_{\mathrm{UMVU}}(\mathbf{X})$ satisfy the information inequality (or Cramer-Rao lower bound) with an equality?