Moore families

Let \( \langle P, \sqsubseteq \rangle \) be a poset with top element \( \top \). A Moore family is \( M \subseteq P \), such that:

- \( \top \in M \)
- If \( X \in \wp(M) \setminus \{\emptyset\} \) then \( \sqcap X \) exists in \( P \) and \( \sqcap X \in M \)
  or equivalently \(^1\)
- If \( X \in \wp(M) \) then \( \sqcap X \) exists in \( P \) and \( \sqcap X \in M \)
  that is \( M \) is closed under meet.

\(^1\) Since \( \emptyset = \bot \).
Moore closure

A Moore closure $\mathcal{M}$ is for the particular case of $\langle \wp(X), \subseteq, \emptyset, X, \cup, \cap \rangle$:

- $X \in \mathcal{M}$
- If $Y \subseteq \mathcal{M}$ then $\cap Y \in \mathcal{M}$

The elements of $\mathcal{M}$ are called Moore closed sets or closed sets or saturated sets, etc., depending on the mathematical context. A Moore closure is also called Moore collection, closed system, $\cap$-structure, etc.

Example of Moore closure

- Let $\equiv$ be an equivalence relation on a set $X$
- Let us define $S \subseteq X$ to be $\equiv$-saturated iff it is a union of equivalence classes:

  $$\forall x_1, x_2 \in X : (x_1 \in S \land x_2 \equiv x_1) \implies (x_2 \in S)$$

- Let $\mathcal{M} = \{S \subseteq X \mid S \text{ is } \equiv \text{-saturated}\}$
- $\mathcal{M}$ is a Moore family in $\langle \wp(X), \subseteq, \emptyset, X, \cup, \cap \rangle$

**Proof.** Assume $M = \{S_\alpha \mid \alpha \in \Delta\} \subseteq \mathcal{M}$. Then $x_1 \in \bigcap M \implies x_1 \in \bigcap_{\alpha \in \Delta} S_\alpha \implies \forall \alpha \in \Delta : x_1 \in S_\alpha \implies (\forall \alpha \in \Delta : (x_1 \in S_\alpha \land x_2 \equiv x_1) \implies (x_2 \in S_\alpha))$

whence $((x_1 \in \bigcap_{\alpha \in \Delta} S_\alpha \land x_2 \equiv x_1) \implies (x_2 \in \bigcap_{\alpha \in \Delta} S_\alpha))$ so $x_1 \in \bigcap M \land x_2 \equiv x_1 \implies x_2 \in \bigcap M$ proving that $\bigcap M \in \mathcal{M}$. □

Example: convex subsets of a poset

Let $\langle P, \leq \rangle$ be a pre-order ($\leq$ is reflexive and transitive). Given $a, b \in P$, define

$$[a, b] \overset{\text{def}}{=} \{x \in P \mid a \leq x \land x \leq b\}$$

Call $S \subseteq P$ to be convex whenever

$$a, b \in S \implies [a, b] \subseteq S$$

Then $\mathcal{M} = \{S \subseteq P \mid S \text{ is convex}\}$ is a Moore family of $\langle \wp(P), \subseteq, \emptyset, P, \cup, \cap \rangle$
Proof. – Let \( S_\alpha, \alpha \in \Delta \) be a family of convex subsets of \( P \) i.e. \( \forall \alpha \in \Delta : S_\alpha \in M \).

If \( a, b \in \bigcap_{\alpha \in \Delta} S_\alpha \) then \( \forall \alpha \in \Delta : a, b \in S_\alpha \) so \( \forall \alpha \in \Delta : [a, b] \subseteq S_\alpha \) (since \( S_\alpha \) is convex) whence \( [a, b] \subseteq \bigcap_{\alpha \in \Delta} S_\alpha \) (def. of glb) proving that \( \bigcap_{\alpha \in \Delta} S_\alpha \in M \).

– If \( \bigcap_{\alpha \in \Delta} S_\alpha = \emptyset \), then \( \emptyset \) is convex, so in that case \( \bigcap_{\alpha \in \Delta} S_\alpha \in M \)

– \( M \) is closed under arbitrary intersections whence is a Moore family.

\[ \square \]

Note that in general, the lub in the Moore family \( M \) is not the same as the lub in the original poset \( P \):

So in general a Moore family of a complete lattice is not a complete sublattice of this complete lattice.

A Moore family in a poset is a complete lattice

**Theorem.** Let \( \langle P, \sqsubseteq, \sqcap, \sqcup \rangle \) be a topped poset and \( M \subseteq P \) be a Moore family then \( \langle M, \sqsubseteq, \sqcap M, \sqcup \rangle \) is a complete lattice.

**Proof.** Since \( \langle P, \sqsubseteq \rangle \) is a poset and \( M \subseteq P \), \( \langle M, \sqsubseteq \rangle \) is a poset. Being a Moore family it is topped and any subset \( S \subseteq M \) has \( \cap S \in M \) so \( \cap \) is the meet in \( M \). It follows that \( M \) is a complete lattice, which lub is:

\[ \sqcup S = \cap \{ y \in M \mid \forall x \in S : x \sqsubseteq y \} \in M \]

The infimum is \( \cap M \in M \).

\[ \square \]

Moore family/complete lattice of safety properties

Let \( \Sigma \) be a set of states and \( \Sigma^\infty \) be the set of finite or infinite sequences on \( \Sigma \). A trace property is \( P \subseteq \Sigma^\infty \). A safety property is \( S \subseteq \Sigma^\infty \) such that:

\[ \forall \sigma \in \Sigma^\infty : (\sigma \not\in S) \iff (\exists i \geq 1 : \sigma \not\prec i \not\in S) \quad (1) \]

where \( \sigma \not\prec i = \sigma_0 \ldots \sigma_{\min\{i,|\sigma|\}-1} \) and \( |\sigma| \) is the length of \( \sigma \).

**Theorem.** The set \( \text{Safe}(\Sigma^\infty) \) of safety properties on \( \langle \rho(\Sigma^\infty), \subseteq, \emptyset, \Sigma^\infty, \cup, \sqcap \rangle \) is a Moore family whence a complete lattice.

\[ \square \]
Proof. – The top element $\Sigma^\omega$ is a safety property since both sides of the implication are false in the definition (1).

Let $P_\alpha$, $\alpha \in \Delta$ be a family of safety properties: $\forall \sigma \in \Sigma^\omega : (\sigma \not\in P_\alpha) \iff (\exists i \geq 1 : (\sigma \not\sqsubset i \not\in P_\alpha))$.

- $\forall \sigma \in \Sigma^\omega : \sigma \not\in \bigcap_{\alpha \in \Delta} P_\alpha \implies \exists \alpha : \sigma \not\in P_\alpha \implies (\exists i \geq 1 : (\sigma \not\sqsubset i \not\in P_\alpha))$.
- Conversely, $\forall \sigma \in \Sigma^\omega : (\exists i \geq 1 : (\sigma \not\sqsubset i \not\in \bigcap_{\alpha \in \Delta} P_\alpha)) \implies (\exists \alpha \in \Delta : (\exists i \geq 1 : (\sigma \not\sqsubset i \not\in P_\alpha))).$

It follows that $\bigcap_{\alpha \in \Delta} P_\alpha$ is a safety property since $\forall \sigma \in \Sigma^\omega : (\sigma \not\in \bigcap_{\alpha \in \Delta} P_\alpha) \iff (\exists i \geq 1 : (\sigma \not\sqsubset i \not\in \bigcap_{\alpha \in \Delta} P_\alpha))$. So $\text{Safe}(\Sigma^\omega)$ contains the top and is closed under intersection whence is a Moore family hence is a complete lattice. \qed

Note that $\text{Safe}(\Sigma^\omega)$ is not a complete sublattice of $\langle \wp(\Sigma^\omega), \subseteq \rangle$ since $\forall n \in \mathbb{N} : \{a^n b\}$ is a safety property whereas $\bigcup_{n \in \mathbb{N}} \{a^n b\} = a^* b$ is not (since it is not closed by limits).

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**Linear (ordinal) sum of posets**

Let $\langle P, \leq \rangle$ and $\langle Q, \subseteq \rangle$ be two posets. Their linear (ordinal) sum is $\langle P, \leq \rangle \oplus \langle Q, \subseteq \rangle = \langle P \cup Q, \leq \rangle$ such that:

- $P \oplus Q \overset{\text{def}}{=} \{ (0, x) \mid x \in P \} \cup \{ (1, y) \mid y \in Q \}$
- $\langle i, x \rangle \leq \langle j, y \rangle \overset{\text{def}}{=} (i = j = 0 \land x \leq y) \lor (i = 0 \land j = 1) \lor (i = j = 1 \land x \subseteq y)$

The linear (ordinal) sum of posets is a poset. Example:

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**Bottom/top lifting**

Given the posets $\langle P, \leq \rangle$, $\langle \bot, = \rangle$ and $\langle \top, = \rangle$.

- **bottom lifting** $P_\bot \overset{\text{def}}{=} \{ \bot \} \oplus P$ adds a bottom to $P$:

- **top lifting** $P_\top \overset{\text{def}}{=} P \oplus \{ \top \}$ adds a top to $P$:
Flat ordering

Given a set $S$, and posets $\langle S, =\rangle$ and $\langle \{\bot\}, =\rangle$, Scott's flat ordering is $S_\bot$. For example:

Smashed linear sum (or smashed ordinal sum) of posets

Let $\langle P, \preceq, \top \rangle$ and $\langle Q, \sqsubseteq, \bot \rangle$ be two posets such that $P$ has a top $\top$ and $Q$ has a bottom $\bot$. Their smashed linear sum (or smashed ordinal sum) is

$$\langle P, \preceq \rangle \oplus_{\bot} \langle Q, \sqsubseteq \rangle \overset{\text{def}}{=} \langle P \setminus \{\top\}, \preceq \rangle \oplus \langle Q \setminus \{\bot\}, \sqsubseteq \rangle$$

(so that it is obtained from the linear sum $\langle P, \preceq \rangle \oplus \langle Q, \sqsubseteq \rangle$ by identifying the top $\top$ of $P$ with the bottom $\bot$ of $Q$).

Example:

Disjoint (cardinal) sum of posets

Let $\langle P, \leq \rangle$ and $\langle Q, \sqsubseteq \rangle$ be two posets. Their disjoint (cardinal) sum is $\langle P, \leq \rangle + \langle Q, \sqsubseteq \rangle \overset{\text{def}}{=} \langle P + Q, \leq \rangle$ such that:

- $P + Q \overset{\text{def}}{=} \{ (0, x) \mid x \in P \} \cup \{ (1, y) \mid y \in Q \}$
- $\langle i, x \rangle \preceq \langle j, y \rangle \overset{\text{def}}{=} (i = j = 0 \land x \leq y) \lor (i = j = 1 \land x \sqsubseteq y)$

- Intuition:
Example 1: \( \langle \{ \bot \}, \bot \rangle \oplus \langle \{ 0 \}, = \rangle \oplus (\langle \{ \bot \}, = \rangle + \langle \{ \top \}, = \rangle) \oplus \langle \{ \top \}, = \rangle \) is (up to an isomorphism):

![Diagram](image1.png)

Example 2: \( \langle \{ 0 \}, = \rangle \oplus (\langle \{ 1 \}, = \rangle + \langle \{ 2 \}, = \rangle) \oplus (\langle \{ 3 \}, = \rangle + \langle \{ 4 \}, = \rangle + \langle \{ 5 \}, = \rangle) \) is (up to an isomorphism):

![Diagram](image2.png)

### Smashed disjoint (cardinal) sum of posets

Let \( \langle P, \leq, \bot_P, \top_P \rangle \) and \( \langle Q, \subseteq, \bot_P, \top_P \rangle \) be two posets. The **smashed disjoint sum** \( \langle P, \leq \rangle + \bot \langle Q, \subseteq \rangle \) is \( \langle P + \bot Q, \leq \rangle \) where:

\[
P + \bot Q \overset{\text{def}}{=} \{ (0, x) \mid x \in P \setminus \{ \bot_P, \top_P \} \} \\
\cup \{ (1, y) \mid y \in Q \setminus \{ \bot_Q, \top_Q \} \} \\
\cup \{ \bot, \top \}
\]

with ordering \( \leq \) such that:

- \( \bot \leq \bot \leq (0, x) \leq \top \leq \top \) for all \( x \in P \setminus \{ \bot_P, \top_P \} \)
- \( \bot \leq (1, y) \leq \top \) for all \( y \in Q \setminus \{ \bot_Q, \top_Q \} \)
- \( (0, x) \leq (0, x') \) iff \( x \leq x' \) and \( x, x' \in P \setminus \{ \bot_P, \top_P \} \)
- \( (1, y) \leq (1, y') \) iff \( y \subseteq y' \) and \( y, y' \in Q \setminus \{ \bot_Q, \top_Q \} \)

### Intuition:

More generally, we can write:

- \( + \) for the cardinal sum
- \( + \bot \) for the \( \bot \)-smashed cardinal sum
- \( + \top \) for the \( \top \)-smashed cardinal sum
- \( + \bot \top \) for the \( \bot \) and \( \top \)-smashed cardinal sum

For example:

![Diagram](image3.png)
The cartesian (cardinal, componentwise) product of posets

Let \( \langle P_1, \leq_1 \rangle, \ldots, \langle P_n, \leq_n \rangle \) be posets. The cartesian product

\[
P_1 \times \ldots \times P_n \overset{\text{def}}{=} \{ (x_1, \ldots, x_n) \mid \bigwedge_{i=1}^{n} x_i \in P_i \}
\]

can be made a poset \( \langle P_1 \times \ldots \times P_n, \leq \rangle \) with the componentwise ordering:

\[
\langle x_1, \ldots, x_n \rangle \leq \langle y_1, \ldots, y_n \rangle \overset{\text{def}}{=} \bigwedge_{i=1}^{n} x_i \leq_i y_i
\]

The componentwise ordering \( \leq \) is sometimes denoted \( \leq_1 \times \ldots \times \leq_n \).

If the relations \( \leq_i, i = 1, \ldots, n \) are reflexive, symmetric, antisymmetric, transitive, a preorder, an equivalence, a directed order or a partial order then so is the componentwise ordering.

Examples:

- Smashed cartesian (cardinal, componentwise) product of posets

Let \( \langle P, \leq \rangle \) and \( \langle Q, \sqsubseteq \rangle \) be posets with infima \( \bot_P, \bot_Q \) and suprema \( \top_P \) and \( \top_Q \).

The \textbf{smashed cartesian product} \( \langle P, \leq \rangle \times_{\bot} \langle Q, \sqsubseteq \rangle \) of \( \langle P, \leq \rangle \) and \( \langle Q, \sqsubseteq \rangle \) is \( \langle P \times_{\bot} Q, \leq \rangle \) such that:

\[
P \times_{\bot} Q \overset{\text{def}}{=} \{ (x, y) \mid x \in P \setminus \{ \bot_P, \top_P \} \land y \in Q \setminus \{ \bot_Q, \top_Q \} \}
\]

where \( \bot, \top \not\in P \cup Q \) and

1. \( \bot \sqsubseteq_x \bot \leq \langle x, y \rangle \leq \top \leq \top \) for all \( x \in P \setminus \{ \bot_P, \top_P \} \) and \( y \in Q \setminus \{ \bot_Q, \top_Q \} \)
2. \( \langle x, y \rangle \leq \langle x', y' \rangle \) iff \( x \leq x' \land y \sqsubseteq y' \) for all \( x, x' \in P \setminus \{ \bot_P, \top_P \} \) and \( y, y' \in Q \setminus \{ \bot_Q, \top_Q \} \)
The lexicographic (ordinal) product of posets

Given posets \( \langle P_1, \leq_1 \rangle, \ldots, \langle P_n, \leq_n \rangle \), the cartesian product

\[
P_1 \times \ldots \times P_n \overset{\text{def}}{=} \{ (x_1, \ldots, x_n) | \bigwedge_{i=1}^{n} x_i \in P_i \}
\]

can be made a poset by the lexicographic ordering \( \leq^n \):

\[
\langle x_1, \ldots, x_n \rangle <^n \langle y_1, \ldots, y_n \rangle \overset{\text{def}}{=} \exists i \in [1, n] : \forall j < i : x_j = y_j \land x_i < y_i
\]

\[
\langle x_1, \ldots, x_n \rangle \leq^n \langle y_1, \ldots, y_n \rangle \overset{\text{def}}{=} \langle x_1, \ldots, x_n \rangle <^n
\]

\[
\bigvee_{i=1}^{n} x_i = y_i
\]

Example:

Pointwise ordering of maps on posets

Let \( f, g \in D \mapsto P \) be maps on the poset \( \langle P, \leq \rangle \). The pointwise ordering between such maps is

\[
f \leq g \overset{\text{def}}{=} \forall x \in D : f(x) \leq g(x)
\]

Example:

\[
f \in \mathbb{N} \mapsto \mathbb{N} f(x) = 2x
\]

\[
g \in \mathbb{N} \mapsto \mathbb{N} g(x) = 3x
\]

We have \( f \leq g \) since \( \forall x \in \mathbb{N} : f(x) = 2x \leq 3x = g(x) \).

If the cartesian product \( P^n = P \times \ldots \times P \) is seen as a \( n \) times map of \( [1, n] \mapsto P \), then the componentwise ordering on \( P^n \) coincide with the pointwise ordering on \( [1, n] \mapsto P \).
Cardinal power of posets

Given a set $X$ and a poset $\langle P, \leq \rangle$, the \textbf{cardinal power} $P^X$ is the poset $\langle X \mapsto P, \leq \rangle$ of maps of $X$ into $P$ for the pointwise ordering $f \leq g \iff \forall x \in X : f(x) \leq g(x)$.

Since $n = \{0, \ldots, n - 1\}$, $P^n$ can be isomorphically viewed as:
- The cardinal product $\{\langle x_0, \ldots, x_n \rangle \mid \land_{i=1}^{n-1} x_i \in P\}$
- The set of maps $n \mapsto P$

Example:

\[ \langle x \mapsto P, \leq \rangle = \begin{array}{c}
\text{x} = \begin{array}{c} x_1 \end{array} \\
\text{P} = \begin{array}{c} P_1 \end{array}
\end{array} \]

Ordinal/cardinal sum/product/power of posets/cpos/lattices/complete lattices

Let $\langle P, \leq \rangle$ and $\langle Q, \sqsubseteq \rangle$ be posets. If $\langle P, \leq \rangle$ and $\langle Q, \sqsubseteq \rangle$ are respectively
- posets
- cpos
- lattices
- complete lattices
then
- the ordinal sum $P \oplus Q$
- the smashed ordinal sum $P \oplus \bot Q$
- the cardinal sum $P + Q$
- the smashed cardinal sum $P + \bot Q$

- the ordinal product $P \otimes Q$
- the cardinal product $Q^P$
is respectively
- a poset
- a cpo
- a lattice
- a complete lattice

\textbf{Proof.} tedious but trivial.
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THE END

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