Evaluation question solutions

Abstract signals in MATLAB and SCILAB

Numerical solutions of some problems are written using a specific language for LTI system modeling, available as part of the smiqc package for MATLAB, available from A. Megretski’s Web site. The package is designed with a complete SCILAB compatibility in mind, though the corresponding SCILAB code is not available at the moment. Here is a brief description of basic smiqc commands.

The main data type in smiqc is “abstract signal” (asgn). An abstract signal \( f \) refers to a sub-vector of the output \( y \), which is related to input \( w \) and state \( x \) according to

\[
y = Cx + Dw, \quad \dot{x} = Ax + Bw.
\]

Matrices \( A, B, C, D \) are stored as hidden global variables, and are updated every time an operation is performed on abstract signals.

- **newmodel**  Enable handling of abstract signals or start a new LTI model from scratch by setting \( A, B, C, D \) to their initial values.
  
  Example: `newmodel`;

- **newsignal**  Define \( f \) as an abstract input signal of dimension \( n \). Accordingly, \( w \) and \( y \) are replaced by \( w_{new} = [w; f] \), \( y_{new} = [y; f] \).
  
  Example: `f=newmodel(n)`;

- **size**  Dimension of signal \( f \).
  
  Example: `n=size(f)`;

- **<=**  Extract the LTI transformation from \( f \) to \( g \). \( f \) must have an inner representation \( f = C_f x + D_f w \) with a left invertible \( D_f \). The components of \( w \) which are orthogonal to the rows of \( D_f \) are assumed to be zero in this operation.
  
  Example: `G=(f<=g)`;

- **+, -, [·, ·]**  Sum, difference, and concatenation of signals.
  
  Example: `y=[f+g; f-g]`;

- **∗**  LTI transformation of signals (the first argument is a matrix, a transfer matrix, or an LTI model).
  
  Example: `g=2*f`;
• **==** Forces an equality constraint between two signals. The difference $\delta$ of signals must have an inner representation $\delta = C_\delta x + D_\delta w$ with a left invertible $D_\delta$.

**Example:** $g == f$;

• **h2hi** H2 and H-Infinity optimization, with four abstract signal input arguments representing disturbance, control, cost, and measurement.

**Example:** $[k_2, k_i, g_2, g_i] = h2hi(w, u, e, y)$;

• **slap** $n$-dimensional “Laplace” $s$, introduced for compatibility with SCILAB.

**Example:** $s = slap(1)$;

• **minlti** Minimal realization of LTI system, introduced for compatibility with SCILAB.

**Example:** $H = minlti(G)$;

In particular, the MATLAB code

```matlab
newmodel; f = newsignal(1); e = newsignal(1); e == 2*f -(1/s)*e; tf(e <= f)
```

should return transfer function $2s/(s + 1)$.

**Question 1.**

**For all values of parameter $a \in \mathbb{R}$, find L2 gain of system**

$$f(t) \mapsto y(t) = f(t) - |f(t - a)|.$$  

**Answer:** for $a \geq 0$, the L2 gain equals 2; for $a < 0$, the L2 gain is $+\infty$.

**Proof.** Since $(a + b)^2 \leq 2(a^2 + b^2)$ for all $a, b$,

$$|f(t) - |f(t - a)||^2 \leq |f(t)|^2 + |f(t - a)|^2.$$  

for $a \geq 0$ this yields

$$\int_0^T \{4|f(t)|^2 - |y(t)|^2\} dt \geq \int_0^T \{2|f(t)|^2 - 2|f(t - a)|^2\} dt \geq - \int_{-a}^0 |f(t)|^2 dt > -\infty.$$  

Hence L2 gain does not exceed 2 for $a \geq 0$. On the other hand, input $f(t) \equiv -1$ produces output $y(t) \equiv -2$, which shows that L2 gain is not smaller than 2.

For $a < 0$, input $f(t) = e^{rt}$ produces output $y(t) = (1 - e^{ra})e^{rt}$. Since $|1 - e^{ra}| \to \infty$ as $r \to +\infty$, the L2 gain is infinite.
Question 2.

Systems \( G \) and \( \Delta \) on block diagram on Figure 1 have \( L_2 \) gains not larger than \( a \geq 0 \). Depending on the value of \( a \), what is the largest possible \( L_2 \) gain from \( f \) to \( y \)?

![Block diagram for Problems 2 and 6.](image)

**Answer:** when \( a < 1 \), the largest possible gain is \( a(1 - a^2)^{-1} \); otherwise, the largest gain is infinity.

**Proof.** For \( a < 1 \), considering the case when \( G \) and \( \Delta \) are LTI systems with constant transfer functions \( G(s) = \Delta(s) = a \) shows that \( a(1 - a^2)^{-1} \) is a lower bound for the answer. When \( a \geq 1 \), using \( G(s) = \Delta(s) = 1 - \epsilon \), where \( \epsilon > 0 \) and \( \epsilon \to 0 \) shows that the only upper bound for the gain is plus infinity.

To show that the gain cannot be larger than \( a(1 - a^2)^{-1} \) consider systems \( G_1 \) and \( \Delta_1 \), where \( G_1 \) maps inputs \( [f_1; w] \) to outputs \( [y_1; v] = [c_3G(c_1f_1 + c_2w); c_4G(c_1f_1 + c_2w)] \), and \( \Delta_1 \) maps \( v \) to \( w = c_2^{-1}\Delta(c_1^{-1}v) \) (the dashed boxes on Figure 2), where \( c_i > 0 \) are parameters.

![Re-scaling in Problem 1.2](image)

It is easy to see that \( L_2 \) power gain of \( G_1 \) is not larger than \( (c_1^2 + c_2^2)^{1/2}(c_3^3 + c_4^3)^{1/2}a \), and power gain of \( \Delta_1 \) is not larger than \( (c_2c_4)^{-1}a \). Therefore, according to the small gain theorem,
L2 power gain in the original system is not larger than \((c_1 c_3)^{-1}\), provided that 
\[c_2 c_4 > a, \quad (c_1^2 + c_2^2)^{1/2} (c_1^3 + c_4^2)^{1/2} < a^{-1} \]
Using \(c_2 = c_4 = b^{1/2}\) and \(c_1 = c_3 = (b^{-1} - b)^{1/2}\), where \(b > a\), shows that the L2 power gain is not larger than \((b^{-1} - b)^{-1}\). As \(b \to a + 0\), this yields the desired upper bound.

**Question 3.**

**Matrices** \(A, B, C\) **are such that**
\[C(sI - A)^{-1} B = (s + 2)^{-200},\]
and \(A\) is a Hurwitz matrix. **For which values of** \(\gamma > 0\) **does there exist a stabilizing solution** \(P = P'\) **of the Riccati equation**
\[C' C + PA + A' P + \gamma^{-1} P BB' P = 0?\]

**Answer:** \(\gamma > 2^{-400}\).

**Proof.** According to the KYP Lemma, the Riccati equation, which can be re-written as 
\[-C' C + P(-A) + (-A)' P = P B \gamma^{-1} B' P,\]
has a stabilizing solution if and only if for all \(\omega \in \mathbb{R}\) the Hermitian form 
\[\sigma(x, w) = \gamma |w|^2 - |Cx|^2\]
is positive definite on the linear subspace defined by 
\[j \omega x = -Ax + Bw.\]
Equivalently, the inequality \(|G(j\omega)|^2 < \gamma\) must be satisfied for all \(\omega\). Since H-Infinity norm of \(G\) is \(2^{-200}\), a stabilizing solution \(P\) exists if and only if \(\gamma > 2^{-400}\).

**Question 4.**

**For all values of parameter** \(a \in \mathbb{R}\) **such that the standard feedback optimization setup**
\[
\begin{align*}
\dot{x} &= ax + u + w_1, \\
z &= x + u, \\
y &= x + w_1 + w_2
\end{align*}
\]
is not singular, find an explicit formula for the H2 optimal feedback controller.

Answer:

\[ u = -x_f, \quad \dot{x}_f = ax_f + u - (a + \sqrt{a^2 - a + 0.5})(x_f - y) \]

for \( a < 1 \), and

\[ u = (1 - 2a)x_f, \quad \dot{x}_f = ax_f + u - (a + \sqrt{a^2 - a + 0.5})(x_f - y) \]

for \( a > 1 \); the problem has a control singularity at \( \omega = 0 \) when \( a = 1 \).

Proof. The abstract H2 optimization problem associated with the full information control part of the setup has the form

\[
\int_0^\infty |x + u|^2 dt \to \min, \quad \text{subject to} \\
\dot{x} = ax + u, \quad x(0) = 1, \quad \lim_{t \to \infty} x(t) = 0.
\]

The corresponding completion of squares takes the form

\[
(x + u)^2 + 2px(ax + u) = (u - Fx)^2,
\]

which yields \( F = -p - 1 \) and the Riccati equation

\[ p^2 = 2p(a - 1). \]

For \( a < 1 \), its stabilizing solution is \( p_u = 0 \), with \( F = -1 \). For \( a > 1 \), its stabilizing solution is \( p_u = 2a - 2 \), with \( F = 1 - 2a \). For \( a = 1 \), its only solution \( p = 0 \) is not a stabilizing one.

The abstract H2 optimization problem associated with the state estimation part of the setup has the form

\[
\int_0^\infty \{(\psi + v)^2 + v^2\} dt \to \min \quad \text{subject to} \\
\dot{\psi} = a\psi + v, \quad \psi(0) = F.
\]

The corresponding completion of squares takes the form

\[
(\psi + v)^2 + v^2 + 2p\psi(a\psi + v) = 2(v - Lv)^2,
\]

which yields \( L = -0.5(p + 1) \) and the Riccati equation

\[ 1 + 2(2a - 1)p = p^2. \]

Its stabilizing solution is

\[ p_y = 2a - 1 + \sqrt{4a^2 - 4a + 2}, \quad L = -a - \sqrt{a^2 - a + 0.5}. \]
The H2 optimal controller has the form

\[ u = F x_f, \quad \dot{x}_f = a x_f + u + L(x_f - y). \]

**Numerical verification.** The following MATLAB code (file q4leuven.m) uses the standard numerical optimization to check correctness of the analytical solution.

```matlab
function q4leuven(a)
% function q4leuven(a)

% checks correctness of analytical solution of Question 4/Leuven 2005

B0=a+sqrt(a^2-a+0.5);
if a<1,
    C0=-1;
    A0=-1-sqrt(a^2-a+0.5);
elseif a>1,
    C0=1-2*a;
    A0=1-2*a-sqrt(a^2-a+0.5);
else
    fprintf('q4leuven: the problem is singular
');
end
s=slap(1);
newmodel;
u=newsignal(1);
w1=newsignal(1);
w2=newsignal(1);
x=(1/(s-a))*(u+w1);
k2=h2hi([w1,w2],u,x+u,x+w1+w2);
fprintf('q4leuven: analytical')
tf(ss(A0,B0,C0,0))
fprintf('q4leuven: numerical')
tf(k2)

**Question 5.**

For all values of parameter \( a > 0 \) find Hankel norm of \( G(s) = (s + a)^{-1} \).
Answer: \((2a)^{-1}\).

**Proof.** A state space model for \(G\) has \(A = -a\), \(B = C = 1\). Accordingly, the controllability and observability Gramians satisfy \(-2aW = 1\), i.e. \(W_c = W_o = (2a)^{-1}\). Hence the Hankel norm of \(G\), being the square root of the largest eigenvalue of \(W_cW_o\), equals \((2a)^{-1}\). □

**Question 6.**

For the block diagram on Figure 1, where

\[
\Delta(s) = \frac{s - 2}{(s - 1)(s - 3)},
\]

find a stabilizing \(G = G(s)\) such that the variance of \(y\), assuming \(f\) is the normalized white noise, is minimal.

**Answer:** the optimal controller

\[
G(s) = \frac{-320s + 384}{s^2 + 12s - 253}
\]

can be found using the following MATLAB script (available in file `q6leuven.m`):

```matlab
% numerical solution of Question 6/Leuven 2005
s=slap(1);
D=(s-2)/((s-1)*(s-3));
newmodel;
F=newsignal(1);
y=newsignal(1);
[k2,ki,g2]=h2hi(f,y,y,D*y+f);
fprintf('}\nq6leuven: minimal variance \%f at',norm(g2)^2);
tf(k2)
```

The resulting variance is quite large, which is not surprising since the plant has an unpleasant combination of unstable zeros and poles.

**Question 7.**

For the block diagram on Figure 3, where

\[
H(s) = \frac{s - 1}{(s + 1)^2},
\]

7
Find a stabilizing controller $K = K(s)$ such that $L2$ gain from $r$ and $f$ to $u$ and $q$ is not larger than 100, and the closed loop bandwidth, defined as the maximal $\omega_0$ such that $|G(j\omega) - 1| \leq 0.1$ for all $\omega \in [0, \omega_0]$, is as large as possible. Here $G$ is the closed loop transfer function from $r$ to $q$.

![Block diagram for Question 7](image)

**Answer:** A bandwidth of $\omega_0 = 0.22$ is achieved by a controller produced by the MATLAB function `q7leuven.m`:

```matlab
function K=q7leuven(w0,d)
    % function K=q7leuven(w0,d)
    %
    % tries to achieve bandwidth w0 in Question 7/Leuven 2005
    if nargin<2, d=5; end
    s=slap(1);
    w1=9*w0;
    k0=sqrt(2)/abs(1+sqrt(2)*j*(w0/w1)-(w0/w1)^2);
    L=k0*(1+sqrt(2)*((s/w1)+(s/w1)^2))/(1+sqrt(2)*((s/w0)+(s/w0)^2));
    P=(s-1)/(s+2)^2;
    newmodel;
    w1=newsignal(1);
    w2=newsignal(1);
    u=newsignal(1);
    q=P*u;
    r=L*w1;
    e1=10*(q-r);
    e2=(0.1/d)*u;
    e3=(0.1/d)*q;
    y=[r;q+(d/10)*w2];
    [k2,K,g2,gi]=h2hi([w1;w2],u,[e1;e2;e3],y);
    fprintf('
q7leuven: bandwidth %f achieved if %f<1\n',w0,norm(gi,Inf));
```


Note that the system specifications practically disallow the use of $q$ measurements in feedback, so that the controller simply pre-conditions the reference input. When the plant equations are changed to unstable, as in $P(s) = (s - 1)/(s - 0.1)^2$, the achievable bandwidth remains essentially the same, but the output measurement is actively used in the feedback.

**Question 8.**

For

$$G(s) = \sum_{k=1}^{100} \frac{1}{s^2 + 2s + k/10},$$

find a good low bound of H-Infinity norm of approximation error $G - \hat{G}$, where the order of $\hat{G}$ is not larger than 10. Also, use Hankel optimal model reduction to find a good 10-th order approximation $\hat{G}$ of $G$.

11-th Hankel singular number of $G$ provides a lower bound for the quality of such approximation. Numerical calculations (file q8leuven.m show that the 10-th order Hankel optimal reduced model is very close to H-Infinity optimality.

```matlab
% script for Question 8/Leuven 2005Gbal,sig]
N=100;
A=zeros(2*N);
B=zeros(2*N,1);
C=zeros(1,2*N);
for k=1:N,
   k2=2*k;
   k1=k2-1;
   A(k1:k2,k1:k2)=[0 1;-k/10 -2];
   B(k1:k2)=[0;1];
   C(k1:k2)=[1 0];
end
G=ss(A,B,C,0);
Gnom=pck(A,B,C,0);
[Gbal,sig]=sysbal(Gnom);
```
Gred=hankmr(Gbal,sig,10,'d');
[Ar,Br,Cr,Dr]=unpck(Gred);
Gr=ss(Ar,Br,Cr,Dr);
fprintf('\nq8leuven: lower bound %d, actual %d\n',sig(11),norm(G-Gr,Inf));
ww=linspace(0,10,300);
g=squeeze(freqresp(G,ww));
gr=squeeze(freqresp(Gr,ww));
close(gcf);
subplot(2,1,1); plot(ww,real(g),ww,real(gr));grid
subplot(2,1,2); plot(ww,imag(g),ww,imag(gr));grid