Probability/Statistics

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Problem

Let $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n$ be independent and identically distributed (i.i.d.) Poisson random variables (RVs) with the following common probability density functions (PDFs)

$$f_{\mathsf{x}}(x;\lambda) = \begin{cases} \frac{\lambda^x}{x!}e^{-\lambda} & \text{if } x = 0,\ 1,\ 2,\ \ldots \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is an unknown parameter. Define $\mathbf{x} := [\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n]^{\mathrm{T}}$.

(a) Find the maximum likelihood estimator (MLE) of λ based on $\mathbf{x}$, i.e., $\hat{\lambda}_{\mathrm{ML}}(\mathbf{x})$. Is $\hat{\lambda}_{\mathrm{ML}}(\mathbf{x})$ an unbiased estimator of λ?

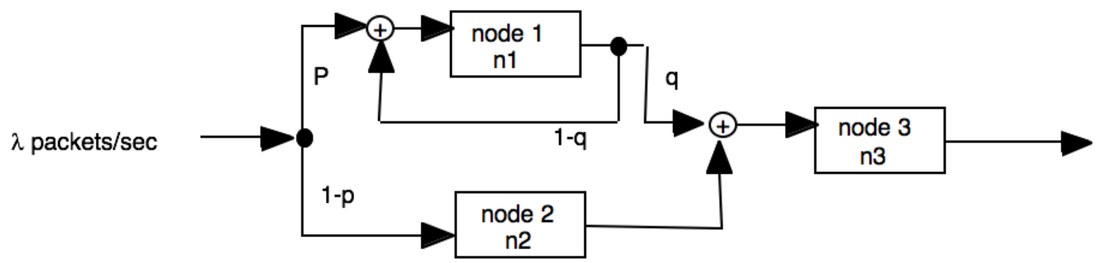
(b) Let $c > 0$ be a known constant. Find the probability of $\{\hat{\lambda}_{\mathrm{ML}}(\mathbf{x}) \geqslant \lambda + c\}$, i.e., $\mathbb{P}\{\hat{\lambda}_{\mathrm{ML}}(\mathbf{x}) \geqslant \lambda + c\}$.

(c) Find the lower bound on the variance of any unbiased estimator of λ based on $\mathbf{x}$ using Information Inequality.

(d) Find the variance of $\hat{\lambda}_{\mathrm{ML}}(\mathbf{x})$. Does it achieve the equality in Information Inequality?

Part 2: Networks

A) Consider the network shown in the figure below. Packets enter the network at a rate of λ packets per second. Each entering packet follows the upper route with probability p, and the lower route with probability 1-p. At the output of node 1, a packet proceeds to node 3 with probability q or with probability 1-q loops back to node 1. The system is in the steady state and the average number of packets at each node is n1, n2, n3, as shown. Assume zero propagation delays between nodes. Expressed in terms of n1, n2, and n3, what is the average amount of time that a packet spends at each node (t1, t2, t3) and in the entire system (ts)?



B) Now assume that external arrivals are Poisson and that the service times of a packet at each node are independent, even if the packet has previously visited the same node, and service times are exponentially distributed with rates μ_1, μ_2, and μ_3, respectively.

i) Under what conditions are each of the queues stable?

ii) What is the steady-state, joint occupancy distribution (distribution of number of packets at each node) for nodes 1, 2 and 3?

P(n1,n2,n3) =

iii) Calculate the steady-state average occupancies (transmitted plus queue) n1, n2, n3 and the average amount of time that a packet spends in the system, ts.

C) Are the assumptions in part A realistic for communication networks?

D) In what sense are the queues in part B stable?