Field Exam — Controls

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Consider a linear, time-invariant, single-input, *unstable*, controllable system, described by the following state-space model:

$$\frac{d}{dt}x(t) = Ax(t) + bu(t), \qquad x \in \mathbb{R}^n, u \in \mathcal{U} \subseteq \mathbb{R}. \tag{1}$$

In this exam, you will be asked to design full-state feedback control systems to stabilize the plant (1), under a variety of constraints on the set $\mathcal{U}$ of allowable values for the control input.

NOTE: IT IS NOT NECESSARY TO COMPLETE PART A TO DO PART B.

A. Unbounded control

Assuming that $\mathcal{U} = \mathbb{R}$,

1. Design a full-state feedback control law for (1) such that the resulting closed-loop system is stable, and the following cost functional is minimized:

$$J(x, u) = \int_0^\infty \left[\|x(t)\|_2^2 + \gamma |u(t)|^2 \right] \, dt. \tag{2}$$

2. Discuss the choice of the parameter γ if it is desired that $|u| < u_{\max}$, and $\|x\|_2 < x_{\max}$.
3. Assume that system (1) is in controllable canonical form, and that all the eigenvalues of A are strictly positive, given by $p_1 \geq \ldots \geq p_n > 0$. Recall that a state-space model in controllable canonical form has the following structure:

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & \ldots & & \ldots & -a_1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

   Compute the control gain corresponding to the *expensive control* solution, i.e., the control gain that would minimize (2) for $\gamma \to +\infty$. (Please compute this explicitly, in terms of the open-loop poles $p_1, \ldots, p_n$)

IN THE FOLLOWING, LET $k \in \mathbb{R}^n$ BE A CONTROL GAIN VECTOR, AND LET $V(x) = x^T P x$ BE A LYAPUNOV FUNCTION FOR THE SYSTEM (1) WITH CONTROL LAW $u = -kx$.

B. Bounded control

Now assume that $\mathcal{U} = [-1, 1]$. Consider the control input obtained by using a control law of the form $u = -\mathrm{sat}(kx)$, for some vector gain k. The saturation function $\mathrm{sat} : \mathbb{R} \to [-1, 1]$ is defined as

$$\mathrm{sat}(z) = \begin{cases} 1 & \text{if } z \geq 1 \\ z & \text{if } z \in (-1, 1) \\ -1 & \text{if } z \leq -1 \end{cases}$$

1. Would a control law of the form $u = -\mathrm{sat}(kx)$ globally stabilize the system (1)? Prove or disprove.
2. Determine a set Ω such that, for all $x(0) \in \Omega$, and under the action of the control law $u = -\mathrm{sat}(kx)$,

$$\lim_{t \to +\infty} x(t) = 0.$$

   Hint: In order to compute such a set, you may find it useful to think about this problem: Given an ellipsoid $\mathcal{E} = \{x : x^T M x \leq \alpha\}$, $M > 0$, what is the maximum value attained by $|kx|$ inside $\mathcal{E}$?
3. How would you choose k and V in such a way that the set Ω is as large as possible, and why?
4. Would such a design have any limitations? If so, how would you address these limitations?
   What techniques would you suggest to design a control law for (1), in the presence of saturation?