Competition in the Supply Option Market[1]

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Abstract

This paper develops a multi-attribute competition model for procurement of commodities, such as electricity. We describe a purchasing process between a buyer and many suppliers for option contracts in a single period supply environment. The parameters of the negotiation are two-dimensional: the option reservation price, or premium, and the option execution price, also referred to as exercise price or strike. These two parameters illustrate the trade-off between total price and flexibility of the contract, and are both important to the buyer. We model the interaction between the suppliers and the buyer as a Stackelberg game in which the suppliers are the leaders and the buyer is the follower. Specifically, suppliers compete to provide supply capacity to the buyer and the buyer optimizes its expected profit by selecting one or more suppliers. We characterize the suppliers' Nash equilibria in pure strategies for a class of customer demand distributions. In particular, we show that this type of interaction gives rise to *cluster competition*. That is, in equilibrium, suppliers tend to be clustered in small groups of two or three suppliers each, such that within the same group all suppliers use similar technologies and offer the same type of contract.

1 Introduction

The development of electronic marketplaces for supply has been a major revolution in the way manufacturers and suppliers interact. Indeed, the success story of start-ups such as Freemarkets in the mid 90s was quickly followed by the blossoming of many competing e-markets. These markets brought together many suppliers and, by forcing competition, reduced the overall buyer's cost.

Of course, in the process, suppliers saw their revenue drop due to the increased price pressure. A common complaint was that the e-market bidding process put too much emphasis on price, and did not value enough suppliers' characteristics such as flexibility, quality or lead time.

Indeed, manufacturers were unable to quantify the value of these attributes, which led to the destruction of the traditional relationships between manufacturers and suppliers. Specifically, manufacturers were incapable of rewarding flexible or high-quality suppliers versus inflexible or low-quality suppliers. Thus, there is a need for models, tools and bidding mechanisms that capture the multiple dimensions of the suppliers' characteristics, force them to compete and differentiate on these multiple dimensions, and hence allow buyers to value attributes other than price.

A first step in this direction was the paper by Martínez-de-Albéniz and Simchi-Levi [12]. They propose a multi-period model for the optimization of a portfolio of contracts. In their framework, a manufacturer can optimally structure its sourcing so that it takes advantage of the flexibility of different long-term or option contracts. As a result, the manufacturer purchases from each competing supplier a share of capacity that reflects the trade-off between price and flexibility offered by that

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supplier. Thus, in their framework, flexibility and price are the two attributes that manufacturers care about.

Evidently, this framework can trigger industry-wide changes, and, in particular, changes in the way suppliers compete in the marketplace. The objective of this paper is to analyze the strategic behavior of suppliers under such purchasing behavior by the buyer. We consider a single period model where many suppliers and a single buyer must negotiate the contract terms for a single component. This negotiation may take the form of a sealed-bid auction or an iterative mechanism. We study the sealed bid negotiation and provide an equivalent iterative mechanism.

In our model, we assume that suppliers have an initial cost for reserving capacity and an additional cost for using the capacity to satisfy the buyer's orders. This is typically due to different technologies used by various suppliers.

For instance, in the electricity industry, different types of power plants exist, from nuclear to coal or gas power plants. Nuclear power plants have a relatively small degree of flexibility in adjusting production level to meet demand and hence all the incurred cost is associated with reserving capacity. On the other hand, gas power plants can adjust the production level rapidly and hence most of the cost is associated with delivering electricity.

In manufacturing, and especially in the plastics, chemicals or semi-conductor industries, buyers reserve capacity with suppliers in advance of production time. Of course, different suppliers may have different costs for reserving capacity and delivering supply, depending on the type of technology (machinery) and their geographical location (labor, transportation).

Finally, our model may also be relevant to the travel and tourism industry, where the service providers may have different cost structures in terms of capacity reservation cost (cost of leasing airplanes or hotels) and variable cost (operating cost).

These cost characteristics obviously impact the negotiation process. In our model, each supplier offers an option contract to the buyer characterized by two pricing parameters, a capacity reservation fee and an execution fee. Consequently, suppliers can become more competitive by pushing in two directions: either lowering the reservation price or the execution price. The trade-off is clear. A supplier that charges mainly a reservation fee (and a small execution fee) competes on price but not flexibility. On the other hand, a supplier that charges mainly an execution fee (and a small reservation fee) typically emphasizes flexibility and not price.

We describe the market equilibrium outcomes of such system, and in particular the behavior of market prices for existing supply options. Interestingly, this model is an extension of the Bertrand price competition model to two dimensions. An important result in one dimension is that, in equilibrium, there is a unique supplier, the least costly supplier, that captures all the orders at a market price that is between its cost and the cost of the second most competitive supplier. We show that this is not the case when two attributes are important to the buyer. Indeed, we demonstrate that in equilibrium, a variety of suppliers coexists, and these suppliers offer different prices. We call this *cluster competition*, since suppliers tend to cluster in small groups of two or three suppliers each, such that within the same group all suppliers use similar technologies and offer the same type of contract.

We start by reviewing in Section 2 the different streams of literature relevant to our research and present the model in Section 3. We then analyze the buyer's behavior in Section 4 and the suppliers' strategies in Section 5. This leads to the study of the equilibria of the negotiation process in Section 6. Finally, we introduce, in Section 7, a mechanism for practical implementation and conclude with managerial insights in Section 8.

2 Literature review

Our starting point for this research is the recent paper by Martínez-de-Albéniz and Simchi-Levi [12]. In their work, they develop a multi-period framework in which buyers optimize their purchasing strategy by carefully balancing price and flexibility. In particular, in their single period version, they provide a closed form expression for the amounts of option capacities that a buyer purchases from a pool of suppliers. We apply this result in the analysis of the behavior of the suppliers in such a setting, where competition is carried through two dimensions: price and flexibility, or equivalently reservation and execution prices.

We relate this research to the literature on supply contracts; for a review see Cachon [4] or Lariviere [10]. In particular, some papers study option contracts, e.g., Barnes-Schuster et al. [1] or Eppen and Iyer [6]. More relevant to our model are papers that analyze the behavior of suppliers in offering options to a buyer, the prelude to introducing competition between suppliers. The existing literature usually models a Stackelberg game where a single buyer is the follower and a single supplier is the leader. Typically, competition in such models is introduced by a spot market. This spot market is the buyer's sourcing alternative and a potential client for the supplier. The focus is on finding conditions for which both players are willing to sign a contract and determining option prices as the outcome of the negotiation process.

The first publication in this stream of literature is by Wu et al. [17]. Motivated by electricity markets, they derive option prices as a function of the cost of the system and the elasticity of demand. Later, Spinler et al. [15] and Golovachkina and Bradley [9] analyze models similar to that of Wu et al. Interestingly, there are no papers that directly analyze competition among suppliers since this implies utilizing the notion of portfolio contracts, developed in Martínez-de-Albéniz and Simchi-Levi [12]. Thus, our paper moves from the traditional models of competition through dual sourcing, i.e., single supplier offering an option contract versus spot market, to a model of pure competition between suppliers offering different types of options.

A second stream of the literature concentrates on analyzing multi-attribute auctions. This research is quite recent and follows the development of online auctions in B2B markets. Typically, the objective is to find conditions that guarantee that an auction mechanism leads to an optimal outcome. An optimal outcome may be defined as social efficiency or utility maximization from the auctioneer's point of view (similarly to the seminal work of Myerson [13] in a one-dimensional auction).

In this line of research, various authors have studied the winner determination problem, where a single supplier is awarded all the orders. This differs from our formulation where all the suppliers may potentially be selected for part of the procurement. For instance, Beil and Wein [2], following Che [5], present a multi-attribute Request For Quotation (RFQ) process where the buyer declares a scoring rule and chooses a winner among many suppliers, the one that obtains the highest score for the declared rule.

In a different direction, some research has been done on mechanism design where many bidders can be awarded orders at the same time. For instance, Schummer and Vohra [14] analyze a class of two-dimensional option auction mechanisms for a set of suppliers confronted with a single buyer. Their formulation is similar to our model but focuses on designing an efficient procurement mechanism where the suppliers truthfully reveal their costs. Because suppliers submit their true costs, their paper does not address competition between suppliers.

3 Assumptions and Notation

Consider a single manufacturer looking for supply of a component that is used in the manufacturing of the final product. This component, e.g., a PC motherboard or a chassis, may be obtained from a variety of suppliers.

3.1 Supply side

We denote by N the number of suppliers in the market, each of which offers an option contract for the component. Such a contract is defined by two parameters, $v \geq 0$, the reservation price, and $w \geq 0$, the execution price. These values are determined by the supplier based on its cost structure as well as on whether the supplier emphasizes price or flexibility.

Given the suppliers' offerings, a buyer specifies the amount of capacity to reserve with each supplier. At the time the buyer executes a contract with a supplier, it can purchase any amount q, where q is no more than the reserved capacity with that supplier.

Specifically, supplier i, $i = 1, \ldots, N$, takes position in the market by offering options at a reservation price v_i and an execution price w_i.

Assumption 1 *Supplier i, $i = 1, \ldots, N$, receives an amount $v_i x_i + w_i q_i$ when a buyer reserves x_i units of capacity and executes q_i units, $q_i \leq x_i$.*

The suppliers' cost structure is assumed to consist of two parts. Each supplier incurs a fixed unit cost for reserving capacity, f_i, $i = 1, \ldots, N$, that can be seen as the unit cost of building a factory of the appropriate size, developing the technology required to produce the component, hiring manpower, or signing its own supply contracts with its suppliers, e.g., the energy provider. In addition, the supplier pays a unit cost, c_i, $i = 1, \ldots, N$, for each unit executed by the buyer. This cost is typically the cost of raw materials and operational costs.

These costs differ from supplier to supplier and may be explained by the use of different technologies (e.g. different type of power plants in electricity generation, with much different fixed costs and variable costs) or management practices. Without loss of generality, we assume that $c_1 \leq \ldots \leq c_N$.

Assumption 2 *Supplier i, $i = 1, \ldots, N$, pays an amount $f_i x_i + c_i q_i$ when a buyer reserves x_i units of capacity and executes q_i units, $q_i \leq x_i$.*

The objective of the suppliers is to maximize their expected profit by selecting (w_i, v_i) optimally.

3.2 Demand side

The buyer uses the component purchased from the suppliers in the manufacturing of the finished product.

Assumption 3 *The manufacturer sells to end customers at a given unit price p fixed in advance.*

Assumption 4 *The total customer demand D follows a distribution defined over an interval $[\underline{d}, \overline{d}] \subset [0, \infty]$. The c.d.f. of the demand $F(\cdot)$ is strictly increasing in $[\underline{d}, \overline{d}]$. We assume that $F(\cdot)$ is a continuous and differentiable function over $(\underline{d}, \overline{d})$. Define $f(\cdot) = F'(\cdot)$ and $\overline{F}(\cdot) = 1 - F(\cdot)$.*

The manufacturer's objective is to maximize expected profit by optimally selecting the amount of capacity to reserve from each supplier.

3.3 Sequence of events

We analyze a two-stage model. In the first stage all the suppliers submit bids that are defined by (w_i, v_i), $i = 1, \ldots, N$. At the same time, and based on these bids, the manufacturer decides on the amount of capacity to reserve with each supplier. In the second period, demand is realized and the manufacturer decides the amount to execute from each contract. If total capacity is not enough, unsatisfied demand is lost.

This is a Stackelberg game in which the suppliers are leaders and the manufacturer is the follower. Thus, there are multiple leaders that compete knowing the reaction of the follower. This type of game, related to backward induction and sub-game perfection concepts, is described in detail in Fudenberg and Tirole [8], pp. 92-96.

Of course, the manufacturer's objective is to maximize profit based on the suppliers bids. Suppliers have complete visibility to the manufacturer decision making process. Therefore, given any N pairs (w_i, v_i), $i = 1, \ldots, N$, each supplier can figure out the amount of capacity that the manufacturer would reserve with each individual supplier as well as the distribution of the amount of supply executed (requested) by the manufacturer. The costs (c_i, f_i), $i = 1, \ldots, N$, are also known by all suppliers.

In addition, we assume that the suppliers submit sealed bids simultaneously, and that they are not allowed to change their bids again. Thus, this is a one-shot game. Every supplier submits a bid that maximizes its expected profit. We are interested in determining the Nash equilibria of this game in pure strategies, i.e., the N-uples (w_i, v_i), $i = 1, \ldots, N$, where no supplier has an incentive to unilaterally change its bid.

4 Manufacturer's Procurement Strategy

Martínez-de-Albéniz and Simchi-Levi [12] present a general framework for supply contracts in which portfolios of options can be analyzed and optimized. In this section, we review the framework in the context of a single period environment.

Consider a manufacturer facing N different options with terms (w_i, v_i), $i = 1, \ldots, N$. Martínez-de-Albéniz and Simchi-Levi show that the manufacturer's expected profit is concave in the quantities $(x_1, \ldots, x_N)$ purchased. Without loss of generality, assume that $w_1 \leq \ldots \leq w_N \leq p$. Define $v_{N+1} = 0$ and $w_{N+1} = p$ and

$$C(x,q) = \sum_{i=1}^{N+1} v_i x_i + \max \sum_{i=1}^{N+1} w_i q_i \quad \text{subject to} \begin{cases} 0 \leq q_i \leq x_i \;\; i = 1, \ldots, N, \\ 0 \leq q_{N+1}, \\ \sum_{i=1}^{N+1} q_i = q \end{cases}$$

Then the manufacturer's profit is $\Pi(x, D) = pD - C(x, D)$. Thus, the expected profit is $\overline{\Pi}(x) = pE[D] - E[C(x, D)]$.

Let $y_0 = 0$ and

$$y_i = x_1 + \ldots + x_i \text{ for } i = 1, \ldots, N. \tag{1}$$

Then, $V(y) = \overline{\Pi}(x)$ satisfies for $i = 1, \ldots, N$, see [12],

$$\frac{dV}{dy_i}(y) = (v_{i+1} - v_i) + (w_{i+1} - w_i)Pr[D \geq y_i]. \tag{2}$$

Equation (2) thus provides the structure of the manufacturer's optimal portfolio which is determined by the c.d.f. of customer demand. In particular, under Assumption 4, the profit is a strictly concave function of $(y_1, \ldots, y_N)$ defined over the set

$$P = \left\{ (y_1, \ldots, y_N) \in \mathbb{R}^N \middle| 0 \le y_1 \le \ldots \le y_N \right\} \tag{3}$$

Strict concavity implies that the optimal solution is unique. Thus, in the Stackelberg game analyzed in this paper, the leaders know exactly how the follower behaves.

To characterize the optimal portfolio, $(x_1^*, \ldots, x_N^*)$, we need the following definitions.

Definition 1 *Supplier i is called* active *if $x_i^* > 0$. Otherwise, it is called* inactive.

Definition 2 *Given a set of t different pairs $\{(a_1, b_1), \ldots, (a_t, b_t)\}$ with $a_1 \le \ldots \le a_t$, the* winning set *is the minimal subset $S = \{i_1, \ldots, i_k\}$ of these points such that:*

(a) $a_{i_1} \le \ldots \le a_{i_k}$;

(b) for $1 \le i < i_1$, $b_i - b_{i_1} \ge -(a_i - a_{i_1})$;

(c) for $j = 2, \ldots, k$, *for* $i_{j-1} < i < i_j$, $b_i - b_{i_j} \ge -\dfrac{b_{i_{j-1}} - b_{i_j}}{a_{i_j} - a_{i_{j-1}}}(a_i - a_{i_j})$;

(d) for $i_k \le i \le t$, $b_i \ge b_{i_k}$.

$i_1, \ldots, i_k$ *are called* winning points *among the t pairs.*

Definition 3 *Given a set of t pairs $\{(a_1, b_1), \ldots, (a_t, b_t)\}$ with $a_1 \le \ldots \le a_t$, the* lower envelope *is the curve $Z^{(\mathbf{a},\mathbf{b})}(\cdot)$ defined as follows*

$$Z^{(\mathbf{a},\mathbf{b})}(u) = \begin{cases} v_1 - (u - w_1) & \text{for } u \le w_1 \\ v_2 - \dfrac{v_1 - v_2}{w_2 - w_1}(u - w_2) & \text{for } w_1 \le u \le w_2 \\ \vdots & \\ v_k - \dfrac{v_{k-1} - v_k}{w_k - w_{k-1}}(u - w_k) & \text{for } w_{k-1} \le u \le w_k \\ 0 & \text{for } w_k \ge u, \end{cases} \tag{4}$$

where $(w_1, v_1), \ldots, (w_k, v_k)$ are the winning points of $\{(a_1, b_1), \ldots, (a_t, b_t)\}$ after sorting the first coordinates in increasing order.

These definitions, together with Equation (2), are used to characterize the optimal portfolio explicitly, as is done in the next proposition.

Proposition 1 *Supplier i, $i = 1, \ldots, N$, is active if and only if i is a winning point of $\{(w_1, v_1), \ldots, (w_{N+1}, v_{N+1})\}$.*

All the proofs are presented in the appendix.

The winning points, i.e., all the active suppliers, can be determined graphically, see Figure 1. Plot the pairs (w_i, v_i) in a graph with the w_i in the x coordinate and the v_i in the y coordinate. Add

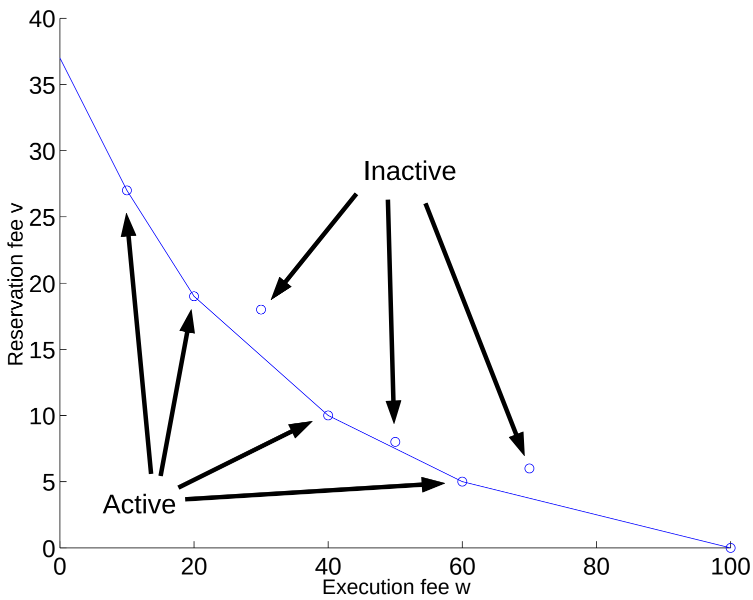

Figure 1: Illustration of active and inactive bids

also the point $(p, 0)$. Determine the convex hull of the points, and in particular find the extreme points on the lower envelope as defined in Definition 3; these are the points $i_1 < \ldots < i_k$.

Hence, the lower envelope is piecewise linear and convex. The segments have increasing slopes or equivalently decreasing negative slopes, that is,

$$\frac{v_{i_1} - v_{i_2}}{w_{i_2} - w_{i_1}} > \ldots > \frac{v_{i_{k-1}} - v_{i_k}}{w_{i_k} - w_{i_{k-1}}}.$$

The buyer's optimal portfolio only includes segments with negative slopes between 0 and 1. To identify these segments, let $l \geq 1$ be such that $\frac{v_{i_{l-1}} - v_{i_l}}{w_{i_l} - w_{i_{l-1}}} \geq 1 > \frac{v_{i_l} - v_{i_{l+1}}}{w_{i_{l+1}} - w_{i_l}}$ or $l = 1$ if all the ratios are below 1. Similarly, find $h \leq k$ such that $\frac{v_{i_{h-1}} - v_{i_h}}{w_{i_h} - w_{i_{h-1}}} > 0 \geq \frac{v_{i_h} - v_{i_{h+1}}}{w_{i_{h+1}} - w_{i_h}}$. Typically $h = N + 1$ since all prices (v_i or w_i) are non-negative and $v_i + w_i \leq p$ usually.

With these definitions, and recalling $y_0 = 0$, the optimal portfolio is defined by

$$y_i^* = \begin{cases} \overline{F}^{-1}\left(\frac{v_{i_j} - v_{i_{j+1}}}{w_{i_{j+1}} - w_{i_j}}\right) \text{ if } i = i_j, \ j = l, \ldots, h-1, \\ y_{i-1}^* \text{ for all others.} \end{cases}$$

The vector $\mathbf{x}^*$ follows directly from $\mathbf{y}^*$. In particular $x_i^* = 0$ for i different than $i_l, \ldots, i_h$.

5 Suppliers' Behavior

Given that the manufacturer uses a portfolio approach as described in the previous section, each supplier will set its reservation and execution price to maximize its expected profit, taking into account the behavior of other suppliers.

Consider the decision of supplier i, $i = 1, \ldots, N$. It is confronted by bids from other suppliers. Let $(\mathbf{w}_{-i}, \mathbf{v}_{-i})$ be the vector representing all other bids with the additional point $(w_{N+1} = p, v_{N+1} = 0)$.

Given the bids in the vector $(\mathbf{w}_{-i}, \mathbf{v}_{-i})$, we can identify the manufacturer optimal procurement strategy. Without loss of generality, assume that there are $k \leq N$ active suppliers, indexed from 1 to k, and $w_1 \leq \ldots \leq w_k$ (one of the suppliers might be the dummy supplier with parameters $(p, 0)$). The manufacturer best procurement strategy is

$$\begin{cases} y_j^{-i} = \overline{F}^{-1}(\dfrac{v_j - v_{j+1}}{w_{j+1} - w_j}) \ j = 1, \ldots, k-1, \\ y_k^{-i} = \overline{F}^{-1}(0). \end{cases} \tag{5}$$

If supplier i places a bid (w_i, v_i), the manufacturer optimal solution may change to take this bid into account. Of course, suppliers that were not active before are not going to be active with the new bid from supplier i. However, it is entirely possible that some suppliers may become inactive when supplier i enters with the bid (w_i, v_i). Finally, supplier i may capture zero capacity if its bid makes it inactive. Clearly, in this case, if supplier i is inactive, we can withdraw it from the pool of bids and consequently the capacities allocated to the other suppliers remain unchanged. This happens when (w_i, v_i) is above the lower envelope which is described by the function $Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(\cdot)$ in Definition 3. Thus, when $v_i \geq Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(w_i)$, supplier i is inactive and its profit is $\Pi = 0$. We define this bidding region which makes i inactive as

$$A_{OUT}^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})} = \Big\{(w, v) \in \mathbb{R}_+^2 | v \geq Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(w)\Big\}.$$

If supplier i's bid is not in that region, then supplier i becomes active. Adding bid (w_i, v_i) to the rest of the bids may change the convex hull of the points in two different ways:

- Supplier i becomes the first active supplier, i.e., there exist $h \in \{1, \ldots, k\}$ such that suppliers $i, h, \ldots, k-1$ are active and suppliers $1, \ldots, h-1$ are inactive. We define this region as $A_{0h}^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}$.

$$A_{0h}^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})} = \left\{ (w, v) \in \mathbb{R}_+^2 \;\middle|\; \begin{array}{l} v - v_1 \leq -(w - w_1) \\ v - v_h \leq -(\dfrac{v_{h-1} - v_h}{w_h - w_{h-1}})(w - w_h) \quad \text{(only if } h > 1) \\ v - v_h \geq -(\dfrac{v_h - v_{h+1}}{w_{h+1} - w_h})(w - w_h) \end{array} \right\} \tag{6}$$

- Supplier i is not the first active supplier, i.e., there exist $l \in \{1, \ldots, k-1\}$ and $h \in \{1, \ldots, k\}$, $h > l$, such that suppliers $i, 1, \ldots, l, h, \ldots, k$ are active and $l+1, \ldots, h-1$ inactive. We define this region as $A_{lh}^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}$.

$$A_{lh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})} = \left\{ (w,v) \in \mathbb{R}_+^2 \;\middle|\; \begin{array}{l} v - v_l \geq -(\dfrac{v_{l-1} - v_l}{w_l - w_{l-1}})(w - w_l) \\ (\text{or } v - v_1 \geq -(w - w_1) \text{ if } l = 1) \\ v - v_l \leq -(\dfrac{v_l - v_{l+1}}{w_{l+1} - w_l})(w - w_l) \\ v - v_h \leq -(\dfrac{v_{h-1} - v_h}{w_h - w_{h-1}})(w - w_h) \\ v - v_h \geq -(\dfrac{v_h - v_{h+1}}{w_{h+1} - w_h})(w - w_h) \end{array} \right\} \quad (7)$$

These regions are nicely illustrated in Figure 2. Intuitively, a bid in region $A_{lh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$ implies that supplier i forces suppliers $l+1,\ldots,h-1$ out of the market, i.e., these suppliers receive zero capacity allocation.

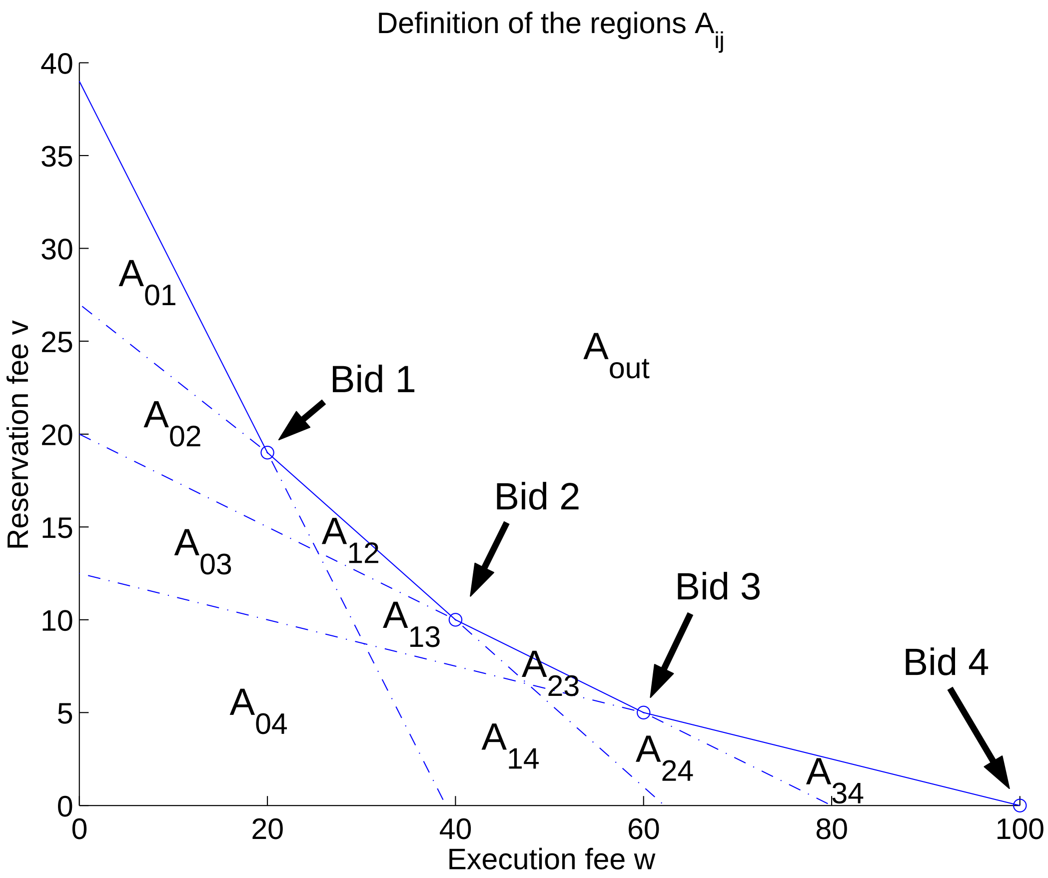

Figure 2: Division of the bidding strategies in different regions

The capacity allocated by the buyer to supplier i, x_i, if i bids in $A_{lh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$, $l > 0$, is $x_i = y_{i+} - y_{i-}$ where y_{i+} and y_{i-} are given by the following set of equations. We drop the sub-index i to simplify notation.

$$\begin{aligned} \frac{v_l - v}{w - w_l} &= \overline{F}(y_-) \\ \frac{v - v_h}{w_h - w} &= \overline{F}(y_+). \end{aligned} \tag{8}$$

The case of $l = 0$ is special. The buyer allocates then capacity such that

$$\begin{aligned} 0 &= y_- \\ \frac{v - v_h}{w_h - w} &= \overline{F}(y_+). \end{aligned}$$

For $l > 0$, since $(w, v) \in A_{lh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$, one must keep in mind that $y_{l-1}^{-i} \leq y_- \leq y_l^{-i}$, $y_{h-1}^{-i} \leq y_+ \leq y_h^{-i}$ and therefore $y_- \leq y_+$ (moreover, if $y_- = y_+$, supplier i is in $A_{OUT}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$ and is thus inactive).

The profit of supplier i in this case is

$$\Pi = (v - f)(y_+ - y_-) + (w - c)\mathbb{E}\Big[\min\{\max(D - y_-, 0), y_+ - y_-\}\Big].$$

Since $E[\min\{\max(D - y_-, 0), y_+ - y_-\}] = \displaystyle\int_{y_-}^{y_+} (u - y_-)f(u)du + (y_+ - y_-)\overline{F}(y_+)$, integration in parts yields

$$\Pi = (v - f)(y_+ - y_-) + (w - c)\int_{y_-}^{y_+} \overline{F}(u)du.$$

Using Equation (8), one can express (w, v) as a function of y_- and y_+ when $y_- < y_+$, since $f(\cdot) > 0$. Specifically,

$$\begin{aligned} v &= v_h + \overline{F}(y_+)\frac{-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)} = v_l - \overline{F}(y_-)\frac{(v_l - v_h) - \overline{F}(y_+)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)} \\ w &= w_h - \frac{-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)} = w_l + \frac{(v_l - v_h) - \overline{F}(y_+)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)}. \end{aligned} \tag{9}$$

This implies that we can express Π using y_- and y_+ instead of v and w. Within $A_{lh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$, $l > 0$,

$$\begin{aligned} \Pi(w, v) = J_{lh}(y_-, y_+) &= \left\{ \begin{array}{l} (v_h - f)(y_+ - y_-) + (w_h - c)\displaystyle\int_{y_-}^{y_+} \overline{F}(u)du \\ -\Big[\dfrac{-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)}\Big]\displaystyle\int_{y_-}^{y_+} \Big(\overline{F}(u) - \overline{F}(y_+)\Big)du \end{array} \right. \\ &= \left\{ \begin{array}{l} (v_l - f)(y_+ - y_-) + (w_l - c)\displaystyle\int_{y_-}^{y_+} \overline{F}(u)du \\ -\Big[\dfrac{(v_l - v_h) - \overline{F}(y_+)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)}\Big]\displaystyle\int_{y_-}^{y_+} \Big(\overline{F}(y_-) - \overline{F}(u)\Big)du \end{array} \right. \end{aligned}$$

When $l = 0$, the transformation described in Equation (9) is not well defined. Instead, we consider

$$
\begin{aligned}
v + w &= v_1 + w_1 \\
\frac{v - v_h}{w_h - w} &= \overline{F}(y_+).
\end{aligned}
\tag{10}
$$

To see why this transformation holds, we observe that for a given y_+, $y_- = 0$, the profit with a bid $w = w_h - t$ and $v = v_h + \overline{F}(y_+)t$, $t \geq 0$, is,

$$
\begin{aligned}
\Pi &= (v_h + \overline{F}(y_+)t - f)y_+ + (w_h - t - c)\int_0^{y_+} \overline{F}(u)du \\
&= (v_h - f)y_+ + (w_h - c)\int_0^{y_+} \overline{F}(u)du - t\int_0^{y_+} [\overline{F}(u) - \overline{F}(y_+)]du.
\end{aligned}
\tag{11}
$$

Thus, to maximize Π it is best for the supplier to select t as small as possible. This justifies the extension of Equation (8) for $l = 0$. Consequently,

$$
\Pi(w, v) = J_{0h}(y_+) \quad = \begin{cases} (v_h - f)y_+ + (w_h - c)\int_0^{y_+} \overline{F}(u)du \\ -\Big[\dfrac{(v_h + w_h) - (v_1 + w_1)}{1 - \overline{F}(y_+)}\Big]\int_0^{y_+} \Big(\overline{F}(u) - \overline{F}(y_+)\Big)du \end{cases}
$$

Finally, the problem faced by supplier i is:

$$
\sup_{(w,v)} \Pi(w, v) = \max\Big(0, \max_{\substack{l = 0, \ldots, k-1 \\ h = l+1, \ldots, k}} \sup_{(y_-, y_+)} J_{lh}(y_-, y_+)\Big)
$$

The optimization problem is defined as a supremum of profit, in terms of either (w, v) or (y_-, y_+). As we shall see later, when optimizing on (w, v), there does not always exist an optimal solution, and the supremum may be obtained by bidding identically to another supplier. However, when using (y_-, y_+) as decision variables, an optimal solution is always obtained.

Since $\overline{F}(\cdot)$ is differentiable over $(\underline{d}, \overline{d})$, the expected profit is differentiable in (y_-, y_+):

$$
\begin{aligned}
\frac{dJ_{lh}}{dy_-} &= (f - v_l) + (c - w_l)\overline{F}(y_-) + f(y_-)\Big[\frac{(v_l - v_h) - \overline{F}(y_+)(w_h - w_l)}{(\overline{F}(y_-) - \overline{F}(y_+))^2}\Big]\int_{y_-}^{y_+} \Big(\overline{F}(u) - \overline{F}(y_+)\Big)du \\
\frac{dJ_{lh}}{dy_+} &= (v_h - f) + (w_h - c)\overline{F}(y_+) - f(y_+)\Big[\frac{-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l)}{(\overline{F}(y_-) - \overline{F}(y_+))^2}\Big]\int_{y_-}^{y_+} \Big(\overline{F}(y_-) - \overline{F}(u)\Big)du
\end{aligned}
\tag{12}
$$

Notice that $(v_l - v_h) - \overline{F}(y_+)(w_h - w_l) \geq 0$ and $-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l) \geq 0$ hold; we shall use this observation later.

6 Game Equilibria

We consider a game in which the suppliers compete for selling capacity. This is a Stackelberg game in which the manufacturer is the follower and the suppliers are the leaders. This section analyzes the Nash equilibria of this game in pure strategies.

As discussed in the previous section, when the strategies are defined only through (w, v), no equilibria might exist since some supplier's problem may not have an optimal solution. This occurs when two suppliers submit the same bid, and as a consequence, the buyer's problem has multiple optimal solutions.

To get rid of this technical problem, we may study Nash ϵ-equilibria, i.e. a set of strategies such that each supplier obtains at least the maximum possible payment minus ϵ, and make ϵ tend to 0. This would make the analysis cumbersome. To simplify the analysis, we choose instead to define a strategy as a bid (w, v) accompanied by, in case of a tie, an agreement on the allocation of capacity to each one of them. That is, when suppliers i and j bid the same reservation and execution prices, they will be in equilibrium when the buyer allocates them a slice of capacity (y_{lower}, y_{higher}), and they agree to split this allocation with supplier i getting the slice (y_{lower}, y_{middle}) and supplier j the slice (y_{middle}, y_{higher}). This can be interpreted as a rationing rule implicitly determined by the suppliers. In practice, this is achieved by each supplier bidding very close to each other such that the targeted allocation to each supplier is obtained. In the rest of the paper, we will obtain equilibria when such agreements in case of ties can be reached.

In this section, we provide necessary conditions for Nash equilibria. We do not analyze the existence of these equilibria, although these can be shown to exist, see Martínez-de-Albéniz [11].

6.1 Border Distributions

Throughout the remainder of the paper, we shall use the following assumption on the customer demand distribution. The customer demand distribution is called *border demand distribution* when it satisfies the following property.

Property 1 *For every supplier with cost (c_i, f_i), given that it is bidding in any region $A_{lh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$, defined by Equations (6) or (7), there is an optimal bid (w_i, v_i) that belongs to the border of the region.*

The property implies that for any supplier bidding in region $A_{lh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$, there is an optimal bid on the boundary of region $A_{lh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$, for all cost parameters. For instance, for a supplier bidding in region A_{12} of Figure 2, there is an optimal bid on the boundary of A_{12} with either A_{02} or A_{13} or A_{OUT}, for all cost parameters. As we demonstrate later on, this property is satisfied, for instance, when demand is uniformly or exponentially distributed.

Hence, under this property, we assume that the suppliers will place their bids in the border of some region. The property allows us to determine their optimal bids.

Observe that border distributions may also allow for optimal bids in the interior of the region, but these are unstable in the sense that a small perturbation in the cost parameters will completely change the optimal bid. Therefore, we consider only optimal bids on the borders of the regions defined in Equations (7) and (6).

The next proposition provides a sufficient condition to guarantee that a distribution is border.

Proposition 2 *The customer demand distribution is a border distribution when, for all $y_m \geq 0$, for all $(a, b) \in \mathbb{R}^2$, the function*

$$a(y_+ - y_-) + b\int_{y_-}^{y_+} \overline{F}(u)du - \Big[\frac{\overline{F}(y_-) - \overline{F}(y_m)}{\overline{F}(y_-) - \overline{F}(y_+)}\Big] \int_{y_-}^{y_+} \Big(\overline{F}(u) - \overline{F}(y_+)\Big)du$$

does not have any interior strict local maximum for (y_-, y_+) in the area

$$\Big\{(y_-, y_+)\Big| 0 \leq y_- \leq y_m \leq y_+\Big\}.$$

6.2 Efficiency

We start by defining the concept of efficiency which leads to a natural and desirable property of Nash equilibria. Note that no assumptions on the demand distribution are made.

Definition 4 *We say that supplier i is efficient when (c_i, f_i) is a winning point in the set $\{(c_1, f_1)$, $\ldots$, (c_N, f_N), $(p, 0)\}$.*

Proposition 3 *Assume that supplier i is efficient. Then, in every Nash equilibrium, supplier i is active.*

6.3 Optimal bids

As we will soon see, it is of particular interest to examine the situation in A_{lh} where there is no active supplier between l and h and both l and h are active. Notice that these are all the regions that share an edge with A_{OUT}.

Consider supplier i bidding in such a region, $A_{lh}^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}$, and define y_m to be the quantity captured by supplier l when i is absent, i.e.,

$$\overline{F}(y_m) = \frac{v_l - v_h}{w_h - w_l}. \tag{13}$$

The constraints for y_- and y_+ are then $y_{l-1}^{-i} \leq y_- \leq y_m \leq y_+ \leq y_h^{-i}$ where y_{l-1}^{-i} and y_h^{-i} are defined in (5).

If the bid of supplier i does not make l or h inactive, we can derive useful properties. In this case, the optimal bid cannot be such that $y_- = y_{l-1}^{-i}$ (because it makes l inactive) or $y_+ = y_h^{-i}$ (h inactive). Therefore, since it is optimal to bid on the border of the region, it must be that $y_- = y_m$ or $y_+ = y_m$ is optimal. These imply that supplier i bids the same pair (w, v) as l or h, respectively. These bids are going to be very close to each other, but we can consider these two bids as identical with the line connecting the two bids having a slope $-\overline{F}(y)$ for some y.

In the first case, i.e., when $y_- = y_m$ is optimal, recall that $-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l) = 0$ so from Equation (12)

$$\frac{dJ_{lh}}{dy_+} = (v_h - f) + (w_h - c)\overline{F}(y_+)$$

and therefore we must have that $c \leq w_h$ and

$$\overline{F}(y_+) = \frac{f - v_h}{w_h - c}.$$

Similarly, when $y_+ = y_m$ is optimal, then $w_l \leq c$ and

$$\frac{dJ_{lh}}{dy_-} = (f - v_l) + (c - w_l)\overline{F}(y_-),$$

$$\overline{F}(y_-) = \frac{v_l - f}{c - w_l}.$$

We summarize these results in the next proposition.

Proposition 4 *Given a border distribution, assume that, for a supplier with costs (c, f), it is optimal to bid in some unique region A_{lh}, $l > 0$, where there is no active supplier between l and h. Define y_m as in Equation (13) and, for some y_0, y_3, we can represent A_{lh} by all the pairs of quantities (y_-, y_+) such that $y_0 \leq y_- \leq y_m \leq y_+ \leq y_3$. Define y_1 and y_2 as follows,*

$$\overline{F}(y_1) = \frac{v_l - f}{c - w_l}, \tag{14}$$

$$\overline{F}(y_2) = \frac{f - v_h}{w_h - c}. \tag{15}$$

Then, one and only one case from the following is true.

- *either $y_0 \leq y_1 \leq y_m$ and $y_2 > y_3$, and $(w^*, v^*) = (w_l, v_l)$, $y_+^* = y_m$ and $y_-^* = y_1$,*
- *or $y_0 > y_1$ and $y_m \leq y_2 \leq y_3$, and $(w^*, v^*) = (w_h, v_h)$, $y_-^* = y_m$ and $y_+^* = y_2$,*
- *or $y_0 \leq y_1 \leq y_m \leq y_2 \leq y_3$; $(w^*, v^*) = (w_l, v_l)$, $y_+^* = y_m$ and $y_-^* = y_1$, only if*

$$\frac{\int_{y_1}^{y_m} [\overline{F}(y_1) - \overline{F}(u)]du}{\overline{F}(y_1) - \overline{F}(y_m)} \geq \frac{\int_{y_m}^{y_2} [\overline{F}(u) - \overline{F}(y_2)]du}{\overline{F}(y_m) - \overline{F}(y_2)}; \tag{16}$$

$(w^*, v^*) = (w_h, v_h)$, $y_-^* = y_m$ and $y_+^* = y_2$, only if

$$\frac{\int_{y_1}^{y_m} [\overline{F}(y_1) - \overline{F}(u)]du}{\overline{F}(y_1) - \overline{F}(y_m)} \leq \frac{\int_{y_m}^{y_2} [\overline{F}(u) - \overline{F}(y_2)]du}{\overline{F}(y_m) - \overline{F}(y_2)}. \tag{17}$$

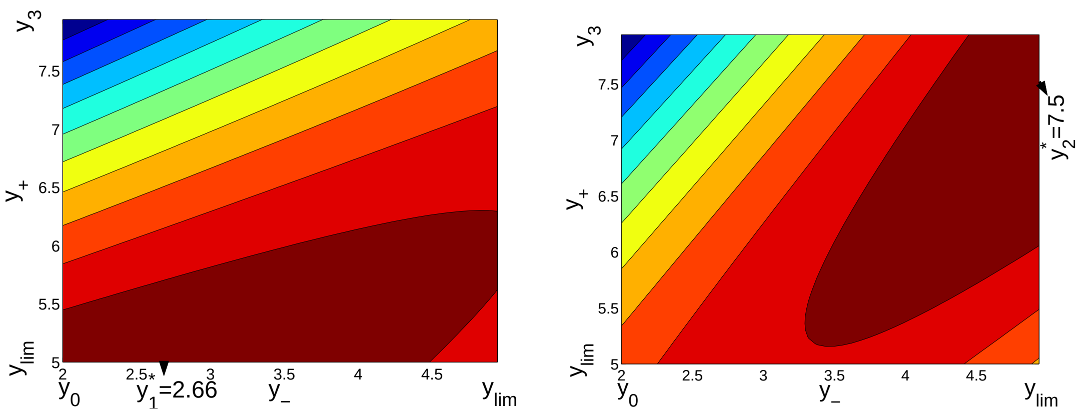

Figure 3: Profit for the different situations, we observe that it is optimal to set $y_- = y_m$ or $y_+ = y_m$ depending on the case

Figure 3 illustrates the two optimal bids discussed in the Proposition. In the figure we show iso-profit curves as a function of y_- and y_+. As you can see, the figure on the left illustrates the first case in which the optimal bid is $y_+^* = y_m$ and $y_-^* = y_1$ whereas the figure on the right illustrates the case in which the optimal bid is $y_-^* = y_m$ and $y_+^* = y_2$. Notice that bidding in the left-side edge of the definition box is equivalent to making supplier l inactive and bidding on the up-side edge to making h inactive.

Again, notice that such optimal bids imply that two suppliers end up submitting the same bid (w, v). The buyer thus orders capacity from both indifferently. In equilibrium, after capacity has been ordered by the buyer, the two suppliers share it such that they obtain the capacity they were aiming at by selecting the quantities (y_-, y_+). That is, if $y_+^* = y_m$ and $y_-^* = y_1$ is optimal for the current supplier, then it obtains the share of capacity between y_1 and y_m, and if $y_-^* = y_m$ and $y_+^* = y_2$ is optimal, then it obtains the share of capacity between y_m and y_2. Proposition 4 allows us to derive a general property of any equilibrium.

Proposition 5 *Consider a border distribution. In any Nash equilibrium, if i and j are active and $(w_i, v_i) = (w_j, v_j)$ then (w_i, v_i) belongs in the segment $[(c_i, f_i); (c_j, f_j)]$.*

6.4 Equilibria with efficient suppliers only

In this section, we study the situation where all suppliers are efficient, that is, all the suppliers receive some capacity order from the buyer when they bid their true cost.

Proposition 6 *Given a border distribution, assume that all suppliers are efficient. Then, in every Nash equilibrium, for every pair (i, j), if $c_i < c_j$ then $w_i \leq w_j$.*

Proposition 6 implies that if all suppliers are efficient, supplier i, $i = 1, \dots, N$, bids in region $A_{i-1\ i+1}^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}$ in every Nash equilibrium. More importantly, this result confirms the intuition on the suppliers' bidding behavior. No supplier will bid an execution fee, w, lower than a competitor's execution fee if the competitor's execution cost is smaller. Put differently, the smaller a supplier's execution cost, c, the lower this supplier's execution bid, w.

Proposition 7 *For a border distribution, assume that all the suppliers are efficient. Then, in every Nash equilibrium, supplier i, $i = 1, \ldots, N$, places its bid (w_i, v_i):*

- *in the segment $[(c_{i-1}, f_{i-1}); (c_i, f_i)]$ if $i > 1$, and then $(w_i, v_i) = (w_{i-1}, v_{i-1})$;*
- *or in the segment $[(c_i, f_i); (c_{i+1}, f_{i+1})]$ (after defining $c_{N+1} = p, f_{N+1} = 0$), and then $(w_i, v_i) = (w_{i+1}, v_{i+1})$.*

The proposition thus implies that supplier bids will be clustered in groups of two or three suppliers. This is true since according to the proposition either two suppliers bid somewhere in the segment connecting their true cost parameters, or one supplier bids its true costs and two other suppliers place a similar bid to this one. Thus, in practice, one will observe less bids than the number of suppliers, roughly half of them.

The type of competition described in this result has some interesting properties. The most striking feature is that more than one supplier will be offering the same bid. One may then wonder whether any supplier in that position should instead reduce its bidding a little bit so that it puts its rival out of the market. The answer provided by the proposition is that this is not the case, and that all suppliers offering the same bid are better off by co-existing with someone else. This allows them to capture, among all the orders allocated to the group of suppliers, those which are better suited for their production costs, (c, f). We call this cluster competition, since in equilibrium the market is divided into stable clusters.

The proposition also suggests that every supplier is competing directly with one of its rival suppliers, i.e., with one of the suppliers whose execution cost, c, is either the smallest among all suppliers with higher c, or the highest among all suppliers with lower c.

An important insight from this observation is that the supplier will care only about competing with its closest competitor, and not competing against other suppliers; this implies that in equilibrium, competition is no longer done on a global basis (among all suppliers) but rather locally (between two or three competing suppliers). For instance, if different suppliers use different technologies, and hence incur different costs, each supplier should focus only on competing with those using similar technologies.

Finally, observe that this result does not rule out the existence of multiple Nash equilibria, and in general the set of Nash equilibria contains multiple possibilities. In any case, this result shows that the possible equilibria are restricted to option prices, (w, v), which belong to the lower envelope of the true suppliers' costs. Such equilibria should satisfy the optimality conditions in Equations (16) and (17). The following example illustrates the multiplicity of Nash equilibria.

Example 1 *Assume that customer demand is uniformly distributed in* $[0,1]$. *Let* $N = 2$ *and the true costs be*

$$(c_1, f_1) = (0, 40), \quad (c_2, f_2) = (40, 20), \quad p = 100.$$

Both suppliers are efficient. For any $w \in (0, 40)$, *the following bids form Nash equilibria:*

$$(w_1, v_1) = (w_2, v_2) = (w, 40 - w/2), \quad y_1 = 0.5, \quad y_2 = 0.5 + \frac{10}{100 - w}.$$

Finally, to conclude this section, we provide a bound on the inefficiencies created by suppliers' competition. We define the total welfare as follows:

$$U = \text{ (PROFIT OF BUYER) } + \sum_{i=1}^{N} \text{ (PROFIT OF SUPPLIER } i).$$

The payments between buyer and suppliers will cancel out, and this quantity will only capture the true revenue from customers minus the costs of production. Thus, we can express the total supply chain welfare as

$$\begin{aligned} U = & \ p \int_0^{y_N} \overline{F}(u)du - \sum_{i=1}^{N} f_i(y_i - y_{i-1}) - \sum_{i=1}^{N} c_i \int_{y_{i-1}}^{y_i} \overline{F}(u)du \\ = & \sum_{i=1}^{N} \Delta c_i \int_0^{y_i} [\overline{F}(u) - \overline{F}(y_i^*)]du. \end{aligned}$$

where $\overline{F}(y_i^*) = \dfrac{f_i - f_{i+1}}{c_{i+1} - c_i}$ and $\Delta c_i = c_{i+1} - c_i$. These quantities are well-defined when all the suppliers are efficient. The social welfare is maximized when $y_i = y_i^*$, $i = 1, \dots, N$. In this case, the optimal welfare is

$$U^* = \sum_{i=1}^{N} \Delta c_i \int_0^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du.$$

When the suppliers compete, the allocation of capacities, y_i, $i = 1, \dots, N$, is not necessarily efficient, in the sense that it is possible that $y_i \neq y_i^*$ for some i. The loss in welfare, due to the suppliers' competition, is equal to

$$\Delta U = \sum_{i=1}^{N} \Delta c_i \int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du.$$

Proposition 8 *Given a border demand distribution and efficient suppliers, in every Nash equilibrium, the allocation of capacities obtains at least* 50% *of the optimal total welfare, i.e.,*

$$\frac{\Delta U}{U^*} \leq \frac{1}{2}.$$

So far, we do not know whether this bound is tight. In any event, this proposition shows that the distortion on the optimal decisions created by competition among suppliers is bounded. This bound is obtained over all demand distributions, but for a specific distribution, the bound might be better. For instance, as we show later, for a uniform demand distribution, the market welfare is at least 3/4 of the optimum, i.e., the loss of optimality due to competition is no more than 25% of the optimal welfare.

6.5 Equilibria with inefficient suppliers

The previous results, characterizing any Nash equilibrium, are obtained under the assumption that all suppliers are efficient. We now investigate the case in which not all suppliers are efficient.

Interestingly, as we demonstrate below, it might happen that a non-efficient supplier is active at a Nash equilibrium. This occurs because bids are only partially linked to the true costs, and a non-efficient supplier may capture market share by positioning itself in a segment of the market with no, or low, competition.

Example 2 *Assume that customer demand is uniformly distributed in* $[0, 1]$. *Let* $N = 3$ *and the true costs be*

$$(c_1, f_1) = (0, 40), \quad (c_2, f_2) = (40, 20), \quad (c_3, f_3) = (70, 11), \quad p = 100.$$

Clearly, supplier 3 *is not efficient. If this was a centralized system, in which the true costs are considered, we would have* $y_1^* = 0.5$, $y_2^* = 0.666$ *and* $y_3^* = 0.666$, *and so the buyer would purchase capacities* $x_1^* = 0.5$, $x_2^* = 0.166$ *and* $x_3^* = 0$.

The following bids form a Nash equilibrium:

$$(w_1, v_1) = (w_2, v_2) = (20, 30), \quad (w_3, v_3) = (100, 0), \quad y_1 = 0.5, \quad y_2 = 0.625, \quad y_3 = 0.633.$$

Thus, a non-efficient supplier captures capacity.

Unfortunately, even the statement in Proposition 6 may not be true when inefficient suppliers exist. This is demonstrated by the following example.

Example 3 *Assume that customer demand is uniformly distributed in* $[0, 1]$. *Let* $N = 3$ *and the true costs be*

$$(c_1, f_1) = (0, 40), \quad (c_2, f_2) = (40, 20), \quad (c_3, f_3) = (70, 11), \quad p = 100.$$

Again supplier 3 *is not efficient. The following bids form a Nash equilibrium:*

$$(w_1, v_1) = (w_2, v_2) = (30, 25), \quad (w_3, v_3) = (0, 55), \quad y_1 = 0.5, \quad y_2 = 0.643, \quad y_3 = 0.$$

Thus, we have $c_2 < c_3$ *and still* $w_3 < w_2$.

The two examples suggest that the presence of inefficient suppliers can lead to counter-intuitive situations. This can be explained by referring to the *Bertrand model* with asymmetric players. Although it is commonly argued that the only Nash equilibrium in pure strategies is such that the most competitive producer captures all the market at a price equal to the second most competitive cost, as in Tirole [16] p.211, this equilibrium is not unique. As noted by Erlei [7], all the prices between the smallest and the second smallest costs are Nash equilibria of the system. This is true since an inefficient player can impact the market price by placing absurd bids knowing that it will not capture any market share.

The next proposition depicts the behavior of the suppliers at equilibrium.

Proposition 9 *For a border distribution, let* $\{(w_1, v_1), \ldots, (w_N, v_N), (p, 0)\}$ *be the bids of the suppliers in a Nash equilibrium. Assume that supplier* i *is active. Then we must have that:*

- *either there is* $j = 1, \ldots, N+1$ *such that supplier* j *is active,* $(w_i, v_i) = (w_j, v_j)$ *and moreover* (w_i, v_i) *belongs in the segment* $[(c_i, f_i); (c_j, f_j)]$*;*
- *or there are* $j, k = 1, \ldots, N+1$ *such that supplier* k *is inactive, supplier* j *is active and* $(w_i, v_i) = (w_k, v_k) + \theta(w_k - w_j, v_k - v_j)$ *for some* $\theta \geq 0$*.*

This proposition adds a new case to what was presented in Proposition 7. This new situation arises when an inactive supplier sets the price of some active supplier, which is similar to entry deterrence pricing in industrial organization models, see Tirole [16]. This reaction keeps the inefficient supplier out of the market by making its entry non-profitable. This is illustrated by the next example.

Example 4 *Assume that customer demand is uniformly distributed in* $[0, 1]$*. Let* $N = 4$ *and the true costs be*

$$(c_1, f_1) = (0, 40), \quad (c_2, f_2) = (40, 20), \quad (c_3, f_3) = (70, 6), \quad (c_4, f_4) = (80, 6), \quad p = 100.$$

Supplier 4 *is not efficient, the rest are. The following bids form a Nash equilibrium*

$$\begin{aligned} (w_1, v_1) = (w_2, v_2) = (20, 30), &\quad (w_3, v_3) = (w_4, v_4) = (80, 4) \\ y_1 = 0.5,\ y_2 = 0.567, &\quad y_3 = 0.8,\ y_4 = 0.8. \end{aligned}$$

Evidently, supplier 4, by placing a bid with which it would never make a positive profit, sets the price of supplier 3, who is efficient and must react to the threat of supplier 4.

6.6 Examples of border distributions

In this section, we show that Property 1 is satisfied for at least two typical distributions, the uniform distribution and the exponential distribution. We start with the uniform distribution.

Property 2 *The demand distribution is uniform, i.e. there exists* $[a, b] \subset [0, \infty)$ *such that*

$$F(t) = \begin{cases} 0 & \text{if } t < a \\ \dfrac{t-a}{b-a} & \text{if } a \le t \le b \\ 1 & \text{if } t > b \end{cases}$$

Given Property 2, we can simplify the profit obtained by a supplier after bidding in A_{lh}. By transforming $\dfrac{y-a}{b-a}$ for $y \in [a, b]$ into a new variable, we obtain an equation where a and b are replaced by $a = 0$ and $b = 1$. Therefore, without loss of generality, one can assume that $a = 0$ and $b = 1$. Thus,

$$J_{lh}(y_-, y_+) = \begin{cases} (v_h - f)(y_+ - y_-) + (w_h - c)(y_+ - y_-)\Big[1 - \dfrac{y_- + y_+}{2}\Big] \\ -\Big[-(v_l - v_h) + (1 - y_-)(w_h - w_l)\Big]\Big[\dfrac{y_+ - y_-}{2}\Big] \end{cases}$$

This is a quadratic function on (y_-, y_+). It can be expressed as

$$J_{lh}(y_-, y_+) = \frac{1}{2}(y_- \ y_+) H \begin{pmatrix} y_- \\ y_+ \end{pmatrix} + (g \ \ -g) \begin{pmatrix} y_- \\ y_+ \end{pmatrix} + \text{constant}$$

where H is the Hessian

$$H = \begin{pmatrix} -(c - w_l) & (w_h - w_l)/2 \\ (w_h - w_l)/2 & -(w_h - c) \end{pmatrix}$$

and

$$g = f + c - \frac{v_l + v_h}{2} - \frac{w_l + w_h}{2}$$

The determinant of H is $(c - w_l)(w_h - c) - (w_h - w_l)^2/4$, which is always non-positive and reaches its maximum, 0, when $c = (w_h + w_l)/2$. Hence, there cannot be a strict local maximum in A_{lh}. Thus, we have shown the next proposition.

Proposition 10 *The uniform distribution is a border distribution.*

We can now specialize Proposition 4 to the uniform distribution.

Proposition 11 *Given a uniform distribution, assume that, for a supplier with costs* (c, f)*, it is optimal to bid in some unique region* A_{lh}*,* $l > 0$*, where there is no active supplier between* l *and* h*. Define* y_m *as in Equation (13) and, for some* y_0, y_3*, we can represent* A_{lh} *by all the pairs of quantities* (y_-, y_+) *such that* $y_0 \le y_- \le y_m \le y_+ \le y_3$*. Define* y_1 *and* y_2 *as in Equations (14) and (15). Then, one and only one case from the following is true.*

- *either* $y_0 \le y_1 \le y_m$ *and* $y_2 > y_3$*, and* $(w^*, v^*) = (w_l, v_l)$*,* $y_+^* = y_m$ *and* $y_-^* = y_1$*,*
- *or* $y_0 > y_1$ *and* $y_m \le y_2 \le y_3$*, and* $(w^*, v^*) = (w_h, v_h)$*,* $y_-^* = y_m$ *and* $y_+^* = y_2$*,*

- *or* $y_0 \le y_1 \le y_m \le y_2 \le y_3$; $(w^*, v^*) = (w_l, v_l)$, $y_+^* = y_m$ *and* $y_-^* = y_1$, *when* $c \le \dfrac{w_l + w_h}{2}$; $(w^*, v^*) = (w_h, v_h)$, $y_-^* = y_m$ *and* $y_+^* = y_2$, *when* $c \ge \dfrac{w_l + w_h}{2}$.

The proposition thus implies that in the uniform case, a supplier execution bid, w, is going to be equal to that of the supplier whose execution bid is the closest to its true execution cost, c.

Proposition 12 *Given a uniform distribution and efficient suppliers, in every Nash equilibrium, the allocation of capacities obtains at least* 75% *of the optimal total welfare, i.e.,*

$$\frac{\Delta U}{U^*} \le \frac{1}{4},$$

and this bound is tight.

This result shows that the inefficiencies created by the market for a uniform distribution are closer to the optimum than the worst-case bound proved in Proposition 8.

As observed earlier the exponential distribution also satisfies Property 1.

Property 3 *The demand distribution is exponential, i.e. there exists* $\lambda > 0$ *such that*

$$\overline{F}(t) = e^{-\lambda t} \text{ for } t \ge 0$$

Proposition 13 *The exponential distribution is a border distribution.*

Finally, we have tested whether the truncated normal distribution is border numerically, and simulation results suggest that it satisfies Proposition 2 as well and hence is border.

Property 4 *The demand distribution is truncated normal, i.e. there exists* $\mu > 0$, $\sigma^2 \ge 0$ *such that*

$$f(t) = \kappa e^{-\dfrac{(t-\mu)^2}{2\sigma^2}} \text{ for } t > 0$$

where $\kappa = \left[\displaystyle\int_0^\infty e^{-\dfrac{(t-\mu)^2}{2\sigma^2}} dt\right]^{-1}$.

Conjecture 1 *The truncated normal distribution is a border distribution.*

7 Practical Implementation

The analysis conducted in this paper so far assumes that the suppliers submit a sealed bid based on perfect information on the costs of each other as well as on customer demand distribution. The existence of multiple equilibria makes this specific mechanism very difficult to implement because the suppliers cannot predict which equilibrium will occur, and thus cannot bid accordingly. This can lead to practical bids being different from the predicted Nash equilibria. The natural way to address this issue would be to study Nash equilibria in mixed strategies, but the complexity of those is tremendous, i.e. the bidding space is a probability distribution in two dimensions. Therefore, for practical purposes, one should develop a mechanism that allows suppliers to obtain some additional information in order to reach a bidding equilibrium.

Also, it is desirable to remove the assumption of perfect information, since in that case the buyer could use this information if it had market power to force prices to be equal to the suppliers' costs.

Thus, it is important to construct a robust bidding mechanism that addresses these practical pitfalls.

When all the participants are efficient suppliers, we propose an iterative mechanism that conserves the structure of the Nash equilibria demonstrated in this paper.

1. Each supplier submits a bid (w, v) to the buyer and the set of bids becomes public information.
2. The buyer publicly declares the allocation of capacity to each supplier and therefore each supplier can estimate its expected profit.
3. If a supplier wants to change its bid, GOTO 1, otherwise TERMINATE.

Intuitively, by imposing technical conditions, such as requesting that a change in a bid should be by at least a given number, ϵ, we can guarantee that the mechanism terminates in a finite time if the buyer and the suppliers optimize their profit as if this was a single-shot game. Such a change in a bid can be made in any direction (either an increase or a decrease) of reservation and execution prices.

Then, when the process terminates, it must be that no supplier has an incentive to change its bid unilaterally. Hence, the final bids should be very close to the predicted Nash equilibria, but may not be equal if we impose a condition of changing bids by at least ϵ. This process can be implemented even if suppliers have only information about their own cost and no information about other suppliers' costs.

8 Conclusions and Research Directions

In this research, we have analyzed the procurement process between a single buyer and multiple suppliers. Suppliers compete on price and flexibility, two attributes that are important to the buyer. Specifically, each supplier is offering a different option contract and the buyer reserves capacities at each supplier so as to maximize expected profit.

We have modeled the process as a single-shot sealed-bid auction where the suppliers submit an offer with a reservation and an execution fee. We have assumed that the costs and the demand distribution are known to the suppliers, although, as discussed in Section 7, the model can be extended to an iterative process with information limited to the bids of each supplier in the current round of bidding together with information on the amount of capacity allocated to each supplier.

Under the assumption of border distribution, satisfied for instance by the uniform or the exponential distributions, we characterized optimality conditions for suppliers' bids and provide necessary conditions for Nash equilibria bids.

Interestingly, Nash equilibria in pure strategies give rise to what we call cluster competition. This provides several insights.

1 It pays to be efficient. *No matter how the competitors bid, when a supplier is efficient, it will capture orders from the buyer and will have a non-negative expected profit.*

In other words, being an efficient supplier means capturing market share, and no other supplier can push an efficient supplier out of business. Thus, efficiency guarantees long-term survival. Notice that our definition of efficiency allows having multiple efficient technologies, because the cost space is two-dimensional. This implies that an inefficient supplier may become efficient by reaching the efficient frontier defined by the lower envelope of the true costs of the other suppliers. Hence, this inefficient supplier does not necessarily have to change technology and copy the same exact cost as other suppliers; what is needed is a local improvement of its costs so as to move to the efficient frontier. This encourages technological variety.

2 It is enough to compete against suppliers with similar technologies. *When all suppliers are efficient, a supplier will compete against another supplier with similar technology, either the one with next lower or next higher execution cost.*

Thus, two suppliers with completely different technologies will never compete against each other. This local competition characteristic leads to our third insight.

3 Competition preserves diversity and segments the market. *At a market equilibrium with efficient suppliers, the suppliers are clustered into small groups of no more than three suppliers and no less than two suppliers. All suppliers within each group offer the same option and share the order from the buyer.*

The market will thus be segmented by groups of similar technologies. Competition will diminish technological variety but will not eliminate it. This is in contrast to market behavior in the price-only competition. Thus, in our model, if at some point a supplier "kills" its competitors in a given niche, i.e., a given cluster, and its competitors exit the market, this supplier will increase its market share by moving to a different niche.

4 Prices are directly related to true cost. *The equilibrium prices of the different options offered by the suppliers lie in the lower envelope of the costs of the system. That is, the reservation and execution equilibrium prices are linked to the true reservation and execution costs and no inflation of prices is stable.*

This insight shows the link between the costs of the system and the option prices available in the market. Specifically, if all suppliers are efficient, this implies a range of possible bids, each of which is along the lower envelope of the true suppliers' costs. However, many equilibria are possible, and hence it is not possible to predict the option prices.

5 Competition leads to a loss of supply chain profit. *While suppliers' prices are related to their true costs, the allocation of capacity can be quite different from the one achieved in a centralized system. However, our analysis indicates that the loss of system profit is no more than 50% of the maximum possible.*

At this point it is appropriate to point out that empirical results from experiments with classes of MBA students confirm that the iterative process presented in Section 7 converges to the Nash equilibria described here. In these experiments, bidders had complete information on both, demand distribution and other suppliers' cost parameters. We are in the process of gathering data from additional in-class experiments; these results, which we plan to publish in a companion paper, will be used to test the behavior predicted by our model.

Finally, this paper will be incomplete if we do not mention important extensions of our model. One possible direction is to allow buyers to purchase products at a spot market in addition to using the contracts signed with the suppliers. In such a model, suppliers and buyers negotiate contracts knowing that additional supply or demand are available in the spot market. Such a model would generalize not only the model in the current paper but also the models presented in Wu et al.[17], Spinler et al. [15] and Golovachkina and Bradley [9]. Another extension is to develop a multi-attribute competition for other factors such as quality or lead time, where the optimal portfolio for the buyer would be found endogenously. All these extensions present significant technical challenges.

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A Proofs

A.1 Proposition 1

Proof. To obtain the optimal portfolio we maximize function $V(\cdot)$ over the feasible region P defined in Equation (3). From Equation (2), we observe that function $V(\cdot)$ is the sum of strictly concave functions of y_i, $i = 1, \ldots, N$. Hence, it is strictly concave jointly in $(y_1, \ldots, y_N)$. The feasible region is a polyhedral cone with non-empty interior. This implies that the Slater conditions hold for this problem and that the Karush-Kuhn-Tucker conditions are necessary and sufficient at optimality (see Bertsekas [3] for details).

Define for every constraint $y_{i-1} - y_i \leq 0, i = 1, \ldots, N$, the associate Lagrange multiplier $\lambda_i \geq 0$. The KKT optimality conditions are, for $i = 1, \ldots, N$, assuming $\lambda_{N+1} = 0$:

$$
\begin{aligned}
(v_{i+1} - v_i) + (w_{i+1} - w_i)\overline{F}(y_i) &= \lambda_{i+1} - \lambda_i \\
\lambda_i(y_{i-1} - y_i) &= 0 \\
y_{i-1} - y_i &\le 0 \\
\lambda_i &\ge 0
\end{aligned} \tag{18}
$$

Let $\{i_1, \ldots, i_k\}$ be the winning set of $\{(w_1, v_1), \ldots, (w_{N+1}, v_{N+1})\}$. Define $y_{i_1}, \ldots, y_{i_{k-1}}$ such that

$$\overline{F}(y_{i_j}) = \frac{v_{i_j} - v_{i_{j+1}}}{w_{i_{j+1}} - w_{i_j}}$$

$\overline{F}(y_{i_k}) = 0$ and for the other variables $y_i = y_{i-1}$ (remember from Equation (1) that $y_0 = 0$). Note that it can happen that $y_i = \infty$ for some i. Define also $\lambda_{i_1} = \ldots = \lambda_{i_k} = 0$ together with:

(i) for $1 \le i < i_1$, $\lambda_i = (v_i - v_{i_1}) + (w_i - w_{i_1})$,

(ii) for $j = 1, \ldots, k-1$, for $i_j < i < i_{j+1}$, $\lambda_i = (v_i - v_{i_j}) + (w_i - w_{i_j})\overline{F}(y_{i_j})$,

(iii) for $i_k < i$, $\lambda_i = (v_i - v_{i_k})$.

It is now sufficient to verify that this solution satisfies the KKT conditions, Equation (18). Evidently, the first three requirements in (18) are satisfied by construction. It remains to verify that $\lambda_i \ge 0$ for all $i = 1, \ldots, N$. To see this, we analyze three different cases:

(i) for $1 \le i < i_1$, $\lambda_i = (v_i - v_{i_1}) + (w_i - w_{i_1}) \ge 0$ from part (b) of Definition 2;

(ii) for $j = 1, \ldots, k-1$, for $i_j < i < i_{j+1}$, $\lambda_i = (v_i - v_{i_j}) + (w_i - w_{i_j})\frac{v_{i_j} - v_{i_{j+1}}}{w_{i_{j+1}} - w_{i_j}} \ge 0$ from part (c);

(iii) for $i_k < i$, $\lambda_i = (v_i - v_{i_k}) \ge 0$ from part (d).

Finally, we see that no inactive point can be winning since this would imply that one of the inactive points is on the segment joining two other points. This would contradict the minimality of the winning set in Definition 2. ■

A.2 Proposition 2

Proof. Assume that the demand is not a border distribution. We show that it cannot satisfy the assumptions of the proposition.

By contradiction, assume that we can find $(c, f) \in {\mathbb{R}_+}^2$ and a region A_{lh} such that there is no optimum of the profit in the border. It is clear that $l = 0$ is not a valid case, since in every region A_{0h} the optimum belongs always in the border of the region as seen in Equation (11).

This implies that there is a strict local maximum inside A_{lh}. We can now expand A_{lh} into the region defined by

$$
\begin{aligned}
& v + w \ge v_l + w_l \\
& v - v_l \le -\frac{v_l - v_h}{w_h - w_l}(w - w_l) \\
& v - v_h \ge 0.
\end{aligned}
$$

We transform the profit expression from being characterized by w and v to (y_-, y_+), as presented in Equation (8). Define

$$\overline{F}(y_m) = \frac{v_l - v_h}{w_h - w_l},$$

and the feasible region in terms of (y_-, y_+) is $0 \leq y_- \leq y_m \leq y_+$. The profit function extended to this region is

$$J_{lh}(y_-, y_+) = \begin{cases} (v_h - f)(y_+ - y_-) + (w_h - c)\displaystyle\int_{y_-}^{y_+} \overline{F}(u)du \\ -\Big[\dfrac{-(v_l - v_h) + \overline{F}(y_-)(w_h - w_l)}{\overline{F}(y_-) - \overline{F}(y_+)}\Big] \displaystyle\int_{y_-}^{y_+} \Big(\overline{F}(u) - \overline{F}(y_+)\Big)du \end{cases}$$

and reaches a strict local maximum (y_-^*, y_+^*) such that $0 < y_-^* < y_m$ and $y_m < y_+^*$. We can now divide this expression by $w_h - w_l$ and obtain

$$\frac{v_h - f}{w_h - w_l}(y_+ - y_-) + \frac{w_h - c}{w_h - w_l}\int_{y_-}^{y_+} \overline{F}(u)du - \Big[\frac{\overline{F}(y_-) - \overline{F}(y_m)}{\overline{F}(y_-) - \overline{F}(y_+)}\Big] \int_{y_-}^{y_+} \Big(\overline{F}(u) - \overline{F}(y_+)\Big)du.$$

By writing $a = \dfrac{v_h - f}{w_h - w_l}$ and $b = \dfrac{w_h - c}{w_h - w_l}$, we find a contradiction to the hypothesis of the proposition. ■

A.3 Proposition 3

Proof. Assume that supplier i is not active in an equilibrium of the game. This implies that its profit is 0. Define the function $Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(\cdot)$ as in Equation (4), the lower envelope made of all bids except i's. If $f_i < Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(c_i)$, then by bidding $(c_i + \epsilon, f_i + \epsilon)$, supplier i achieves some positive profit for ϵ small enough. This contradicts the previous hypothesis and therefore we must have $f_i \geq Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(c_i)$.

Construct the lower envelope $C^{-i}(\cdot)$ of the costs $(c_1, f_1), \ldots, (c_{i-1}, f_{i-1}), (c_{i+1}, f_{i+1}), \ldots, (c_N, f_N)$, $(p, 0)$. That is, $C^{-i}(\cdot) = Z^{(\mathbf{c}_{-i}, \mathbf{f}_{-i})}(\cdot)$. Assume that $C^{-i}(\cdot)$ is not a lower bound on the function $Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(\cdot)$. This implies that there is an active bid (w_j, v_j) such that $v_j < C^{-i}(w_j)$. $j \neq i$ since i is not active and is not defining the function $Z^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}(\cdot)$. We claim that if supplier j bids in some region $A_{lh}^{(\mathbf{w}_{-j}, \mathbf{v}_{-j})}$, it cannot be at equilibrium. Indeed, we can use Equation (12), in particular,

$$\frac{dJ_{lh}}{dy_{j-}} \geq (f_j - v_l) + (c_j - w_l)\overline{F}(j-),$$

$$\frac{dJ_{lh}}{dy_{j+}} \leq (v_h - f_j) + (w_h - c_j)\overline{F}(y_{j+}).$$

If this is an equilibrium, then j's bid in $A_{lh}^{(\mathbf{w}_{-j}, \mathbf{v}_{-j})}$ must be such that $(f_j - v_l) + (c_j - w_l)\overline{F}(y_{j-}) \leq 0$ and $(v_h - f_j) + (w_h - c_j)\overline{F}(y_{j+}) \geq 0$. This is equivalent to

$$f_j \leq v_l + (v_j - v_l)\frac{c_j - w_l}{w_j - w_l},$$

or if $l = 0$, $f_j + c_j \leq v_k + w_k$ for some k, and

$$f_j \leq v_h + (v_j - v_h)\frac{w_h - c_j}{w_h - w_j}.$$

But if all this feasible area is not strictly below $C^{-i}(\cdot)$, we can find some other supplier k bidding next to j that also satisfies $v_k < C^{-i}(w_k)$. By repeating the argument, we must find a third supplier l satisfying $v_l < C^{-i}(w_l)$ that is not j (so no cycling possible). When we reach the supplier with the smallest w or with the biggest w (the dummy supplier, $N+1$), we reach a contradiction: for the smallest w, we cannot find a different supplier satisfying the condition, for the dummy supplier $v_{N+1} = f_{N+1} = 0 = C^{-i}(c_{N+1}) = C^{-i}(w_{N+1}) = C^{-i}(p)$. Hence j cannot be in equilibrium, and this is a contradiction.

Therefore, the function $C^{-i}(\cdot)$ lies below the function $Z^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}(\cdot)$. This implies that i cannot be efficient, since it is not needed to define the function $C^{-i}(\cdot)$, and thus is not a winning point of $\{(c_1, f_1), \ldots, (c_N, f_N), (p, 0)\}$. ■

A.4 Proposition 4

Proof. We have explained previously that under the assumptions of the proposition, either $y_+^* = y_m$ and $y_-^* = y_1$ or $y_+^* = y_2$ and $y_-^* = y_m$. Otherwise, it would be optimal to bid in some other region $A_{l'h'}$ in addition to A_{lh}. Since this is a contradiction to the hypothesis, it implies that the two possible optimal bids are either (w_l, v_l) or (w_h, v_h).

If $y_1 > y_m$ or $y_2 < y_m$, from Equation (12) it is clear that it is not optimal for the supplier to bid in this particular region A_{lh}, because it has an incentive to bid in A_{OUT} instead of A_{lh}. Similarly, if $y_1 < y_0$ and $y_2 > y_3$, neither one of the bids is admissible, and therefore there is an optimum outside A_{lh}. We can now partition the remaining possibilities into the three cases presented in the proposition.

In the two first cases, since only one of the two bids is admissible, it must be optimal. In the third case, it implies that $(c, f) \in A_{lh}$. Bidding (w_h, v_h) is better than (w_l, v_l) when

$$\Pi_2 = (v_h - f)(y_2 - y_m) + (w_h - c)\int_{y_m}^{y_2} \overline{F}(u)du \geq \Pi_1 = (v_l - f)(y_m - y_1) + (w_l - c)\int_{y_1}^{y_m} \overline{F}(u)du.$$

Using Equations (14) and (15), this is equivalent to

$$(w_h - c)\int_{y_m}^{y_2} [\overline{F}(u) - \overline{F}(y_2)]du \geq (c - w_l)\int_{y_1}^{y_m} [\overline{F}(y_1) - \overline{F}(u)]du.$$

But also, we have that, similarly to Equation (9),

$$c = w_h - (w_h - w_l)\frac{\overline{F}(y_1) - \overline{F}(y_m)}{\overline{F}(y_1) - \overline{F}(y_2)} = w_l + (w_h - w_l)\frac{\overline{F}(y_m) - \overline{F}(y_2)}{\overline{F}(y_1) - \overline{F}(y_2)}.$$

Therefore, we can rewrite the previous condition as

$$\frac{\overline{F}(y_1)-\overline{F}(y_m)}{\overline{F}(y_1)-\overline{F}(y_2)}\int_{y_m}^{y_2}[\overline{F}(u)-\overline{F}(y_2)]du \geq \frac{\overline{F}(y_m)-\overline{F}(y_2)}{\overline{F}(y_1)-\overline{F}(y_2)}\int_{y_1}^{y_m}[\overline{F}(y_1)-\overline{F}(u)]du.$$

After simplifying this expression, we obtain Equations (16) and (17). ■

A.5 Proposition 5

Proof. In a Nash equilibrium, if i and j submit the same bid and are both active, then i must bid in some region $A_{lj}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$ or $A_{jh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$; similarly, j must bid in some region $A_{l'i}^{(\mathbf{w}_{-j},\mathbf{v}_{-j})}$ or $A_{ih'}^{(\mathbf{w}_{-j},\mathbf{v}_{-j})}$. Indeed, if i (resp. j) bids in a different region, then i (resp. j) makes j (resp. i) inactive.

We can now apply Proposition 4. If i bids in $A_{lj}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$ then j must bid in $A_{ih'}^{(\mathbf{w}_{-j},\mathbf{v}_{-j})}$. Therefore by using the optimality equations (14) and (15), we have that the slope between (c_i, f_i) and (w_i, v_i), and (w_i, v_i) and (c_j, f_j) must be the same, since this is an equilibrium and $y_{i+} = y_{j-}$. Similarly, if i bids in $A_{jh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$ then j bids in $A_{l'i}^{(\mathbf{w}_{-j},\mathbf{v}_{-j})}$ and again the slopes (c_j, f_j) and (w_j, v_j), and (w_j, v_j) and (c_i, f_i) must be the same. In both cases, we have that (w_i, v_i) belongs to the segment $[(c_i, f_i); (c_j, f_j)]$. ■

A.6 Proposition 6

Proof. Using Proposition 3, we know that every supplier is active.

If the proposition was false, we could find suppliers i and j such that $c_i < c_j$ and $w_i > w_j$. We may furthermore assume without loss of generality that these are consecutive bidders, i.e. there is no bid (w, v) with $w_j < w < w_i$. To see this, assume that the active suppliers are indexed such that $w_1 \leq \ldots \leq w_t$ and in case of a tie, sorted by increasing execution cost c.

Select a pair (i, j) such that $i + 1 < j$ with $w_i < w_j$ and $c_i > c_j$. One of the following three cases is possible.

- The pair $(i, i + 1)$ satisfies $w_i < w_{i+1}$ and $c_i > c_{i+1}$ and then $(i, i + 1)$ are consecutive bidders.
- $w_i < w_{i+1}$ and $c_i \leq c_{i+1}$. Then, it is the pair $(i + 1, j)$ that satisfies $c_{i+1} > c_j$ and $w_{i+1} < w_j$. Hence, we can iterate this argument until we find consecutive bidders i and j such that $c_i < c_j$ and $w_i > w_j$.
- $w_i = w_{i+1}$ but then, by construction, $c_i \leq c_{i+1}$. Hence, similarly to the previous case, we iterate the argument with the pair $(i + 1, j)$.

Since $w_i > w_j$ and i and j are consecutive bidders, the bid of supplier j must be in the border of some region A_{li} (where there is no active supplier between l and i because if there was one it would not be active), where supplier l is active. Also, $w_i > w_j$ implies that $w_j = w_l$ and $v_j = v_l$ is optimal, from Proposition 4. But applying Proposition 9 yields that (w_j, v_j) belongs in the segment $[(c_l, f_l); (c_j, f_j)]$. Similarly, supplier i bids in some region $A_{jh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$, with supplier h active and no active suppliers between j and h. With the same argument as before, we have that $(w_i, v_i) = (w_h, v_h)$ and this bid belongs in the segment $[(c_i, f_i); (c_h, f_h)]$.

Define,

$$\overline{F}(y_m) = \frac{v_j - v_i}{w_i - w_j},$$
$$\overline{F}(y_{j1}) = \frac{v_j - f_j}{c_j - w_j},$$
$$\overline{F}(y_{i2}) = \frac{v_i - f_i}{c_i - w_i},$$

and we have that $y_{j1} < y_m < y_{i2}$ because i and j are active.

We can also define

$$\overline{F}(y_{i1}) = \frac{v_j - f_i}{c_i - w_j},$$
$$\overline{F}(y_{j2}) = \frac{f_j - v_i}{w_i - c_j}.$$

Since $c_i < c_j$, and supplier i is efficient, we must have that $f_i \leq f_j + (c_j - c_i)\overline{F}(y_{j1})$ because $\overline{F}(y_{j1})$ is the slope of the line joining (c_l, f_l) to (c_j, f_j). Similarly, $f_j \leq f_i - (c_j - c_i)\overline{F}(y_{i2})$. This implies that $y_{i1} \leq y_{j1} < y_m < y_{i2} \leq y_{j2}$ as can be seen from Figure 4.

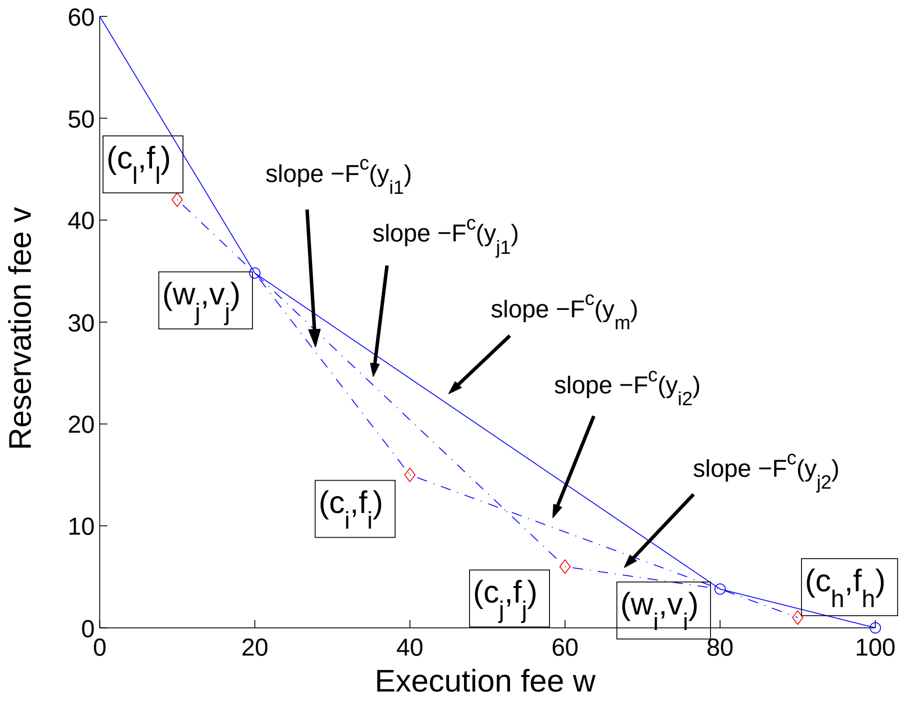

Figure 4: Geometric situation of costs (c_i, f_i) and (c_j, f_j) in region A_{lh}

Finally, we apply Proposition 4. For this purpose, define the functions

$$\phi(y_1) = \frac{\int_{y_1}^{y_m} [\overline{F}(y_1) - \overline{F}(u)]du}{\overline{F}(y_1) - \overline{F}(y_m)}$$

and

$$\psi(y_2) = \frac{\displaystyle\int_{y_m}^{y_2} [\overline{F}(u) - \overline{F}(y_2)]du}{\overline{F}(y_m) - \overline{F}(y_2)}.$$

Taking derivatives, we have, for $y_1 > y_m$ and $y_2 < y_m$,

$$\phi'(y_1) = -f(y_1)\frac{\displaystyle\int_{y_1}^{y_m} [\overline{F}(u) - \overline{F}(y_m)]du}{[\overline{F}(y_1) - \overline{F}(y_m)]^2} < 0$$

$$\psi'(y_2) = f(y_2)\frac{\displaystyle\int_{y_m}^{y_2} [\overline{F}(y_m) - \overline{F}(u)]du}{[\overline{F}(y_m) - \overline{F}(y_2)]^2} > 0.$$

Hence, $\phi(\cdot)$ is non-increasing and $\psi(\cdot)$ is non-decreasing.

We now apply the last case of Proposition 4. Since supplier i bids (w_i, v_i) and not (w_j, v_j), we have $\phi(y_{i1}) \leq \psi(y_{i2})$. Similarly, for j, $\phi(y_{j1}) \geq \psi(y_{j2})$. $y_{i1} \leq y_{j1} < y_m < y_{i2} \leq y_{j2}$ yields $\phi(y_{i1}) \geq \phi(y_{j1}) \geq \psi(y_{j2}) \geq \psi(y_{i2})$, and hence $\phi(y_{i1}) \leq \psi(y_{i2})$ implies that all inequalities are in fact equalities. Therefore $c_i = c_j$ which is a contradiction. ■

A.7 Proposition 7

Proof. Consider supplier $1 < i \leq N$. From Propositions 3 and 6, we know that at equilibrium it will be bidding in region $A_{i-1\ i+1}$ because otherwise one of the suppliers would be inactive or they would not be sorted in the correct order. Let $l = i-1$ and $h = i+1$. Supplier i will in particular bid in the border of this region, with $y_{i-} = y_m$ or $y_{i+} = y_m$ as established in Proposition 4. $y_{i-} = y_m$ is equivalent to saying that it is bidding $w_i = w_{i+1}$ and $v_i = v_{i+1}$ and $\overline{F}(y_{i+}) = \dfrac{f_i - v_{i+1}}{w_{i+1} - c_i}$. In this case, applying Proposition 5 yields that (v_i, w_i) belongs in the segment $[(c_i, f_i); (c_{i+1}, f_{i+1})]$. Similarly, $y_{i+} = y_m$ implies that (v_i, w_i) belongs in the segment $[(c_{i-1}, f_{i-1}); (c_i, f_i)]$, and this is of course possible only if $i > 1$. For $i = 1$, only the first case can occur, i.e., $w_1 = w_2$, $v_1 = v_2$, $y_{1-} = 0$ and y_{1+} such that $\overline{F}(y_{1+}) = \dfrac{f_1 - v_2}{w_2 - c_1}$. Again, Proposition 5 implies that (v_1, w_1) belongs in the segment $[(c_1, f_1); (c_2, f_2)]$. ■

A.8 Proposition 8

Proof. As we mentioned before, the loss in welfare occurs for every supplier i when $(w_i, v_i) = (w_{i-1}, v_{i-1})$ and $(w_i, v_i) \neq (w_{i+1}, v_{i+1})$. For all other cases, we have that $y_i = y_i^*$. We have two different possible cases.

(A) The market allocation is such that $y_i < y_i^*$.

(B) The market allocation is such that $y_i > y_i^*$.

In case (A), Equation (16) holds since $(w_i, v_i) = (w_{i-1}, v_{i-1})$. Therefore, using $y_1 = y_{i-1}^*$, $y_m = y_i$ and $y_2 \geq y_i^*$,

$$\frac{\int_{y_{i-1}^*}^{y_i} [\overline{F}(y_{i-1}^*) - \overline{F}(u)]du}{\overline{F}(y_{i-1}^*) - \overline{F}(y_i)} \geq \frac{\int_{y_i}^{y_2} [\overline{F}(u) - \overline{F}(y_2)]du}{\overline{F}(y_i) - \overline{F}(y_2)} \geq \frac{\int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du}{\overline{F}(y_i) - \overline{F}(y_i^*)},$$

where we used the fact that the function $\frac{\int_{y_i}^{y_2} [\overline{F}(u) - \overline{F}(y_2)]du}{\overline{F}(y_i) - \overline{F}(y_2)}$ is non-decreasing in y_2 and $y_2 \geq y_i^*$.

Integration in parts yields

$$\int_a^b [\overline{F}(a) - \overline{F}(u)]du = -\int_a^b uf(u)du + b[\overline{F}(a) - \overline{F}(b)],$$

and

$$\int_a^b [\overline{F}(u) - \overline{F}(b)]du = \int_a^b uf(u)du - a[\overline{F}(a) - \overline{F}(b)],$$

and hence

$$-\frac{\int_{y_{i-1}^*}^{y_i} uf(u)du}{\overline{F}(y_{i-1}^*) - \overline{F}(y_i)} + y_i \geq \frac{\int_{y_i}^{y_i^*} uf(u)du}{\overline{F}(y_i) - \overline{F}(y_i^*)} - y_i.$$

Examine now the loss created by supplier i.

$$\begin{aligned}
\int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du &= \int_{y_i}^{y_i^*} uf(u)du - y_i[\overline{F}(y_i) - \overline{F}(y_i^*)] \\
&\leq \int_{y_i}^{y_i^*} uf(u)du - \frac{1}{2}\Big[\overline{F}(y_i) - \overline{F}(y_i^*)\Big]\Big[\frac{\int_{y_{i-1}^*}^{y_i} uf(u)du}{\overline{F}(y_{i-1}^*) - \overline{F}(y_i)} + \frac{\int_{y_i}^{y_i^*} uf(u)du}{\overline{F}(y_i) - \overline{F}(y_i^*)}\Big] \\
&\leq \frac{1}{2}\int_{y_i}^{y_i^*} uf(u)du \\
&\leq \frac{1}{2}\int_0^{y_i^*} uf(u)du.
\end{aligned}$$

Since $\int_0^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du = \int_0^{y_i^*} uf(u)du$, we have that

$$\Delta c_i \int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du \leq \frac{1}{2}\Delta c_i \int_0^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du.$$

In case (B), it must be that $i < N$. Since $(w_i, v_i) \neq (w_{i+1}, v_{i+1})$, Proposition 7 implies that $(w_{i+1}, v_{i+1}) = (w_{i+2}, v_{i+2})$, and this means that $w_i \leq c_i \leq c_{i+1} \leq w_{i+1} \leq c_{i+2}$, $y_{i+1} = y_{i+1}^*$ and $y_i \leq y_{i+1}^*$. We can now use Equation (17) for supplier $i+1$ in order to derive a bound on the loss. Here, $y_m = y_i$, $y_2 = y_{i+1}^*$ and $y_1 \leq y_i^*$,

$$\frac{\int_{y_i}^{y_{i+1}^*} [\overline{F}(u) - \overline{F}(y_{i+1}^*)]du}{\overline{F}(y_i) - \overline{F}(y_{i+1}^*)} \geq \frac{\int_{y_1}^{y_i} [\overline{F}(y_1) - \overline{F}(u)]du}{\overline{F}(y_1) - \overline{F}(y_i)} \geq \frac{\int_{y_i^*}^{y_i} [\overline{F}(y_i^*) - \overline{F}(u)]du}{\overline{F}(y_i^*) - \overline{F}(y_i)}.$$

This is equivalent to

$$-\frac{\int_{y_i^*}^{y_i} uf(u)du}{\overline{F}(y_i^*) - \overline{F}(y_i)} + y_i \leq \frac{\int_{y_i}^{y_{i+1}^*} uf(u)du}{\overline{F}(y_i) - \overline{F}(y_{i+1}^*)} - y_i.$$

The loss created by supplier i involves

$$\begin{aligned}
\int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du &= \int_{y_i^*}^{y_i} [\overline{F}(y_i^*) - \overline{F}(u)]du \\
&= -\int_{y_i^*}^{y_i} uf(u)du + y_i[\overline{F}(y_i^*) - \overline{F}(y_i)] \\
&\leq -\int_{y_i^*}^{y_i} uf(u)du + \frac{1}{2}\Big[\overline{F}(y_i^*) - \overline{F}(y_i)\Big]\Big[\frac{\int_{y_i^*}^{y_i} uf(u)du}{\overline{F}(y_i^*) - \overline{F}(y_i)} + \frac{\int_{y_i}^{y_{i+1}^*} uf(u)du}{\overline{F}(y_i) - \overline{F}(y_{i+1}^*)}\Big] \\
&\leq \frac{1}{2}\Big[\frac{\overline{F}(y_i^*) - \overline{F}(y_i)}{\overline{F}(y_i) - \overline{F}(y_{i+1}^*)}\Big]\Big[\int_{y_i}^{y_{i+1}^*} uf(u)du\Big]
\end{aligned}$$

Now, note that

$$\begin{aligned}
(w_{i+1} - w_i)\overline{F}(y_i) &= (c_{i+1} - w_i)\overline{F}(y_1) + (w_{i+1} - c_{i+1})\overline{F}(y_{i+1}^*) \\
&\geq (c_{i+1} - w_i)\overline{F}(y_i^*) + (w_{i+1} - c_{i+1})\overline{F}(y_{i+1}^*),
\end{aligned}$$

where the inequality is justified by $c_{i+1} \geq w_i$ and $y_1 \leq y_i^*$. This, together with $\Delta c_i \leq c_{i+1} - w_i$ and $w_{i+1} - c_{i+1} \leq \Delta c_{i+1}$, implies that

$$\begin{aligned}
\Delta c_i[\overline{F}(y_i^*) - \overline{F}(y_i)] &\leq (c_{i+1} - w_i)[\overline{F}(y_i^*) - \overline{F}(y_i)] \\
&\leq (w_{i+1} - c_{i+1})[\overline{F}(y_i) - \overline{F}(y_{i+1}^*)] \\
&\leq \Delta c_{i+1}[\overline{F}(y_i) - \overline{F}(y_{i+1}^*)].
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta c_i \int_{y_i}^{y_i^*} [\overline{F}(u) - \overline{F}(y_i^*)]du &\leq \frac{1}{2}\Delta c_i\Big[\frac{\overline{F}(y_i^*) - \overline{F}(y_i)}{\overline{F}(y_i) - \overline{F}(y_{i+1}^*)}\Big]\Big[\int_{y_i}^{y_{i+1}^*} uf(u)du\Big] \\
&\leq \frac{1}{2}\Delta c_{i+1} \int_{y_i}^{y_{i+1}^*} uf(u)du \\
&\leq \frac{1}{2}\Delta c_{i+1} \int_{0}^{y_{i+1}^*} [\overline{F}(u) - \overline{F}(y_{i+1}^*)]du.
\end{aligned}$$

Since $y_{i+1} = y_{i+1}^*$, this completes the proof of the bound for any border distribution. ■

A.9 Proposition 9

Proof. Supplier i is active in this Nash equilibrium. Since the demand follows a border distribution, supplier i bids in the boundary of some region $A_{lh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$ constructed with all the bids except i's. If the bid (w_i, v_i) belongs to more than one region, choose $A_{lh}^{(\mathbf{w}_{-i},\mathbf{v}_{-i})}$ with l and h active. We must consider two cases, either there is no supplier in the lower envelope between the bids of l and h, or there is one.

In the first case, there is j, j being l or h, such that j is active, and $(w_i, v_i) = (w_j, v_j)$, from Proposition 4. From Proposition 5, (w_i, v_i) belongs in the segment $[(c_i, f_i); (c_j, f_j)]$.

In the second case, there is one supplier, k on the lower envelope between l and h such that the bid (w_i, v_i) is in the border of $A_{lh}^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}$ and $A_{lk}^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}$ or $A_{lh}^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}$ and $A_{kh}^{(\mathbf{w}_{-i}, \mathbf{v}_{-i})}$. k is thus inactive because of supplier i, and either the bids of l, k and i are aligned, or those of i, k and h. Such a situation is depicted in Figure 5. Hence, we find j, j being l or h, active, such that (w_i, v_i) is equal to $(w_j, v_j) + \theta(w_k - w_j, v_k - v_j)$ for some non-negative θ. ∎

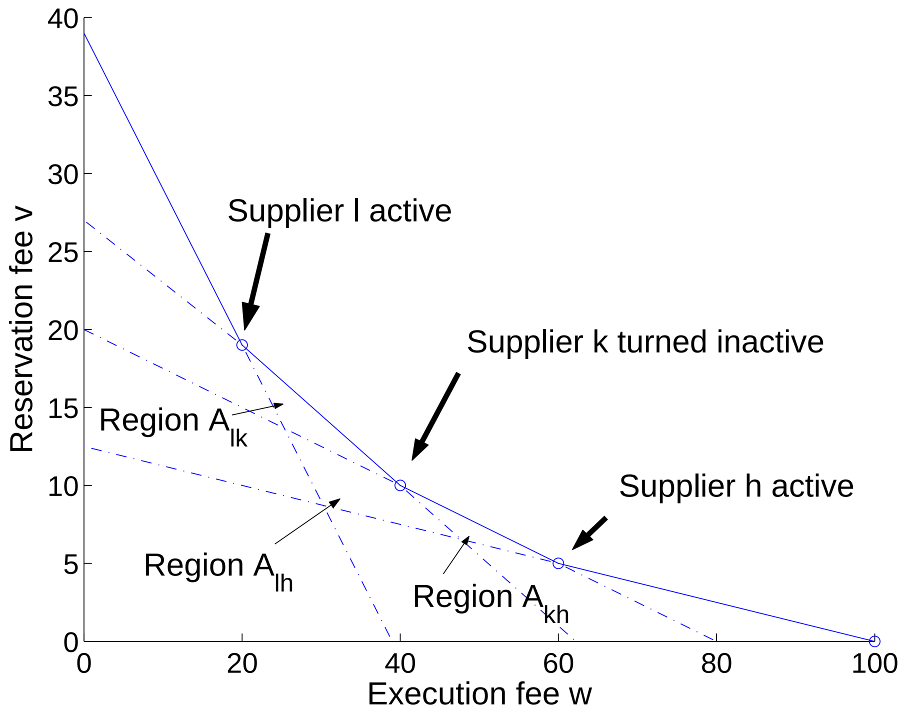

Figure 5: Suppliers l and h are active and supplier k is turned inactive by supplier i's bid

A.10 Proposition 11

Proof. Equation (17) in Proposition 4 reduces to

$$\frac{y_m - y_1}{2} \leq \frac{y_2 - y_m}{2}$$

in the uniform case. Since also $c = w_h - (w_h - w_l)\dfrac{y_m - y_1}{y_2 - y_1}$, the condition is equivalent to

$$c \geq \frac{w_l + w_h}{2}.$$

Similarly, Equation (16) is equivalent to $c \leq \dfrac{w_l + w_h}{2}$. ∎

A.11 Proposition 12

Proof. For the uniform distribution, we have that

$$\frac{\Delta U}{U^*} = \frac{\sum_{i=1}^{N} \Delta c_i (y_i^* - y_i)^2}{\sum_{i=1}^{N} \Delta c_i y_i^{*2}}.$$

As in the proof of Proposition 8, we analyze two different possible cases.

(A) The market allocation is such that $y_i < y_i^*$.

(B) The market allocation is such that $y_i > y_i^*$.

In case (A), Equation (16) in the uniform case becomes, with $y_1 = y_{i-1}^*$, $y_m = y_i$ and $y_2 \geq y_i^*$,

$$y_i - y_{i-1}^* \geq y_2 - y_i \geq y_i^* - y_i \geq 0.$$

Thus, $(y_i - y_i^*)^2 \leq \frac{1}{4} y_i^{*2}$.

In case (B) (again it must be that $i < N$), Equation (17) for supplier $i+1$ becomes, with $y_m = y_i$, $y_2 = y_{i+1}^*$ and $y_1 \leq y_i^*$,

$$0 \leq y_i - y_i^* \leq y_i - y_1 \leq y_{i+1}^* - y_i.$$

This implies that $(y_i - y_i^*)^2 \leq \frac{1}{4} y_{i+1}^{*2}$.

To show that the bound is tight, consider the uniform $[0, 1]$ distribution with $N = 2$ suppliers. Define the following data, with ϵ close to 0.

$$(c_1, f_1) = (0, 50 - \epsilon), \quad (c_2, f_2) = (50, 0), \quad p = 100$$

The optimal allocation is $y_1^* = \epsilon/50$ and $y_2^* = 1$. The bids $w_1 = w_2 = 0$, $v_1 = v_2 = 50 - \epsilon$, together with the allocation $y_1 = \epsilon/50$, $y_2 = 1/2 + \epsilon/100$, form a Nash equilibrium. When ϵ approaches 0, the welfare loss clearly approaches $\frac{1}{4}$. ∎

A.12 Proposition 13

Proof. We use Proposition 2 to prove the proposition. Take $a, b \in \mathbb{R}$ and $y_m \geq 0$. Show that the function

$$a(y_+ - y_-) + b \int_{y_-}^{y_+} \overline{F}(u) du - \left[\frac{\overline{F}(y_-) - \overline{F}(y_m)}{\overline{F}(y_-) - \overline{F}(y_+)} \right] \int_{y_-}^{y_+} \left(\overline{F}(u) - \overline{F}(y_+) \right) du$$

$$= a(y_+ - y_-) + b \Big(\frac{e^{-\lambda y_-} - e^{-\lambda y_+}}{\lambda} \Big) - \Big(\frac{e^{-\lambda y_-} - e^{-\lambda y_m}}{e^{-\lambda y_-} - e^{-\lambda y_+}} \Big) \Big(\frac{e^{-\lambda y_-} - e^{-\lambda y_+}}{\lambda} - e^{-\lambda y_+} (y_+ - y_-) \Big),$$

does not have a strict local maximum such that $0 < y_-^* < y_m$ and $y_+^* > y_m$ is sufficient to apply Proposition 2.

For this purpose, we change variables and define $\Delta = y_+ - y_-$, and hence the function becomes

$$a\Delta + \frac{b}{\lambda}e^{-\lambda y_-}\Big(1 - e^{-\lambda\Delta}\Big) - \frac{1}{\lambda}\Big(e^{-\lambda y_-} - e^{-\lambda y_m}\Big) + \Big(\frac{e^{-\lambda y_-} - e^{-\lambda y_m}}{1 - e^{-\lambda\Delta}}\Big)e^{-\lambda\Delta}\Delta$$

This is clearly differentiable in y_-, and the derivative is

$$-e^{-\lambda y_-}b\Big(1 - e^{-\lambda\Delta}\Big) + e^{-\lambda y_-} - \Big(\frac{e^{-\lambda y_-}}{1 - e^{-\lambda\Delta}}\Big)e^{-\lambda\Delta}\lambda\Delta$$

$$= e^{-\lambda y_-}\Big[1 - b(1 - e^{-\lambda\Delta}) - \frac{\lambda\Delta e^{-\lambda\Delta}}{1 - e^{-\lambda\Delta}}\Big]$$

We see that the sign of the derivative is independent of y_-. This implies that if the derivative is positive, we can increase the profit by increasing y_- and keeping Δ constant until it hits the border of the region. If it is non-positive, we can obtain at least the same profit by decreasing y_- and keeping Δ constant. Therefore, there can be no interior strict local maximum. Proposition 2 implies that the distribution is border. ■