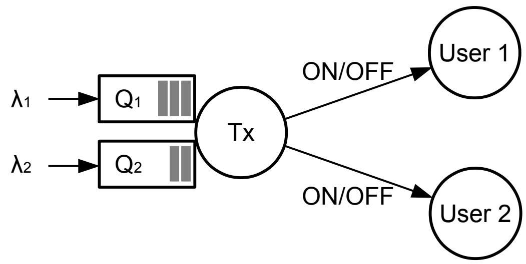
Opportunistic Scheduling. Consider a wireless downlink system where two users receive packets from a transmitter (Tx)-see picture. At every slot, $A_i[t]$ packets destined to user i arrive at the transmitter and become stored at the corresponding queue Q_i. Assume that $A_i[t]$ are independent across time and users, and identically distributed with mean $E\{A_i[t]\} = \lambda_i$ and finite second moment.



The packets are transmitted through two independent ON-OFF channels, where at every time slot each channel is ON with probability $\frac{1}{2}$. During a slot where a channel is ON, one packet may be transmitted through it. No packets can be transmitted through an OFF channel. Moreover, at most one user can be served at each time slot. There is a scheduler that decides which user is served at each time slot. For example, if both channels are ON, the scheduler can choose $(R_1[t], R_2[t]) = (1, 0)$ to allow a transmission of a user 1 packet.

Note that when the scheduler chooses to serve an empty queue, dummy packets are sent through the channel. Therefore the actual number of served packets might be smaller than $R_i[t]$.

1. Determine the necessary conditions on arrival rates for the system to be stable (no proof required - just explain the intuition). Plot the area of rate vectors that satisfy the conditions.

2. Given arrival rates (λ_1, λ_2) satisfying the conditions in step 1 with strict inequality, propose a randomized scheduling policy whose time average allocated service $(\overline{R}_1, \overline{R}_2)$ dominates the arrival rates (λ_1, λ_2) element-wise.

   Can such a policy provide the sufficiency of the conditions of step 1, and therefore complete the proof for the system stability region?

3. Propose a dynamic throughput optimal control policy for this system.

   Hint: To prove the policy optimality via the Lyapunov Drift theorem, start with the following steps:

   - Write the equations that describe the evolution of queue backlogs as a function of $A_i[t], R_i[t]$.
   - Then, derive an expression for the conditional expectation

   $$E\left\{L(\mathbf{Q}[t+1]) - L(\mathbf{Q}[t]) \;|\mathbf{Q}[t]\right\},$$

     where $L(\mathbf{Q}[t]) = \frac{1}{2}(Q_1[t]^2 + Q_2[t]^2)$. A useful inequality is $[(x-b)^+ + a]^2 \leq a^2 + b^2 + x^2 - 2x(b-a)$.
   - Find an upper bound for this expression that involves the product of queue backlogs $Q_i[t]$ and scheduling functions $R_i[t]$.

Probability

Consider n independent Gamma random variables $X_1, X_2, \ldots, X_n$ with the probability density function of X_k given by

$$f_{X_k}(x) = \begin{cases} \frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)} x^{\alpha_k - 1} e^{-\beta x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Show that $S = \sum_{i=k}^{n} X_k$ is a Gamma random variable, and find the probability density function of S.

[*Hint*: The characteristic function for X_k is

$$\phi_{X_k}(t) = \left(1 - \frac{it}{\beta}\right)^{-\alpha_k}.$$

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