Question 1

Consider a continuous-time, LTI, single-input system described by the following state-space model:
\[ \dot{x} = Ax + bu. \]

The matrix $A$ has a single positive eigenvalue $\lambda_1$, all other eigenvalues have negative real part.

1. What is the minimum energy necessary to bring the system to rest at the origin, starting from initial condition $x(0) = x_0$? In other words, please compute the value of
\[ J^*_0(x_0) = \min_u \int_0^{+\infty} |u(t)|^2 \, dt \]
subject to $\lim_{t \to +\infty} x(t) = 0$.

2. It is desired to stabilize the system with a digital computer implementing a sample-and-hold control strategy with sampling time $\Delta$, while keeping the cost of the control system to a minimum. Write down a state-space model describing the evolution of the system between sampling times.

3. Considering that fast computers and strong actuators are expensive, it is desired to minimize a cost functional of the form
\[ J(x, u) = \frac{cJ^*_0(x_0)}{\Delta} + \int_0^{+\infty} |u(t)|^2 \, dt. \]

What is the optimal sampling time $\Delta$, as a function of $A$, $b$, and $c$?
Question 2

Consider a stochastic first-order system with linear dynamics and nonlinear measurement given by

\[
\dot{x} = f(x) + w
\]
\[
y = x + v
\]

where

\[
f(x) = x - x^3
\]

The process noise \( w \) and measurement noise \( v \) are independent, zero mean, Gaussian white noise processes with intensities \( W \) and \( R \), respectively. A plot of \( f(x) \) is below.

Consider three different initial states \( x(0) = 0, x(0) = -1, \) and \( x(0) = 1 \). Describe how you would develop a state estimator for this problem, and the problems you might encounter. You might consider some limiting cases:

1. \( W \ll 1, R \ll W \)
2. \( W \ll 1, R \gg W \)
3. \( W \gg 1, R \ll W \)
4. \( W \gg 1, R \gg W \)