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Attenuation of long interfacial waves over a randomly rough seabed

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The effects of randomly irregular bathymetry on the propagation of interfacial gravity waves are studied. Modelling sea water by a two-layer fluid of different densities, weakly nonlinear waves much longer than the sea depth but comparable to the horizontal scale of bathymetry are treated by Boussinesq approximation and multiple-scale analysis. For transient wave pulses, the governing equation for the pulse profile is shown to be an integro-differential equation combining KdV and Burgers terms. Quantitative and qualitative effect of disorder on the attenuation of wave amplitude, reduction of wave speed and change of wave profile are examined numerically and analytically based on the asymptotic approximation. For time-harmonic waves, mode-coupling equations are derived and examined for the competition between diffusion by random scattering, steepening by nonlinearity and frequency dispersion for a broad range of depth ratios.

1. Introduction

The oceanographic importance of and the literature on the dynamics of internal gravity waves have been reviewed by Miles (1980), Ostrovsky & Stepanyants (1989), Akylas (1994) and Grimshaw (1997, 2002). Helfrich & Melville (2006) have given a comprehensive survey of advances in theory as well as laboratory and field observations. Among several key mechanisms vital to the dissipation process, a full understanding of radiation damping appears still to be lacking.

For weakly nonlinear surface waves, an apparent damping mechanism by disorder has been explored by Belzons, Guazzelli & Parodi (1988) and Devillard, Dunlop & Souillard (1988), who refer to the spatial attenuation as localization because of its common root of random multiple scattering with the Anderson localization in solid-state physics. Unlike Bragg scattering by a periodically modulated seabed where strong reflection occurs for a discrete set of frequencies satisfying the Bragg resonance condition, spatial attenuation takes place at nearly all wave frequencies in disordered media. For water of constant density, Kawahara (1976) first derived modifications to the nonlinear Schrödinger equation for narrow-banded waves in water of finite depth and to the Korteweg–de Vriès (KdV) equation for long pulses in shallow water, but did not examine the physical consequences of disorder. For the case of finite depth, Mei & Hancock (2003) examined the effects of one-dimensional disorder on the linear and nonlinear regimes of side-band instability. Extensions to two-dimensional disorder have been made by Pihl, Mei & Hancock (2002) with emphasis on the combined diffraction and localization of Stokes waves. For time-periodic long waves in shallow water, Grataloup & Mei (2003) studied the effects of such radiation damping on harmonic generation. For solitary pulses in shallow water, Mei & Li (2004) derived an integro-differential equation similar to that of Kawahara which combines features of both KdV and Burgers equations, and further examined various physical ramifications of localization including the phenomenon of soliton fission in a one-layered sea over a rough bed of finite extent. Theories for variable depth have been reported by Nachbin & Papanicolaou (1992), Nachbin (1995) and Nachbin & Solna (2003), for linear waves. For weak fluctuations and weakly nonlinear long waves Rosales & Papanicolaou (1983) gave an asymptotic theory on the change of phase speed by randomness.

Gas extraction from the North Sea is a current project in the new gas field of Ormen Lange on the Norwegian continental shelf.† A 1200 km pipe line of 0.75 m diameter is to be laid on the seabed connecting the processing plant in Nyhamna, Norway to Easington, UK. The seabed can be as deep as 800 to 1100 m and is known to be exceedingly rough with a typical height of 20 m and length of 200–1000 m. Since the pipe line will be suspended between, and anchored at, the successive crests of the undersea terrain, much of its length will be exposed to the dynamic forces of current and waves. For safe design, the possible impact of long-period internal waves is of engineering concern and motivates this study.

For interfacial waves in a two-layered sea, Chen & Liu (1996) have considered slowly varying random depth where the bathymetric length scale is much longer than the characteristic wavelength. Attention was focused on a special case where the depths of the two layers are nearly the same so that KdV approximation must be modified by including cubic nonlinearity. In this paper, we describe a theory of interfacial waves in a two-layered sea with a different focus. Specifically the bathymetric length is assumed to be the same order as the wavelength, and the ratio of layer depths is in the broader range away from the special value where higher-order nonlinearity is required. Under the usual assumption of small density difference, $R = \rho'/\rho \approx 1$, where primes are used to distinguish quantities associated with the upper (lighter) layer from those of the lower layer (unprimed), the rigid-lid approximation will be made from the start so that the free-surface displacement is ignored. The Boussinesq assumption is made so that nonlinearity and dispersion are both weak and comparable,

$$\epsilon \equiv \frac{A}{H'} \sim \frac{A}{H} \ll 1, \quad \mu \equiv \frac{H'}{\ell} \sim \frac{H}{\ell} \sim (kH', kH) \ll 1 \quad \text{but} \quad \epsilon = O(\mu^2), \tag{1.1}$$

where ℓ is the characteristic length of the bathymetric variations and k the characteristic wavenumber. As in Grataloup & Mei (2003) and Mei & Li (2004) we shall consider the fluctuations of depth h from its constant mean H_o to be weak but slightly stronger than wave nonlinearity so that

$$h = H_o(1 + \mu b(x)), \tag{1.2}$$

where $b(x)$ is the dimensionless depth fluctuation. Extensions to slowly varying mean depth is straightforward by ray approximation. An integro-differential equation of the KdV–Burgers type will be derived. The evolution of solitons is first studied. Effects on the harmonic generation in time-periodic waves will also be examined.

Since the derivation and some of the analysis follow closely our recent works cited in the references, many details are omitted here.

† http://www.hydro.com/ormenlange/, http://en.wikipedia.org/wiki/Ormen-Lange.

2. Boussinesq approximation for random bathymetry

From the linearized water-wave theory, it is well known that, under the rigid-lid approximation the dispersion relation between the frequency ω and wavenumber k of a sinusoidal wavetrain on the interface of two shallow layers of immiscible fluids is

$$\omega^2 = gk^2(1-R)\frac{H'H}{H'+RH}, \quad \text{where} \quad kH', kH \ll 1, \quad R \equiv \frac{\rho'}{\rho}. \tag{2.1}$$

Guided by this dispersion relation, we first cite the Boussinesq approximate equations for weakly nonlinear waves. Using overbars to denote physical variables, let us introduce the following normalization:

$$x = \frac{\bar{x}}{\ell}, \quad z = \frac{\bar{z}}{H'}, \quad t = \frac{\bar{t}}{\ell}\sqrt{gH'(1-R)}, \tag{2.2a–c}$$

$$h = \frac{H}{H'}, \quad \eta = \frac{\bar{\eta}}{A}, \quad (\phi', \phi) = \frac{H'(\bar{\phi}', \bar{\phi})}{A\ell\sqrt{gH'(1-R)}}. \tag{2.2d–f}$$

where η denotes the interface displacement and (ϕ', ϕ) are the velocity potentials in the (upper, lower) layer. The normalization scale ℓ for horizontal distances will be specified later as a characteristic scale of bathymetry. Following the standard procedure of Boussinesq approximation, the asymptotic equations of mass and momentum conservation can be derived. Let the depth-averaged horizontal velocities be defined by

$$u'(x,t) \equiv \frac{1}{1-\epsilon\eta}\int_{\epsilon\eta}^{1} \phi'_{,x}\,\mathrm{d}z, \quad u(x,t) \equiv \frac{1}{h+\epsilon\eta}\int_{-h}^{\epsilon\eta} \phi_{,x}\,\mathrm{d}z, \tag{2.3}$$

where $h(x) = H(\bar{x})/H'$ is the normalized still-water depth of the lower layer.

We assume that the still-water depth of the seabed fluctuates from the constant mean h_o

$$h = h_o - \mu b(x), \qquad h_o = H_o/H' = \text{constant}, \tag{2.4}$$

where $b(x)$ is random in x with zero mean: $\langle b(x)\rangle = 0$ and is of order unity. Keeping terms up to order $O(\epsilon) = O(\mu^2)$ only, we obtain from the laws of mass and momentum conservation

$$-(1-R)\eta_t + u'_{,x} = \epsilon(\eta u')_{,x}, \tag{2.5}$$

$$(1-R)\eta_t + h_o u_{,x} = \mu(bu)_{,x} - \epsilon(\eta u)_{,x}, \tag{2.6}$$

$$R(u'_{,t} + \eta_x) - (u_{,t} + \eta_x) = \frac{\epsilon}{1-R}(uu_{,x} - Ru'u'_{,x}) + \frac{\mu^2}{3}\left(Ru'_{,xxt} - h_o^2 u_{,xxt}\right). \tag{2.7}$$

Equation (2.7) combines the conservation of horizontal momentum in the upper layer and pressure continuity at the interface. In the limit of $R = 0$ and $h_o = 1$, the equations reduce to those for a homogeneous layer of shallow water. These three equations can be combined to give a single Boussinesq equation valid to the accuracy of $O(\epsilon, \mu^2)$:

$$\begin{aligned}\eta_{tt} - C^2\eta_{xx} = &-\frac{\mu}{(1+Rh_o)^2}(b\eta_x)_{,x} - \frac{\mu^2 R}{(1+Rh_o)^3}(b^2\eta_x)_{,x} \\ &+ \frac{\mu^2}{3}\frac{h_o^2(R+h_o)}{(1+Rh_o)^2}\eta_{0,xxxx} + \frac{\epsilon h_o}{2(1-R)^2(1+Rh_o)}(u^2 - Ru'^2)_{,xx} \\ &+ \frac{\epsilon\left(1-Rh_o^2\right)}{2h_o(1+Rh_o)}(\eta^2)_{,tt} + \frac{\epsilon\left(1-Rh_o^2\right)}{2(1-R)^2}(u^2)_{,tt},\end{aligned} \tag{2.8}$$

where

$$C = \left(\frac{h_o}{1 + Rh_o}\right)^{1/2} \tag{2.9}$$

is the dimensionless phase speed of the interfacial wave in the linearized limit. Since $b(x)$ is a random function of x, (2.8) is a nonlinear stochastic differential equation. As a model of natural seas the density difference is small, i.e. $1 - R = (\rho - \rho')/\rho \ll 1$ within a narrow range. The depth ratio h_o is the key parameter characterizing the interfacial waves.

As in the case of a homogeneous fluid (Grataloup & Mei 2003; Mei & Li 2004), the accumulated effects of random scattering are expected to induce slow changes over a long distance on the scale inversely proportional to the mean square of the depth fluctuations. We therefore introduce the fast and slow coordinates x and $X = \mu^2 x$, and expand η, u and u' as power series of μ. The following perturbation equations are obtained at $O(\mu^0)$, $O(\mu)$ and $O(\mu^2)$:

$$\eta_{0,tt} - C^2\eta_{0,xx} = 0, \tag{2.10}$$

$$\eta_{1,tt} - C^2\eta_{1,xx} = -\frac{1}{(1 + Rh_o)^2}(b\eta_{0,x})_{,x}, \tag{2.11}$$

$$\begin{aligned} \eta_{2,tt} - C^2\eta_{2,xx} = &-\frac{1}{(1 + Rh_o)^2}(b\eta_{1,x})_{,x} + 2c^2\eta_{0,xX} + \frac{1}{3}\frac{h_o^2(R + h_o)}{(1 + Rh_o)^2}\eta_{0,xxxx} \\ &+ \frac{U_r h_o}{2(1 - R)^2(1 + Rh_o)}\left(u_0^2 - Ru_0'^2\right)_{,xx} + \frac{U_r\left(1 - Rh_o^2\right)}{2h_o(1 + Rh_o)}\left(\eta_0^2\right)_{,tt} \\ &+ U_r\frac{1 - Rh_o^2}{2(1 - R)^2}\left(u_0^2\right)_{,tt} - \frac{R}{(1 + Rh_o)^3}(b^2\eta_{0,x})_{,x}, \end{aligned} \tag{2.12}$$

where $U_r = O(1)$ is Ursell's number defined as the ratio of nonlinearity to dispersion:

$$U_r = \frac{\epsilon}{\mu^2} = \frac{A\ell^2}{H'^3}. \tag{2.13}$$

We first treat long and transient pulses and then time-harmonic waves.

3. Unidirectional propagation of long pulses

At the zeroth-order, η_0 is governed by the homogeneous wave equation, hence it represents the coherent motion unaffected by randomness on the short scale. By limiting to right-going waves, the formal solution is of the form

$$\eta_0(x, X; t) = \eta_0(\sigma, X) \ \text{ where } \sigma = xCt \tag{3.1}$$

is the coordinate moving with the linearized phase speed. At order μ, the equation for η_1 is an inhomogeneous wave equation and can be formally solved in terms of Green's function,

$$G(x, t; x', t') = \frac{1}{2c}H[C(t - t') - |x - x'|], \tag{3.2}$$

where $H(z)$ is the Heaviside step function, with the result,

$$\eta_1(x, X; t) = \frac{1}{(1 + Rh_o)^2}\int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x' G(x, t; x', t')\frac{\partial}{\partial x'}\left(b(x')\frac{\partial \eta_0(x' - Ct', X')}{\partial x'}\right). \tag{3.3}$$

Clearly, η_1 is a random function of zero mean because of b. When use is made of the following results from the first-order approximation of (2.5) and (2.6),

$$u_0'(\sigma, X) = -C(1-R)\eta_0, \quad u_0(\sigma, X) = C(1-R)\frac{\eta_0}{h_o}, \tag{3.4}$$

we obtain from the ensemble average of the second-order equation,

$$\begin{aligned}\langle\eta_2\rangle_{,tt} - C^2\langle\eta_2\rangle_{,xx} = &-\frac{1}{(1+Rh_o)^2}\langle b\eta_{1,x}\rangle_{,x} \\ &+ w_1\eta_{0,xX} + U_r w_2\left(\eta_0^2\right)_{,tt} + w_3\eta_{0,xxxx} - w_4\left(\langle b^2\rangle\eta_{0,x}\right)_{,x},\end{aligned} \tag{3.5}$$

where

$$w_1 = \frac{2h_o}{(1+Rh_o)}, \quad w_2 = \frac{3}{2h_o}\frac{\left(1-Rh_o^2\right)}{(1+Rh_o)}, \quad w_3 = \frac{1}{3}\frac{h_o^2(R+h_o)}{(1+Rh_o)^2}, \quad w_4 = \frac{R}{(1+Rh_o)^3}. \tag{3.6}$$

We now assume that the depth fluctuations are statistically homogeneous over the short scale, with known autocorrelation

$$\langle b(x), b(x')\rangle \equiv \Gamma(\xi) \quad \text{where } \xi = x' - x. \tag{3.7}$$

From now on, we denote η_0 by ζ for brevity. The first term on the right-hand side of (3.5) can be shown as in Mei & Li (2004) to be

$$\begin{aligned}\langle b\eta_{1,x}\rangle &= -\frac{1}{(1+Rh_o)^2}\int_{-\infty}^{\infty}\mathrm{d}t'\int_{-\infty}^{\infty}\mathrm{d}x'\left[\frac{\partial}{\partial x'}\left(\langle b(x)b(x')\rangle\frac{\partial\zeta}{\partial x'}\right)\right]\frac{\partial G}{\partial x} \\ &= \frac{h_o(1+Rh_o)}{\pi}\int_{-\infty}^{\infty}\beta(k, X)\hat{\zeta}(k, X)\,\mathrm{e}^{\mathrm{i}k(x-Ct)}\,\mathrm{d}k,\end{aligned} \tag{3.8}$$

where $\hat{\zeta}(k, X)$ is the exponential Fourier transform of $\zeta(\sigma, X)$ with respect to σ and

$$\beta = \frac{\mathrm{i}k}{4Ch_o(1+Rh_o)^3}\int_{-\infty}^{\infty}\frac{\partial}{\partial\xi}\left[\Gamma(\xi)\,\mathrm{e}^{\mathrm{i}k\xi}\right]\mathrm{sgn}(\xi)\,\mathrm{e}^{\mathrm{i}k|\xi|}\,\mathrm{d}\xi \tag{3.9}$$

is independent of σ. Since $\zeta_{,t} = -C\zeta_{,\sigma}$, $\zeta_{,x} = \zeta_{,\sigma}$, all terms on the right-hand side of (3.5) are functions of σ. To ensure boundedness of $\langle\eta_2\rangle$ the entire right-hand side must vanish. This solvability condition for $\langle\eta_2\rangle$ is a partial differential equation for ζ, which can be integrated with respect to σ once to give

$$\zeta_{,X} + \lambda_1\zeta\zeta_\sigma + \lambda_2\zeta_{,\sigma\sigma\sigma} = \lambda_3\zeta_{,\sigma} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k\sigma}\beta(k, X)\hat{\zeta}(k, X)\,\mathrm{d}k, \tag{3.10}$$

where

$$\lambda_1 = \tfrac{3}{2}U_r\,\frac{1-Rh_o^2}{h_o(1+Rh_o)}, \quad \lambda_2 = \frac{1}{6}\,\frac{h_o(R+h_o)}{(1+Rh_o)}, \quad \lambda_3 = \frac{\Gamma(0)}{2}\,\frac{R}{h_o(1+Rh_o)^2}, \tag{3.11}$$

and $\Gamma(0) = \langle b^2\rangle$. Equation (3.10) is an extension of the KdV equation governing the evolution of ζ in the moving coordinate σ. The variable X plays the role of time. The coefficients λ_1 and λ_2 are the same as those in existing theories for interfacial long waves over a flat bottom (see e.g. Helfrich, Melville & Miles 1984). In the limit of $R \to 0$ and $h_o = 1$, equation (23) of Mei & Li (2004) for a one-layered sea is recovered. These coefficients are plotted in figure 1 for $R = 0.97$, $\Gamma(0) = 1$, $U_r = 1$ and a wide range of h_o. It is important to note that $\lambda_1 \gtrless 0$ if $h_o \lessgtr 1/\sqrt{R}$. Since $\Gamma(0) > 0$,

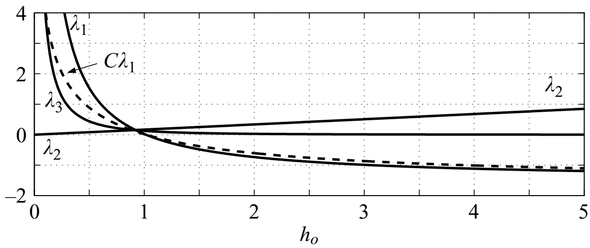

FIGURE 1. Variation of coefficients in (3.11) for $R = 0.97$, $\Gamma(0) = 1$ and $U_r = 1$.

λ_3 is positive; the first term on the right-hand side indicates that randomness reduces the wave speed in the moving frame of reference.

The integral in (3.9) can be transformed to a more revealing form by partial integration as in Mei & Li (2004),

$$\beta = \frac{k^2}{4Ch_o(1+Rh_o)^3}\left\{\tfrac{1}{2}\hat{\Gamma}(0) + \tfrac{1}{2}\hat{\Gamma}(2k) + \mathrm{i}k\hat{P}(2k)\right\} - \frac{\mathrm{i}k\Gamma(0)}{2Ch_o(1+Rh_o)^3}, \qquad (3.12)$$

where

$$P(\xi) = \int_{|\xi|}^{\infty} \Gamma(\xi)\,\mathrm{d}\xi,\ \hat{P}(2k) = \int_{-\infty}^{\infty} P(\xi)\,\mathrm{e}^{-\mathrm{i}2k\xi}\,\mathrm{d}\xi. \qquad (3.13)$$

Substituting (3.12) into (3.10), taking the inverse Fourier transform and invoking the convolution theorem, we obtain the following governing equation

$$\zeta_{,X} + \lambda_1\zeta\zeta_\sigma + \lambda_2\zeta_{,\sigma\sigma\sigma} = \frac{1}{Ch_o(1+Rh_o)^3}\left\{\frac{\Gamma(0)}{2}\gamma\zeta_{,\sigma} + \frac{\hat{\Gamma}(0)}{8}\zeta_{,\sigma\sigma} + \frac{1}{16}\int_{-\infty}^{\infty}\Gamma\left(\frac{\sigma-\sigma'}{2}\right)\zeta_{,\sigma'\sigma'}\,\mathrm{d}\sigma' + \frac{1}{8}\int_{-\infty}^{\infty}P\left(\frac{\sigma-\sigma'}{2}\right)\zeta_{,\sigma'\sigma'\sigma'}\,\mathrm{d}\sigma'\right\} \qquad (3.14)$$

where

$$\gamma = 1 + RC(1+Rh_o) > 0. \qquad (3.15)$$

Since $\Gamma(0) > 0$, the first term on the right-hand side shows that roughness reduces the phase speed. The second and third terms represent diffusion by wave radiation. Note that Γ and $\hat{\Gamma}(0)$ which is the total area under the correlation curve, are both positive. Hence randomness gives rise to diffusion. The last term alters the dispersiveness. Thus, the governing integro-differential equation combines the features of KdV and Burgers equations. In particular, the diffusion terms should lead to attenuation, i.e. localization. Although (3.10) is of the same form as that for long waves in a one-layered fluid (Mei & Li 2004), the coefficients now depend on the depth ratio h_o which affects the wave dynamics quantitatively. Moreover, the possibility of sign reversal of λ_1 (figure 1) leads to physical features unique to interfacial waves.

4. Soliton attenuation by Gaussian disorder

To examine greater details we take the correlation function to be Gaussian and specify the horizontal scale ℓ for normalization (cf. (2.2)) to be the correlation length. Hence, in dimensionless form, the correlation function reads,

$$\Gamma(\xi) = D^2\exp(-\xi^2), \qquad (4.1)$$

where D is the normalized root-mean-square amplitude of disorder. The governing equation becomes,

$$\zeta_{,X} + \lambda_1 \zeta \zeta_\sigma + \lambda_2 \zeta_{,\sigma\sigma\sigma} = D^2 \frac{1}{Ch_o(1+Rh_o)^3} \left\{ \frac{\gamma}{2}\zeta_{,\sigma} + \frac{\sqrt{2\pi}}{8}\zeta_{,\sigma\sigma} \right.$$

$$\left. + \frac{1}{16}\int_{-\infty}^{\infty} \exp\left(-\frac{|\sigma-\sigma'|^2}{8}\right)\zeta_{,\sigma'\sigma'}\,\mathrm{d}\sigma' + \frac{\sqrt{2\pi}}{16}\int_{-\infty}^{\infty} \operatorname{erfc}\left(\frac{|\sigma-\sigma'|}{2\sqrt{2}}\right)\zeta_{,\sigma'\sigma'\sigma'}\,\mathrm{d}\sigma' \right\}. \quad (4.2)$$

Let us examine the effects of roughness on the propagation of a soliton after entering a semi-infinite region of disorder. For reference, the well-known interfacial soliton over a smooth seabed ($D=0$) may be first recalled,

$$\zeta = \operatorname{sgn}(\lambda_1)\,\operatorname{sech}^2\left[\sqrt{\frac{|\lambda_1|}{12\lambda_2}}\left(\sigma - \frac{\lambda_2}{3}X\right)\right], \quad \text{where } \operatorname{sgn}(\lambda_1) = \operatorname{sgn}\left(1 - Rh_o^2\right). \quad (4.3)$$

The wave is an elevation if $h_o < 1/\sqrt{R}$ and a depression if $h_o > 1/\sqrt{R}$. For $R=0.97$ which is typical in oceans, the threshold is $1/\sqrt{R} = 1.0153$. The total area under the wave of unit amplitude is $4\sqrt{3\lambda_2/|\lambda_1|}$ which is large for $|\lambda_1| \ll 1$. Since $\lambda_2 > 0$, the soliton propagates forward in the moving frame of reference. The expression

$$\sqrt{\frac{12\lambda_2}{|\lambda_1|}} = \sqrt{\frac{3}{4}\frac{U_r|1-Rh_o^2|}{h_o(R+h_o)}} \quad (4.4)$$

can be regarded as the dimensionless wavelength of the soliton. If $h_o \ll 1/\sqrt{R}$, the elevation soliton is short, hence the profile is sharp. As the depth ratio h_o increases, the length increases and the wave flattens. When $h_o = 1/\sqrt{R}$, the length is infinite. If h_o increases above $1/\sqrt{R}$, the depression soliton at first becomes shorter. After reaching a minimum, the wavelength increases again with increasing h_o. For a given R, the minimum wavelength for the depression soliton can be found by extremizing $|\lambda_1|/\lambda_2$ with respect to h_o, yielding the cubic equation

$$Rh_o^3 - 3h_o - 2R = 0. \quad (4.5)$$

The critical depth $\widetilde{h}_o$ corresponding to the minimum wavelength is the only positive root of the cubic equation,

$$\widetilde{h}_o = \frac{W}{R} + \frac{1}{W} \quad \text{with} \quad W = \left[\left(R - \sqrt{\frac{1-R^3}{R}}\right)R^2\right]^{1/3} \quad (4.6)$$

For $R=0.97$, we find $\widetilde{h}_o = 2.02$.

With this background we first examine numerically a soliton entering from the left of a semi-infinite region of disorder, i.e. $D=0$, $X<0$ and $D=\text{constant}$, $0<X<\infty$. Equation (4.2) is solved by the Fourier spectral method for a domain of σ, centred around the initial soliton. Periodic boundary conditions are imposed. The computational domain is sufficiently large to contain all disturbances. Integration in X is performed by a fourth-order Runge–Kutta method.

4.1. *Computed evolution of a soliton*

Numerical solutions of (4.2) are obtained only for $R=0.97$. Figure 2 compares the effects of seabed roughness on the evolution of a relatively strong ($U_r=3$) elevation

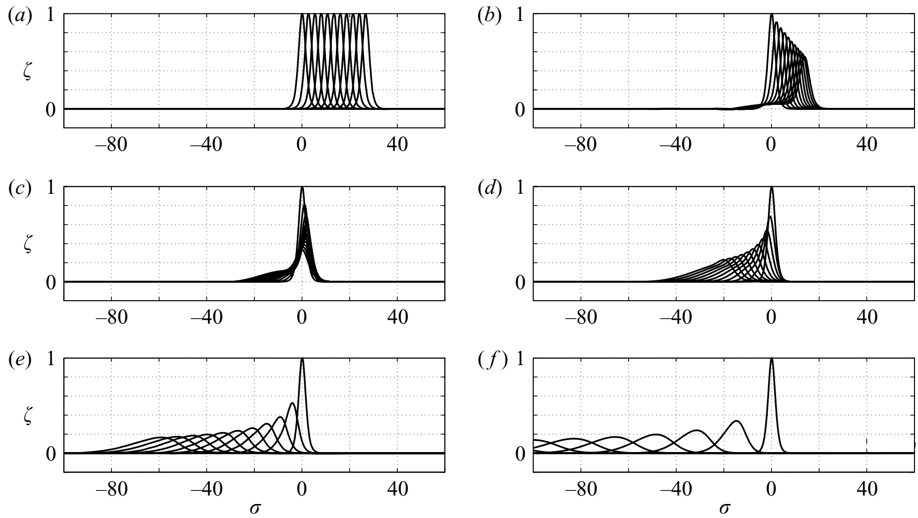

FIGURE 2. Effect of roughness amplitude on soliton evolution over a random seabed in a two-layer stratified fluid in the case of hump for $R = 0.97$ and $h_o = 0.80$. The total travel distance is 200 ($0 \leqslant X \leqslant 200$). Wave profiles are shown at every $\Delta X = 20$. $U_r = 1$. (*a*) $D^2 = 0$, (*b*) $D^2 = 0.1$, (*c*) $D^2 = 0.25$, (*d*) $D^2 = 0.5$, (*e*) $D^2 = 1$, (*f*) $D^2 = 2.5$.

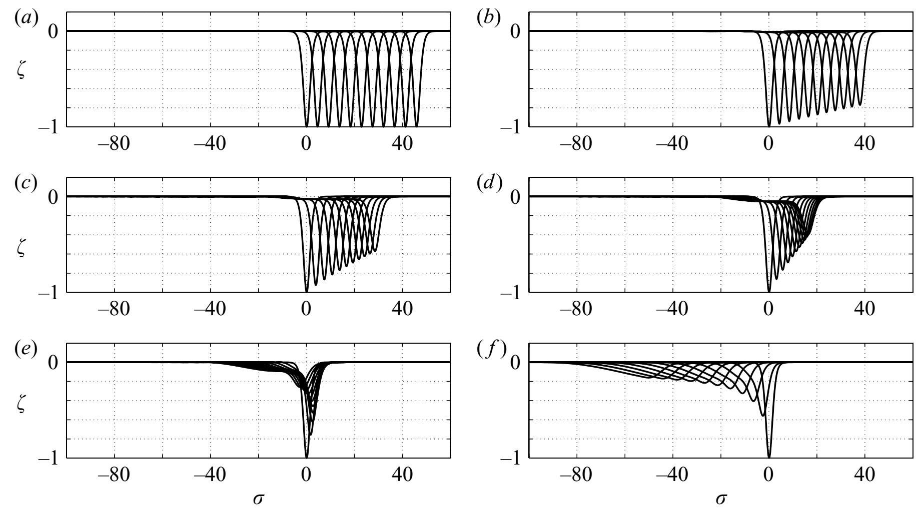

FIGURE 3. As for figure 2 but for $h_o = 1.2$ and $U_r = 3$.

soliton with a subcritical depth ratio $h_o = 0.8$. For $D = 0$ (smooth bed), the soliton advances without changing its profile. For a larger D^2, the crest becomes lower, the profile flatter, and the speed is reduced. For the largest $D^2 = 1$, the speed drops beneath the linear phase speed. For a supercritical depth ratio $h_o = 1.2$, similar comparisons for a depression soliton are made in figure 3 for the same $U_r = 3$ and different D. Again, for increasingly large D, the reduction of trough depth and the flattening of profiles are more rapid. The reduction of propagation speed is also more pronounced.

We have also examined the transformation of a soliton passing over a long but finite region of disorder ($0 < X < X_0$) for several different roughness heights D. Over the rough region, the profile lowers and flattens with X, hence is no longer a soliton upon exit. After re-entering the smooth region, the flat pulse undergoes fission into several new solitons, as is expected from the inverse scattering theory. For both elevation and depression solitons crossing a finite region of rough seabed, the number of disintegrated new solitons increases by either increasing the roughness height D, or the length X_0. These features are qualitatively the same as those found for a homogeneous layer by Mei & Li (2004), hence plots are omitted. The new feature of interfacial waves is that the soliton can be either an elevation or a depression, depending on the depth ratio.

Focusing on the case of semi-infinite disorder we now examine analytically the asymptotic behaviour for large X, in particular the attenuation (localization) length, along the lines of Mei & Li (2004).

4.2. *Large X behaviour and the length of localization*

It is physically expected that the reduction of crest (trough) amplitude with X must be compensated by an increase in profile length for mass conservation. Hence, higher derivatives of ζ suffer greater loss in magnitude. By comparing the diffusion and dispersion terms with the inertia term in (4.2), we expect that $\zeta_{,\sigma\sigma\sigma}/\zeta\zeta_{,\sigma}$ diminishes with increasingly large X while $\zeta_{,\sigma\sigma}/\zeta\zeta_{,\sigma}$ remains comparable. The third term on the right-hand side of (4.2) can be approximated by

$$\frac{1}{16}\int_{-\infty}^{\infty}\exp\left(-\frac{|\sigma-\sigma'|^2}{8}\right)\zeta_{,\sigma'\sigma'}\,\mathrm{d}\sigma' \approx \frac{1}{16}\zeta_{,\sigma\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma'^2}{8}\right)\mathrm{d}\sigma' = \frac{\sqrt{2\pi}}{8}\zeta_{,\sigma\sigma}. \quad (4.7)$$

Consequently, the asymptotic form of (4.2) for $X \gg 1$ reduces to

$$\zeta_{,X} + \lambda_1\zeta\zeta_{,\sigma} = \frac{D^2}{Ch_o(1+Rh_o)^3}\left(\frac{\gamma}{2}\zeta_{,\sigma} + \frac{\sqrt{2\pi}}{4}\zeta_{,\sigma\sigma}\right) \quad (4.8)$$

which can be transformed to the Burgers equation:

$$\zeta_{,X} + \lambda_1\zeta\zeta_{,\rho} = \frac{D^2\sqrt{2\pi}}{4Ch_o(1+Rh_o)^3}\zeta_{,\rho\rho} \quad (4.9)$$

after changing to a new coordinate moving at a lower speed than the linearized phase velocity,

$$\rho = \sigma + \left(\frac{D^2\gamma}{2Ch_o(1+Rh_o)^3}\right)X. \quad (4.10)$$

We now use the analytical solution to an initial-value problem by Cole and Hopf (see Whitham 1974) for the initial data of a concentrated pulse represented by the δ-function. Let us choose the total area of the pulse to be the same as that of a soliton of unit amplitude, i.e.

$$\zeta(\rho, 0) = \mathrm{sgn}(\lambda_1)4\sqrt{\frac{3\lambda_2}{|\lambda_1|}}\delta(\rho). \quad (4.11)$$

The Cole–Hopf theory gives for large X,

$$\zeta \approx \mathrm{sign}(\lambda_1)\sqrt{\frac{\nu}{X}}\mathscr{F}(\overline{\rho}) \;\; (X \gg 1), \quad (4.12)$$

where

$$\mathscr{F}(\overline{\rho}) = \frac{N}{\sqrt{\pi}}\,\frac{\exp(-\overline{\rho}^2)}{1 + (N/2)\,\mathrm{erfc}\overline{\rho}}, \tag{4.13}$$

with

$$\overline{\rho} = \frac{\rho}{2|\lambda_1|\sqrt{\nu X}}, \quad N = \exp\left(\frac{2}{|\lambda_1|\nu}\sqrt{\frac{3\lambda_2}{|\lambda_1|}}\right) - 1, \quad \nu = \frac{D^2\sqrt{2\pi}}{4Ch_o(1+Rh_o)^3\lambda_1^2}. \tag{4.14}$$

Clearly, the wave (elevation or depression) advances in the moving coordinate σ at the negative speed $D^2/2Ch_o(1 + Rh_o)^3$. In the stationary frame of reference, the wave speed is less than that of the linearized wave over a smooth bed. For large D^2, i.e. strong disorder,

$$N \approx \frac{2}{|\lambda_1|\nu}\sqrt{\frac{3\lambda_2}{|\lambda_1|}} \ll 1, \tag{4.15}$$

the Cole–Hopf result (4.12) can be further approximated by

$$\zeta \approx \mathrm{sgn}(\lambda_1)\sqrt{\frac{12\lambda_2}{\pi\nu X|\lambda_1|^3}}\,\exp(\bar{\rho}^2). \tag{4.16}$$

The profile is Gaussian hence symmetric and the amplitude decays as $X^{-1/2}$. The algebraic rate of decay is a feature of nonlinearity, similar to what was found by Devillard & Souillard (1986) for a nonlinear Schrödinger equation with a random potential. This is different from linear waves which decay exponentially with the propagation distance.

4.3. *Small mean-square height*

Similar to the damping of solitary waves due to weak viscous dissipation in the boundary layer at the bottom (Keulegan 1949; see also Mei, Stiassnie & Yue 2005), analytical approximation is possible for small mean square height $D^2 \ll O(1)$. Since the attenuation rate is expected to be slow, we introduce an additional slow variable $X_1 = D^2 X$ and expand ζ as a power series of D^2 which is characteristic of the mean-square height. The perturbation analysis is similar to that in Mei & Li (2004) for a homogeneous layer, and is sketched in Appendix A. The final result is an ordinary differential equation for the soliton amplitude $A(X_1)$

$$\frac{\partial A}{\partial X_1} = -\frac{\sqrt{2\pi}|\lambda_1|}{90\lambda_2 Ch_o(1+Rh_o)^3}A^2 - \frac{1}{16\sqrt{3}Ch_o(1+Rh_o)^3}\sqrt{\frac{|\lambda_1|}{\lambda_2}}A^{3/2}F\left(\sqrt{\frac{2|\lambda_1|}{3\lambda_2}}A^{1/2}\right), \tag{4.17}$$

where

$$F(u) = 2\int_0^\infty \mathrm{d}p\,\mathrm{sech}^2 p\int_0^\infty (3\,\mathrm{sech}^4 q - 2\,\mathrm{sech}^2 q) \times (\exp-((q-p)^2/u^2) + \exp(-(q+p)^2/u^2))\,\mathrm{d}q. \tag{4.18}$$

The function $F(u)$ has been computed by Mei & Li (2004). Qualitatively, $F(u)$ increases from 0 when $u = 0$ to about 0.53 near $u = 1$. Then $F(u)$ decreases monotonically with increasing u. We have integrated (4.17) numerically for $A(X_1)$

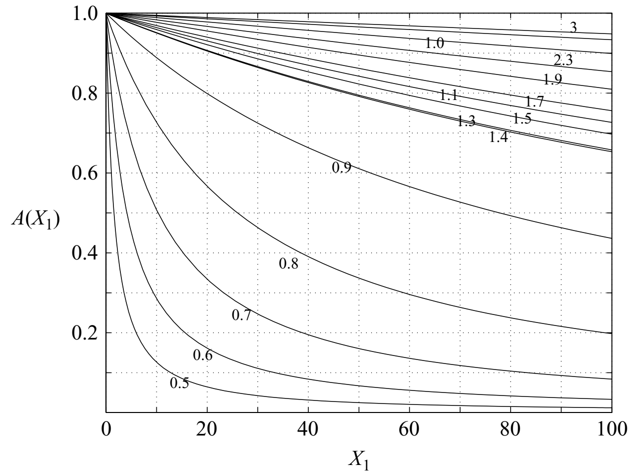

FIGURE 4. Decay of $A(X_1)$ according to (4.17) for different depth ratios and $U_r = 1$. The numbers on curves are the depth ratios h_o.

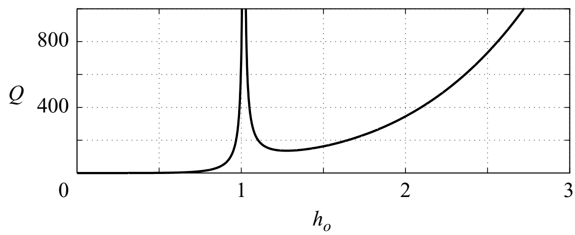

FIGURE 5. Behaviour of coefficient Q for small mean-square height, $U_r = 1$.

for different depth ratios (figure 4). Generally $A(X_1)$ decays with X_1. For $h_o < 1/\sqrt{R}$, the decay rate increases with h_o. However for $h_o > 1/\sqrt{R}$, the decay rate decreases to a minimum at about $h_o = 1.4$ then increases again with h_o.

To understand the last result better we approximate $F(u)$ for large X_1, hence small A and u. The second integral in (4.18) is then dominated by contributions near $q = p$, hence $F(u)$ can be approximated by

$$F(u) \approx 2\int_0^{\infty} \operatorname{sech}^4 p(3\operatorname{sech}^2 p - 2)\,\mathrm{d}p \int_{-\infty}^{\infty} \exp(-\xi^2/u^2)\,\mathrm{d}\xi = \frac{8\sqrt{\pi}}{15}u. \tag{4.19}$$

Now (4.17) can be approximated by

$$\frac{\partial A}{\partial X_1} = -\frac{\sqrt{2\pi}}{45Ch_o(1+Rh_o)^3}\frac{|\lambda_1|}{\lambda_2}A^2. \tag{4.20}$$

Therefore for large X_1, A decays again algebraically

$$A(X_1) = \frac{A_0}{1 + \dfrac{\sqrt{2\pi}lA_0}{45Ch_o(1+Rh_o)^3}\dfrac{|\lambda_1|}{\lambda_2}X_1} \approx \frac{45Ch_o(1+Rh_o)^3}{\sqrt{2\pi}}\frac{\lambda_2}{|\lambda_1|}\frac{1}{X_1} \equiv \frac{Q}{X_1}. \tag{4.21}$$

The dependence of the coefficient Q on the depth ratio is given in figure 5. This approximation is poor near the threshold where $|\lambda_1|$ or $h_o = 1/\sqrt{R}$ where Q is singular. For $h_o < 1/\sqrt{R}$, Q is small and rises with h_o slowly except near $h_o \sim 1$. Localization occurs at a relatively short distance. At the same X, A is larger for larger

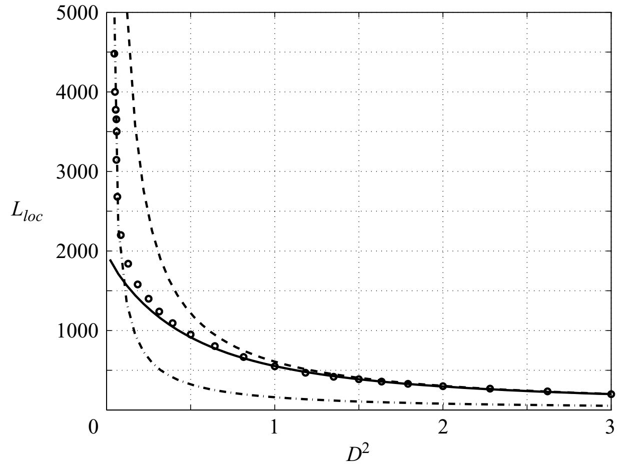

FIGURE 6. Localization length by approximate formulae for $R = 0.97$, $h_o = 2$, $U_r = 1$. —, (4.25) for $D^2 = O(1)$; ---, (4.26) for large D^2; $-\cdot-$, (4.27) for small D^2. ∘, results by spectral computation of (3.14).

h_o. Between the critical depth and $h_o \approx 1.4$, Q decreases with h_o hence at the same X, A decreases with h_o. Beyond $h_o \approx 1.4$, Q, hence A, increases with h_o again.

4.4. *Localization length*

Let us define the localization (or attenuation) length as the distance over which the height of the soliton decreases to a certain small fraction of its initial height. Here the fraction is arbitrarily chosen to be 1/10, i.e.

$$A(X = L_{loc}) = \frac{1}{10} A(X = 0). \tag{4.22}$$

For finite disorder $D^2 = O(1)$, we first locate the crest from (4.12)

$$\sqrt{\frac{\nu}{X}} \ \max(\mathscr{F}) = \frac{1}{10}. \tag{4.23}$$

The zero of $\mathrm{d}\mathscr{F}/\mathrm{d}X$ occurs at the root ρ_0' of the following transcendental equation

$$\rho_0'(2 + N \,\mathrm{erfc}\rho_0') = \frac{N}{\pi} \exp\left(-\rho_0'^2\right). \tag{4.24}$$

Upon substituting this result into (4.13) we find $\max(F) = 2\rho_0'$, therefore

$$L_{loc} \approx 400 \ \nu \ \rho_0'^2 \quad \text{for} \quad D^2 = O(1). \tag{4.25}$$

For large D^2, a more explicit result can be obtained from (4.16):

$$L_{loc} \approx \frac{1200\lambda_2}{\pi\nu|\lambda_1|^3} \quad \text{for } D^2 \gg 1. \tag{4.26}$$

On the other hand, for small D^2, we obtain from (4.21) for a unit initial height,

$$L_{loc} = \frac{450 C h_o (1 + R h_o)^3}{\sqrt{2\pi} D^2} \frac{\lambda_2}{|\lambda_1|} \quad \text{for } D^2 \ll 1. \tag{4.27}$$

Figure 6 shows the dependence of the localization length L_{loc} as a function of disorder D^2 for $h_o = 2$ only, according to the three approximations above. Results by direct

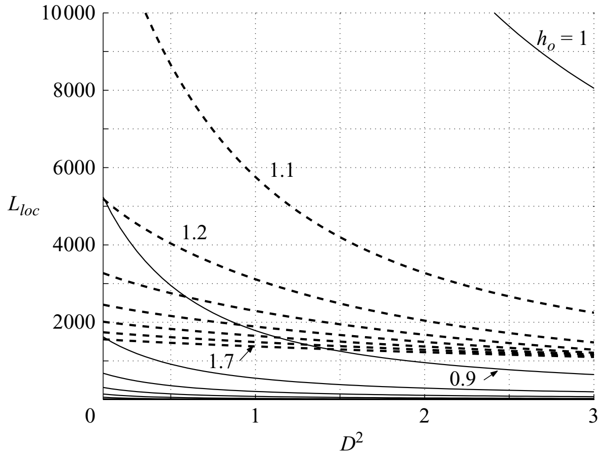

FIGURE 7. Localization length for various depth ratios based on (4.25). —, $h_o < 1/\sqrt{R}$ and h_o increases from bottom to top with steps $\Delta h_o = 0.1$; - - -, $h_o > 1/\sqrt{R}$ and h_o increases from top to bottom. $R = 0.97$.

numerical computations of (3.14) confirm the approximations in their respective domains. Of course (4.25) has the widest scope of validity, and is therefore used to calculate the effect of the depth ratio h_o on the localization length, as displayed in figure 7. It can be seen that the localization length is very large for $h_o \approx 1/\sqrt{R}$ where the nonlinear term is very small. For more accurate prediction, the extended KdV approximation with a cubic nonlinearity is required.

5. Localization and harmonic generation

We now turn to the physics of harmonic generation, a phenomenon of well-known importance in optics (Armstrong *et al.* 1962). It is also known that, in surface waves, if progressive long waves of single frequency are generated in shallow water over a smooth bed, higher harmonics emerge with the distance of propagation, followed by spatially periodic exchanges of energy between higher and lower harmonics (Goda 1967; Mei & Ünlüata 1972; Bryant 1973). The effects of a randomly rough bed on harmonic generation in a homogeneous layer have been investigated by Grataloup & Mei (2003). Their theory is extended here for a two-layered fluid.

5.1. *Coupled-mode equations for harmonics*

Consider a train of progressive waves propagating from $X = 0$ towards $X \sim \infty$. The train is composed of many harmonics with amplitudes $A_m(X)$, frequencies ω_m, and wavenumbers k_m, with $m = 1, 2 \ldots,$

$$\eta_0 = \frac{1}{2} \sum_{m=-\infty}^{\infty} A_m(X) \exp(\mathrm{i}\theta_m), \quad X > 0 \quad \text{where} \quad \theta_m = k_m x - \omega_m t, \tag{5.1}$$

is the wave phase, with

$$k_m = mk, \quad \omega_m = m\omega = Ck_m \tag{5.2}$$

and $A_{-m} = A_m^*$ where A^* denotes the complex conjugate of A. The harmonic amplitudes vary slowly in $X = \mu^2 x$ because of nonlinear interactions and random scattering. By proper definition of the mean depth, we set $A_0 = 0$.

To find the solution to the governing equation (2.11) for η_1, we expand the right-hand side in time harmonics

$$(b\eta_{0,x})_{,x} = \sum_{m=-\infty}^{\infty} F_m \exp(-\mathrm{i}\omega_m t) \tag{5.3}$$

where

$$F_m = \tfrac{1}{2}\,\mathrm{i}k_m A_m(X)[b(x)\exp(\mathrm{i}k_m x)]_{,x} \tag{5.4}$$

is a random function of x. We seek a solution of (2.11) in the form

$$\eta_1 = \sum_{m=-\infty}^{\infty} \eta_1^{(m)} \exp(-\mathrm{i}\omega_m t), \quad \eta_1^{(0)} = 0; \tag{5.5}$$

where η_m can be solved by a Green's function method

$$\eta_1^{(m)} = \frac{A_m(X)}{4h_o(1+Rh_o)} \int_{-\infty}^{\infty} \exp(\mathrm{i}k_m|x-x'|)\frac{\mathrm{d}}{\mathrm{d}x'}[b(x')\exp(\mathrm{i}k_m x')]\,\mathrm{d}x'. \tag{5.6}$$

so that

$$\eta_1 = \frac{1}{4h_o(1+Rh_o)} \sum_{m=-\infty}^{\infty} A_m(X) \times \exp(-\mathrm{i}\omega_m t)\int_{-\infty}^{\infty} \exp(\mathrm{i}k_m|x-x'|)\frac{\mathrm{d}}{\mathrm{d}x'}[b(x')\exp(\mathrm{i}k_m x')]\,\mathrm{d}x'. \tag{5.7}$$

We next turn to the governing equation for $\langle\eta_2\rangle$, which is still (3.5). Using η_0 and η_1 derived in this section, the forcing terms on the right-hand side are worked out in Appendix B. As a consequence, (3.5) becomes

$$\begin{aligned}\langle\eta_2\rangle_{,tt} - C^2\langle\eta_2\rangle_{,xx} = {} & \frac{1}{(1+Rh_o)^2}\sum_{m=1}^{\infty}\mathrm{i}k_m A_m(X)\beta_m\exp(\mathrm{i}\theta_m) + \frac{w_1}{2}\sum_{m=1}^{\infty}\mathrm{i}k_m\frac{\mathrm{d}A_m}{\mathrm{d}X}\exp(\mathrm{i}\theta_m)\\ & - \frac{U_r w_2}{4}\sum_{m=1}^{\infty}\omega_m^2\exp(\mathrm{i}\theta_m)\left[\sum_{l=1}^{\infty}2A_l^* A_{m+l} + \sum_{l=1}^{[m/2]}\epsilon_l A_l A_{m-l}\right]\\ & + \frac{w_3}{2}\sum_{m=1}^{\infty}k_m^4 A_m\exp(\mathrm{i}\theta_m) + \frac{w_4}{2}\sum_{m=1}^{\infty}k_m^2 A_m\exp(\mathrm{i}\theta_m) + *.\end{aligned} \tag{5.8}$$

Because $\theta_m = k_m(x - Ct)$, then $\exp(\mathrm{i}\theta_m)$ is a homogeneous solution of the forced wave equation. All harmonic coefficients on the right-hand side must vanish to avoid secularity. It follows that

$$\begin{aligned}&\frac{\mathrm{d}A_m}{\mathrm{d}X} + \beta_m^{\dagger}A_m(X) - \lambda_2\mathrm{i}k_m^3 A_m + \lambda_3\mathrm{i}k_m A_m\\ &\quad + \frac{C\lambda_1}{2}\mathrm{i}\omega_m\left[\sum_{l=1}^{\infty}2A_l^* A_{m+l} + \sum_{l=1}^{[m/2]}\varepsilon_l A_l A_{m-l}\right] = 0 \quad (m = 1, 2, 3, \cdots).\end{aligned} \tag{5.9}$$

where

$$\beta_m^{\dagger} = \frac{\mathscr{I}_m}{h_o^2(1+Rh_o)^2} \tag{5.10}$$

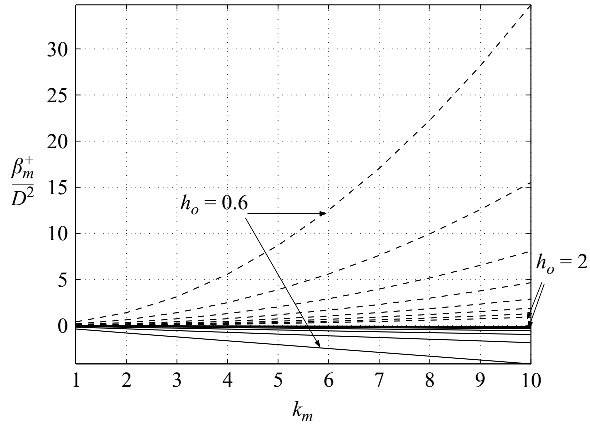

FIGURE 8. Real (dash line) and imaginary (solid line) parts of $\beta_m^\dagger/D^2$ for various depth ratios h_o from 0.6 to 2 at the interval $\Delta h_o = 0.2$.

with

$$\mathscr{I}_m \equiv \frac{D^2}{4} \mathrm{i} k_m \int_{-\infty}^{\infty} \operatorname{sgn}(\xi) \left(\frac{\mathrm{d}\Gamma}{\mathrm{d}\xi} + \mathrm{i} k_m \Gamma \right) \exp(\mathrm{i} k_m (|\xi| + \xi))\, \mathrm{d}\xi \tag{5.11}$$

depending only on X. Equation (5.9) is an infinite set of mode-coupling equations for the harmonic envelopes. This result is an extension of the theory of Armstrong *et al.* (1962) on second-harmonic generation in optics and of Bryant (1976) for long water waves, both for homogeneous and uniform media, and of Grataloup & Mei (2003) for long waves in a homogeneous fluid over a random seabed. The linear terms multiplied by the coefficients $\beta_m^\dagger$ represent the effects of incoherent scattering. These equations can be solved numerically for given initial data $A_m(0), m = 1, 2, 3, \ldots$. Dependences of coefficients λ_2, λ_3 and $C\lambda_1$ on h_o are shown in figure 1. Again let us take the correlation function to be Gaussian according to (4.1). The integral above can be evaluated to give

$$\mathscr{I}_m = D^2 \left\{ k_m^2 \frac{\sqrt{2\pi}}{8} \left(1 + \exp\left(-2k_m^2\right)\right) - \mathrm{i} \frac{k_m}{2} \left(1 - \frac{k_m \sqrt{\pi}}{2\sqrt{2}} \exp\left(-2k_m^2\right) \operatorname{erfi}(\sqrt{2} k_m) \right) \right\}, \tag{5.12}$$

as in Grataloup & Mei for a homogeneous layer. From (5.9), the following energy relation can easily be shown:

$$\frac{\mathrm{d}}{\mathrm{d}X} \sum_{m=1}^{\infty} |A_m|^2 = -2 \sum_{m=1}^{\infty} \operatorname{Re}(\beta_m^\dagger) |A_m|^2. \tag{5.13}$$

The fact that $\operatorname{Re}\beta_m^\dagger$ is always positive implies energy dissipation and amplitude localization of all interfacial harmonics. Figure 8 shows the dependence of the real and imaginary parts of $\beta_m^\dagger$ on the depth ratio h_o and the normalized wavenumber k_m. Clearly, the damping rate increases with the wavenumber k_m and decreases with the depth ratio h_o. The physical effect of the nonlinear products in the square brackets of (5.9) is to enable energy exchange between modes and is reflected in slow oscillations of modal amplitudes over X. This modulational wavelength, i.e. the recurrence distance, decreases with increasing depth ratio through the phase speed C in λ_1.

5.2. *Energy budget of incoherent scattering*

Before presenting the numerical solutions of (5.9), it is useful to examine more closely the physics of energy removal from the coherent wave by incoherent scattering. Since the effect is represented by linear terms in (5.9) and is not the direct result of nonlinear coupling, it suffices to consider just one harmonic, say the fundamental, from the linearized part of the preceding theory. Let us rewrite the total interface displacement as the sum of the ensemble mean $\langle\eta\rangle$ and the random fluctuation $\tilde{\eta}$,

$$\eta = \langle\eta\rangle + \mu\tilde{\eta}. \tag{5.14}$$

Clearly, they are related to the perturbation solutions by

$$\langle\eta\rangle \cong \eta_0 + \mu^2\langle\eta_2\rangle + \cdots, \quad \tilde{\eta} = \eta_1 + \cdots \tag{5.15}$$

to the present degree of accuracy. The following solutions for the fundamental harmonic are just the first harmonic parts of (5.1) and (5.6),

$$\eta_0 = A\mathrm{e}^{\mathrm{i}kx}, \quad \eta_1 = \frac{A}{4h_o(1+Rh_o)}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k|x-x'|}\frac{\mathrm{d}}{\mathrm{d}x'}[b(x')\,\mathrm{e}^{\mathrm{i}kx'}]\,\mathrm{d}x'. \tag{5.16}$$

Without the artifice of fast and slow coordinates, we substitute (5.14) into (2.8) with only the linear terms kept, then take the ensemble average to obtain

$$\langle\eta\rangle_{,xx} + k^2\langle\eta\rangle = \frac{\mu^2}{h_o(1+Rh_o)}\langle b\eta_{1,x}\rangle_{,x} + \frac{\mu^2 R}{h_o(1+Rh_o)^2}\langle b^2\rangle\eta_{0,xx} - \frac{\mu^2}{3}\frac{h_o(R+h_o)}{(1+Rh_o)}\eta_{0,xxxx}, \tag{5.17}$$

which reduces to

$$\langle\eta\rangle_{,xx} + K^2\langle\eta\rangle = \mu^2\alpha_1\langle b\eta_{1,x}\rangle_{,x} \tag{5.18}$$

where

$$\alpha_1 = \frac{1}{h_o(1+Rh_o)}, \quad K^2 = k^2 + \frac{\mu^2 k^4 h_o(R+h_o)}{3(1+Rh_o)} + \frac{\mu^2 R k^2 \Gamma(0)}{h_o(1+Rh_o)^2}. \tag{5.19}$$

Applying Green's formula to $\langle\eta\rangle$ and its complex conjugate $\langle\eta^*\rangle$ over a large region of disorder of $L = O(1/\mu^2 k)$, we obtain

$$\int_0^L \mathrm{d}x\,[\langle\eta\rangle^*(\langle\eta\rangle_{,xx} + K^2\langle\eta\rangle) - \langle\eta\rangle(\langle\eta\rangle^*_{,xx} + K^2\langle\eta\rangle^*)] = [\langle\eta\rangle^*\langle\eta\rangle_x - \langle\eta\rangle\langle\eta\rangle^*_x]_0^L. \tag{5.20}$$

If both sides are multiplied by $-\mathrm{i}$, the right-hand side of (5.20) can be written as

$$2\mathrm{Re}[\langle\eta\rangle^*(-\mathrm{i}\langle\eta\rangle_x)]_0^L \approx 2\mathrm{Re}[\eta_0^*(-\mathrm{i}\eta_{0,x})]_0^L + O(\mu^2), \tag{5.21}$$

which is, except for the constant factor $\rho g^2/\omega$, the net power out-flux of the leading-order coherent wave through the ends. By using (5.18), the left-hand side of (5.20) is

$$\mu^2\int_0^L \mathrm{d}x\,[\langle\eta\rangle^*(-\mathrm{i}\alpha_1\langle b\eta_{1,x}\rangle)_x + \langle\eta\rangle(-\mathrm{i}\alpha_1\langle b\eta_{1,x}\rangle)^*_x] \approx 2\mu^2\mathrm{Re}\int_0^L \mathrm{d}x\;\eta_0^*(-\mathrm{i}\alpha_1\langle b\eta_{1,x}\rangle)_x.$$

Since $\mu b(x)$ is the random fluctuation of depth and $-\mathrm{i}\mu\eta_{1,x}$ in the integrand on the right-hand side is the induced random fluctuation of the horizontal velocity, $-\mathrm{i}\mu^2\langle b\eta_{1,x}\rangle$ represents the averaged adjustment of volume flux. Thus, the last integral above is the rate of pressure-working by the coherent pressure through the fluctuating flux over the region of disorder. Since the total extent is very large $L = O(1/k\mu^2)$, the integral above is of the same order as (5.21). It follows that, to leading order,

$$\mathrm{Re}[\eta_0^*(-\mathrm{i}\eta_{0,x})]_0^L = \mu^2 \mathrm{Re}\int_0^L \mathrm{d}x\, \eta_0^*(-\mathrm{i}\alpha_1\langle b\eta_{1,x}\rangle)_x, \tag{5.22}$$

which means that the coherent motion loses energy by making a random adjustment of velocity in response to random fluctuations of the seabed. As an analogy, a horse running on a rough road must slow down by having to spend energy to make small and random adjustments of its speed. Equation (5.22) can be further confirmed by explicit calculation. To the leading order, we can use the first harmonic results (5.16) derived in Appendix B to obtain

$$\eta_0 = A\exp(\mathrm{i}Kx) = A_0\exp(-\mu^2\beta_r^\dagger x)\exp(\mathrm{i}\kappa x), \tag{5.23}$$

where $(\beta_r^\dagger) = \mathrm{Re}(\beta_1^\dagger)$ and $\kappa = K - \mu^2\,\mathrm{Im}\beta_1^\dagger)$. Hence, the left-hand side of (5.22) is

$$\kappa|A(0)|^2\exp(-2\mu^2\beta_r^\dagger x)\Big]_0^L = -\kappa|A(0)|^2[1-\exp(-2\mu^2\beta_r^\dagger L)] < 0. \tag{5.24}$$

From (B 4) we have

$$\begin{aligned}\langle b\eta_{1,x}\rangle &= -2\beta A(0)\exp(-\mu^2\beta_r^\dagger x)\exp(\mathrm{i}\kappa x)\\ &= -2\beta^\dagger[h_o(1+Rh_o)]A(0)\exp(-\mu^2\beta_r^\dagger x)\exp(\mathrm{i}\kappa x),\end{aligned} \tag{5.25}$$

therefore the right-hand side of (5.22) is, to the leading order

$$-2\mu^2\beta_r^\dagger\kappa|A(0)|^2\int_0^L \mathrm{d}x\exp(-2\mu^2\beta_r^\dagger x) = -\kappa|A(0)|^2[1-\exp(-2\mu^2\beta_r^\dagger L)] < 0, \tag{5.26}$$

which is indeed equal to (5.24).

5.3. *Numerical results*

We present below numerical results computed for the initial condition that only the first harmonic has non-zero energy at $X = 0$, i.e. $A_1(0) = 1,\ A_m(0) = 0,\ m = 2, 3, 4, \ldots$. For reference, we first show in figure 9 the evolution of interfacial harmonics on a smooth seabed with $D = 0$ for fixed $R = 0.97$ and $U_r = 3$ and various depth ratios. It can be seen that as long as $h_o \neq 1/\sqrt{R}$, energy exchange between harmonics is strong. Near the threshold, nonlinearity is weak; energy transfer is practically only in one direction from the first to higher harmonics. For subcritical depths, the modulational distance decreases with increasing h_o (deeper lower layer). For a relatively weak disorder with $D = 0.4$, envelope harmonics are shown in figure 10 for several subcritical and supercritical depths. Energy exchange among harmonics is modified by significant localization by disorder in all cases. For a strong disorder with $D = 1$, figure 11 shows that all harmonics are heavily attenuated. Attenuation is, of course, more rapid for shallower lower layers (smaller h_o). Harmonic generation by nonlinear interaction can be overwhelmed by radiation damping from random scattering.

6. Concluding remarks

We have investigated the effects of seabed roughness on the propagation of weakly nonlinear interfacial waves of wavelength much greater than the sea depth, but comparable to the horizontal scale of depth variation. The slow attenuation of a

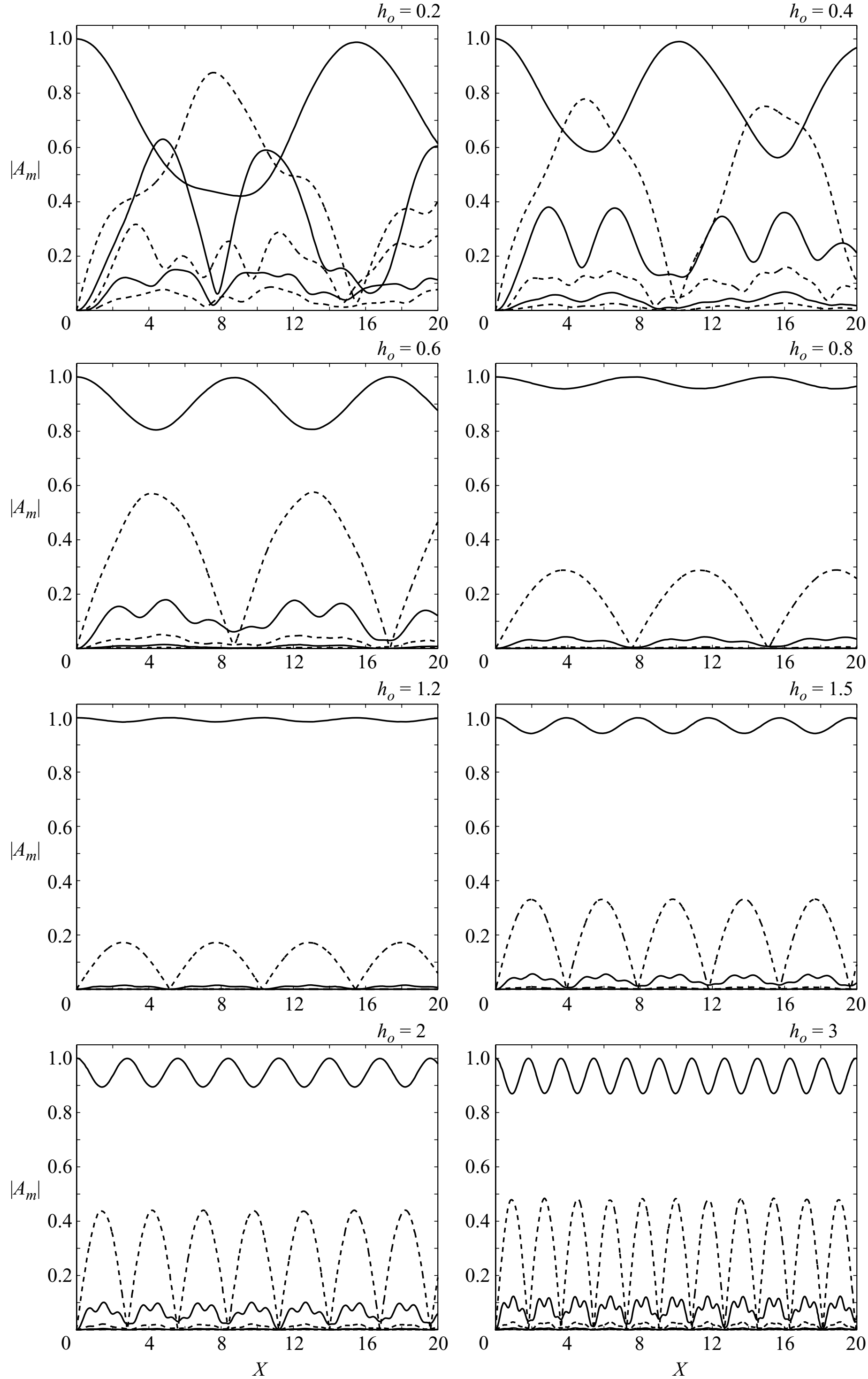

FIGURE 9. Harmonic generation for interfacial waves when there is no disorder ($D = 0$) for depth ratios $h_o = 0.2, 0.4, 0.6, 0.8, 1.2, 1.5, 2$ and 3. Other input parameters are $k_1 = 1$, $R = 0.97$, $U_r = 3$. Curves from top down represent first, second, third, … harmonics.

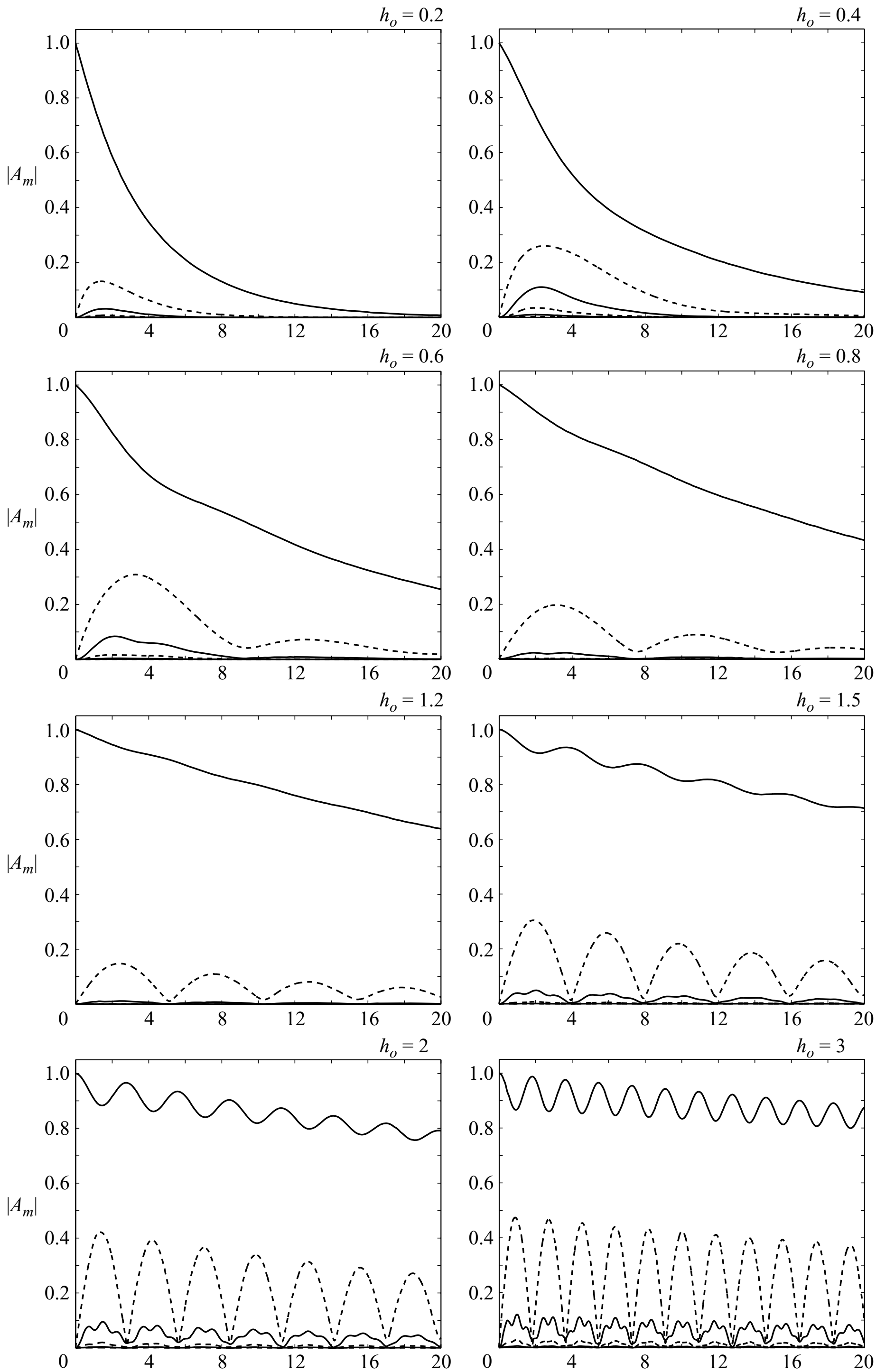

FIGURE 10. Evolution of harmonics over a semi-infinite range of weak disorder ($D = 0.4$). Other values as for figure 9.

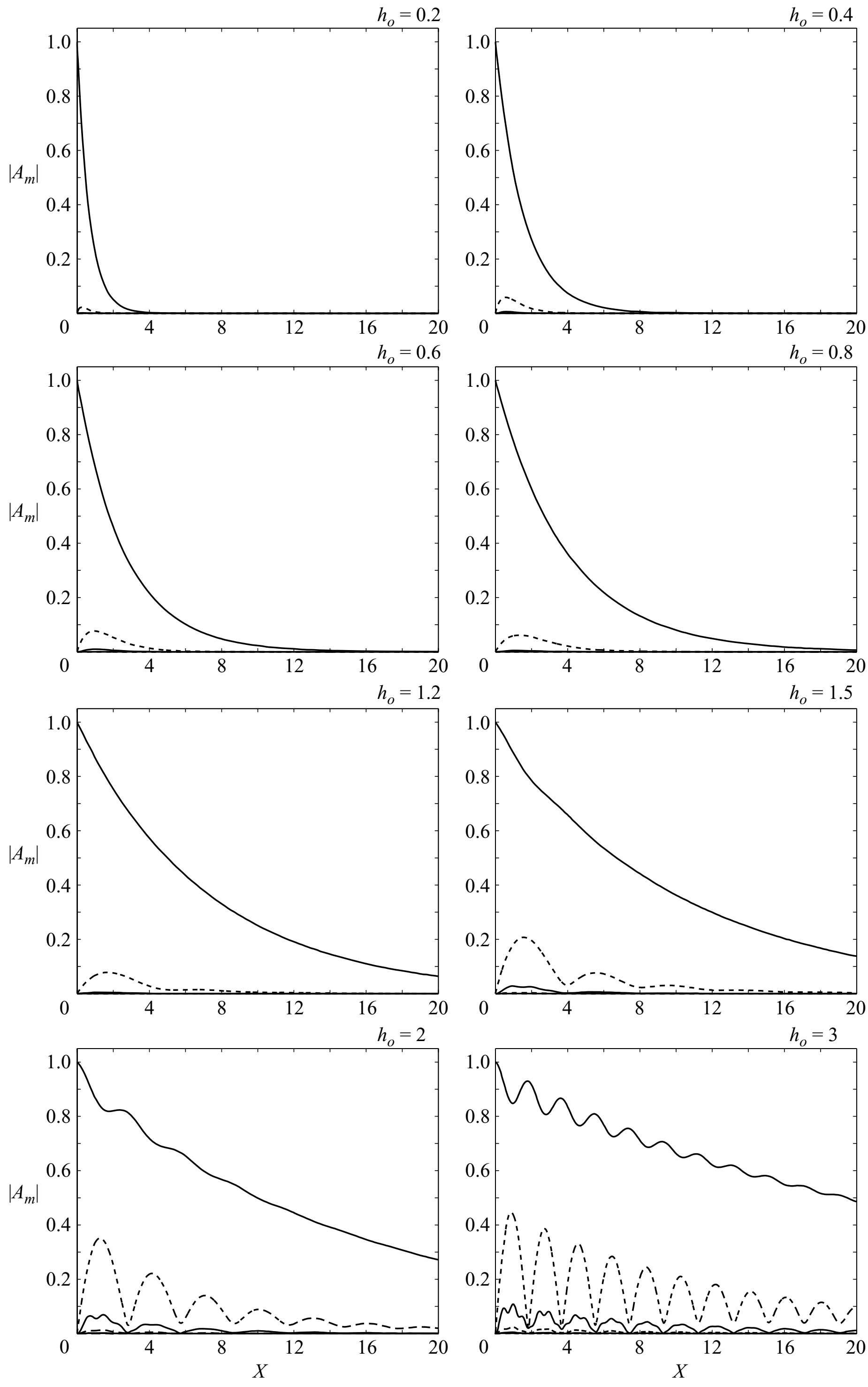

FIGURE 11. Evolution of harmonics over a semi-infinite range of strong disorder ($D = 1$). Other values as for figure 9.

transient solitary pulse and periodic waves passing over a long region of disorder is examined. Emphasis is on the case where wave nonlinearity $\epsilon = kA$ and dispersion $\mu^2 = H'^2/\ell^2$ are of the same order and the seabed roughness height is slightly greater and of the order of μH. A broad range of depth ratio not close to the critical value, $H/H' = \sqrt{\rho'/\rho}$, is examined. The localization of a transient pulse is shown to be governed by an integro-differential equation generalizing both KdV and Burgers equations. Localization is studied by both analytical approximations and numerical computations of the generalized equation. For time-harmonic waves, the mode-coupling equations are derived with new damping terms representing effects of disorder, for various depth ratios.

For the small neighbourhood of the threshold depth ratio $H/H' = \sqrt{\rho'/\rho}$, the remedy is, in principles to extend the KdV approximation with additional nonlinear terms of higher order, as in the theory for a smooth bed (see e.g. Helfrich, Melville & Miles (1984). In recent theories of two-layered fluids at constant depth, more accurate equations accounting for strong nonlinearity have been proposed as the basis of numerical computations (Choi & Camassa 1999; Ostrovsky & Grue 2003). In the present theory for weak nonlinearity and dispersion, weak disorder modifies the KdV equation or the mode-coupling equations by linear diffusive and dispersive terms. It is likely that, even for strongly nonlinear interfacial waves, the effects of weak disorder would still be represented by similar linear terms. The quantitative consequences are worthy of further studies.

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Appendix A. Perturbation analysis for small roughness

Introducing the multi-scale expansion $\zeta = \zeta^{(0)}(X, X_1, \sigma) + D^2\zeta^{(1)}(X, X_1, \sigma) + \ldots$ into (4.2), the following perturbation equations are obtained

$$\zeta^{(0)}_{,X} + \lambda_1 \zeta^{(0)}\zeta^{(0)}_{\sigma} + \lambda_2 \zeta^{(0)}_{,\sigma\sigma\sigma} = 0, \tag{A 1}$$

$$\zeta^{(1)}_{,X} + \lambda_1 \zeta^{(0)}\zeta^{(1)}_{\sigma} + \lambda_1 \zeta^{(1)}\zeta^{(0)}_{\sigma} + \lambda_2 \zeta^{(1)}_{,\sigma\sigma\sigma} = -\zeta^{(0)}_{,X_1} + \frac{1}{Ch_o(1+Rh_o)^3}\left\{\frac{\gamma}{2}\zeta^{(0)}_{,\sigma} + \frac{\sqrt{2\pi}}{8}\zeta^{(0)}_{,\sigma\sigma}\right.$$
$$\left. + \frac{1}{16}\int_{-\infty}^{\infty} \exp\left(-\frac{|\sigma-\sigma'|^2}{8}\right)\zeta^{(0)}_{,\sigma'\sigma'}\,\mathrm{d}\sigma' + \frac{\sqrt{2\pi}}{16}\int_{-\infty}^{\infty} \operatorname{erfc}\left(\frac{|\sigma-\sigma'|}{2\sqrt{2}}\right)\zeta^{(0)}_{,\sigma'\sigma'\sigma'}\,\mathrm{d}\sigma'\right\}. \tag{A 2}$$

Denoting $\sigma^\dagger = \sigma - UX$, equations (A 1) and (A 2) are transformed to two adjoint equations

$$\frac{\partial}{\partial\sigma^\dagger}\left\{-U + \begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}\lambda_1\zeta^{(0)} + \lambda_2\frac{\partial^2}{\partial\sigma^{\dagger 2}}\right\}\begin{bmatrix}\zeta^{(0)}\\ \zeta^{(1)}\end{bmatrix} = \begin{bmatrix}0\\ \text{r.h.s.(A 2)}\end{bmatrix}. \tag{A 3}$$

The solution for $\zeta^{(0)}$ is the classical internal soliton given by (4.3) with $U = \lambda_2/3$, multiplied by the amplitude $A(X_1)$. The solvability condition of the problem for $\zeta^{(1)}$ can be obtained by applying Green's identity to the two adjoint equations above, yielding,

$$\int_{-\infty}^{\infty} \mathrm{d}\sigma\, \zeta^{(0)}\,\text{r.h.s.(A 2)} = \int_{-\infty}^{\infty} \mathrm{d}\sigma\, \zeta^{(0)} \left\{ -\zeta^{(0)}_{,X_1} + \frac{1}{Ch_o(1+Rh_o)^3} \left[\frac{\gamma}{2}\zeta^{(0)}_{,\sigma} + \frac{\sqrt{2\pi}}{8}\zeta^{(0)}_{,\sigma\sigma} \right.\right.$$
$$\left.\left. + \frac{1}{16}\int_{-\infty}^{\infty} \exp\left(-\frac{|\sigma-\sigma'|^2}{8}\right) \zeta^{(0)}_{,\sigma'\sigma'}\, \mathrm{d}\sigma' + \frac{\sqrt{2\pi}}{16}\int_{-\infty}^{\infty} \operatorname{erfc}\left(\frac{|\sigma-\sigma'|}{2\sqrt{2}}\right) \zeta^{(0)}_{,\sigma'\sigma'\sigma'}\mathrm{d}\sigma' \right]\right\}. \quad \text{(A 4)}$$

Inside the curly brackets, the second and fifth terms do not contribute to the integral because of their oddness in σ. We obtain by partial integration

$$-\frac{1}{2}\frac{\partial}{\partial X_1}\int_{-\infty}^{\infty} \zeta^{(0)^2}\,\mathrm{d}\sigma - \frac{\sqrt{2\pi}}{8Ch_o(1+Rh_o)^3}\int_{-\infty}^{\infty}\left(\frac{\partial \zeta^{(0)}}{\partial\sigma}\right)^2 \mathrm{d}\sigma$$
$$+ \frac{1}{16Ch_o(1+Rh_o)^3}\int_{-\infty}^{\infty} \mathrm{d}\sigma\, \zeta^{(0)} \int_{-\infty}^{\infty} \frac{\partial^2\zeta^{(0)}}{\partial\sigma'^2} \exp(-(\sigma-\sigma')^2/8)\,\mathrm{d}\sigma' = 0. \quad \text{(A 5)}$$

After using the soliton solution for $\zeta^{(0)}$, we obtain the amplitude evolution equation (4.17).

Appendix B. Second-order forcing terms for the harmonic generation problem

Referring to the right-hand side of (3.5), the mth harmonic of the first forcing term is

$$\left\langle b\eta^{(m)}_{1,x}\right\rangle = \left\langle b\ \mathrm{i}k_m \operatorname{sgn}(x-x')\eta^{(m)}_1\right\rangle = \mathrm{i}k_m \frac{A_m(X)}{4h_o(1+Rh_o)}$$
$$\times \int_{-\infty}^{\infty} \operatorname{sgn}(x-x')\exp(\mathrm{i}k_m|x-x'|)\frac{\mathrm{d}}{\mathrm{d}x'}[\langle b(x)b(x')\rangle \exp(\mathrm{i}k_m x')]\,\mathrm{d}x'. \quad \text{(B 1)}$$

Assuming again spatial homogeneity of the depth disorder over the short scale as in (3.7), it then follows that

$$\left\langle b\eta^{(m)}_{1,x}\right\rangle = -\,\mathrm{i}k_m \frac{A_m(X)D^2}{4h_o(1+R)} \exp(\mathrm{i}k_m x)\int_{-\infty}^{\infty} \operatorname{sgn}(\xi)\exp(\mathrm{i}k_m|\xi|)\frac{\mathrm{d}}{\mathrm{d}\xi}[\Gamma(\xi)\exp(\mathrm{i}k_m\xi)]\,\mathrm{d}\xi. \quad \text{(B 2)}$$

Since

$$\left\langle b\eta^{(-m)}_{1,x}\right\rangle = \left\langle b\eta^{(m)}_{1,x}\right\rangle^*, \quad \text{(B 3)}$$

where f^* denotes the complex conjugate of f, we have

$$-\frac{\partial}{\partial x}\left[\sum_{m=-\infty}^{\infty} \exp(-\mathrm{i}\omega_m t)\left\langle b\eta^{(m)}_{1,x}\right\rangle\right] = \sum_{m=1}^{\infty} \mathrm{i}k_m A_m(X)\beta_m \exp(\mathrm{i}\theta_m) + c.c. \quad \text{(B 4)}$$

where

$$\beta_m = \frac{\mathscr{I}_m}{h_o(1+Rh_o)}, \quad \text{(B 5)}$$

with

$$\mathscr{I}_m \equiv \frac{D^2}{4}\mathrm{i}k_m \int_{-\infty}^{\infty} \operatorname{sign}(\xi)\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}\xi} + \mathrm{i}k_m\Gamma\right)\exp(\mathrm{i}k_m(|\xi|+\xi))\,\mathrm{d}\xi \quad \text{(B 6)}$$

depends only on X. The nonlinear forcing term in (3.5) can be transformed to

$$w_2\left(\eta_0^2\right)_{,tt} = -\frac{w_2}{4}\sum_{m=1}^{\infty}\omega_m^2\exp(\mathrm{i}\theta_m)\left[\sum_{l=1}^{\infty}2A_l^*A_{m+l} + \sum_{l=1}^{[m/2]}\epsilon_l A_l A_{m-l}\right] + *, \qquad (\mathrm{B}\,7)$$

where $[m/2]$ is the integer part of $m/2$, and ϵ_l is the Jacobi symbol equal to unity for $l=[m/2]$ and equal to two for $l=2, 3, 4, \ldots$ otherwise. Further details can be found in Grataloup & Mei (2003). The last two forcing terms in (3.5) are

$$w_3\eta_{0,xxxx} - w_4\Gamma(0)\eta_{0,xx} = \frac{w_3}{2}\sum_{m=1}^{\infty}k_m^4 A_m\exp(\mathrm{i}\theta_m) + \frac{w_4}{2}\sum_{m=1}^{\infty}k_m^2 A_m\exp(\mathrm{i}\theta_m) + c.c \qquad (\mathrm{B}\,8)$$

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