Procedural abstraction and recursion
6.037 - Structure and Interpretation of Computer Programs

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Lecture 1

http://web.mit.edu/alexmv/6.037/
Class Structure

- TR, 7-9PM, through end of IAP
- E-mail: 6.001-zombies@mit.edu
- Five projects: due on the 10th, 15th, 17th, 24th, and 1st.
- Graded P/D/F
- Taking the class for credit is zero-risk!
- E-mail list sign-up on the website
Goals of the Class

- This is not a class to teach Scheme
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- This is not a class to teach Scheme
- Nor really a class about programming at all
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- This is a course about Computer Science
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- ...which isn’t about computers
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- This is a course about *Computer Science*
- ...which isn’t about computers
- ...nor actually a science
Goals of the Class

- This is not a class to teach Scheme
- Nor really a class about programming at all
- This is a course about Computer Science
- ...which isn’t about computers
- ...nor actually a science
- This is actually a class in computation
Prerequisites

- High confusion threshold
- Some programming clue
- A copy of Racket (Formerly PLT Scheme / DrScheme)  
  http://www.racket-lang.org/
- Free time
Project 0

- Project 0 is out today
- Due on Thursday!
- Mail to 6.037-psets@mit.edu
- Collaboration with current students is fine, as long as you note it
Some History

- Lisp invented in 1959 by John McCarthy (R.I.P. 2010)

- Scheme invented in 1975 by Guy Steele and Gerald Sussman

- Hardware Lisp machines, around 1978

- 6.001 first taught in 1980

- SICP published in 1984 and 1996

- R6RS in 2007

- 6.001 last taught in 2007

- 6.037 first taught in 2009
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The Book ("SICP")

- **Structure and Interpretation of Computer Programs**
  by Harold Abelson and Gerald Jay Sussman
- [http://mitpress.mit.edu/sicp/](http://mitpress.mit.edu/sicp/)
- Not required reading
- Useful as study aid and reference
- Roughly one lecture per chapter
Key ideas

- Procedural and data abstraction
- Conventional interfaces & programming paradigms
  - Type systems
  - Streams
  - Object-oriented programming
- Metalinguistic abstraction
  - Creating new languages
  - Evaluators
Key ideas

- **Procedural and data abstraction**
- **Conventional interfaces & programming paradigms**
  - Type systems
  - Streams
  - Object-oriented programming
- **Metalinguistic abstraction**
  - Creating new languages
  - Evaluators
1. Syntax of Scheme, procedural abstraction, and recursion
2. Data abstractions, higher order procedures, symbols, and quotation
3. Mutation, and the environment model
4. Interpretation and evaluation
5. Debugging
6. Language design and implementation
7. Continuations, concurrency, lazy evaluation, and streams
8. 6.001 in perspective, and the Lambda Calculus
## Projects

0. Basic Scheme warm-up  
   Thursday 1/10

1. Higher-order procedures and symbols  
   Tuesday 1/15

2. Mutable objects and procedures with state  
   Thursday 1/17

3. Meta-circular evaluator  
   Thursday 1/24

4. OOP evaluator (The Adventure Game)  
   Friday 2/1*
“How to” knowledge

To approximate $\sqrt{x}$ (Heron’s Method):

- Make a guess $G$
- Improve the guess by averaging $G$ and $\frac{x}{G}$
- Keep improving until it is good enough
“How to” knowledge

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$x = 2 \quad G = 1$
“How to” knowledge

To approximate $\sqrt{x}$ (Heron’s Method):

- Make a guess $G$
- Improve the guess by averaging $G$ and $\frac{x}{G}$
- Keep improving until it is good enough

\[
\begin{align*}
x &= 2 & G &= 1 \\
\frac{x}{G} &= 2 & G &= \frac{(1+2)}{2} = 1.5
\end{align*}
\]
“How to” knowledge

To approximate $\sqrt{x}$ (Heron’s Method):
- Make a guess $G$
- Improve the guess by averaging $G$ and $\frac{x}{G}$
- Keep improving until it is good enough

\[
x = 2 \\
\frac{x}{G} = 2 \\
\frac{x}{G} = \frac{4}{3}
\]

\[
G = \frac{(1+2)}{2} = 1.5 \\
G = \frac{(\frac{3}{2} + \frac{4}{3})}{2} = 1.4166
\]
“How to” knowledge

To approximate $\sqrt{x}$ (Heron’s Method):
- Make a guess $G$
- Improve the guess by averaging $G$ and $\frac{x}{G}$
- Keep improving until it is good enough

$x = 2$

$\frac{x}{G} = 2$

$G = \frac{1+2}{2} = 1.5$

$\frac{x}{G} = \frac{4}{3}$

$G = \frac{\left(\frac{3}{2} + \frac{4}{3}\right)}{2} = 1.4166$

$\frac{x}{G} = \frac{24}{17}$

$G = \frac{\left(\frac{17}{12} + \frac{24}{17}\right)}{2} = 1.4142$
“How to” knowledge

Could just store tons of “what is” information
“How to” knowledge

Could just store tons of “what is” information

Much more useful to capture “how to” knowledge – a series of steps to be followed to deduce a value – a procedure.
Describing “How to” knowledge

Need a language for describing processes:

- Vocabulary – basic primitives
Describing “How to” knowledge

Need a language for describing processes:

- Vocabulary – basic primitives
- Rules for writing compound expressions – syntax
Describing “How to” knowledge

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- Rules for writing compound expressions – syntax
- Rules for assigning meaning to constructs – semantics
Describing “How to” knowledge

Need a language for describing processes:

- Vocabulary – **basic primitives**
- Rules for writing compound expressions – **syntax**
- Rules for assigning meaning to constructs – **semantics**
- Rules for capturing process of evaluation – **procedures**
Representing basic information

- Numbers
Representing basic information

- Numbers
  - As floating point values
Representing basic information

- Numbers
  - As floating point values
  - In IEEE 754 format
Representing basic information

- **Numbers**
  - As floating point values
  - In IEEE 754 format
  - Stored in binary
Representing basic information

- Numbers
  - As floating point values
  - In IEEE 754 format
  - Stored in binary
  - In registers
Representing basic information

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  - Stored in flip-flops
  - Made of logic gates
Representing basic information

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  - Made of logic gates
  - Implemented by transistors
Representing basic information

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  - Stored in flip-flops
  - Made of logic gates
  - Implemented by transistors
  - In silicon wells
Representing basic information

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  - Stored in binary
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  - Made up of bits
  - Stored in flip-flops
  - Made of logic gates
  - Implemented by transistors
  - In silicon wells
  - With electrical potential
Representing basic information

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  - Of individual electrons
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  - With mass, charge, spin, and chirality
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  - With mass, charge, spin, and chirality
  - Whose mass is imparted by interaction with the Higgs field
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...
Assuming a basic level of abstraction

- We assume that our language provides us with a basic set of data elements:
  - Numbers
Assuming a basic level of abstraction

- We assume that our language provides us with a basic set of data elements:
  - Numbers
  - Characters
Assuming a basic level of abstraction

- We assume that our language provides us with a basic set of data elements:
  - Numbers
  - Characters
  - Booleans
Assuming a basic level of abstraction

- We assume that our language provides us with a basic set of data elements:
  - Numbers
  - Characters
  - Booleans
- It also provides a basic set of operations on these primitive elements
Assuming a basic level of abstraction

- We assume that our language provides us with a basic set of data elements:
  - Numbers
  - Characters
  - Booleans

- It also provides a basic set of operations on these primitive elements

- We can then focus on using these basic elements to construct more complex processes
Legal expressions have rules for constructing from simpler pieces – the syntax.
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(Almost) every expression has a value, which is “returned” when an expression is “evaluated.”
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(Almost) every expression has a value, which is “returned” when an expression is “evaluated.”

Every value has a type.
Legal expressions have rules for constructing from simpler pieces – the **syntax**.

(ALmost) every expression has a **value**, which is “returned” when an expression is “evaluated.”

Every value has a **type**.

The latter two are the **semantics** of the language.
Self-evaluating primitives – value of expression is just object itself:

**Numbers** 29, −35, 1.34, 1.2e5
Self-evaluating primitives – value of expression is just object itself:

**Numbers**  29, −35, 1.34, 1.2e5

**Strings**  “this is a string” “odd #$@%#$ thing number 35”
Self-evaluating primitives – value of expression is just object itself:

**Numbers**  29, −35, 1.34, 1.2e5

**Strings**  “this is a string” “odd #$@%#$ thing number 35”

**Booleans**  #t, #f
Built-in procedures to manipulate primitive objects:

- **Numbers**: +, −, *, /, >, <, >=, <=, =
- **Strings**: string-length, string=?
- **Booleans**: and, or, not
Names for built-in procedures

- +, −, *, /, =, ...  

- What is the **value** of them?
Language elements – primitives

Names for built-in procedures

- +, −, *, ‌/, =, …
- What is the value of them?
- + → #<procedure:+>
Language elements – primitives

Names for built-in procedures

- +, -, *, /, =, ...
- What is the value of them?
- + → #<procedure:+>
- Evaluate by looking up value associated with the name in a special table – the environment.
Language elements – combinations

- How to we create expressions using these procedures?
- \((+ \ 2 \ 3)\)
Language elements – combinations

- How to we create expressions using these procedures?
- (+ 2 3)
  - Open paren

This type of expression is called a combination. Evaluate it by getting values of sub-expressions, then applying the operator to values of arguments.
How to we create expressions using these procedures?

(+ 2 3)

- Open paren
- Expression whose value is a procedure
How to we create expressions using these procedures?

+ 2 3

Open paren
Expression whose value is a procedure
Other expressions
How to we create expressions using these procedures?

\[(+ \ 2 \ 3)\]

- Open paren
- Expression whose value is a procedure
- Other expressions
- Close paren

This type of expression is called a combination. Evaluate it by getting values of sub-expressions, then applying the operator to values of arguments.

You now know all there is to know about Scheme syntax!

(almost)
Language elements – combinations

How to we create expressions using these procedures?

(+ 2 3)

- Open paren
- Expression whose value is a procedure
- Other expressions
- Close paren

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You now know all there is to know about Scheme syntax!
How to create expressions using these procedures?

(+ 2 3)
- Open paren
- Expression whose value is a procedure
- Other expressions
- Close paren

This type of expression is called a combination
Evaluate it by getting values of sub-expressions, then applying operator to values of arguments.

You now know all there is to know about Scheme syntax! (almost)
Note the recursive definition – can use combinations as expressions to other combinations:

\[(+ (* 2 3) 4) \rightarrow \]
Note the recursive definition – can use combinations as expressions to other combinations:

\[ (+ \ (* \ 2 \ 3) \ 4) \rightarrow 10 \]
Note the recursive definition – can use combinations as expressions to other combinations:

\[(+ (* 2 3) 4) \rightarrow 10\]
\[(* (+ 3 4) (+ 8 2)) \rightarrow\]
Note the recursive definition – can use combinations as expressions to other combinations:

\((+ (* 2 3) 4)\) \rightarrow 10

\((* (+ 3 4) (- 8 2))\) \rightarrow 42
In order to abstract an expression, need a way to give it a name
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(\texttt{define score 23})
In order to abstract an expression, need a way to give it a name

(define score 23)

This is a special form

- Does not evaluate the second expression
- Rather, it pairs the name with the value of the third expression
In order to abstract an expression, need a way to give it a name:

(define score 23)

This is a special form:
- Does not evaluate the second expression
- Rather, it pairs the name with the value of the third expression

The return value is unspecified
Language elements – abstractions

To get the value of a name, just look up pairing in the environment

(\texttt{define score 23}) →
To get the value of a name, just look up pairing in the environment

(define score 23) → undefined
To get the value of a name, just look up pairing in the environment

```scheme
(define score 23)
(score) → undefined
```
To get the value of a name, just look up pairing in the environment.

\[(\text{define score } 23)\]
\[\text{score} \rightarrow \text{undefined} \]
\[\text{score} \rightarrow 23 \]
To get the value of a name, just look up pairing in the environment.

\[
\begin{align*}
(\text{define score 23}) & \quad \rightarrow \quad \text{undefined} \\
\text{score} & \quad \rightarrow \quad 23 \\
(\text{define total (+ 12 13)}) & \quad \rightarrow \\
\end{align*}
\]
To get the value of a name, just look up pairing in the environment

\[
\text{(define score 23)} \\
\text{score} \\
\text{(define total (+ 12 13))}
\]

→ undefined
→ 23
→ undefined
To get the value of a name, just look up pairing in the environment

```
(define score 23)  →  undefined
score              →  23
(define total (+ 12 13)) →  undefined
(* 100 (/ score total)) →
```
To get the value of a name, just look up pairing in the environment

\[(\text{define } \text{score} \ 23)\]  →  undefined
\[\text{score}\]  →  23
\[(\text{define } \text{total} \ (+ \ 12 \ 13))\]  →  undefined
\[(\times \ 100 \ (/ \ \text{score} \ \text{total}))\]  →  92
Language elements – common errors

\[(5 + 6)\]
(5 + 6)

=> procedure application: expected procedure,
given: 5; arguments were: #<procedure:+> 6
(5 + 6)
  =>  procedure application: expected procedure, given: 5; arguments were: #<procedure:+> 6

((+ 5 6))
(5 + 6)
  => procedure application: expected procedure, given: 5; arguments were: #<procedure:+> 6

((+ 5 6))
  => procedure application: expected procedure, given: 11 (no arguments)
Language elements – common errors

(5 + 6)
   => procedure application: expected procedure,
       given: 5; arguments were: #<procedure:+> 6

((+ 5 6))
   => procedure application: expected procedure,
       given: 11 (no arguments)

(* 100 (/ score totla))
Language elements – common errors

\[(5 + 6)\]

=> procedure application: expected procedure, given: 5; arguments were: #<procedure:+> 6

\[((+ 5 6))\]

=> procedure application: expected procedure, given: 11 (no arguments)

\[(* 100 (/ score totala))\]

=> reference to undefined identifier: totala
Rules for evaluation:
- If self-evaluating, return value
Rules for evaluation:

- If self-evaluating, return value
- If a name, return value associated with name in environment
Rules for evaluation:

- If self-evaluating, return value
- If a name, return value associated with name in environment
- If a special form, do something special
Rules for evaluation:

- If **self-evaluating**, return value
- If a **name**, return value associated with name in environment
- If a **special form**, do something special
- If a **combination**, then
  - **Evaluate** all of the sub-expressions, in any order
Rules for evaluation:

- If **self-evaluating**, return value
- If a **name**, return value associated with name in environment
- If a **special form**, do something special
- If a **combination**, then
  - Evaluate all of the sub-expressions, in any order
  - Apply the operator to the values of the operands and return the result
Mathematical operators are just names

\[(+ \ 3 \ 5) \rightarrow 8\]
Mathematical operators are just names

(+ 3 5) → 8
(define fred +) → undefined
Mathematical operators are just names

(+ 3 5) → 8
(define fred +) → undefined
(fred 3 6) → 9
Mathematical operators are just names

(+ 3 5) → 8
(define fred +) → undefined
(fred 3 6) → 9

- + is just a name
- + is bound to a value which is a procedure
- line 2 binds the name fred to that same value
All names are names

\[(+ 3 5) \rightarrow 8\]

There's nothing "special" about the operators you take for granted, either!

Of course, this is generally a horrible idea
All names are names

\[(+ 3 5) \rightarrow 8\]
\[(\text{define} + *) \rightarrow \text{undefined}\]
All names are names

(+ 3 5) → 8
(define + *) → undefined
(+ 3 5) →
All names are names

(+ 3 5) \rightarrow 8
(define + *) \rightarrow undefined
(+ 3 5) \rightarrow 15
All names are names

(+ 3 5) \rightarrow 8
(define + *) \rightarrow undefined
(+ 3 5) \rightarrow 15

- There’s nothing “special” about the operators you take for granted, either!
All names are names

(+ 3 5) → 8
(define + *) → undefined
(+ 3 5) → 15

There’s nothing “special” about the operators you take for granted, either!

Their values can be changed using define just as well
All names are names

\[(+ 3 5) \rightarrow 8\]
\[(\text{define } + *) \rightarrow \text{undefined}\]
\[(+ 3 5) \rightarrow 15\]

- There's nothing "special" about the operators you take for granted, either!
- Their values can be changed using \texttt{define} just as well
- Of course, this is generally a horrible idea
Making our own procedures

To capture a way of doing things, create a procedure:
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\[ (\text{lambda} \ (x) \ (\ast \ x \ x)) \]
Making our own procedures

- To capture a way of doing things, create a procedure:
  \[(\text{lambda } (x) (* x x))\]
- \((x)\) is the list of parameters
To capture a way of doing things, create a procedure:

\[
\text{(lambda} \ (x) \ (* \ x \ x))
\]

(x) is the list of parameters

(* x x) is the body
To capture a way of doing things, create a procedure:

\[ \text{(lambda (x) (* x x))} \]

(x) is the list of parameters

(* x x) is the body

lambda is a special form: create a procedure and returns it
Use this anywhere you would use a built-in procedure like `+:

\[(\text{lambda}\ (x)\ (*\ x\ x))\ 5\]
Use this anywhere you would use a built-in procedure like +:

```
(lambda (x) (* x x)) 5
```

**Substitute** the value of the provided arguments into the body:

```
(* 5 5)
```
Substitution

- Use this anywhere you would use a built-in procedure like `+`:
  \[
  \left( \text{lambda} \ (x) \ (\ast \ x \ x) \right) \ 5
  \]

- **Substitute** the value of the provided arguments into the body:
  \[
  \ast \ 5 \ 5
  \]

- Can also give it a name:
  \[
  \text{(define square} \ (\text{lambda}(x) \ (\ast \ x \ x)))
  \]

\[
\text{(square} \ 5) \rightarrow 25
\]
Substitution

- Use this anywhere you would use a built-in procedure like `+`:
  
  ```lisp
  (lambda (x) (* x x)) 5
  ```

- **Substitute** the value of the provided arguments into the body:
  
  ```lisp
  (* 5 5)
  ```

- Can also give it a name:
  
  ```lisp
  (define square (lambda(x) (* x x)))
  (square 5) → 25
  ```

- This creates a loop in our system, where we can create a complex thing, name it, and treat it as a primitive like `+`
Rules for evaluation:
- If self-evaluating, return value
- If a name, return value associated with name in environment
- If a special form, do something special.
- If a combination, then
  - Evaluate all of the sub-expressions, in any order
  - Apply the operator to the values of the operands and return the result

Rules for applying:
- If primitive, just do it
- If a compound procedure, then substitute each formal parameter with the corresponding argument value, and evaluate the body
Interaction of define and lambda

\[(\text{lambda} \ (x) \ (* \ x \ x))\]

\[\Rightarrow \ #\langle\text{procedure}\rangle\]
(lambda (x) (* x x))
   => #<procedure>
(define square (lambda (x) (* x x)))
   => undefined
(lambda (x) (* x x))
   => #<procedure>
(define square (lambda (x) (* x x)))
   => undefined
(square 4)
(lambda (x) (* x x))
    => #<procedure>
(define square (lambda (x) (* x x)))
    => undefined
(square 4)
    => (* 4 4)
Interaction of define and lambda

(lambda (x) (* x x))
   => #<procedure>
(define square (lambda (x) (* x x)))
   => undefined
(square 4)
   => (* 4 4)
   => 16
Interaction of `define` and `lambda`

```
(lambda (x) (* x x))
   => #<procedure>
(define square (lambda (x) (* x x)))
   => undefined
(square 4)
   => (* 4 4)
   => 16
```

“Syntactic sugar”:

```
(define (square x) (* x x))
   => undefined
```
Lambda special form

- **Syntax:** `(lambda (x y) (/ (+ x y) 2))`
Lambda special form

- Syntax: `(lambda (x y) (/ (+ x y) 2))`
- 1st operand is the parameter list: `(x y)`
  - a list of names (perhaps empty)
  - determines the number of operands required
Lambda special form

- **Syntax:** \( \text{lambda} \ (x \ y) \ (\div \ (+ \ x \ y) \ 2) \)

- 1st operand is the parameter list: \( (x \ y) \)
  - a list of names (perhaps empty)
  - determines the number of operands required

- 2nd operand is the **body:** \( (\div \ (+ \ x \ y) \ 2) \)
  - may be any expression
  - not evaluated when the lambda is evaluated
  - evaluated when the procedure is applied
Meaning of a lambda

\[(\text{define } x (\lambda () (+ 3 2))) \quad \rightarrow \quad \text{undefined}\]

The value of a lambda expression is a procedure.
Meaning of a lambda

(\text{define } x (\text{lambda } () (+ 3 2))) \rightarrow \text{undefined}
Meaning of a lambda

\[
\text{(define } x \text{ (lambda () (+ 3 2)))} \quad \rightarrow \quad \text{undefined}
\]

\[
\text{x} \quad \rightarrow
\]
Meaning of a lambda

(define x (lambda () (+ 3 2)))

\[x\]

\[\rightarrow\text{ undefined}\]

\[\rightarrow\text{ #<procedure>}\]
Meaning of a lambda

```
(define x (lambda () (+ 3 2)))
x
(x)
```

→ undefined
→ #<procedure>
→
Meaning of a lambda

\[
\text{(define } x \text{ (lambda () (+ 3 2)))}
\]

\[
x
\]

\[
(x)
\]

→ undefined
→ #<procedure>
→ 5
Meaning of a lambda

\[(\text{define } x (\lambda () (+ 3 2)))\]  
\[x\]
\[(x)\]  
→ undefined
→ #<procedure>
→ 5

The value of a lambda expression is a procedure
What does a procedure describe?

Capturing a common pattern:

- \((\times 3 3)\)
- \((\times 25 25)\)
- \((\times \text{foobar foobar})\)
What does a procedure describe?

Capturing a common pattern:
- \((* 3 3)\)
- \((* 25 25)\)
- \((* \text{foobar} \text{foobar})\)

\((\text{lambda} \ (x) \ (* \ x \ x))\)

Name for the thing that changes
What does a procedure describe?

Capturing a common pattern:

- \((* \ 3 \ 3)\)
- \((* \ 25 \ 25)\)
- \((* \ \text{foobar} \ \text{foobar})\)
- \((\text{lambda} (x) \ (** \ x \ x))\)

Common pattern to capture
Modularity of common patterns

Here is a common pattern:

\[
\sqrt{\left(3 \times 3 + 4 \times 4\right)}
\]
\[
\sqrt{\left(9 \times 9 + 16 \times 16\right)}
\]
\[
\sqrt{\left(4 \times 4 + 4 \times 4\right)}
\]
Here is a common pattern:

\[
\sqrt{3^2 + 4^2}
\]
Here is a common pattern:

- \((\sqrt{+(*33)(*44)})\)
- \((\sqrt{+(*99)(*1616)})\)
Modularity of common patterns

Here is a common pattern:

- \((\sqrt{+(*33)(*44)})\)
- \((\sqrt{+(*99)(*1616)})\)
- \((\sqrt{+(*44)(*44)})\)
Here is a common pattern:

- \((\sqrt{+ (\times \ 3 \ 3) (\times \ 4 \ 4)})\)
- \((\sqrt{+ (\times \ 9 \ 9) (\times \ 16 \ 16)})\)
- \((\sqrt{+ (\times \ 4 \ 4) (\times \ 4 \ 4)})\)
Here is a common pattern:

- \((\text{sqrt} \ (+ \ (\ast \ 3 \ 3) \ (\ast \ 4 \ 4)))\)
- \((\text{sqrt} \ (+ \ (\ast \ 9 \ 9) \ (\ast \ 16 \ 16)))\)
- \((\text{sqrt} \ (+ \ (\ast \ 4 \ 4) \ (\ast \ 4 \ 4)))\)
Modularity of common patterns

Here is a common pattern:

- \((\text{sqrt } (+ \ (* \ 3 \ 3) \ (* \ 4 \ 4)))\)
- \((\text{sqrt } (+ \ (* \ 9 \ 9) \ (* \ 16 \ 16)))\)
- \((\text{sqrt } (+ \ (* \ 4 \ 4) \ (* \ 4 \ 4)))\)

Here is one way to capture this pattern:

\[
\text{(define pythagoras}
  \text{(lambda} \ (x \ y)
    \text{(sqrt } (+ \ (* \ x \ x) \ (* \ y \ y))))
\]
Modularity of common patterns

Here is a common pattern:

- \((\sqrt{3 \times 3 + 4 \times 4})\)
- \((\sqrt{9 \times 9 + 16 \times 16})\)
- \((\sqrt{4 \times 4 + 4 \times 4})\)

Here is a better way to capture this pattern:

```scheme
(define square (lambda (x) (* x x)))
(define pythagoras
  (lambda (x y)
    (sqrt (+ (square x) (square y))))))
```
Why?

- Breaking computation into modules that capture commonality

Enables reuse in other places (e.g., square)
Isolates (abstracts away) details of computation within a procedure from use of the procedure
May be many ways to divide up:
Why?

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```
(define square (lambda (x) (* x x)))
```

```
(define pythagoras
  (lambda (x y)
    (sqrt (+ (square x) (square y)))))
```
Why?

- Breaking computation into modules that capture commonality
- Enables reuse in other places (e.g. `square`)
- Isolates (abstracts away) details of computation within a procedure from use of the procedure
- May be many ways to divide up:

```
(define square (lambda (x) (* x x)))
(define sum-squares
  (lambda (x y) (+ (square x) (square y))))
(define pythagoras
  (lambda (x y)
    (sqrt (sum-squares x y))))
```
A more complex example

To approximate $\sqrt{x}$:

1. Make a guess $G$
A more complex example

To approximate $\sqrt{x}$:

1. Make a guess $G$
2. Improve the guess by averaging $G$ and $\frac{x}{G}$:
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Sub-problems:

- When is “close enough”?
A more complex example

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Sub-problems:
- When is “close enough”?
- How do we create a new guess?
A more complex example

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1. Make a guess $G$
2. Improve the guess by averaging $G$ and $\frac{x}{G}$:
3. Keep improving until it is good enough

Sub-problems:

- When is “close enough”?
- How do we create a new guess?
- How do we control the process of using the new guess in place of the old one?
“When the square of the guess is within 0.001 of the value”
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(define close-enough? (lambda (guess x) (< (abs (- (square guess) x)) 0.001)))
“When the square of the guess is within 0.001 of the value”

(define close-enough? (lambda (guess x)
    (< (abs (- (square guess) x))
      0.001)))

Note the use of the square procedural abstraction from earlier!
(define average
  (lambda (a b) (/ (+ a b) 2)))
(define average
  (lambda (a b) (/ (+ a b) 2)))

(define improve
  (lambda (guess x)
    (average guess (/ x guess)))))
Why this modularity?

- `average` is something we are likely to want to use again
- Abstraction lets us separate implementation details from use
  - Originally:
    
    ```scheme
    (define average
      (lambda (a b) (/ (+ a b) 2)))
    ```

    Could redefine as:
    
    ```scheme
    (define average
      (lambda (x y) (* (+ x y) 0.5)))
    ```

    There's actually a difference between those in Racket (exact vs inexact numbers)

    No other changes needed to procedures that use `average`

    Also note that parameters are internal to the procedure – cannot be referred to by name outside of the lambda
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  - There’s actually a difference between those in Racket (exact vs inexact numbers)
  - No other changes needed to procedures that use `average`
  - Also note that parameters are internal to the procedure – cannot be referred to by name outside of the lambda
Controlling the process

- Given $x$ and guess, want $(\text{improve guess } x)$ as new guess.
Controlling the process

- Given $x$ and guess, want $(\text{improve guess } x)$ as new guess
- But only if the guess isn’t good enough already
Controlling the process

- Given \( x \) and \( \text{guess} \), want (improve \( \text{guess} \) \( x \)) as new \( \text{guess} \)
- But only if the guess isn’t good enough already
- We need to make a decision – for this, we need a new special form
  (if \( \text{predicate} \) \( \text{consequent} \) \( \text{alternative} \))
The *if* special form

```
(if predicate consequent alternative)
```

- Evaluator first evaluates the `predicate` expression.
- If it returns a true value (`#t`), then the evaluator evaluates and returns the value of the `consequent` expression.
- Otherwise, it evaluates and returns the value of the `alternative` expression.
The if special form

(if predicate consequent alternative)

- Evaluator first evaluates the `predicate` expression
- If it returns a true value (`#t`), then the evaluator evaluates and returns the value of the `consequent` expression
- Otherwise, it evaluates and returns the value of the `alternative` expression
- Why must this be a special form? Why can’t it be implemented as a regular `lambda` procedure?
So the heart of the process should be:

(define (sqrt-loop guess x)
  (if (close-enough? guess x)
      guess
      (improve guess x)
    ))

But somehow we need to use the value returned by improve as the new guess, keep the same x, and repeat the process.
So the heart of the process should be:

\[
\text{(define (sqrt-loop \text{guess} x)}
\begin{align*}
\text{(if (close-enough? guess x)} & \text{guess} \\
& \text{(improve guess x)} \quad \text{}})
\end{align*}
\]

But somehow we need to use the value returned by improve as the new guess, keep the same x, and repeat the process.

Call the sqrt-loop function again and reuse it!
Using if

- So the heart of the process should be:
  
  (define (sqrt-loop guess x)
      (if (close-enough? guess x)
          guess
          (sqrt-loop (improve guess x) x)))

- But somehow we need to use the value returned by `improve` as the new `guess`, keep the same `x`, and repeat the process.

- Call the `sqrt-loop` function again and reuse it!
Now we just need to kick the process off with an initial guess:

```
(define sqrt
  (lambda (x)
    (sqrt-loop 1.0 x)))
```

```
(define (sqrt-loop guess x)
  (if (close-enough? guess x)
      guess
      (sqrt-loop (improve guess x) x)))
```
Testing the code

How do we know it works?
Testing the code

- How do we know it works?
- Fall back to rules for evaluation from earlier
Substitution model

Rules for evaluation:
- If self-evaluating, return value
- If a name, return value associated with name in environment
- If a special form, do something special.
- If a combination, then
  - Evaluate all of the sub-expressions, in any order
  - Apply the operator to the values of the operands and return the result

Rules for applying:
- If primitive, just do it
- If a compound procedure, then substitute each formal parameter with the corresponding argument value, and evaluate the body
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The substitution model of evaluation
Rules for evaluation:

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Rules for applying:

- If primitive, just do it
- If a compound procedure, then substitute each formal parameter with the corresponding argument value, and evaluate the body

The substitution model of evaluation

... is a lie and a simplification, but a useful one!
(sqrt 2)
(sqrt 2)
((lambda (x) (sqrt-loop 1.0 x)) 2)
\[(\sqrt{2})\]

\[((\lambda (x) (sqrt\text{-}\text{loop} 1.0 x)) 2)\]
(sqrt 2)
((lambda (x) (sqrt-loop 1.0 x)) 2)
(sqrt-loop 1.0 2)
(sqrt 2)
((lambda (x) (sqrt-loop 1.0 x)) 2)
(sqrt-loop 1.0 2)
((lambda (guess x)
   (if (close-enough? guess x)
       guess
       (sqrt-loop (improve guess x) x))) 1.0 2)
(sqrt 2)
((lambda (x) (sqrt-loop 1.0 x)) 2)
(sqrt-loop 1.0 2)
((lambda (guess x)  
  (if (close-enough? guess x)  
    guess  
    (sqrt-loop (improve guess x) x))) 1.0 2)
(if (close-enough? 1.0 2)  
  1.0  
  (sqrt-loop (improve 1.0 2) 2))
(sqrt 2)
((lambda (x) (sqrt-loop 1.0 x)) 2)
(sqrt-loop 1.0 2)
((lambda (guess x)
   (if (close-enough? guess x)
       guess
       (sqrt-loop (improve guess x) x))) 1.0 2)
(if (close-enough? 1.0 2)
  1.0
  (sqrt-loop (improve 1.0 2) 2))
(sqrt-loop (improve 1.0 2) 2)
(sqrt 2)
(((lambda (x) (sqrt-loop 1.0 x)) 2)
(sqrt-loop 1.0 2)
(((lambda (guess x)
   (if (close-enough? guess x)
       guess
       (sqrt-loop (improve guess x) x))) 1.0 2)
(if (close-enough? 1.0 2)
  1.0
  (sqrt-loop (improve 1.0 2) 2))
(sqrt-loop (improve 1.0 2) 2)
(sqrt-loop ((lambda (a b) (/ (+ a b) 2)) 1.0 2) 2)
(sqrt 2)
((lambda (x) (sqrt-loop 1.0 x)) 2)
(sqrt-loop 1.0 2)
((lambda (guess x)
    (if (close-enough? guess x)
        guess
        (sqrt-loop (improve guess x) x))) 1.0 2)
(if (close-enough? 1.0 2)
    1.0
    (sqrt-loop (improve 1.0 2) 2))
(sqrt-loop (improve 1.0 2) 2)
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(sqrt 2)
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    1.0
    (sqrt-loop (improve 1.0 2) 2))
(sqrt-loop (improve 1.0 2) 2)
(sqrt-loop ((lambda (a b) (/ (+ a b) 2)) 1.0 2) 2)
(sqrt-loop (/ (+ 1.0 2) 2) 2)
(sqrt 2)
((lambda (x) (sqrt-loop 1.0 x)) 2)
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(sqrt-loop (improve 1.0 2) 2)
(sqrt-loop ((lambda (a b) (/ (+ a b) 2)) 1.0 2) 2)
(sqrt-loop (/ (+ 1.0 2) 2) 2)
(sqrt-loop 1.5 2)
(sqrt 2)
((lambda (x) (sqrt-loop 1.0 x)) 2)
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  1.0
  (sqrt-loop (improve 1.0 2) 2))
(sqrt-loop (improve 1.0 2) 2)
(sqrt-loop ((lambda (a b) (/ (+ a b) 2)) 1.0 2) 2)
(sqrt-loop (/ (+ 1.0 2) 2) 2)
(sqrt-loop 1.5 2)
...
(sqrt-loop 1.4166 2)
...
A canonical example

- Compute $n$ factorial, defined as:
  $$n! = n(n - 1)(n - 2)(n - 3) \ldots 1$$
- How can we capture this in a procedure, using the idea of finding a common pattern?
Recursive algorithms

1. Wishful thinking
2. Decompose the problem
3. Identify non-decomposable (smallest) problems
Recursive algorithms

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2. Decompose the problem
3. Identify non-decomposable (smallest) problems

Wishful thinking
- Assume the desired procedure exists
Recursive algorithms

1. Wishful thinking
2. Decompose the problem
3. Identify non-decomposable (smallest) problems

Wishful thinking
- Assume the desired procedure exists
- Want to implement $\text{factorial}$? Assume it exists.
Wishful thinking

1. Wishful thinking
2. Decompose the problem
3. Identify non-decomposable (smallest) problems

Wishful thinking

- Assume the desired procedure exists
- Want to implement \texttt{factorial}? Assume it exists.
- But, it only solves a smaller version of the problem
Recursive algorithms

1. Wishful thinking
2. Decompose the problem
3. Identify non-decomposable (smallest) problems

**Wishful thinking**
- Assume the desired procedure exists
- Want to implement $\text{factorial}$? Assume it exists.
- But, it only solves a smaller version of the problem
- This is just finding the common pattern; but here, solving the bigger problem involves the same pattern in a smaller problem
Recursive algorithms

1. Wishful thinking
2. Decompose the problem
3. Identify non-decomposable (smallest) problems

Decompose the problem
- Solve a smaller instance
Recursive algorithms

1. Wishful thinking
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3. Identify non-decomposable (smallest) problems

Decompose the problem
- Solve a smaller instance
- Convert that solution into desired solution

\[ n! = n(n-1)(n-2) \ldots = n[(n-1)(n-2) \ldots] = n \times (n-1)! \]
Recursive algorithms

1. Wishful thinking
2. Decompose the problem
3. Identify non-decomposable (smallest) problems

Decompose the problem

- Solve a smaller instance
- Convert that solution into desired solution

\[ n! = n(n-1)(n-2) \ldots = n[(n-1)(n-2) \ldots] = n \times (n-1)! \]

(define fact (lambda (n) (* n (fact (- n 1)))))
(define fact
  (lambda (n) (* n (fact (- n 1)))))
(define fact
  (lambda (n) (* n (fact (- n 1)))))

(fact 2)
(define fact
    (lambda (n) (* n (fact (- n 1))))
)

(fact 2)
(* 2 (fact 1))
(define fact
  (lambda (n) (* n (fact (- n 1)))))

(fact 2)
(* 2 (fact 1))
(* 2 (* 1 (fact 0)))
(define fact
    (lambda (n) (* n (fact (- n 1))))))

(fact 2)
(* 2 (fact 1))
(* 2 (* 1 (fact 0)))
(* 2 (* 1 (* 0 (fact -1)))))
(define fact
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(fact 2)
(* 2 (fact 1))
(* 2 (* 1 (fact 0)))
(* 2 (* 1 (* 0 (fact -1))))
(* 2 (* 1 (* 0 (* -1 (fact -2))))))
(define fact
    (lambda (n) (* n (fact (- n 1))))))

(fact 2)
(* 2 (fact 1))
(* 2 (* 1 (fact 0)))
(* 2 (* 1 (* 0 (fact -1))))
(* 2 (* 1 (* 0 (* -1 (fact -2)))))
::
Recursive algorithms

1. Wishful thinking
2. Decompose the problem
3. Identify non-decomposable (smallest) problems

Identify non-decomposable problems
   - Must identify the “smallest” problems and solve explicitly
Recursive algorithms

1. Wishful thinking
2. Decompose the problem
3. Identify non-decomposable (smallest) problems

Identify non-decomposable problems
- Must identify the “smallest” problems and solve explicitly
- Define 1! to be 1
Recursive algorithms

Have a test, a base case, and a recursive case

(\texttt{define fact}
 \begin{align*}
 (\texttt{lambda} \ (n) \\
 (\texttt{if} \ (= \ n \ 1) \\
 1 \ \\
 (* \ n \ (\texttt{fact} \ (- \ n \ 1)))))))
\end{align*}
)
Recursive algorithms

- Have a test, a base case, and a recursive case

```scheme
(define fact
  (lambda (n)
    (if (= n 1)
      1
      (* n (fact (- n 1))))))
```
Recursive algorithms

- Have a test, a **base case**, and a recursive case

\[
\text{(define fact}
\quad \text{(lambda (n)}
\quad \text{(if (= n 1)}
\quad \quad \quad \quad 1
\quad \quad \quad \quad 1
\quad \quad \quad \quad \text{(\*) n (fact (- n 1)))})\})
\]

More complex algorithms may have multiple base cases or multiple recursive cases.
Recursive algorithms

- Have a test, a base case, and a recursive case

```
(define fact
  (lambda (n)
    (if (= n 1)
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      (* n (fact (- n 1))))))
```
Recursive algorithms

- Have a test, a base case, and a recursive case
  
  \[
  \text{(define fact}
  
  \quad \text{(lambda (n)}
  
  \quad \text{(if (= n 1)}
  
  \quad \quad \text{1)}
  
  \quad \quad \text{1)}
  
  \quad \quad \text{(* n (fact (- n 1))))))}
  
  \]

- More complex algorithms may have multiple base cases or multiple recursive cases
(define fact (lambda (n)
  (if (= n 1) 1 (* n (fact (- n 1))))))

(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
(* 3 (fact (- 3 1))
(* 3 (fact 2))
(* 3 (* 2 (fact (- 2 1))))
(* 3 (* 2 (fact 1))
(* 3 (* 2 1))
(* 3 2)
6)
(define fact (lambda (n)
  (if (= n 1) 1 (* n (fact (- n 1))))))

(fact 3)
(define fact (lambda (n)
    (if (= n 1) 1 (* n (fact (- n 1)))))

(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1)))))
(define fact (lambda (n)
    (if (= n 1) 1 (* n (fact (- n 1))))))

(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
(if #f 1 (* 3 (fact (- 3 1))))
(define fact (lambda (n)
  (if (= n 1) 1 (* n (fact (- n 1))))))

(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
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(* 3 (fact (- 3 1)))
(define fact (lambda (n)
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(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
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(* 3 (fact (- 3 1)))
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(define fact (lambda (n)
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(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
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(* 3 (fact (- 3 1))
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(* 3 (if (= 2 1) 1 (* 2 (fact (- 2 1))))))
(define fact (lambda (n)
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(fact 3)
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(* 3 (fact (- 3 1))
(* 3 (fact 2))
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(define fact (lambda (n)
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(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
(if #f 1 (* 3 (fact (- 3 1))))
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(* 3 (fact 2))
(* 3 (if (= 2 1) 1 (* 2 (fact (- 2 1)))))
(* 3 (if #f 1 (* 2 (fact (- 2 1)))))
(* 3 (* 2 (fact (- 2 1))))
(* 3 (* 2 (fact 1)))
(define fact (lambda (n)
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(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
(if #f 1 (* 3 (fact (- 3 1))))
(* 3 (fact (- 3 1)))
(* 3 (fact 2))
(* 3 (if (= 2 1) 1 (* 2 (fact (- 2 1)))))
(* 3 (if #f 1 (* 2 (fact (- 2 1)))))
(* 3 (* 2 (fact (- 2 1))))
(* 3 (* 2 (fact 1)))
(* 3 (* 2 (if (= 1 1) 1 (* 1 (fact (- 1 1))))))
(define fact (lambda (n)
    (if (= n 1) 1 (* n (fact (- n 1))))))

(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
(if #f 1 (* 3 (fact (- 3 1))))
(* 3 (fact (- 3 1)))
(* 3 (fact 2))
(* 3 (if (= 2 1) 1 (* 2 (fact (- 2 1))))))
(* 3 (if #f 1 (* 2 (fact (- 2 1))))))
(* 3 (* 2 (fact (- 2 1))))
(* 3 (* 2 (fact 1)))
(* 3 (* 2 (if (= 1 1) 1 (* 1 (fact (- 1 1)))))))
(* 3 (* 2 (if #t 1 (* 1 (fact (- 1 1)))))))
(define fact (lambda (n)
    (if (= n 1) 1 (* n (fact (- n 1))))))

(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
(if #f 1 (* 3 (fact (- 3 1))))
(* 3 (fact (- 3 1)))
(* 3 (fact 2))
(* 3 (if (= 2 1) 1 (* 2 (fact (- 2 1))))))
(* 3 (if #f 1 (* 2 (fact (- 2 1))))))
(* 3 (* 2 (fact (- 2 1))))
(* 3 (* 2 (fact 1)))
(* 3 (* 2 (if (= 1 1) 1 (* 1 (fact (- 1 1))))))
(* 3 (* 2 (if #t 1 (* 1 (fact (- 1 1))))))
(* 3 (* 2 1))
(define fact (lambda (n)
    (if (= n 1) 1 (* n (fact (- n 1))))))

(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
(if #f 1 (* 3 (fact (- 3 1))))
(* 3 (fact (- 3 1)))
(* 3 (fact 2))
(* 3 (if (= 2 1) 1 (* 2 (fact (- 2 1)))))
(* 3 (if #f 1 (* 2 (fact (- 2 1)))))
(* 3 (* 2 (fact (- 2 1))))
(* 3 (* 2 (fact 1)))
(* 3 (* 2 (if (= 1 1) 1 (* 1 (fact (- 1 1))))))
(* 3 (* 2 (if #t 1 (* 1 (fact (- 1 1))))))
(* 3 (* 2 1))
(* 3 2)
(define fact (lambda (n)
    (if (= n 1) 1 (* n (fact (- n 1)))))

(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
(if #f 1 (* 3 (fact (- 3 1))))
(* 3 (fact (- 3 1)))
(* 3 (fact 2))
(* 3 (if (= 2 1) 1 (* 2 (fact (- 2 1)))))
(* 3 (if #f 1 (* 2 (fact (- 2 1)))))
(* 3 (* 2 (fact (- 2 1))))
(* 3 (* 2 (fact 1)))
(* 3 (* 2 (if (= 1 1) 1 (* 1 (fact (- 1 1))))))
(* 3 (* 2 (if #t 1 (* 1 (fact (- 1 1))))))
(* 3 (* 2 1))
(* 3 2)
6
(define fact (lambda (n)
    (if (= n 1) 1 (* n (fact (- n 1))))))

(fact 3)
(if (= 3 1) 1 (* 3 (fact (- 3 1))))
(if #f 1 (* 3 (fact (- 3 1))))
(* 3 (fact (- 3 1)))

(* 3 (fact 2))
(* 3 (if (= 2 1) 1 (* 2 (fact (- 2 1))))))
(* 3 (if #f 1 (* 2 (fact (- 2 1))))))
(* 3 (* 2 (fact (- 2 1))))

(* 3 (* 2 (fact 1)))
(* 3 (* 2 (if (= 1 1) 1 (* 1 (fact (- 1 1)))))
(* 3 (* 2 (if #t 1 (* 1 (fact (- 1 1)))))
(* 3 (* 2 1))
(* 3 2)
6
Effects of recursive algorithms

Recursive algorithms consume more space with bigger operands!

(fact 4)
Effects of recursive algorithms

Recursive algorithms consume more space with bigger operands!

\[(\text{fact } 4)\]
\[(\ast \ 4 \ (\text{fact } 3))\]
Recursive algorithms consume more space with bigger operands!

\[(\text{fact } 4)\]
\[(\ast \ 4 \ (\text{fact } 3))\]
\[(\ast \ 4 \ (\ast \ 3 \ (\text{fact } 2)))\]
Effects of recursive algorithms

Recursive algorithms consume more space with bigger operands!

\[(\text{fact } 4)\]
\[(\ast \ 4 \ (\text{fact } 3))\]
\[(\ast \ 4 \ (\ast \ 3 \ (\text{fact } 2)))\]
\[(\ast \ 4 \ (\ast \ 3 \ (\ast \ 2 \ (\text{fact } 1))))\]
Effects of recursive algorithms

Recursive algorithms consume more space with bigger operands!

(fact 4)
(* 4 (fact 3))
(* 4 (* 3 (fact 2)))
(* 4 (* 3 (* 2 (fact 1))))
(* 4 (* 3 (* 2 1)))
Effects of recursive algorithms

Recursive algorithms consume more space with bigger operands!

\[(\text{fact } 4)\]
\[(\ast \ 4 \ (\text{fact } 3))\]
\[(\ast \ 4 \ (\ast \ 3 \ (\text{fact } 2)))\]
\[(\ast \ 4 \ (\ast \ 3 \ (\ast \ 2 \ (\text{fact } 1))))\]
\[(\ast \ 4 \ (\ast \ 3 \ (\ast \ 2 \ 1)))\]

\[
\ldots
\]

24
Effects of recursive algorithms

Recursive algorithms consume more *space* with bigger operands!

\[(\text{fact } 8)\]
Recursive algorithms consume more space with bigger operands!

(fact 8)
(* 8 (fact 7))
Effects of recursive algorithms

Recursive algorithms consume more space with bigger operands!

(fact 8)
(* 8 (fact 7))
(* 8 (* 7 (fact 6)))
Recursive algorithms consume more space with bigger operands!

(fact 8)
(* 8 (fact 7))
(* 8 (* 7 (fact 6)))
(* 8 (* 7 (* 6 (fact 5)))))
Effects of recursive algorithms

Recursive algorithms consume more space with bigger operands!

(fact 8)
(\* 8 (fact 7))
(\* 8 (\* 7 (fact 6)))
(\* 8 (\* 7 (\* 6 (fact 5))))
...

Recursive algorithms consume more space with bigger operands!

\[(\text{fact } 8)\]
\[(*\ 8\ (\text{fact } 7))\]
\[(*\ 8\ (*\ 7\ (\text{fact } 6)))\]
\[(*\ 8\ (*\ 7\ (*\ 6\ (\text{fact } 5))))\]
\[
...\]
\[(*\ 8\ (*\ 7\ (*\ 6\ (*\ 5\ (*\ 4\ (*\ 3\ (*\ 2\ (\text{fact } 1))))))))\]
Recursive algorithms consume more space with bigger operands!

(fact 8)
(* 8 (fact 7))
(* 8 (* 7 (fact 6)))
(* 8 (* 7 (* 6 (fact 5))))
...
(* 8 (* 7 (* 6 (* 5 (* 4 (* 3 (* 2 (fact 1))))))))
(* 8 (* 7 (* 6 (* 5 (* 4 (* 3 (* 2 1)))))))
Recursive algorithms consume more space with bigger operands!

(fact 8)
(* 8 (fact 7))
(* 8 (* 7 (fact 6)))
(* 8 (* 7 (* 6 (fact 5))))
...
(* 8 (* 7 (* 6 (* 5 (* 4 (* 3 (* 2 (fact 1))))))))
(* 8 (* 7 (* 6 (* 5 (* 4 (* 3 (* 2 1)))))))
(* 8 (* 7 (* 6 (* 5 (* 4 (* 3 2))))))
Effects of recursive algorithms

Recursive algorithms consume more space with bigger operands!

```
(fact 8)
(* 8 (fact 7))
(* 8 (* 7 (fact 6)))
(* 8 (* 7 (* 6 (fact 5))))
...
(* 8 (* 7 (* 6 (* 5 (* 4 (* 3 (* 2 (fact 1)))))))))
(* 8 (* 7 (* 6 (* 5 (* 4 (* 3 (* 2 1)))))))))
(* 8 (* 7 (* 6 (* 5 (* 4 (* 3 2)))))))
...
```
Effects of recursive algorithms

Recursive algorithms consume more space with bigger operands!

\[(\text{fact } 8)\]
\[(* \ 8 \ (\text{fact } 7))\]
\[(* \ 8 \ (* \ 7 \ (\text{fact } 6)))\]
\[(* \ 8 \ (* \ 7 \ (* \ 6 \ (\text{fact } 5))))\]
\[
\ldots
\]
\[(* \ 8 \ (* \ 7 \ (* \ 6 \ (* \ 5 \ (* \ 4 \ (* \ 3 \ (* \ 2 \ (\text{fact } 1))))))))\]
\[(* \ 8 \ (* \ 7 \ (* \ 6 \ (* \ 5 \ (* \ 4 \ (* \ 3 \ (* \ 2 \ 1))))))\]
\[(* \ 8 \ (* \ 7 \ (* \ 6 \ (* \ 5 \ (* \ 4 \ (* \ 3 \ 2))))))\]
\[
\ldots
\]
40320
An alternative

- Try computing $101!$
  \[101 \times 100 \times 99 \times 98 \times 97 \times 96 \times \ldots \times 2 \times 1\]
- How much space do we consume with pending operations?
An alternative

- Try computing 101!
  \[101 \times 100 \times 99 \times 98 \times 97 \times 96 \times \ldots \times 2 \times 1\]

- How much space do we consume with pending operations?
- Better idea: count up, doing one multiplication at a time
  - Start with 1 as the answer
An alternative

- Try computing 101!
  \[101 \times 100 \times 99 \times 98 \times 97 \times 96 \times \ldots \times 2 \times 1\]

- How much space do we consume with pending operations?

- Better idea: count up, doing one multiplication at a time
  - Start with 1 as the answer
  - Multiply by 2, store 2 as the current answer, remember we’ve done up to 2
An alternative

- Try computing $101!$
  $101 \times 100 \times 99 \times 98 \times 97 \times 96 \times \ldots \times 2 \times 1$

- How much space do we consume with pending operations?
- Better idea: count up, doing one multiplication at a time
  - Start with 1 as the answer
  - Multiply by 2, store 2 as the current answer, remember we’ve done up to 2
  - Multiply by 3, store 6, remember we’re done up to 3
An alternative

- Try computing 101!
  \[101 \times 100 \times 99 \times 98 \times 97 \times 96 \times \cdots \times 2 \times 1\]

- How much space do we consume with pending operations?

- Better idea: count up, doing one multiplication at a time
  - Start with 1 as the answer
  - Multiply by 2, store 2 as the current answer, remember we’ve done up to 2
  - Multiply by 3, store 6, remember we’re done up to 3
  - Multiply by 4, store 24, remember we’re done up to 4
An alternative

- Try computing 101!
  \[101 \times 100 \times 99 \times 98 \times 97 \times 96 \times \ldots \times 2 \times 1\]

- How much space do we consume with pending operations?

- Better idea: count up, doing one multiplication at a time
  - Start with 1 as the answer
  - Multiply by 2, store 2 as the current answer, remember we’ve done up to 2
  - Multiply by 3, store 6, remember we’re done up to 3
  - Multiply by 4, store 24, \textit{remember we’re done up to 4}
  - \ldots
An alternative

- Try computing 101!
  - $101 \times 100 \times 99 \times 98 \times 97 \times 96 \times \ldots \times 2 \times 1$

- How much space do we consume with pending operations?
- Better idea: count up, doing one multiplication at a time
  - Start with 1 as the answer
  - Multiply by 2, store 2 as the current answer, remember we’ve done up to 2
  - Multiply by 3, store 6, remember we’re done up to 3
  - Multiply by 4, store 24, remember we’re done up to 4
  - ...
  - Multiply by 101, get
    - 9425947759838359420851623124482936749562
    - 312794702543768327889353416977599316221476503087
    - 861591808346911623490003549599583369706302603264
    - 00000000000000000000000000
  - Realize we’re done up to the number we want, and stop
An alternative

- Try computing 101!
  \[ 101 \times 100 \times 99 \times 98 \times 97 \times 96 \times \ldots \times 2 \times 1 \]

- How much space do we consume with pending operations?
- Better idea: count up, doing one multiplication at a time
  - Start with 1 as the answer
  - Multiply by 2, store 2 as the current answer, remember we’ve done up to 2
  - Multiply by 3, store 6, remember we’re done up to 3
  - Multiply by 4, store 24, remember we’re done up to 4
  - \ldots
  - Multiply by 101, get
    \[ 9425947759838359420851623124482936749562 \]
    \[ 312794702543768327889353416977599316221476503087 \]
    \[ 861591808346911623490003549599583369706302603264 \]
    \[ 00000000000000000000000000000000 \]
  - Realize we’re done up to the number we want, and stop
- This is an iterative algorithm – it uses constant space
Iterative algorithms as tables

<table>
<thead>
<tr>
<th>product</th>
<th>done</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

- **First row** handles $1!$ cleanly
Iterative algorithms as tables

<table>
<thead>
<tr>
<th>product</th>
<th>done</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

- First row handles 1! cleanly
Iterative algorithms as tables

<table>
<thead>
<tr>
<th>product</th>
<th>done</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

- First row handles 1! cleanly
- **product** becomes
  - product * (done + 1)
Iterative algorithms as tables

<table>
<thead>
<tr>
<th>product</th>
<th>done</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>24</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

- First row handles 1! cleanly
- \textbf{product} becomes \textbf{product} * (\textbf{done} + 1)
- \textbf{done} becomes \textbf{done} + 1
Iterative algorithms as tables

<table>
<thead>
<tr>
<th>product</th>
<th>done</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>24</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>120</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

- First row handles 1! cleanly
- **product** becomes $product \times (done + 1)$
- **done** becomes $done + 1$
- The answer is **product** when $done = max$
(define (ifact-helper product done max)

)

The helper has one argument per column
(define (ifact n) (ifact-helper 1 1 n))

(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1))
                    (+ done 1)
                    max))
)

- The helper has one argument per column
- Which is called by ifact
(define (ifact n) (ifact-helper 1 1 n))

(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1)) (+ done 1) max)))

- The helper has one argument per column
- Which is called by \texttt{ifact}
- Which provides the \textit{values for the first row}
(define (ifact n) (ifact-helper 1 1 n))

(define (ifact-helper product done max)

  (ifact-helper (* product (+ done 1))
                (+ done 1)
                max))

- The helper has one argument per column
- Which is called by \texttt{ifact}
- Which provides the values for the first row
- The recursive call to \texttt{ifact-helper}
(define (ifact n) (ifact-helper 1 1 n))

(define (ifact-helper product done max)

    (ifact-helper (* product (+ done 1))
    (+ done 1)
    max) )

- The helper has one argument per column
- Which is called by ifact
- Which provides the values for the first row
- The recursive call to ifact-helper computes the next row
(define (ifact n) (ifact-helper 1 1 n))

(define (ifact-helper product done max)
  (if (= done max)

    (ifact-helper (* product (+ done 1))
     (+ done 1)
     max))

- The helper has one argument per column
- Which is called by ifact
- Which provides the values for the first row
- The recursive call to ifact-helper computes the next row
- And the if statement checks the end condition
(define (ifact n) (ifact-helper 1 1 n))

(define (ifact-helper product done max)
  (if (= done max) product
    (ifact-helper (* product (+ done 1))
      (+ done 1)
      max)))

- The helper has one argument per column
- Which is called by ifact
- Which provides the values for the first row
- The recursive call to ifact-helper computes the next row
- And the if statement checks the end condition and output value
(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1))
          (+ done 1)
          max))))
(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1))
                    (+ done 1)
                    max)))

(ifact 4)
(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1))
                    (+ done 1)
                    max)))

(ifact 4)
(ifact-helper 1 1 4)
(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1))
                    (+ done 1)
                    max)))

(ifact 4)
(ifact-helper 1 1 4)
(if (= 1 4) 1 (ifact-helper (* 1 (+ 1 1)) (+ 1 1) 4))
(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1))
        (+ done 1)
        max)))

(ifact 4)
(ifact-helper 1 1 4)
(if (= 1 4) 1 (ifact-helper (* 1 (+ 1 1)) (+ 1 1) 4))
(ifact-helper 2 2 4)
(if (= 2 4) 2 (ifact-helper (* 2 (+ 2 1)) (+ 2 1) 4))
(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1))
                   (+ done 1)
                   max)))

(ifact 4)
(ifact-helper 1 1 4)
(if (= 1 4) 1 (ifact-helper (* 1 (+ 1 1)) (+ 1 1) 4))
(ifact-helper 2 2 4)
(if (= 2 4) 2 (ifact-helper (* 2 (+ 2 1)) (+ 2 1) 4))
(ifact-helper 6 3 4)
(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1))
        (+ done 1)
        max)))

(ifact 4)
(ifact-helper 1 1 4)
(if (= 1 4) 1 (ifact-helper (* 1 (+ 1 1)) (+ 1 1) 4))
(ifact-helper 2 2 4)
(if (= 2 4) 2 (ifact-helper (* 2 (+ 2 1)) (+ 2 1) 4))
(ifact-helper 6 3 4)
(if (= 3 4) 6 (ifact-helper (* 6 (+ 3 1)) (+ 3 1) 4))
(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1)) (+ done 1) max)))

(ifact 4)
(ifact-helper 1 1 4)
(if (= 1 4) 1 (ifact-helper (* 1 (+ 1 1)) (+ 1 1) 4))
(ifact-helper 2 2 4)
(if (= 2 4) 2 (ifact-helper (* 2 (+ 2 1)) (+ 2 1) 4))
(ifact-helper 6 3 4)
(if (= 3 4) 6 (ifact-helper (* 6 (+ 3 1)) (+ 3 1) 4))
(ifact-helper 24 4 4)
(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1))
                    (+ done 1)
                    max)))

(ifact 4)
(ifact-helper 1 1 4)
(if (= 1 4) 1  (ifact-helper (* 1 (+ 1 1)) (+ 1 1) 4))
(ifact-helper 2 2 4)
(if (= 2 4) 2  (ifact-helper (* 2 (+ 2 1)) (+ 2 1) 4))
(ifact-helper 6 3 4)
(if (= 3 4) 6  (ifact-helper (* 6 (+ 3 1)) (+ 3 1) 4))
(ifact-helper 24 4 4)
(if (= 4 4) 24  (ifact-helper (* 24 (+ 4 1)) (+ 4 1) 4))
(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1)) (+ done 1) max)))

(ifact 4)
(ifact-helper 1 1 4)
(if (= 1 4) 1 (ifact-helper (* 1 (+ 1 1)) (+ 1 1) 4))
(ifact-helper 2 2 4)
(if (= 2 4) 2 (ifact-helper (* 2 (+ 2 1)) (+ 2 1) 4))
(ifact-helper 6 3 4)
(if (= 3 4) 6 (ifact-helper (* 6 (+ 3 1)) (+ 3 1) 4))
(ifact-helper 24 4 4)
(if (= 4 4) 24 (ifact-helper (* 24 (+ 4 1)) (+ 4 1) 4))
24
(define (ifact-helper product done max)
  (if (= done max)
      product
      (ifact-helper (* product (+ done 1))
                    (+ done 1)
                    max)))

(ifact 4)
(ifact-helper 1 1 4)
(if (= 1 4) 1 (ifact-helper (* 1 (+ 1 1)) (+ 1 1) 4))
(ifact-helper 2 2 4)
(if (= 2 4) 2 (ifact-helper (* 2 (+ 2 1)) (+ 2 1) 4))
(ifact-helper 6 3 4)
(if (= 3 4) 6 (ifact-helper (* 6 (+ 3 1)) (+ 3 1) 4))
(ifact-helper 24 4 4)
(if (= 4 4) 24 (ifact-helper (* 24 (+ 4 1)) (+ 4 1) 4))
24
Recursive algorithms have pending operations

**Recursive factorial:**

\[
\text{(define (fact n)} \\
\quad (\text{if (= n 1) 1)} \\
\quad \quad (* n (fact (- n 1))))
\]

\[
\text{(fact 4)} \\
\quad (* 4 (fact 3)) \\
\quad (* 4 (* 3 (fact 2))) \\
\quad (* 4 (* 3 (* 2 (fact 1))))
\]

**Pending operations make the expression grow continuously.**
Iterative algorithms have no pending operations

- **Iterative factorial:**

  (define (ifact n) (ifact-helper 1 1 n))
  (define (ifact-helper product done max)
    (if (= done max)
        product
        (ifact-helper (* product (+ done 1))
                      (+ done 1)
                      max)))

  (ifact-helper 1 1 4)
  (ifact-helper 2 2 4)
  (ifact-helper 6 3 4)
  (ifact-helper 24 4 4)

- **Fixed space because no pending operations**
Iterative processes

- Iterative algorithms have constant space

To develop an iterative algorithm:

1. Figure out a way to accumulate partial answers
2. Write out a table to analyze:
   - initialization of first row
   - update rules for other rows
   - how to know when to stop
3. Translate rules into Scheme

- Iterative algorithms have no pending operations
Lambdas allow us to create procedures which capture processes
Procedural abstraction creates building blocks for complex processes
Recursive algorithms capitalize on “wishful thinking” to reduce problems to smaller subproblems
Iterative algorithms similarly reduce problems, but based on data you can express in tabular form
Reminders

- Project 0 is due Thursday
- Submit to 6.037-psets@mit.edu
- http://web.mit.edu/alexmv/6.037/
- E-mail: 6.001-zombies@mit.edu