Massively Parallel Domain Decomposition Preconditioner for the High-Order Galerkin Least Squares Finite Element Method

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Outline

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1. Introduction
2. High-Order Galerkin Least Squares
3. Balancing Domain Decomposition by Constraint
4. Results
5. Conclusion and Future Work

Motivation

Discretization Requirements

Discretization should support

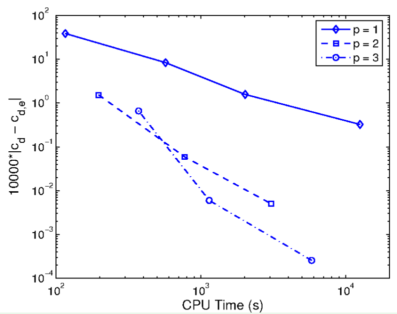
- **High-Order**: Simulation of unsteady flows with multiple scales benefits from higher-order methods
    - Turbulence (DNS/LES)
    - Acoustics / wave propagation
    - Even in steady aerodynamics, high-order methods are more efficient in terms of error/DOF [Barth, 1997][Venkatakrishnan, 2003]
- **Unstructured Mesh**: Handling complex geometries

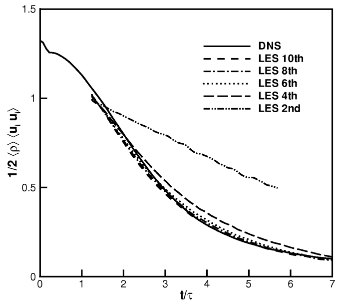
Selected Previous Work on High-Order Methods

- High-order schemes have been designed for various frameworks
    - Finite difference [Lele, 1992]
    - Finite volume [Barth, 1993][Wang, 2004]
    - Finite element [Babuska, 1981][Patera, 1984]

Motivation

Example: High-Order Discretization





- Drag convergence for NACA 0012 [Fidkowski, 2005]
- Decay of turbulent kinetic energy in LES modeling [Kosovic, 2000]

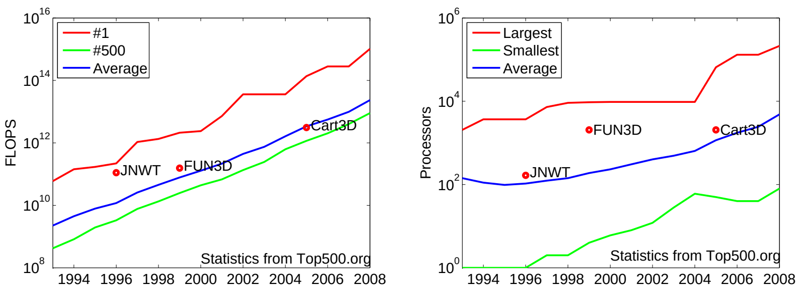
- High-order schemes can significantly improve time to achieve engineering required accuracy.

Motivation

Massively Parallel Algorithms

- Modern supercomputers are massively parallel (1000+ procs).
- Parallelization is essential for problems of practical interests.
  - Flow over large, complex geometries
  - Unsteady simulations (e.g. DNS/LES)

Trend of High Performance Computing in Past 15 Years



Statistics from Top500.org

Statistics from Top500.org

Motivation

High-Performance Computing in Aeronautics

- Typical large jobs remain in $\mathcal{O}(100)$ procs.
- Range of scales present in viscous flows requires *implicit* solvers
- *“The scalability of most of [CFD] codes tops out around 512 cpus ...”* [Mavriplis, 2007]
- Exceptions:
  - FUN3D inviscid matrix-free Newton-Krylov-Schwarz solver using 2048 procs [1999]
  - Cart3D inviscid multigrid-accelerated Runge-Kutta solver using 2048 procs [2005]
  - NSU3D RANS multigrid solver with line implicit solver in boundary layers using 2048 procs [2005]

- Need highly scalable *implicit* solver to take advantage of the future computers with 10,000+ procs.

Objectives and Approaches

Objectives

1. High-order discretization on unstructured mesh
2. Highly scalable solution algorithm for massively parallel systems

Approaches

1. Galerkin Least Squares (GLS)
    - Study behavior of high-order GLS discretization.
    - Use h/p scaling artificial viscosity for subcell shock capturing.
2. Domain Decomposition (DD) Preconditioner
    - Solve Schur complement system using preconditioned GMRES based on DD.
    - Test the preconditioner for high-order discretization of advection-dominated flows.

Outline

Galerkin Least Squares Discretization

Background

- Stabilized FEM for advection-dominated flows
- SUPG developed by Hughes [1982]; analyzed by Johnson [1984]
- SUPG extended to GLS by Hughes [1989]

GLS Discretization for Advection-Diffusion

- Differential operator and bilinear form

$$\mathcal{L}u \equiv \beta \cdot \nabla u - \nabla \cdot (\nu \nabla u) = f$$

$$a(u,\phi) = (\beta \cdot \nabla u, \phi)_\Omega + (\nu \nabla u, \nabla \phi)_\Omega, \quad \forall u, \phi \in V \subset H^1(\Omega)$$

- Find $u \in V$ s.t.

$$a(u,\phi) + (\mathcal{L}u, \tau \mathcal{L}\phi)_{\Omega,\mathcal{T}_h} = (f,\phi)_\Omega + (f, \tau\mathcal{L}\phi)_{\Omega,\mathcal{T}_h}\,, \quad \forall \phi \in V$$

where $(\cdot,\cdot)_{\Omega,\mathcal{T}_h} = \sum_K \int_K \cdot \cdot \, dK$ and $\tau \sim \begin{cases} h, & P_e \gg 1 \\ h^2, & P_e \ll 1 \end{cases}$.

Stabilization Matrix

Stability and *A Priori* Error Estimate for Hyperbolic Case ($\nu = 0$)

$$a(u,u) \geq h\|\beta \cdot \nabla u\|^2_{L_2(\Omega)}$$

$$\sqrt{h}\|\beta \cdot \nabla(u - u_h)\|_{L_2(\Omega)} + \|u - u_h\|_{L_2(\Omega)} \leq Ch^{p+\frac{1}{2}}\|u\|_{H^{p+1}(\Omega)}$$

High-Order Stabilization Matrix

- In order to keep $\frac{\mathrm{cond}(\mathcal{L}_{\mathrm{stabilized}})}{\mathrm{cond}(\mathcal{L}_{\mathrm{unstabilized}})} = \mathcal{O}(1)$ as $p \to \infty$ or $h \to 0$,

$$\tau \sim h^2/p^2, \quad P_e \ll 1$$

- One choice of τ in $\mathbb{R}^d$

$$\tau^{-1} = \sum_{i=1}^{d+1} \left|\beta \cdot \nabla \xi^i\right| + \frac{p^2}{d} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_j}{\partial x_l} \nu_{kl}$$

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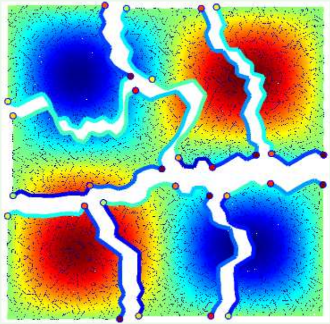
Domain Decomposition Preconditioners

Motivation

- Performance of ILU degrades with number of processors even with line partitioning [Diosady, 2007].
- Data parallelism alone is not sufficient to obtain good performance in massively parallel environment (1000+ procs.)

Selected Previous Work (Elliptic)

- Bourgat proposes Neumann-Neumann method for Schur complement system [1988].
- Mandel improves scalability with Balancing DD [1993].
- Dorhmann extends BDD to BDDC [2003].



Schur Complement System

Discrete Harmonic Extension (Elliptic Equation)

- Decompose Ω into non-overlapping domains Ω_i and define $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$ and $\Gamma = \cup_{i=1}^{N} \Gamma_i$. Decompose $V_h \subset H^1(\Omega)$ into

$$V_h(\Omega \setminus \Gamma) = \{v \in V_h : v|_\Gamma = 0\}$$

$$V_h(\Gamma) = \{v \in V_h : a(v, \phi) = 0, \forall \phi \in V_h(\Omega \setminus \Gamma)\}$$

- $V_h(\Gamma)$ is called the space of discrete harmonic extensions.

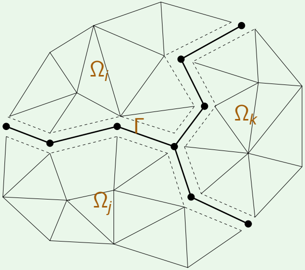
Variational Problem

Find $u = \mathring{u} + \bar{u} \in V_h(\Omega \setminus \Gamma) \oplus V_h(\Gamma)$ s.t.

$$a(\mathring{u}, \psi) = (f, \psi)_\Omega, \quad \forall \psi \in V_h(\Omega \setminus \Gamma)$$

$$a(\bar{u}, \phi) = (f, \phi)_\Omega, \quad \forall \phi \in V_h(\Gamma)$$

- $\mathring{u}$ requires local Dirichlet solves
- $\bar{u}$ requires global interface solve



Schur Complement System

Schur Complement Operator

Schur complement operator $S : V_h(\Gamma) \to V_h(\Gamma)'$ defined by

$$\langle Sv, \phi \rangle = a(v, \phi), \quad \forall v, \phi \in V_h(\Gamma)$$

where, $\langle \cdot, \cdot \rangle : W' \times W \to \mathbb{R}$ s.t. $\langle F, v \rangle \equiv F(v)$, $\forall F \in W', \forall v \in W$.

Schur Complement Problem

Find $u \in V_h(\Gamma)$ s.t.

$$\langle Su, \phi \rangle = (f, \phi)_\Omega \,, \quad \forall \phi \in V_h(\Gamma)$$

- Forming S is expensive; action of S on $v \in V_h(\Gamma)$ is computed via local Dirichlet solves and minimal communication.
- Problem solved using a Krylov space method (e.g. GMRES)
- Scalable preconditioner needed to accelerate convergence

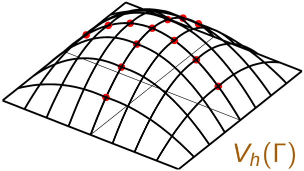
BDDC Spaces

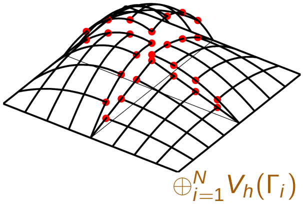
Features of BDDC

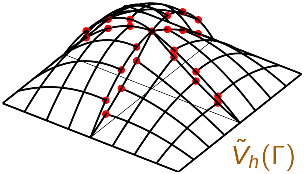
- Equipped with a coarse space that
  - makes subdomain problems wellposed
  - provides global communication
- Example of primal constrains
  - Values at corners of Ω_i
  - Averages on the edges of Ω_i

Spaces

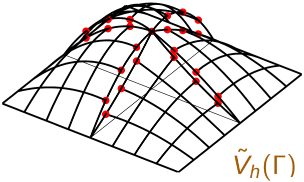
$$
\begin{aligned}
V_h(\Gamma) &= \text{global discrete harmonic extension} \\
\oplus_{i=1}^{N} V_h(\Gamma_i) &= \text{collection of local discrete} \\
&\quad \text{harmonic extensions} \\
&\quad \text{(i.e. discontinuous across } \Gamma) \\
\tilde{V}_h(\Gamma) &= \{ v \in \oplus_{i=1}^{N} V_h(\Gamma_i) : \\
&\quad v \text{ continuous on primal constraints} \}
\end{aligned}
$$







BDDC Spaces

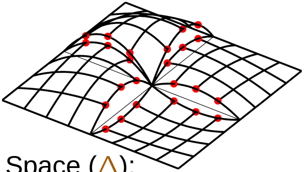


$\tilde{V}_h(\Gamma)$

Dual and Primal Spaces

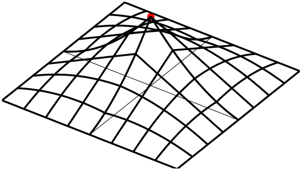
Decompose $\tilde{V}_h(\Gamma)$ into $\oplus_i V_{h,\Delta}(\Gamma_i)$ and $V_{h,\Pi}(\Gamma)$.

- Dual problems are decoupled
- Primal problem has DOF of $\mathcal{O}(N)$



Dual Space (Δ):

$$V_{h,\Delta}(\Gamma_i) = \{v \in V_h(\Gamma_i) : v = 0 \text{ on primal constraint}\}$$



Primal Space (Π):

$$V_{h,\Pi}(\Gamma) = \{v \in \tilde{V}_h(\Gamma) : \sum_{i=1}^{N} \tilde{a}_i(v|_{\Omega_i}, \phi_\Delta|_{\Omega_i}) = 0, \forall \phi_\Delta \in \oplus_{i=1}^{N} V_{h,\Delta}(\Gamma_i)\}$$

BDDC Preconditioner

Primal and Dual Schur Complement

- Primal Schur complement: $S_\Pi : V_{h,\Pi}(\Gamma) \to V'_{h,\Pi}(\Gamma)$

$$\langle S_\Pi v_\Pi, \phi_\Pi \rangle = \sum_{i=1}^N \tilde{a}_i(v|_{\Omega_i}, \phi|_{\Omega_i}), \quad \forall v_\Pi, \phi_\Pi \in V_{h,\Pi}(\Gamma)$$

- Local dual Schur complement: $S_{\Delta,i} : V_{h,\Delta}(\Gamma_i) \to V'_{h,\Delta}(\Gamma_i)$

$$\langle S_{\Delta,i} v_{\Delta,i}, \phi_{\Delta,i} \rangle = \tilde{a}_i(v_{\Delta,i}, \phi_{\Delta,i}) \quad \forall v_{\Delta,i}, \phi_{\Delta,i} \in V_{h,\Delta}(\Gamma_i)$$

BDDC Preconditioner

$$M_{BDDC}^{-1} = \sum_{i=1}^N R_{D,i}^T (T_{\text{sub},i} + T_{\text{coarse}}) R_{D,i}$$

where $T_{\text{sub},i} = R_{\Gamma\Delta,i}^T S_{\Delta,i}^{-1} R_{\Gamma\Delta,i}$ and $T_{\text{coarse}} = R_{\Gamma\Pi}^T S_\Pi^{-1} R_{\Gamma\Pi}$

Condition Number Estimate for Coercive, Symmetric $a(\cdot,\cdot)$

$$\kappa(M_{BDDC}^{-1} S) \le C(1 + \log^2(H/h))$$

Robin-Robin Interface Condition (IC)

Background

- Adaptation of N-N condition to nonsymmetric bilinear form
- Proposed by Achdou for BDD [1997]
- Applied to FETI [Toselli, 2001] and BDDC [Tu, 2008]

Robin-Robin IC

- Subtract $\int_{\Gamma_i} \frac{1}{2}\beta \cdot \hat{n}_i u \phi$ from interfaces.
- Modified local bilinear form is

$$\tilde{a}_i(u,\phi) = (\nu \nabla u, \nabla \phi)_{\Omega_i} + \tfrac{1}{2}(\beta \cdot \nabla u, \phi)_{\Omega_i} - \tfrac{1}{2}(\beta \cdot \nabla \phi, u)_{\Omega_i} + \tfrac{1}{2}(u, v\beta \cdot \hat{n})_{\partial\Omega_i \cap \partial\Omega_N}$$

- $\tilde{a}_i(\cdot,\cdot)$ is positive in local space $V_h(\Omega_i)$: $\tilde{a}_i(u,u) \geq \nu |u|_{H^1(\Omega_i)}$.
- Resulting IC on Γ_i is Robin type

$$\left(\nu \nabla u - \tfrac{1}{2}\beta\right) \cdot \hat{n} = 0 \quad \text{on } \Gamma_i$$

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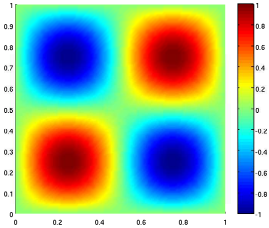
Poisson Equation

Poisson Equation Objectives

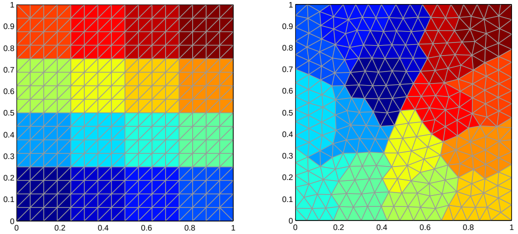
Test scaling with :

- Number of subdomains
- Size of subdomain
- *p* for different primal constraints

Solution



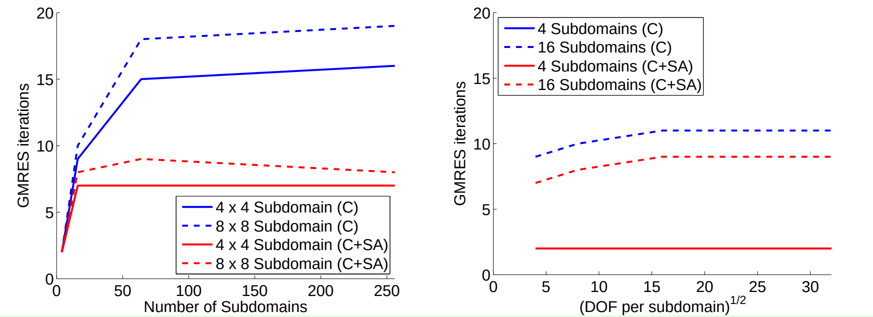
Partitioning Examples



Poisson Equation

Scaling with Number of Subdomains and Subdomain Size

- GMRES iteration is independent of number of subdomains and size of subdomain ($p = 1$).
- Corner + state average (C+SA) primal constraint requires approx. half the iterations of just corner constraint (C).



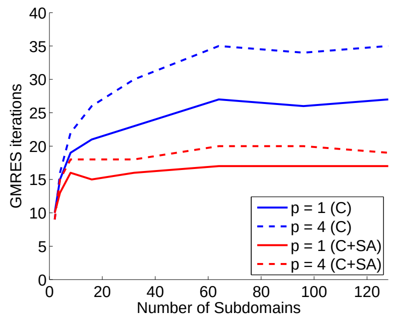
Poisson Equation

Scaling with p

- Poisson equation solved on unstructured mesh $\sim$ 8000 elem.
- Number of iterations nearly independent of p when C+SA primal constraints are used.

Size of Primal Problem

- Corners only: $\approx N_{\text{subdomain}}$
- Corners + Edge State Average: $\approx 3 \times N_{\text{subdomain}}$



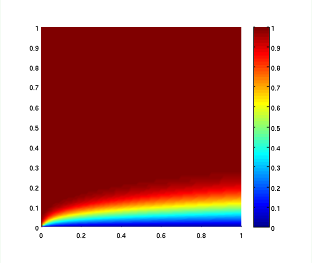
Advection Diffusion: Boundary Layer

Boundary Layer Equation Objectives

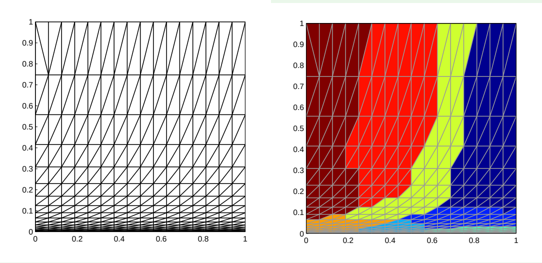
Study effect of:

- interface conditions (Neumann-Neumann vs. Robin-Robin)
- τ scaling (h^2 vs. h^2/p^2)

Solution ($\nu = 10^{-3}$)



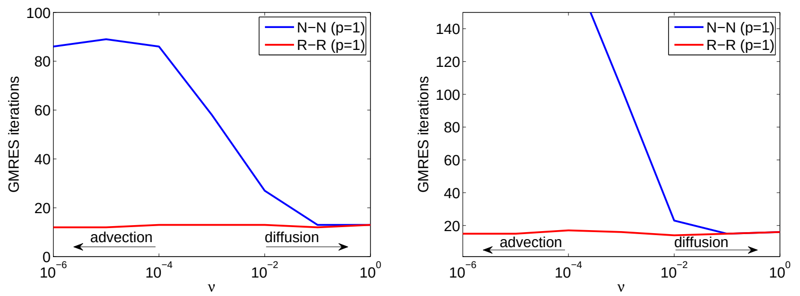
Anisotropic Mesh and Partitioning



Advection Diffusion: Boundary Layer

Interface Conditions (IC)

- Robin-Robin IC performs significantly better than Neumann-Neumann IC in advection dominated cases.
- Interface conditions are identical as $\nu \to \infty$.

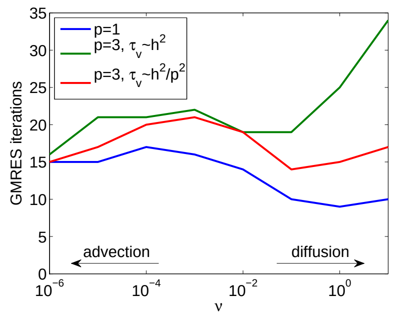


$N_{\text{subdomain}} = 16$ $N_{\text{subdomain}} = 64$

Advection Diffusion: Boundary Layer

τ Matrix Scaling

- Performance of preconditioner degrades for $p > 1$ if $\tau \sim h^2$ in diffusion dominated cases.
- With $\tau \sim h^2/p^2$, the preconditioner performs similar to $p = 1$ case.



$N_{\text{subdomain}} = 64$

Outline

Conclusion and Future Work

Conclusion

- BDDC preconditioner shows good scalability for:
  - Both diffusion-dominated and advection-dominated flows with Robin-Robin interface condition
  - All ranges of interpolation order p with proper choices of τ and primal constraints

Future Work

- Extension to 3D
- Inexact solvers for subdomain problems
- Complex geometries and highly anisotropic mesh

Questions

Supplemental Slides

Instability of the Standard Galerkin Method

Discretization

Advection-diffusion equation

$$\mathcal{L}u \equiv \beta \cdot \nabla u - \nabla \cdot (\nu \nabla u) = f$$

Galerkin: Find $u \in V \subset H^1(\Omega)$ s.t.

$$a(u, \phi) = (f, \phi)_\Omega, \quad \forall \phi \in V$$

where $a(u, \phi) = \int_\Omega \phi \beta \cdot \nabla u + \nu \nabla \phi \cdot \nabla u$.

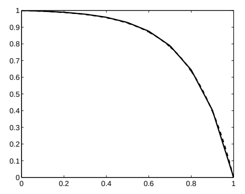
Stability and *A Priori* Error Estimate

$$a(u, u) \geq \nu \|\nabla u\|^2_{L_2(\Omega)} = \nu |u|^2_{H^1(\Omega)}$$

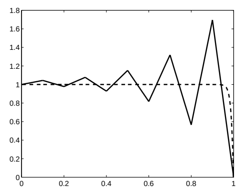
$$\nu |u - u_h|_{H^1(\Omega)} \leq C h^p \|u\|_{H^{p+1}(\Omega)}$$

Not coercive in $H^1(\Omega)$ as $\nu \to 0$.

1D Boundary Layer



$$P_e = \frac{\beta h}{2\nu} = 1/4$$



$$P_e = 5$$

Stabilized Finite Element Methods

Stabilized Methods for Advection-Dominated Flows

- Streamline-Upwind Petrov-Galerkin (SUPG) [Hughes 1982]
- Galerkin Least-Squares (GLS) [Hughes 1989]
- Residual Free Bubbles (RFB) [Brezzi 1994]
- Variational Multiscale based on local Green's function [Hughes 1995]

Finite Element vs. Finite Volume

Advantages of Finite Element Method

- Maintains element-wise compact stencil for high-order discretization.
    - Ease of boundary condition treatment.
    - Reduces communication volume for DD.
- Straightfoward treatment of elliptic operator.
- Rigorous mathematical framework for *a priori* error estimation and DD convergence estimation.

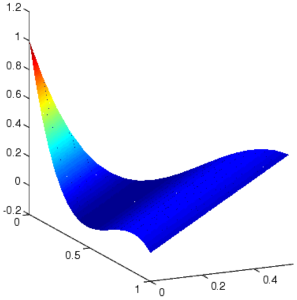
Disadvantages of Finite Element Method

- Requires stabilization term for $H^1(\Omega)$ stability.
- DOF increases with the solution order.

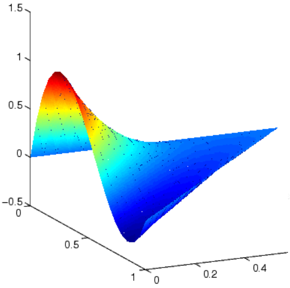
High-Order Galerkin Basis

Basis Function Types

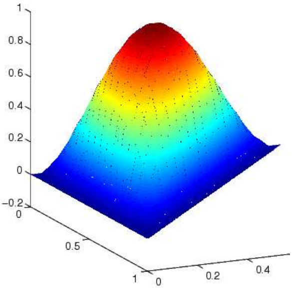
- Three types of basis functions (in 2D): Node, Edge, Element.
- Basis type defined by its support.
- The continuity constraint must be enforced on nodal and edge basis during assembly.



Node



Edge



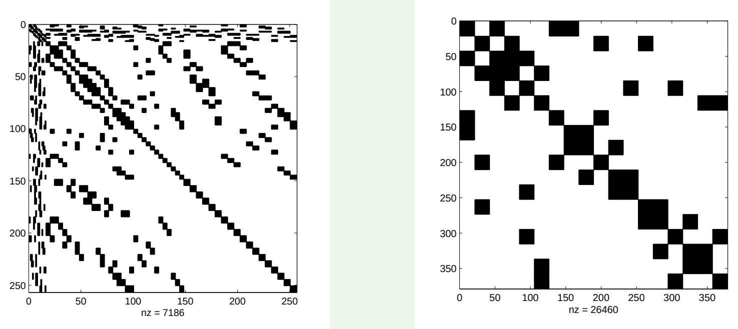
Element

High-Order Galerkin: Matrix Storage

Object-based Block Storage

- Jacobian stored blockwise, with the varying size of blocks
- Matrix converted to CRS format before solved with UMFPACK

Jacobian of 3 × 3 Mesh (18 elem, $p = 5$, arbitrary basis)



Galerkin DG (with compact lifting)

Comparison with Discontinuous Galerkin

Features (common to DG)

- High-order discretization on unstructured mesh
- $H^1(\Omega)$ stability for advection-dominated flows

Advantages (compared to DG)

- Straightforward treatment of elliptic operators (no lifting required)
- Fewer DOF required to represent a solution in $H^1(\Omega)$

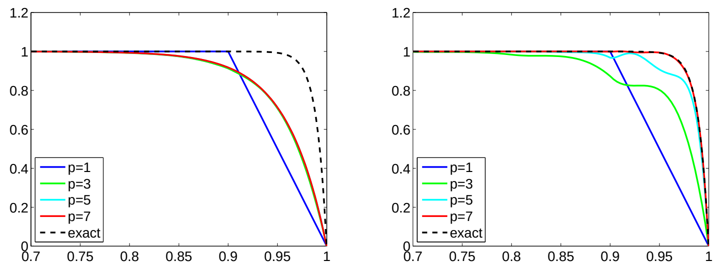
Disadvantages

- Potentially expensive calculation of stabilization terms
- Absence of block-wise compact stencil (more complex preconditioning strategy required)

High-Order SUPG vs. GLS

p-Refinement with Traditional τ

- If $h \gg \nu / \|\beta\|$, SUPG p-refinement fails to converge.
- Asymptotic convergence rates as $h \to 0$ are identical ($\|e\|_{H^1(\Omega)} \sim h^p$)



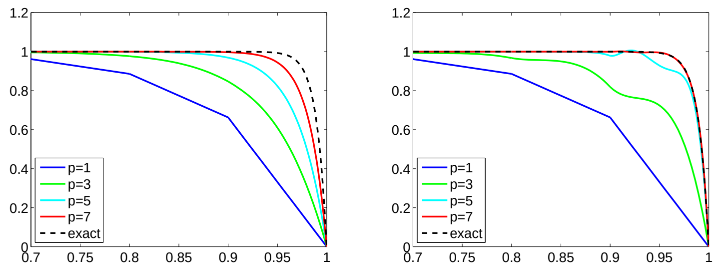
SUPG GLS

- Solution to advection-diffusion equation ($h = 0.1$, $\nu / \|\beta\| = 0.01$)

High-Order SUPG vs. GLS

p-Refinement with High-Order Modified τ

- With the high-order modified τ, both SUPG and GLS p-refinement converge to the exact solution.
- GLS converges much rapidly than SUPG with p.



SUPG GLS

- Solution to advection-diffusion equation ($h = 0.1$, $\nu/\|\beta\| = 0.01$)

Shock Capturing

Background

- Artificial viscosity can regularize underresolved features.
- Method applied to DG by Persson [2006] and Barter [2007] to achieve subcell shock capturing.

Resolution Indicator and h/p-Scaling Artificial Viscosity

- high-order resolution indicator based on orthogonal polynomial expansion [Persson, 2006]

  $$S_e = \frac{(u - \Pi^{p-1}u, u - \Pi^{p-1}u)_K}{(u, u)_K}, \quad u \in \mathcal{P}_p(K)$$

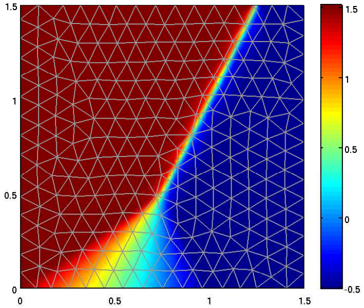
  where Π^{p-1} is the $L_2(K)$ projection onto $\mathcal{P}_{p-1}(K)$.
- If $\log_{10}(S_e) \geq s_0$, add piecewise constant viscosity $\nu \sim h/p$ to the element.

Shock Capturing: Result

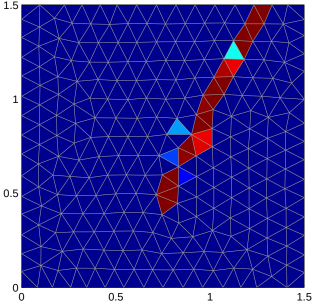
Burger’s Equation

- Burger’s equation solved on ≈ 450 element mesh.

$$\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) + \frac{\partial u}{\partial y} + \nabla\cdot\left(\nu_{\text{artificial}}(u)\nabla u\right) = 0$$



Solution ($p = 5$)



Resolution Indicator

Euler Equation: Gaussian Bump

Formulation

- Symmetric form using Hughes' entropy variables [1986]

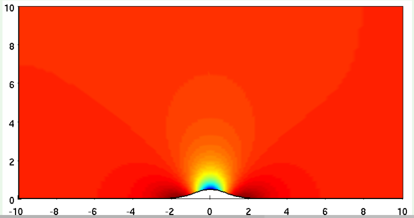
$$V = \left(\frac{-s+\gamma+1}{\gamma-1} - \frac{\rho E}{p}, \quad \frac{\rho u_i}{p}, \quad -\frac{\rho}{p} \right)$$

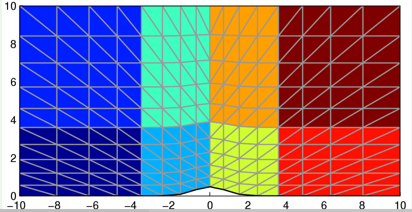
- The governing equation becomes

$$\tilde{A}_0 V_{,t} + \tilde{A}_i V_{,x_i} = 0$$

where A_0 is SPD and A_i is symmetric (K_{ij} is SPSD for N-S).

Gaussian Bump: Pressure and Partitioning

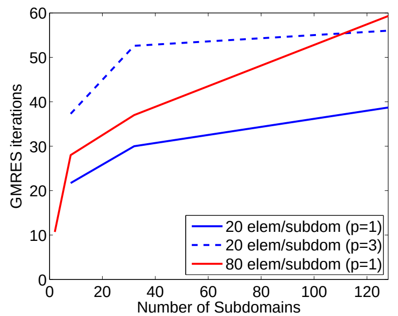




Euler Equation: Gaussian Bump

Euler Equation

- BDDC with R-R IC performs well for Euler equation.
- With N-N IC, 400+ GMRES iteration required for 32 subdomains.

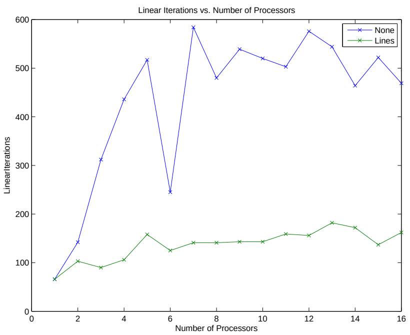


- Average of GMRES iterations required for last three Newton steps

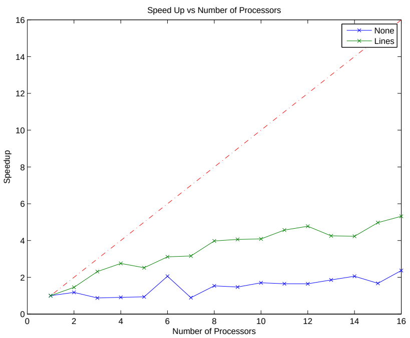
Parallel Scalability of ILU

Scalability of ILU for DG Discretization [Diosady 2007]

- Navier-Stokes equation
- Partitioning based on line connectivity
- Line-ordered ILU with *p*-Multigrid within each subdomain



Iterations vs. Processors



Parallel Speed-Up

Schur Complement for Non-Self-Adjoint Operator

Discrete Harmonic Extension (Non-Self-Adjoint)

Decompose $V_h \subset H^1(\Omega)$ into

$$V_h(\Omega \setminus \Gamma) = \{v \in V_h : v|_\Gamma = 0\}$$
$$V_h(\Gamma) = \{v \in V_h : a(v,\phi) = 0, \forall \phi \in V_h(\Omega \setminus \Gamma)\}$$
$$V_h^*(\Gamma) = \{\psi \in V_h : a(w,\psi) = 0, \forall w \in V_h(\Omega \setminus \Gamma)\}$$

Find $u = \mathring{u} + \bar{u} \in V_h(\Omega \setminus \Gamma) \oplus V_h(\Gamma)$ s.t.

$$a(\mathring{u},\phi) = (f,\phi)_\Omega, \quad \forall \phi \in V_h(\Omega \setminus \Gamma)$$
$$a(\bar{u},\psi) = (f,\psi)_\Omega, \quad \forall \psi \in V_h^*(\Gamma)$$

Schur Complement Operator

Schur complement $S : V_h(\Gamma) \to V_h^*(\Gamma)'$ s.t.

$$\langle Sv, \psi \rangle = a(v,\psi), \qquad \forall v \in V_h(\Gamma), \quad \forall \psi \in V_h^*(\Gamma)$$

Neumann-Neumann Preconditioner

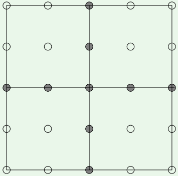
Local Spaces and Local Schur Complement

- Decompose $V_h(\Omega_i) \subset H^1(\Omega_i)$ of Ω_i into

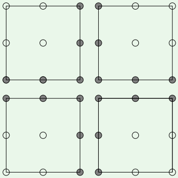
  $V_h(\Omega_i \setminus \Gamma_i) = \{v_i \in V_h(\Omega_i) : v|_{\partial\Omega_i} = 0\}$

  $V_h(\Gamma_i) = \{v_i \in V_h(\Omega_i) : \tilde{a}_i(v_i, \phi) = 0, \forall \phi \in V_h(\Omega_i \setminus \Gamma_i)\}$
- Local Schur complement: $S_i : V_h(\Gamma_i) \to V_h(\Gamma_i)'$

$$\langle S_i v_i, \phi_i \rangle = \tilde{a}_i(v_i, \phi_i), \quad \forall v_i, \phi_i \in V_h(\Gamma_i)$$



$V_h(\Gamma)$



$\oplus_{i=1}^N V_h(\Gamma_i)$

Interpolation and Weighting Function

- Interpolation: $R_i^T : V_h(\Gamma_i) \to V_h(\Gamma)$ s.t.

$$\oplus_{i=1}^N R_i^T V_h(\Gamma_i) = V_h(\Gamma)$$

- Weighting Function: $D_i \in V_h(\Gamma_i)$ s.t.

$$D_i = 1/(\# \text{ of } \Omega_i \text{ sharing DOF on } \Gamma)$$

Note, $v = \sum_{i=1}^N R_i^T D_i v|_{\Omega_i}, \forall v \in V_h(\Gamma)$

Neumann-Neumann Preconditioner

Neumann-Neumann (N-N) Preconditioner

- Precondition S by applying $S_i^{-1} : V'(\Gamma_i) \to V(\Gamma_i)$ on each Ω_i and interpolate the result back to $V(\Gamma)$

$$M_{NN}^{-1} = \sum_{i=1}^{N} R_i^T D_i S_i^{-1} D_i R_i$$

- For $\tilde{a}_i(u, \phi) = (\nabla u, \nabla \phi)_{\Omega_i}$, application of S_i^{-1} corresponds to solving a local problem with Neumann interface condition on Γ_i

$$(\nabla u) \cdot \hat{n} = 0 \quad \text{on } \Gamma_i$$

Condition Number Estimate for Coercive, Symmetric $a(\cdot, \cdot)$

$$\kappa(M_{NN}^{-1} S) \leq \frac{C}{H^2} (1 + \log(H/h))^2$$

Balancing Domain Decompositions

Problems of Neumann-Neumann Preconditioner

- Local Schur complement S_i may be singular
- Limited scalability due to lack of coarse space

Coarse Space

Introduce coarse space $V_{h,0}$ that

- makes subdomain problems well-posed
- provides global communication
- $\mathrm{DOF}_{\mathrm{coarse}} = \mathcal{O}(N)$

Examples of DD with a Coarse Space

- BDD: Balancing Domain Decomposition [Mandel, 1993]
- BDDC: BDD by Constraints [Dohrmann, 2003]

Balancing Domain Decomposition

Balancing Domain Decomposition

- Define a coarse space: $V_{h,0}(\Omega) = \text{span}\left\{R_i^T \delta_i^\dagger\right\}$, $\text{Kernel}(S_i) \subset V_{h,0}$
- Define projection $P_0 : V_h(\Omega) \to V_{h,0}(\Omega)$, $P_0 = R_0^T S_0^{-1} R_0 S$
- Apply Neumann-Neumann preconditioner to balanced problem

$$M_{BDD}^{-1} = R_0^T S_0^{-1} R_0 + (I - P_0) \left(\sum_{i=1}^N R_i^T D_i S_i^{-1} D_i R_i S \right) (I - P_0)$$

Examples of DD with a Coarse Space

- BDD: Balancing Domain Decomposition [Mandel, 1993]
  - $V_{h,0} = \oplus_{i=1}^N \text{Kernel}(S_i)$.
  - Apply *multiplicative* coarse correction such that residual of subdomain problem is in $\text{range}(S_i)$. (i.e. 'balanced' problem)
- BDDC: BDD by Constraints [Dohrmann 2003]
  - $V_{h,0} = \{\Psi_i\}$, harmonic extensions satisfying primal constraints.
  - Solve *additive* coarse and constrained subdomain problems.

Comparison of Primal and Dual Substructuring Methods

Primal Type

Preconditions the Schur complement system by solving Neumann problems using the flux jumps.

- Neumann-Neumann method [Bourgat 1988]
- Balancing Domain Decomposition (BDD) [Mandel, 1993]
- BDD by Constraints (BDDC) [Dohrmann 2003]

Dual Type

Preconditions the flux equations by solving Dirichlet problems using function jumps.

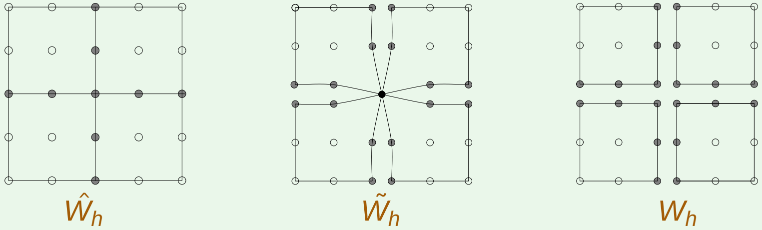
- Finite Element Tearing and Interconnecting (FETI) [Farhat 1991]
- Dual-Primal FETI (FETI-DP) [Farhat 2001]
  - Spectrum of BDDC and FETI-DP preconditioned operators are identical [Mandel 2005][Li 2005]

BDDC Spaces

Partially Assembled Spaces

- Collection of $V_h(\Gamma_i)$: $W_h \equiv \oplus_{i=1}^{N} V_h(\Gamma_i) \supset V_h(\Gamma)$
- Fully assembled FE space: $\hat{W}_h \equiv V_h(\Gamma) = \oplus_{i=1}^{N} R_i^T V_h(\Gamma_i)$
- Partially assembled space: $\tilde{W}_h \equiv V_{h,0}(\Gamma) + \oplus_{i=1}^{N} R_i^T R_{\Gamma\Delta,i}^T V_h(\Gamma_i)$

Idea: BDDC preconditioner applies inverse of $\tilde{S} : \tilde{W}_h \rightarrow \tilde{W}_h'$ to Shucr complement system.



Schur Complement System: Matrix Form

Schur Complement

Decompose local stiffness matrix and load vector into

$$A^{(i)} = \begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \quad \text{and} \quad f^{(i)} = \begin{pmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} \end{pmatrix}$$

Schur complement system is given by

$$\hat{S} u_\Gamma = \hat{g}$$

where

$$\hat{S} = \sum_{j=1}^{N} R^{(j),T} S^{(j)} R^{(j)}, \quad S^{(j)} = A_{\Gamma\Gamma}^{(j)} - A_{\Gamma I}^{(j)} \left(A_{II}^{(j)} \right)^{-1} A_{I\Gamma}^{(j)}$$

$$\hat{g} = \sum_{j=1}^{N} R^{(j),T} g^{(j)}, \quad g^{(j)} = f_\Gamma^{(j)} - A_{\Gamma I}^{(j)} \left(A_{II}^{(j)} \right)^{-1} f_I^{(j)}$$

Application of Schur Complement

Application of $S^{(j)}$

Calculation of $u_\Gamma^{(j)} = S^{(j)} v^{(j)}$ corresponds to solving a local Dirichlet problem:

$$S^{(j)} v^{(j)} = A_{\Gamma\Gamma}^{(j)} v^{(j)} - A_{\Gamma I}^{(j)} \left(A_{II}^{(j)} \right)^{-1} A_{I\Gamma}^{(j)} v^{(j)}$$

Application of $\left(S^{(j)} \right)^{-1}$

Calculation of $u_\Gamma^{(j)} = (S^{(j)})^{-1} v_\Gamma^{(j)}$ corresponds to solving a local Neumann problem:

$$\begin{pmatrix} A_{II}^{(j)} & A_{I\Gamma}^{(j)} \\ A_{\Gamma I}^{(j)} & A_{\Gamma\Gamma}^{(j)} \end{pmatrix} \begin{pmatrix} u_I^{(j)} \\ u_\Gamma^{(j)} \end{pmatrix} = \begin{pmatrix} 0 \\ v_\Gamma^{(j)} \end{pmatrix}$$

BDDC: Coarse Correction

Coarse Correction

Coarse-level correction operator is given by

$$T_{\text{coarse}} = \Psi(\Psi^T S \Psi)^{-1} \Psi^T$$

where

$$\Psi \equiv \left(\Psi^{(1),T}, \dots, \Psi^{(N),T} \right)^T \in \mathbb{R}^{\text{DOF}(\oplus V_h(\Gamma_i)) \times N_{\text{primal}}}$$

Coarse Basis

The local coarse basis, $\Psi^{(i)}$ is the harmonic extension to Ω_i that satisfies the primal constraints, i.e.

$$\begin{pmatrix} S^{(i)} & C^{(i),T} \\ C^{(i)} & 0 \end{pmatrix} \begin{pmatrix} \Psi^{(i)} \\ \Lambda^{(i)} \end{pmatrix} = \begin{pmatrix} 0 \\ R_{\Pi}^{(i)} \end{pmatrix}$$

where $C^{(i)}$ enforces primal constraints and $\Lambda^{(i)}$ is Lagrange multiplier.

BDDC: Subdomain Correction

Subdomain Correction

Subdomain correction is given by

$$T_{\text{sub}} = \sum_{i=1}^{N} \begin{pmatrix} R_\Gamma^{(i),T} & 0 \end{pmatrix} \begin{pmatrix} S^{(i)} & C^{(i),T} \\ C^{(i)} & 0 \end{pmatrix}^{-1} \begin{pmatrix} R_\Gamma^{(i)} \\ 0 \end{pmatrix}$$

- Provides subdomain corrections for which all coarse level, primal variables vanish.
- Primal constraints ensure the local problem is invertible.

BDDC: Change of Basis

Idea [Klawonn, 2004]

Change basis functions on interface to make primal constraints explicit.

- Removes the need for the Lagrange multipliers as primal continuity constraints are satisfied by construction.

Example: State Average

Change of basis $\psi_k \rightarrow \psi_l$ such that

$$\begin{array}{ll} \text{Dual Variables:} & \int_{\text{edge}} \psi_l(\xi) d\xi = 0, \quad l = 1, \dots, n-1 \\ \text{Primal Variable:} & \psi_n(\xi) = 1, \quad \forall \xi \in \text{edge} \end{array}$$

Let T_{COB} be the weighting matrix that maps from ψ to ϕ, then new stiffness matrix is

$$A_{\psi,ij} = a(\psi_i, \psi_j) = (T_{\text{COB}}^T A_\phi T_{\text{COB}})_{i,j}$$

Abstract Additive Schwarz Preconditioner

Assumption

Consider symmetric, coercive $a(\cdot,\cdot) : V \to V$ and $a_i(\cdot,\cdot) : V_i \to V_i$. Assume $\exists C_0$, ω, $E = (\epsilon_{ij})_{i,j=1}^N$ such that

- Stable Decomposition

$$\forall v \in V, \exists v_i \in V_i : \sum_{i=0}^N a_i(v_i, v_i) \le C_0 a(v, v)$$

- Local Stability

$$\forall i = 1, \dots, N, \forall v_i \in V_i : a(v_i, v_i) \le \omega a_i(v_i, v_i)$$

- Strengthened Cauchy-Shwarz Inequality

$$\forall i, j = 1, \dots, N, \forall u_i \in V_i, \forall v_j \in V_j : |a(u_i, v_j)| \le \epsilon_{ij} \sqrt{a(u_i, u_i) a(v_j, v_j)}$$

Conditioner Number Estimate [Dryja, 1995]

$$\kappa \le C_0 \omega \left(1 + \rho(E)\right)$$

BDDC Condition Number Estimate

Rayleigh Quotient

Applying Rayleigh quotient formula to the inner product $\langle M_{\text{BDDC}} \cdot, \cdot \rangle$,

$$\lambda_{\min}\left(M_{\text{BDDC}}^{-1} S\right)$$

$$= \min_{v \neq 0} \frac{\langle Sv, v \rangle}{\min_{\substack{v = R_D^T R_{\Gamma\Pi}^T v_\Pi + \sum_{i=1}^N R_{D,i}^T R_{\Gamma\Delta,i}^T v_{\Delta,i} \\ v_\Pi \in V_{h,\Pi}(\Gamma), v_{\Delta,i} \in V_{h,\Delta}(\Gamma_i)}} \left(\langle S_\Pi v_\Pi, v_\Pi \rangle + \sum_{i=1}^N \langle S_{\Delta,i} v_{\Delta,i}, v_{\Delta_i} \rangle \right)}$$

$$\lambda_{\max}\left(M_{\text{BDDC}}^{-1} S\right)$$

$$= \max_{v \neq 0} \frac{\langle Sv, v \rangle}{\min_{\substack{v = R_D^T R_{\Gamma\Pi}^T v_\Pi + \sum_{i=1}^N R_{D,i}^T R_{\Gamma\Delta,i}^T v_{\Delta,i} \\ v_\Pi \in V_{h,\Pi}(\Gamma), v_{\Delta,i} \in V_{h,\Delta}(\Gamma_i)}} \left(\langle S_\Pi v_\Pi, v_\Pi \rangle + \sum_{i=1}^N \langle S_{\Delta,i} v_{\Delta,i}, v_{\Delta_i} \rangle \right)}$$

Eigenvalues: Lower Bound

Schur Complement on $\tilde{V}_h(\Gamma)$

$$\langle \tilde{S} v, \phi \rangle = \sum\nolimits_{i=1}^{N} \tilde{a}_i \left(v|_{\Omega_i}, \phi|_{\Omega_i} \right), \quad \forall v, \phi \in \tilde{V}_h(\Omega)$$

Derivation

$$\begin{aligned}
\langle S_\Pi v_\Pi, v_\Pi \rangle + \sum\nolimits_{i=1}^{N} \langle S_{\Delta,i} v_{\Delta,i}, v_{\Delta,i} \rangle &= \langle \tilde{S} v_\Pi, v_\Pi \rangle + \sum\nolimits_{i=1}^{N} \tilde{a}_i (v_{\Delta,i}, v_{\Delta,i}) \\
&= \langle \tilde{S} v_\Pi, v_\Pi \rangle + \langle \tilde{S} v_\Delta, v_\Delta \rangle \\
&= \langle \tilde{S}(v_\Pi + v_\Delta), (v_\Pi + v_\Delta) \rangle = \langle S v, v \rangle
\end{aligned}$$

$$\langle S v, v \rangle \geq \min_{\substack{v = R_D^T R_{\Gamma\Pi}^T v_\Pi + \sum_{i=1}^{N} R_{D,i}^T R_{\Gamma\Delta,i}^T v_{\Delta,i} \\ v_\Pi \in V_{h,\Pi}(\Gamma), v_{\Delta,i} \in V_{h,\Delta}(\Gamma_i)}} \left(\langle S_\Pi v_\Pi, v_\Pi \rangle + \sum_{i=1}^{N} \langle S_{\Delta,i} v_{\Delta,i}, v_{\Delta_i} \rangle \right)$$

$$\lambda_{\min} \left(M_{\mathrm{BDDC}}^{-1} S \right) \geq 1$$

Eigenvalues: Upper Bound

Derivation

$$
\begin{aligned}
\langle Sv, v\rangle &\le 2[\langle S R_D^T R_{\Gamma\Pi}^T v_\Pi, R_D^T R_{\Gamma\Pi}^T v_\Pi\rangle \\
&\qquad + \langle S(\textstyle\sum_{i=1}^N R_{D,i}^T R_{\Gamma\Delta,i}^T v_{\Delta,i}), (\textstyle\sum_{i=k}^N R_{D,k}^T R_{\Gamma\Delta,k}^T v_{\Delta,k})\rangle] \\
&\lesssim \langle S R_D^T R_{\Gamma\Pi}^T v_\Pi, R_D^T R_{\Gamma\Pi}^T v_\Pi\rangle + \textstyle\sum_{i=1}^N \langle S(R_{D,i}^T R_{\Gamma\Delta,i}^T v_{\Delta,i}), (R_{D,i}^T R_{\Gamma\Delta,i}^T v_{\Delta,i})\rangle
\end{aligned}
$$

For symmetric, coercive bilinear form with $p = 1$,

$$
\langle S R_D^T R_{\Gamma\Pi}^T v_\Pi, R_D^T R_{\Gamma\Pi}^T v_\Pi\rangle \lesssim (1 + \log^2(H/h))\langle S_\Pi v_\Pi, v_\Pi\rangle
$$

$$
\langle S(R_{D,i}^T R_{\Gamma\Delta,i}^T v_{\Delta,i}), (R_{D,i}^T R_{\Gamma\Delta,i}^T v_{\Delta,i})\rangle \lesssim (1 + \log^2(H/h))\langle S_{\Delta,i} v_{\Delta,i}, v_{\Delta,i}\rangle
$$

Thus,

$$
\langle Sv, v\rangle \lesssim \left(1 + \log^2\left(\frac{H}{h}\right)\right) \min_{\substack{v = R_D^T R_{\Gamma\Pi}^T v_\Pi + \sum_{i=1}^N R_{D,i}^T R_{\Gamma\Delta,i}^T v_{\Delta,i} \\ v_\Pi \in V_{h,\Pi}(\Gamma), v_{\Delta,i} \in V_{h,\Delta}(\Gamma_i)}} \left(\langle S_\Pi v_\Pi, v_\Pi\rangle + \sum_{i=1}^N \langle S_{\Delta,i} v_{\Delta,i}, v_{\Delta_i}\rangle\right)
$$

$$
\lambda_{\max}\left(M_{\mathrm{BDDC}}^{-1} S\right) \le C\left(1 + \log^2(H/h)\right)
$$

BDDC: GMRES Convergence Estimate

GMRES Residual Bound [Eisenstat, 1983]

Let c anc C^2 be such that

$$c_0\langle u,u\rangle \leq \langle u,Tu\rangle$$

$$\langle Tu,Tu\rangle \leq C_0^2\langle u,u\rangle$$

Then,

$$\frac{\|r_m\|}{\|r_o\|} \leq \left(1-\frac{c_0^2}{C_0^2}\right)^{m/2}$$

Advection-Diffusion Convergence Estimate [Tu, 2008]

With state and flux average as primal constraints, BDDC (R-R) satisfies

$$c_0 = 1 - CH(H/h)(1+\log(H/h))$$

$$C_0 = C(1+\log(H/h))^4$$

Nonlinear Equation Solution Scheme

Discrete form using Galerkin Method

$$\mathcal{M}_h \frac{d\mathbf{U}_h}{dt} + R_h(\mathbf{U}_h(t)) = 0$$

Time Stepping Scheme

$$\mathbf{U}_h^{m+1} = \mathbf{U}_h^m - \left(\frac{1}{\Delta t}\mathcal{M}_h + \frac{\partial R_h}{\partial \mathbf{U}_h} \right)^{-1} R_h(\mathbf{U}_h^m)$$

For steady problems $\Delta t \to \infty$.

Linear System

$A_h \mathbf{x}_h = b_h$ must be solved at each time step

$$A = \frac{1}{\Delta t}\mathcal{M}_h + \frac{\partial R_h}{\partial \mathbf{U}_h} \qquad \mathbf{x} = \Delta \mathbf{U}_h^m \qquad \mathbf{b} = -R_h(\mathbf{U}_h^m)$$

Parallel GLS Solver

GLS Solver Features

- Many ideas borrowed from ProjectX (Discontinuous Galerkin solver developed at ACDL since 2002)
- High-order GLS discretization on unstructured mesh
- h/p-scaling artificial viscosity for shock capturing.
- Parallel communication via MPI
- Local direct solve using UMFPACK
- Equation set: Poisson, advection-diffusion, Burger, Euler
- $\sim 70,000$ lines of C code

BDDC Spaces

Primal and Dual Spaces

Decompose $\tilde{V}_h(\Gamma) = (V_{h,\Pi}(\Gamma)) \oplus (\oplus_{i=1}^N V_{h,\Delta}(\Gamma_i))$

$$\text{Local Dual:} \quad V_{h,\Delta}(\Gamma_i) = \{v \in V_h(\Gamma_i) : v = 0 \text{ on primal constraint}\}$$

$$\text{Primal:} \quad V_{h,\Pi}(\Gamma) = \{v \in \tilde{V}_h(\Gamma) : \sum_{i=1}^N \tilde{a}_i(v|_{\Omega_i}, \phi_\Delta|_{\Omega_i}) = 0,$$
$$\forall \phi_\Delta \in \oplus_{i=1}^N V_{h,\Delta}(\Gamma_i)\}$$

- Dual space $V_{h,\Delta}(\Gamma) = \oplus_{i=1}^N V_{h,\Delta}(\Gamma_i)$ is completely localized
- Primal space has DOF of $\mathcal{O}(N)$

Primal and Dual Schur Complement

- Primal Schur complement: $S_\Pi : V_{h,\Pi}(\Gamma) \to V'_{h,\Pi}(\Gamma)$

$$\langle S_\Pi v_\Pi, \phi_\Pi \rangle = \sum_{i=1}^N \tilde{a}_i(v|_{\Omega_i}, \phi|_{\Omega_i}), \quad \forall v_\Pi, \phi_\Pi \in V_{h,\Pi}(\Gamma)$$

- Local dual Schur complement: $S_{\Delta,i} : V_{h,\Delta}(\Gamma_i) \to V'_{h,\Delta}(\Gamma_i)$

$$\langle S_{\Delta,i} v_{\Delta,i}, \phi_{\Delta,i} \rangle = \tilde{a}_i(v_{\Delta,i}, \phi_{\Delta,i}) \quad \forall v_{\Delta,i}, \phi_{\Delta,i} \in V_{h,\Delta}(\Gamma_i)$$

BDDC Preconditioner

Injections and Averaging Operator

- $R^T_{\Gamma\Pi} : V_{h,\Pi}(\Gamma) \rightarrow \tilde{V}_h(\Gamma)$
- $R^T_{\Gamma\Delta} : V_{h,\Delta}(\Gamma_i) \rightarrow \tilde{V}_h(\Gamma)$
- $R^T_{D,i} : \tilde{V}_h(\Gamma) \rightarrow V_h(\Gamma)$ s.t. $v = \sum_{i=1}^{N} R^T_{D,i} v|_{\Omega_i}, \forall v \in V_h(\Gamma)$

BDDC Preconditioner

BDDC preconditioner is given by

$$M^{-1}_{BDDC} = \sum_{i=1}^{N} R^T_{D,i}(T_{\text{sub},i} + T_{\text{coarse}}) D_i R_{D,i}$$

where $T_{\text{sub},i} = R^T_{\Gamma\Delta,i} S^{-1}_{\Delta,i} R_{\Gamma\Delta,i}$ and $T_{\text{coarse}} = R_{\Gamma\Pi} S^{-1}_{\Pi} R^T_{\Gamma\Pi}$

Condition Number Estimate for Coercive, Symmetric $a(\cdot,\cdot)$

$$\kappa(M^{-1}_{BDDC} S) \leq C(1 + \log^2(H/h))$$

Galerkin Least Squares Discretization

Background

- SUPG developed by Hughes [1982]; analyzed by Johnson [1984]
- SUPG extended to GLS by Hughes [1989]

GLS Discretization

Find $u \in V \subset H^1(\Omega)$ s.t.

$$a(u,\phi) + (\mathcal{L}u, \tau\mathcal{L}\phi)_{\Omega,\mathcal{T}_h} = (f,\phi)_\Omega + (f, \tau\mathcal{L}\phi)_{\Omega,\mathcal{T}_h}, \quad \forall \phi \in V$$

where $(\cdot,\cdot)_{\Omega,\mathcal{T}_h} = \sum_K \int_K \cdot\, \cdot\, dK$ and $\tau \sim \begin{cases} h, & P_e \gg 1 \\ h^2, & P_e \ll 1 \end{cases}$.

Stability and *A Priori* Error Estimate for Hyperbolic Case ($\nu = 0$)

$$a(u,u) \geq h\|\beta \cdot \nabla u\|^2_{L_2(\Omega)}$$

$$\sqrt{h}\|\beta \cdot \nabla(u - u_h)\|_{L_2(\Omega)} + \|u - u_h\|_{L_2(\Omega)} \leq Ch^{p+\frac{1}{2}}\|u\|_{H^{p+1}(\Omega)}$$

Stabilization Matrix

High-Order Stabilization Matrix

- In order to keep $\frac{\text{cond}(\mathcal{L}_{\text{stabilized}})}{\text{cond}(\mathcal{L}_{\text{unstabilized}})} = \mathcal{O}(1)$ as $p \to \infty$ and $h \to 0$,

$$\tau \sim h^2/p^2, \quad P_e \ll 1$$

- One choice of τ in $\mathbb{R}^d$

$$\tau^{-1} = \sum_{i=1}^{d+1} \left| \beta \cdot \nabla \xi^i \right| + \frac{p^2}{d} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_j}{\partial x_l} \nu_{kl}$$

Topics Studied in High-Order GLS

- Comparison of SUPG and GLS
- Subcell-shock capturing using h/p scaling artificial viscosity and resolution based shock indicator
- GLS for highly anisotropic mesh

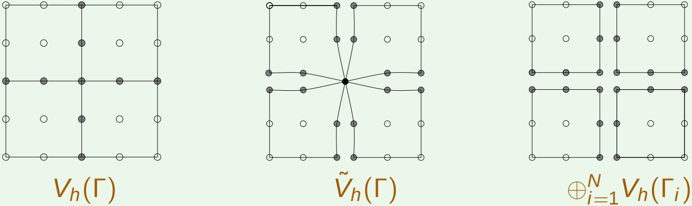
BDDC Preconditioner

Features of BDDC

- Equipped with a coarse space (primal space) that provides global communication and makes subdomain problems well-posed.
- Operates on partially assembled space: $\tilde{V}_h(\Gamma)$

$$\tilde{V}_h(\Gamma) = \{v \in L_2(\Omega) : v \in \oplus_{i=1}^{N} V_h(\Gamma_i) \text{ and } v \text{ is continuous on primal constraints}\}$$

  - Examples of primal constraints
    - Values at the corners of Ω_i
    - State averages on the edges of Ω_i



$V_h(\Gamma)$ $\tilde{V}_h(\Gamma)$ $\oplus_{i=1}^{N} V_h(\Gamma_i)$

BDDC Spaces

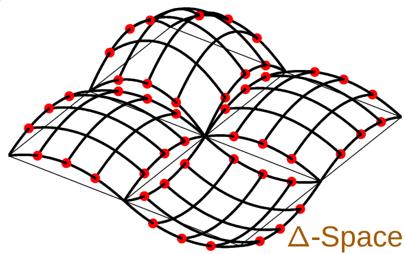
Primal and Dual Spaces

Decompose $\tilde{V}_h(\Gamma) = (V_{h,\Pi}(\Gamma)) \oplus (\oplus_{i=1}^{N} V_{h,\Delta}(\Gamma_i))$

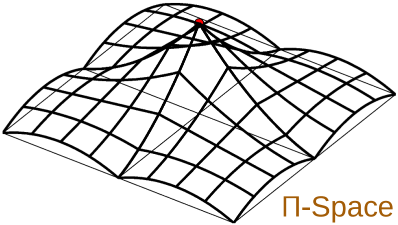
$$\text{Local Dual:} \quad V_{h,\Delta}(\Gamma_i) = \{v \in V_h(\Gamma_i) : v = 0 \text{ on primal constraint}\}$$

$$\text{Primal:} \quad V_{h,\Pi}(\Gamma) = \{v \in \tilde{V}_h(\Gamma) : \sum_{i=1}^{N} \tilde{a}_i(v|_{\Omega_i}, \phi_\Delta|_{\Omega_i}) = 0, \\ \forall \phi_\Delta \in \oplus_{i=1}^{N} V_{h,\Delta}(\Gamma_i)\}$$

- Dual space $V_{h,\Delta}(\Gamma) = \oplus_{i=1}^{N} V_{h,\Delta}(\Gamma_i)$ is completely localized
- Primal space has DOF of $\mathcal{O}(N)$



Δ-Space



Π-Space