Department of Aeronautics and Astronautics
Controls Field Exam, January 30–31, 2012

Part 1

Consider the linear system with dynamics given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The goal of the problem is to move the system from its initial state, $x(0)$, to the origin in minimum time, subject to the constraint $|u(t)| \leq 1$.

1. Show that the optimal controller is a bang-bang controller, that is, that the control is $u(t) = \pm 1$ for all t.

2. Find the switching curves that define the optimal control strategy.

Part 2

Consider the linear system of Part 1, but with cost function defined by

$$J = \int_0^\infty \left(x_1^2 + x_2^2 + u^2\right)\, dt$$

and with bounded control $|u(t)| \leq 1$.

1. What are the necessary conditions for the optimal control history? How would you solve for the optimal solution for a specific initial condition? How would you develop an optimal control strategy for general initial conditions?

2. Now suppose that the dynamics are instead given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v$$

   where w and v are independent, zero-mean, Gaussian white noise process with intensities W and V, respectively. Further, the state is not available for feedback, so that some form of output feedback must be used. Suggest a feedback strategy that optimizes or approximately optimizes the performance index

$$J = E\left[\int_0^\infty \left(x_1^2 + x_2^2 + u^2\right)\, dt\right]$$