Decomposing Properties into Safety and Liveness using Predicate Logic†

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Decomposing Properties into Safety and Liveness

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ABSTRACT

A new proof is given that every property can be expressed as a conjunction of safety and liveness properties. The proof is in terms of first-order predicate logic.

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1. Introduction

Two classes of properties are of particular interest when considering programs: safety properties and liveness properties. Informally, a safety property stipulates that "bad things" do not happen during execution of a program and a liveness property stipulates that "good things" do happen (eventually) [2]. Distinguishing between safety and liveness properties is useful because knowing whether a property is safety or liveness helps when deciding how to prove that the property holds for a program.

In [1], formal definitions of safety and liveness are given and it is proved that every property can be expressed as the conjunction of a safety property and a liveness property. The formal definitions of safety and liveness are given in terms of first-order predicate logic, but the proof that every property can be decomposed into safety and liveness is not—it uses topology. The purpose of this paper is to give a proof of this theorem using only first-order predicate logic.

2. Specifying Properties

A program state is a mapping from variables to values. An execution of a concurrent program can be viewed as an infinite sequence of program states

$$\sigma = s_0 s_1 \ldots,$$

which we call a history. In a history, $s_0$ is an initial state of the program and each subsequent state results from executing a single atomic action in the preceding state. (For a terminating execution, an infinite sequence is obtained by repeating the final state.) A property is a set of such sequences.

One way to specify a property is by using first-order predicate logic. For a state $s$, define $s.v$ to be the value of variable $v$ in that state. A formula of first-order predicate logic where $s$ is the only free variable defines a set of states. For example,

$$\forall i: 1 \leq i < N: s.a[i] \leq s.a[i+1]$$

specifies the set of states in which the elements of array $a[1:N]$ are sorted. Usually "$s." is implicit and therefore left out of such a formula, resulting in the more familiar use of first-order predicate logic as an assertion language.

A set of sequences of states—a property—can also be defined using first-order predicate logic. To facilitate such specifications, for any sequence $\sigma = s_0 s_1 \ldots$ define for $0 \leq i$:

$$\sigma[i] = s_i.$$  
$$\sigma[-1] = s_0 s_1 \ldots s_{i-1}.$$  
$|\sigma|$ = the length of $\sigma$ ($\infty$ if $\sigma$ is infinite).

A formula of first-order predicate logic in which $\sigma$ is the only free variable defines the set of sequences that satisfy the formula and therefore specifies a property. For example,

$$\forall i: 0 \leq i: \sigma[i].v = 0$$

specifies the property in which the value of $v$ remains 0 throughout execution.
We write $\alpha \models P$ if $\alpha \in S^\omega$ is in the property specified by $P$. Thus,
\[
\alpha \models P = P^\alpha.
\]
\[
\alpha \nvdash P = \neg P^\alpha.
\]

3. Safety and Liveness

According to [1], a property $P$ is a safety property provided
\[
\text{Safety: } (\forall \sigma; \; \sigma \in S^\omega; \; \sigma \nvdash P \Rightarrow (\exists t; \; 0 \leq t; \; (\forall \beta; \; \beta \in S^\omega; \; \sigma[\ldots t\ldots] \nvdash P))), \tag{3.1}
\]
where $S$ is the set of program states, $S^*$ the set of finite sequences of states, $S^\omega$ the set of infinite sequences of states, and juxtaposition is used to denote concatenation of sequences. A property $P$ is a liveness property provided
\[
\text{Liveness: } (\forall \sigma; \; \sigma \in S^*; \; (\exists \beta; \; \beta \in S^\omega; \; \sigma \nvdash \beta \models P)). \tag{3.2}
\]

Given a property $P$, we are interested in defining properties $\text{Safe}(P)$ and $\text{Live}(P)$ such that
- $\text{Safe}(P)$ is a safety property,
- $\text{Live}(P)$ is a liveness property, and
- $P = \text{Safe}(P) \land \text{Live}(P)$.

Observe that if
\[
\text{Safe}(P) = P \lor M_P
\]
\[
\text{Live}(P) = P \lor \neg M_P
\]
then
\[
\text{Safe}(P) \land \text{Live}(P) = (P \lor M_P) \land (P \lor \neg M_P)
\]
\[
= (P \land P) \lor (P \land \neg M_P) \lor (M_P \land P) \lor (M_P \land \neg M_P)
\]
\[
= P
\]

Hence, we have only to look for an $M_P$ that makes $P \lor M_P$ (i.e. $\text{Safe}(P)$) a safety property and $P \lor \neg M_P$ (i.e. $\text{Live}(P)$) a liveness property.

It turns out that using
\[
M_P: (\forall t; \; 0 \leq t; \; (\exists \beta; \; \beta \in S^\omega; \; \sigma[\ldots t\ldots] \models P))
\]
suffices. First, we show formally that $\text{Safe}(P)$ satisfies definition (3.1) of safety. The proof that follows is a sequence of first-order predicate logic formulas with explanations interspersed (and delimited by « and ») of how each formula is derived from its predecessor.

Choose any $\sigma \in S^\omega$:
\[
\sigma \nvdash \text{Safe}(P)
\]
«by definition of $\text{Safe}(P)$»

$= \sigma\#(P \vee (\forall i: 0 \leq i: (\exists \beta: \beta \in S^o: \sigma[i] \beta \models P)))$

«by definition of $\#$»

$= \neg(P \vee (\forall i: 0 \leq i: (\exists \beta: \beta \in S^o: \sigma[i] \beta \models P)))^*$

«by substitution»

$= \neg(P \vee (\forall i: 0 \leq i: (\exists \beta: \beta \in S^o: \sigma[i] \beta \models P)))$

«by De Morgan’s Laws»

$= \neg P \land (\exists i: 0 \leq i: (\forall \beta: \beta \in S^o: \sigma[i] \beta \models P))$

«$A \land B \Rightarrow B$»

$= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^o: \sigma[i] \beta \models P))$

«because $(\forall x :: A) = (\forall x :: \neg A) \lor (\forall x :: A)$»

$= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^o: \sigma[i] \beta \models P \land (\forall \gamma: \gamma \in S^o: \sigma[i] \gamma \models P)))$

«because $\text{true} \land P = P$ and $(\sigma[i] \beta)[i_] = \sigma[i]$»

$= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^o: \sigma[i] \beta \models P \land (i=i) \land (\forall \gamma: \gamma \in S^o: (\sigma[i] \beta)[i_] \gamma \models P)))$

«by substitution»

$= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^o: \sigma[i] \beta \models P \land k=k) \land (\forall \gamma: \gamma \in S^o: (\sigma[i] \beta)[k] \gamma \models P)))$

«by $\exists$-Generalization»

$\Rightarrow (\exists i: 0 \leq i: (\forall \beta: \beta \in S^o: \sigma[i] \beta \models P \land (\exists k: k=i: (\forall \gamma: \gamma \in S^o: (\sigma[i] \beta)[k] \gamma \models P)))$

«by Range Widening»

$\Rightarrow (\exists i: 0 \leq i: (\forall \beta: \beta \in S^o: \sigma[i] \beta \models P \land (\exists k: 0 \leq k: (\forall \gamma: \gamma \in S^o: (\sigma[i] \beta)[k] \gamma \models P)))$

«by De Morgan’s Law»

$= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^o: \sigma[i] \beta \models P \land \neg(\forall k: 0 \leq k: (\exists \gamma: \gamma \in S^o: (\sigma[i] \beta)[k] \gamma \models P)))$

«by definition of $\#$»

$= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^o: \sigma[i] \beta \models P \land \neg(\forall k: 0 \leq k: (\exists \gamma: \gamma \in S^o: (\sigma[k] \gamma \models P))))$

«because $\alpha \# A \land \alpha \# B = \alpha \# (A \lor B)$»

$= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^o: \sigma[i] \beta \models P \land (\forall k: 0 \leq k: (\exists \gamma: \gamma \in S^o: \sigma[k] \gamma \models P)))$

«by definition of $\text{Safe}(P)$»

$= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^o: \sigma[i] \beta \models \text{Safe}(P)))$

It is not surprising that $\text{Safe}(P)$ is a safety property. If $\sigma \# \text{Safe}(P)$ then, by definition, $\sigma \# M_P$. However, this means there exists an $i$ such that

$(\forall \beta: \beta \in S^o: \sigma[i] \beta \models P)$.

We could consider prefix $\sigma[i]$ to be a “bad thing”. Thus, $\sigma$ violates a safety property whenever $\sigma \# \text{Safe}(P)$.

We now show formally that $\text{Live}(P)$ satisfies definition (3.2) of liveness.

$(\forall \alpha: \alpha \in S^* : \text{true})$

«since true = $A \lor \neg A$»

$= (\forall \alpha : \alpha \in S^* : (\exists \beta: \beta \in S^o: \alpha \beta \models P) \lor \neg(\exists \beta: \beta \in S^o: \alpha \beta \models P))$

«renaming bound variable $\beta$ to $\gamma$»

$= (\forall \alpha : \alpha \in S^* : (\exists \beta: \beta \in S^o: \alpha \beta \models P) \lor \neg(\exists \gamma: \gamma \in S^o: \alpha \gamma \models P))$

«since $\beta$ is not free in $(\exists \gamma: \gamma \in S^o: \alpha \gamma \models P)$»

$= (\forall \alpha : \alpha \in S^* : (\exists \beta: \beta \in S^o: \alpha \beta \models P) \lor \neg(\exists \gamma: \gamma \in S^o: \alpha \gamma \models P))$

«by De Morgan’s Law»

$= (\forall \alpha : \alpha \in S^* : (\exists \beta: \beta \in S^o: \alpha \beta \models P \lor (\forall \gamma: \gamma \in S^o: \alpha \gamma \models P)))$
\[
\begin{align*}
\text{«since true } \land A &= A \text{»} \\
= (\forall \alpha \in S^a): (\exists \beta: \beta \in S^o: \alpha \beta = P \lor (|i| = |\alpha| \land (\forall \gamma: \gamma \in S^o: \alpha \gamma \not= P))) \\
&\text{by substitution, since } (\alpha \beta)[i][\alpha] = \alpha \\
= (\forall \alpha \in S^a): (\exists \beta: \beta \in S^o: \alpha \beta = P \lor ((i = |\alpha|)[i]_1 \land (\forall \gamma: \gamma \in S^o: (\alpha \beta)[i][\gamma] \not= P))) \\
&\text{by } \exists\text{-Generalization} \\
\Rightarrow (\forall \alpha \in S^a): (\exists \beta: \beta \in S^o: \alpha \beta = P \lor (\exists i: i = |\alpha|: (\forall \gamma: \gamma \in S^o: (\alpha \beta)[i][\gamma] \not= P))) \\
&\text{by Range Widening} \\
\Rightarrow (\forall \alpha \in S^a): (\exists \beta: \beta \in S^o: \alpha \beta = P \lor -(\forall i: 0 \leq i: (\forall \gamma: \gamma \in S^o: (\alpha \beta)[i][\gamma] = P))) \\
&\text{by De Morgan's Law} \\
= (\forall \alpha \in S^a): (\exists \beta: \beta \in S^o: \alpha \beta = P \lor -(\forall i: 0 \leq i: (\forall \gamma: \gamma \in S^o: (\alpha \beta)[i][\gamma] = P))) \\
&\text{by definition of } \alpha \beta = A \\
= (\forall \alpha \in S^a): (\exists \beta: \beta \in S^o: \alpha \beta = P \lor (\forall i: 0 \leq i: (\exists \gamma: \gamma \in S^o: (\gamma \in \text{Live}(P)))) \\
&\text{because } \alpha \beta = A \lor \alpha \beta = B = \alpha \beta = (A \lor B) \\
= (\forall \alpha \in S^a): (\exists \beta: \beta \in S^o: \alpha \beta = P \lor -(\forall i: 0 \leq i: (\exists \gamma: \gamma \in S^o: \gamma \in \text{Live}(P)))) \\
&\text{by definition of Live}(P) \\
= (\forall \alpha \in S^a): (\exists \beta: \beta \in S^o: \alpha \beta = \text{Live}(P)) \\
&\text{by Liveness definition (3.2)} \\
= \text{Live}(P) \text{ is liveness.}
\end{align*}
\]

An informal justification that \text{Live}(P) \text{ is liveness is the following. If } \sigma \not= \text{Live}(P) \text{ then, by definition, } \sigma = M_P. \text{ From, } \sigma = M_P, \text{ we conclude that it always remains possible for some "good thing" (i.e. } \beta \text{ in } M_P \text{) to happen. This is the defining characteristic of liveness, so } \sigma \text{ violates a liveness property whenever } \sigma \not= \text{Live}(P).