Supplementary Information

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After substituting our analytical solutions (Eqs. [**14**], [**15**], [**16**], and [**17**]) into the nondimensionalized, modified sickle cell model (Eqs. [**10**], [**11**], [**12**], and [**13**]) and dropping negligible terms (Eq. [**19**]), we are able to extract a system of autonomous nonlinear ordinary differential equations (Eqs. [**20**], [**21**], and [**22**]) and a system of nonlinear algebraic equations (Eq. [**25**]).The variables $\psi_1, \psi_2,$ and ψ_3 are undefined functions of τ. The system of nonlinear ordinary differential equations that we will obtain expresses a dynamic relationship among the variables ψ_1, ψ_2, ψ_3; and the system of nonlinear algebraic equations expresses a relationship among the free parameters (see [**18**] and [**24**]). In the following analysis, we will separately examine each equation in the nondimensionalized, sickle cell model (see [**9**], [**10**], [**11**] and [**12**]) and then combine our results in the last section.

Equation [9].
Substituting [**13**] and [**14**] into [**9**] yields

$$\psi_1 e^{k_1 r' + k_2 z'}(A_{11}{r'}^2 + A_{12}{r'}^3) = 0 \qquad [25]$$

where $A_{1i\ 1\le i\le 2}$ are nonlinear algebraic equations. These nonlinear algebraic equations are functions of the free parameters in [**18**] such that

$$A_{1i} = A_{1i}(k_1, k_2, a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6).$$

For [**25**] to hold true, then

$$A_{11} = 0 \ \text{ and } \ A_{12} = 0, \qquad [26]$$

because $\psi_1 = 0$ only admits trivial solutions to v'_r and v'_z.

Equation [10].
Substituting [**13**] and [**14**] into [**10**] yields

$$\begin{aligned}&\frac{d\psi_1}{d\tau}\left[e^{k_1 r'+k_2 z'}\left(B_{11}r' + B_{12}{r'}^2 + B_{13}{r'}^3\right)\right] \\ &+{\psi_1}^2 e^{k_1 r'+k_2 z'}\left(B_{21}r' + B_{22}{r'}^2 + B_{23}{r'}^3\right) \\ &+ \psi_1^2 e^{2k_1 r'+2k_2 z'}\left(B_{32}{r'}^2 + B_{33}{r'}^3\right) \\ &= l_1\psi_1 e^{k_1'+k_2 z'}\left(B_{41}r' + B_{42}{r'}^2 + B_{43}{r'}^3\right) \qquad [27]\end{aligned}$$

where $l_1 = \frac{\mu T}{\rho L^2}$ and $B_{ij\ 1\le i,j\le 4}$ are nonlinear algebraic equations. These nonlinear algebraic equations are functions of the free parameters in [**18**] such that

$$B_{ij} = B_{ij}(k_1, k_2, a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6).$$

According to the Galerkin method, we can arbitrarily assume relations between the B_{ij}'s. Lets set

$$\begin{array}{ll} B_{2i} = B_{1i} & \text{for } 1 \le i \le 3, \\ B_{3i} = 0 & \text{for } 2 \le i \le 3, \\ B_{4i} = B_{1i} & \text{for } 1 \le i \le 3. \end{array} \qquad [28]$$

Substituting [**28**] into [**27**] yields

$$\begin{aligned}&\frac{d\psi_1}{d\tau}\left[e^{k_1 r'+k_2 z'}\left(B_{11}r' + B_{12}{r'}^2 + B_{13}{r'}^3\right)\right] \\ &+{\psi_1}^2 e^{k_1 r'+k_2 z'}\left(B_{11}r' + B_{12}{r'}^2 + B_{13}{r'}^3\right) \\ &+ \psi_1^2 e^{2k_1 r'+2k_2 z'}\left(0\cdot{r'}^2 + 0\cdot{r'}^3\right) \\ &= l_1\psi_1 e^{k_1 r'+k_2 z'}\left(B_{11}r' + B_{12}{r'}^2 + B_{13}{r'}^3\right). \qquad [29]\end{aligned}$$

[**29**] can be factored and then rewritten as

$$\left(\frac{d\psi_1}{d\tau} + {\psi_1}^2 - l_1\psi_1\right) e^{k_1 r'+k_2 z'}\left(B_{11}r' + B_{12}{r'}^2 + B_{13}{r'}^3\right) = 0. \qquad [30]$$

The above relation implies that

$$\frac{d\psi_1}{d\tau} + {\psi_1}^2 - l_1\psi_1 = 0, \qquad [31]$$

because having $B_{11}, B_{12},$ and B_{13} equal to zero would yield trivial solutions for v'_r and v'_z. [**31**] can then be rewritten as

$$\frac{d\psi_1}{d\tau} = l_1\psi_1 - {\psi_1}^2 \qquad [32]$$

Equation [11].

Substituting [**13**], [**14**], [**15**] and [**16**] into [**11**] yields

$$\begin{aligned}&\frac{d\psi_2}{d\tau}\left[e^{k_1 r'+k_2 z'}(C_{113}{r'}^3) + C_{103}{r'}^3 + C_{100}\right] \\ &+ \psi_1\psi_2\left[e^{k_1 r'+k_2 z'}(C_{213}{r'}^3)\right] \\ &+\psi_1\left[e^{k_1 r'+k_2 z'}(C_{313}{r'}^3 + C_{312}{r'}^2) + C_{303}{r'}^3 + C_{302}{r'}^2 + C_{301}r' + C_{300}\right] \\ &= \psi_2\left[e^{k_1 r'+k_2 z'}(C_{411}r' + C_{412}{r'}^2 + C_{413}{r'}^3)\right. \\ &\qquad \left. + \left(C_{403}{r'}^3 + C_{402}{r'}^2 + C_{401}r' + C_{400}\right)\right] \\ &+{\psi_2}^2\left[e^{k_1 r'+k_2 z'}(C_{513}{r'}^3) + C_{503}{r'}^3 + C_{500}\right] \\ &+\psi_3\left[e^{k_1 r'+k_2 z'}(C_{613}{r'}^3) + C_{603}{r'}^3 + C_{600}\right] \\ &+\psi_2\psi_3\left[e^{k_1 r'+k_2 z'}(C_{713}{r'}^3) + C_{703}{r'}^3 + C_{700}\right] \\ &+l_2{\psi_2}^2\psi_3\left[e^{k_1 r'+k_2 z'}(C_{813}{r'}^3) + C_{803}{r'}^3 + C_{800}\right] \\ &+ C_{003}{r'}^3 + C_{000} \qquad [33]\end{aligned}$$

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where $l_2 = \frac{k^{Hb}[Hb]P_cT}{\alpha p^{\mathrm{Hb}}_{O_2,50\%}{}^2}$ and $C_{ijk\ 0\leq i\leq 12, 0\leq j\leq 1, 0\leq k\leq 3}$ are nonlinear algebraic equations. These nonlinear algebraic equations are functions of the free parameters in **[18]** such that

$$C_{ijk} = C_{ijk}(k_1, k_2, a_1, a_2, a_3, b_1, .., b_6, c_1, .., c_{23}, d_1, .., d_{23}).$$

According to the Galerkin method, we can assume arbitrary relations between the C_{ijk}'s. Lets set

$$C_{ijk} = \begin{cases} \tilde{e}_{i-2}\cdot C_{1jk} & \text{for } i=3, 0\leq j\leq 1, k=\{0,3\} \\ -\tilde{e}_{i-2}\cdot C_{1jk} & \text{for } 4\leq i\leq 5, 0\leq j\leq 1, k=\{0,3\} \\ \tilde{e}_{i-2}\cdot C_{1jk} & \text{for } 6\leq i\leq 8, 0\leq j\leq 1, k=\{0,3\} \end{cases} \quad [34]$$

and also let

$$\begin{array}{l} C_{213}=0, C_{312}=0, C_{302}=0, C_{301}=0, C_{412}=0, \\ C_{411}=0, C_{402}=0, C_{401}=0, C_{003}=0, C_{000}=0. \end{array} \quad [35]$$

where $\tilde{e}_{i\ 1\leq i\leq 6}$ are undefined, proportionality constants. Substituting **[34]** and **[35]** into **[33]** yields

$$\begin{aligned} &\frac{d\psi_2}{d\tau}\left[e^{k_1r'+k_2z'}(C_{113}r'^3)+C_{103}r'^3+C_{100}\right] \\ &\quad+\left[\psi_1\psi_2e^{k_1r'+k_2z'}(0\cdot r'^3)\right] \\ &\quad\quad+\psi_1\left[e^{k_1r'+k_2z'}(\tilde{e}_1C_{113}r'^3+0\cdot r'^2)\right. \\ &\quad\quad\quad\left.+\tilde{e}_1C_{103}r'^3+0\cdot r'^2+0\cdot r'+\tilde{e}_1C_{100}\right] \\ &=\psi_2\left[e^{k_1r'+k_2z'}(0\cdot r'+0\cdot r'^2-\tilde{e}_2C_{113}r'^3)\right. \\ &\quad\quad\left.+\left(-\tilde{e}_2C_{103}r'^3+0\cdot r'^2+0\cdot r'-\tilde{e}_2C_{100}\right)\right] \\ &\quad+\psi_2{}^2\left[-e^{k_1r'+k_2z'}(\tilde{e}_3C_{113}r'^3)-\tilde{e}_3C_{103}r'^3-\tilde{e}_3C_{100}\right] \\ &\quad+\psi_3\left[e^{k_1r'+k_2z'}(\tilde{e}_4C_{113}r'^3)+\tilde{e}_4C_{103}r'^3+\tilde{e}_4C_{100}\right] \\ &\quad+\psi_2\psi_3\left[e^{k_1r'+k_2z'}(\tilde{e}_5C_{113}r'^3)+\tilde{e}_5C_{103}r'^3+\tilde{e}_5C_{100}\right] \\ &\quad+l_2\psi_2{}^2\psi_3\left[e^{k_1r'+k_2z'}(\tilde{e}_6C_{113}r'^3)+\tilde{e}_6C_{103}r'^3+\tilde{e}_6C_{100}\right] \\ &\quad\quad+0\cdot r'^3+0 \end{aligned} \quad [36]$$

and, after simplifying **[36]**, we obtain

$$\begin{aligned} &\frac{d\psi_2}{d\tau}\left[e^{k_1r'+k_2z'}(C_{113}r'^3)+C_{103}r'^3+C_{100}\right] \\ &\quad+\tilde{e}_1\psi_1\left[e^{k_1r'+k_2z'}(C_{113}r'^3)+C_{103}r'^3+C_{100}\right] \\ &=-\tilde{e}_2\psi_2\left[e^{k_1r'+k_2z'}(C_{113}r'^3)+C_{103}r'^3+C_{100}\right] \\ &\quad-\tilde{e}_3\psi_2{}^2\left[e^{k_1r'+k_2z'}(C_{113}r'^3)+C_{103}r'^3+C_{100}\right] \\ &\quad+\tilde{e}_4\psi_3\left[e^{k_1r'+k_2z'}(C_{113}r'^3)+C_{103}r'^3+C_{100}\right] \\ &\quad+\tilde{e}_5\psi_2\psi_3\left[e^{k_1r'+k_2z'}(C_{113}r'^3)+C_{103}r'^3+C_{100}\right] \\ &\quad+\tilde{e}_6l_2\psi_2{}^2\psi_3\left[e^{k_1r'+k_2z'}(C_{113}r'^3)+C_{103}r'^3+C_{100}\right]. \end{aligned} \quad [37]$$

Factoring **[37]** yields

$$\begin{aligned} &\left(\frac{d\psi_2}{d\tau}+\tilde{e}_1\psi_1+\tilde{e}_2\psi_2+\tilde{e}_3\psi_2{}^2-\psi_3\left(\tilde{e}_4+\tilde{e}_5\psi_2\right.\right. \\ &\quad\left.\left.+\tilde{e}_6l_2\psi_2{}^2\right)\right)\cdot\left[e^{k_1r'+k_2z'}(C_{113}r'^3)+C_{103}r'^3+C_{100}\right]=0 \end{aligned} \quad [38]$$

The above relation implies that

$$\frac{d\psi_2}{d\tau}+\tilde{e}_1\psi_1+\tilde{e}_2\psi_2+\tilde{e}_3\psi_2{}^2-\psi_3(\tilde{e}_4+\tilde{e}_5\psi_2+\tilde{e}_6l_2\psi_2{}^2)=0, \quad [39]$$

because having $C_{113}, C_{103},$ and C_{100} equal to zero would yield trivial solutions for v'_r, v'_z, S' and P'. Equation **[39]** can then be rewritten as

$$\frac{d\psi_2}{d\tau}=-\tilde{e}_1\psi_1-\left(\tilde{e}_2\psi_2+\tilde{e}_3\psi_2{}^2\right)+\psi_3(\tilde{e}_4+\tilde{e}_5\psi_2+\tilde{e}_6l_2\psi_2{}^2). \quad [40]$$

Equation [12].
Substituting **[13]**,**[14]**,**[15]**, and **[16]** into **[12]** yields

$$\begin{aligned} &\frac{d\psi_3}{d\tau}\left[e^{k_1r'+k_2z'}(D_{113}r'^3)+D_{103}r'^3+D_{100}\right] \\ &\quad+\left[\psi_1\psi_3e^{k_1r'+k_2z'}(D_{213}r'^3)\right] \\ &\quad\quad+\psi_1\left[e^{k_1r'+k_2z'}(D_{313}r'^3+D_{312}r'^2)+D_{303}r'^3+D_{302}r'^2+D_{301}r'+D_{300}\right] \\ &\quad\quad=\psi_2\left[e^{k_1r'+k_2z'}(D_{411}r'+D_{412}r'^2+D_{413}r'^3)\right. \\ &\quad\quad\quad\left.+\left(D_{403}r'^3+D_{402}r'^2+D_{401}r'+D_{400}\right)\right] \\ &\quad\quad+\psi_2{}^2\left[e^{k_1r'+k_2z'}(D_{513}r'^3)+D_{503}r'^3+D_{500}\right] \\ &\quad\quad+\psi_3\left[e^{k_1r'+k_2z'}(D_{613}r'^3)+D_{603}r'^3+D_{600}\right] \\ &\quad\quad+\psi_2\psi_3\left[e^{k_1r'+k_2z'}(D_{713}r'^3)+D_{703}r'^3+D_{700}\right] \\ &\quad\quad+l_3\psi_2{}^2\psi_3\left[e^{k_1r'+k_2z'}(D_{813}r'^3)+D_{803}r'^3+D_{800}\right] \\ &\quad\quad\quad+D_{003}r'^3+D_{000} \end{aligned} \quad [41]$$

where $l_3 = \frac{k^{Hb}P_c^2T}{p^{\mathrm{Hb}}_{O_2,50\%}{}^2}$ and $D_{ijk\ 0\leq i\leq 8, 0\leq j\leq 1, 0\leq k\leq 3}$ are nonlinear algebraic equations. These nonlinear algebraic equations are functions of the free parameters in **[18]** such that

$$D_{ijk} = D_{ijk}(k_1, k_2, a_1, a_2, a_3, b_1, .., b_6, c_1, .., c_{23}, d_1, .., d_{23}).$$

According to the Galerkin method, we can assume arbitrary relations between the D_{ijk}'s. Lets set

$$D_{ijk} = \begin{cases} \tilde{e}_{i+4}\cdot D_{1jk} & \text{for } 3\leq i\leq 5, 0\leq j\leq 1, k=\{0,3\} \\ -\tilde{e}_{i+4}\cdot D_{1jk} & \text{for } 6\leq i\leq 8, 0\leq j\leq 1, k=\{0,3\} \end{cases} \quad [42]$$

and

$$\begin{array}{l} D_{213}=0, D_{312}=0, D_{302}=0, D_{301}=0, D_{412}=0, \\ D_{411}=0, D_{402}=0, D_{401}=0, D_{003}=0, D_{000}=0. \end{array} \quad [43]$$

where $\tilde{e}_{i\ 7\leq i\leq 12}$ are undefined, arbitrary constants. Substituting **[42]** and **[43]** into **[41]** yields

$$\frac{d\psi_3}{d\tau}\left[e^{k_1r'+k_2z'}(D_{113}{r'}^3)+D_{103}{r'}^3+D_{100}\right]$$
$$+\left[\psi_1\psi_3e^{k_1r'+k_2z'}(0\cdot{r'}^3)\right]$$
$$+\psi_1\left[e^{k_1r'+k_2z'}(\tilde{e}_7D_{113}{r'}^3+0\cdot{r'}^2)\right.$$
$$\left.+\tilde{e}_7D_{103}{r'}^3+0\cdot{r'}^2+0\cdot r'+\tilde{e}_7D_{100}\right]$$
$$=\psi_2\left[e^{k_1r'+k_2z'}(0\cdot r'+0\cdot{r'}^2+\tilde{e}_8D_{113}{r'}^3)\right.$$
$$\left.+\left(\tilde{e}_8D_{103}{r'}^3+0\cdot{r'}^2+0\cdot r'+\tilde{e}_8D_{100}\right)\right]$$
$$+{\psi_2}^2\left[e^{k_1r'+k_2z'}(\tilde{e}_9D_{113}{r'}^3)+\tilde{e}_9D_{103}{r'}^3+\tilde{e}_9D_{100}\right]$$
$$+\psi_3\left[-e^{k_1r'+k_2z'}(\tilde{e}_{10}D_{113}{r'}^3)-\tilde{e}_{10}D_{103}{r'}^3-\tilde{e}_{10}D_{100}\right]$$
$$+\psi_2\psi_3\left[-e^{k_1r'+k_2z'}(\tilde{e}_{11}D_{113}{r'}^3)-\tilde{e}_{11}D_{103}{r'}^3-\tilde{e}_{11}D_{100}\right]$$
$$+l_3{\psi_2}^2\psi_3\left[-e^{k_1r'+k_2z'}(\tilde{e}_{12}D_{113}{r'}^3)-\tilde{e}_{12}D_{103}{r'}^3-\tilde{e}_{12}D_{100}\right]$$
$$+0\cdot{r'}^3+0 \quad \mathbf{[44]}$$

and, after simplifying **[44]**, we obtain

$$\frac{d\psi_3}{d\tau}\left[e^{k_1r'+k_2z'}(D_{113}{r'}^3)+D_{103}{r'}^3+D_{100}\right]$$
$$+\tilde{e}_7\psi_1\left[e^{k_1r'+k_2z'}(D_{113}{r'}^3)+D_{103}{r'}^3+D_{100}\right]$$
$$=\tilde{e}_8\psi_2\left[e^{k_1r'+k_2z'}(D_{113}{r'}^3)+D_{103}{r'}^3+D_{100}\right]$$
$$+\tilde{e}_9{\psi_2}^2\left[e^{k_1r'+k_2z'}(D_{113}{r'}^3)+D_{103}{r'}^3+D_{100}\right]$$
$$-\tilde{e}_{10}\psi_3\left[e^{k_1r'+k_2z'}(D_{113}{r'}^3)+D_{103}{r'}^3+D_{100}\right]$$
$$-\tilde{e}_{11}\psi_2\psi_3\left[e^{k_1r'+k_2z'}(D_{113}{r'}^3)+D_{103}{r'}^3+D_{100}\right]$$
$$-\tilde{e}_{12}l_3{\psi_2}^2\psi_3\left[e^{k_1r'+k_2z'}(D_{113}{r'}^3)+D_{103}{r'}^3+D_{100}\right]. \quad \mathbf{[45]}$$

Factoring **[45]** yields

$$\left(\frac{d\psi_3}{d\tau}+\tilde{e}_7\psi_1-\tilde{e}_8\psi_2-\tilde{e}_9{\psi_2}^2+\psi_3\left(\tilde{e}_{10}+\tilde{e}_{11}\psi_2\right.\right.$$
$$\left.\left.+\ \tilde{e}_{12}l_3{\psi_2}^2\right)\right)\cdot\left[e^{k_1r'+k_2z'}(D_{113}{r'}^3)+D_{103}{r'}^3+D_{100}\right]=0.$$

The above relation implies that

$$\frac{d\psi_3}{d\tau}+\tilde{e}_7\psi_1-\tilde{e}_8\psi_2-\tilde{e}_9{\psi_2}^2+\psi_3(\tilde{e}_{10}+\tilde{e}_{11}\psi_2+\tilde{e}_{12}l_3{\psi_2}^2)=0, \quad \mathbf{[46]}$$

because having D_{113}, D_{103} and D_{100} equal to zero yields trivial solutions for v'_r, v'_z, S' and P'. Equation **[46]** can then be rewritten as

$$\frac{d\psi_3}{d\tau}=-\tilde{e}_7\psi_1+\tilde{e}_8\psi_2+\tilde{e}_9{\psi_2}^2-\psi_3(\tilde{e}_{10}+\tilde{e}_{11}\psi_2+\tilde{e}_{12}l_3{\psi_2}^2). \quad \mathbf{[47]}$$

Results

We can combine the results from each subsection, which yields a system of nonlinear ordinary differential equations (Eqs. **[32]**, **[40]**, **[47]**) and a system of nonlinear algebraic equations (Eqs. **[26]**, **[28]**, **[34]**, **[35]**, **[42]**, **[43]**) which are shown below. The system of nonlinear algebraic equations is written in standard form such that the right hand side of each algebraic equation is zero. In **[25]**, the system of $F_{i\ 1\le i\le 78}$ (such that $F_i=0$) represent the system of nonlinear algebraic equations in **[51]**,**[52]**, **[53]**,**[54]**, and **[55]**.

System of Ordinary Differential Equations

$$\frac{d\psi_1}{d\tau}=l_1\psi_1-{\psi_1}^2 \quad \mathbf{[48]}$$

$$\frac{d\psi_2}{d\tau}=-\tilde{e}_1\psi_1-\left(\tilde{e}_2\psi_2+\tilde{e}_3{\psi_2}^2\right)+\psi_3(\tilde{e}_4+\tilde{e}_5\psi_2+\tilde{e}_6l_2{\psi_2}^2) \quad \mathbf{[49]}$$

$$\frac{d\psi_3}{d\tau}=-\tilde{e}_7\psi_1+\tilde{e}_8\psi_2+\tilde{e}_9{\psi_2}^2-\psi_3(\tilde{e}_{10}+\tilde{e}_{11}\psi_2+\tilde{e}_{12}l_3{\psi_2}^2) \quad \mathbf{[50]}$$

where

$$l_1=\frac{\mu T}{\rho L^2}, l_2=\frac{k^{Hb}[Hb]P_cT}{\alpha\cdot {p^{\mathrm{Hb}}_{O_2,50\%}}^2}, l_3=\frac{k^{Hb}P_c^2T}{{p^{\mathrm{Hb}}_{O_2,50\%}}^2},$$

and $\tilde{e}_{i\ 1\le i\le 12}$ are free parameters.

System of Nonlinear Algebraic Equations

$$A_{11}=0 \text{ and } A_{12}=0 \quad \mathbf{[51]}$$

$$\begin{aligned} B_{2i}-B_{1i}&=0 && \text{for } 1\le i\le 3\\ B_{3i}&=0 && \text{for } 2\le i\le 3\\ B_{4i}-B_{1i}&=0 && \text{for } 1\le i\le 3\end{aligned} \quad \mathbf{[52]}$$

$$\begin{aligned} C_{ijk}-\tilde{e}_{i-2}\cdot C_{1jk}&=0 && \text{for } i=3, 0\le j\le 1, k=\{0,3\}\\ C_{ijk}+\tilde{e}_{i-2}\cdot C_{1jk}&=0 && \text{for } 4\le i\le 5, 0\le j\le 1, k=\{0,3\}\\ C_{ijk}-\tilde{e}_{i-2}\cdot C_{1jk}&=0 && \text{for } 6\le i\le 8, 0\le j\le 1, k=\{0,3\}\end{aligned} \quad \mathbf{[53]}$$

$$\begin{aligned} &C_{213}=0, C_{312}=0, C_{302}=0, C_{301}=0, C_{412}=0,\\ &C_{411}=0, C_{402}=0, C_{401}=0, C_{003}=0, C_{000}=0.\end{aligned} \quad \mathbf{[54]}$$

$$\begin{aligned} D_{ijk}-\tilde{e}_{i+4}\cdot D_{1jk}&=0 && \text{for } 3\le i\le 5, 0\le j\le 1, k=\{0,3\}\\ D_{ijk}+\tilde{e}_{i+4}\cdot D_{1jk}&=0 && \text{for } 6\le i\le 8, 0\le j\le 1, k=\{0,3\}\end{aligned} \quad \mathbf{[55]}$$

$$\begin{aligned} &D_{213}=0, D_{312}=0, D_{302}=0, D_{301}=0, D_{412}=0,\\ &D_{411}=0, D_{402}=0, D_{401}=0, D_{003}=0, D_{000}=0.\end{aligned} \quad \mathbf{[56]}$$