1. **ACE Core**

The second-order wave equation with forcing is given by,

$$\phi_{tt} - c^2\phi_{xx} = s(x,t) \tag{1}$$

where c is the wave propagation speed (and assumed to be constant) and $s(x,t)$ is a forcing term. Consider a spatial domain $-1 \le x \le 1$ with initial values given by,

$$\phi(x,0) = f^0(x) \tag{2}$$
$$\phi_t(x,0) = g^0(x) \tag{3}$$

and boundary values given by,

$$\phi(-1,t) = f_L(t) \tag{4}$$
$$\phi(+1,t) = f_R(t) \tag{5}$$

Consider the following finite difference discretization,

$$\frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{\Delta t^2} - c^2\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2} = s_j^n \tag{6}$$

where Δt is the timestep and Δx is the spatial size and the sub/superscript notation corresponds to $s_j^n = s(x_j, t^n)$ and similarly v_j^n is intended to be an approximation for $\phi(x_j, t^n)$.

(a) Assuming the exact solution is sufficiently smooth, determine the leading order terms of the truncation error of the discretization in Equation (6) as $\Delta x \to 0$ while keeping $\Delta t/\Delta x =$ constant.

(b) The initial and boundary conditions on ϕ given in Equations (2), (4) and (5) are straightforward to impose. The imposition of Equation (3) is less obvious. Consider the following implementation:

$$v_j^1 = f_j^0 + g_j^0\,\Delta t \tag{10}$$

Would this initial condition implementation decrease the accuracy of the finite difference algorithm? Can you propose another method to impose the initial condition which would have improve accuracy?

(c) Equation (1) has wave solutions that propagate at a speed $\pm c$. Determine the largest Δt (as a function of c and Δx) that satisfies the CFL condition for the discretization in Equation (6).

2. **Numerical Linear Algebra**

Linear potential flow problems which must be solved for a large number of freestream conditions and control deflections typically require the solution to a linear set of equations of the form

$$Ax = \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_M b_M$$

where A is the known $N \times N$ matrix which depends only on the geometry and x is an N-long vector of aerodynamic flow unknowns (e.g. potential values, panel strengths, etc.). One the righthand side we have $b_1, b_2 \ldots b_M$ which are N-long vectors which depend only on the geometry and are considered fixed. Their scalar coefficients $\alpha_1, \alpha_2 \ldots \alpha_M$ depend on the freestream velocities, rotation rates, control deflections. For most problems, $N \sim 1000 \ldots 10000$ and $M \sim 10$ are typical.

A large number of α coefficient sets might be specified in a design study

$$\{\alpha_1, \alpha_2, \ldots \alpha_M\}_1$$
$$\{\alpha_1, \alpha_2, \ldots \alpha_M\}_2$$
$$\vdots$$
$$\{\alpha_1, \alpha_2, \ldots \alpha_M\}_K$$

so that a total of K flow solutions x are to be obtained. Values of $K \sim 100$ or more is not unusual. Note that A and $b_1, b_2 \ldots$ are the same for all these solutions.

(a) Assuming A is dense, estimate the minimum total operation count required to obtain the K solutions via an LU decomposition method.

(b) If the matrix A is sparse, rather than using a direct LU method we would likely use an iterative method. For iterative methods a typical operation count is βN^2, where β is known (this would strongly depend on the sparsity of A). Estimate the minimum total operation count required to obtain the K solutions.

3. **Optimization**

Consider the least squares solution of a set of linear equations, formulated as an optimization problem:

$$\begin{aligned} \text{minimize} \quad & x^T x \\ \text{subject to} \quad & Ax = b \end{aligned}$$

where $A \in \mathbb{R}^{n \times m}$, and $b \in \mathbb{R}^n$ are given.

(a) Under what condition(s) on A and b is this problem feasible?

(b) What is the Lagrange dual function $g(\nu)$ for this optimization problem? Along with your answer, state the dimensionality of ν, and whether $g(\nu)$ is convex, concave, or neither.

(c) Arrive at an analytical expression for x^* in terms of A and b.

(d) Now consider a sparsity-inducing version of the problem, where the least squares objective is replaced by a sum of absolute values:

$$\begin{aligned} \text{minimize} \quad & ||x||_1 \\ \text{subject to} \quad & Ax = b \end{aligned}$$

Formulate this problem as a linear program. How many rows of equality and inequality constraints does the LP have?