Evaluation question solutions

Abstract signals in MATLAB and SCILAB

Numerical solutions of some problems are written using a specific language for LTI system modeling, available as part of the `smiqc` package for MATLAB, available from A. Megretski's Web site. The package is designed with a complete SCILAB compatibility in mind, though the corresponding SCILAB code is not available at the moment. Here is a brief description of basic `smiqc` commands.

The main data type in `smiqc` is "abstract signal" (`asgn`). An abstract signal f refers to a sub-vector of the *output* y, which is related to *input* w and *state* x according to

$$y = Cx + Dw, \quad \dot{x} = Ax + Bw.$$

Matrices A, B, C, D are stored as hidden global variables, and are updated every time an operation is performed on abstract signals.

- **newmodel** Enable handling of abstract signals or start a new LTI model from scratch by setting A, B, C, D to their initial values.

  EXAMPLE: `newmodel;`

- **newsignal** Define f as an abstract input signal of dimension n. Accordingly, w and y are replaced by $w_{new} = [w; f]$, $y_{new} = [y; f]$.

  EXAMPLE: `f=newmodel(n);`

- **size** Dimension of signal f.

  EXAMPLE: `n=size(f);`

- **<=** Extract the LTI transformation from f to g. f must have an inner representation $f = C_f x + D_f w$ with a left invertible D_f. The components of w which are orthogonal to the rows of D_f are assumed to be zero in this operation.

  EXAMPLE: `G=(f<=g);`

- **$+, -, [\cdot, \cdot]$** Sum, difference, and concatenation of signals.

  EXAMPLE: `y=[f+g;f-g];`

- **$*$** LTI transformation of signals (the first argument is a matrix, a transfer matrix, or an LTI model).

  EXAMPLE: `g=2*f;`

- **==** Forces an equality constraint between two signals. The difference δ of signals must have an inner representation $\delta = C_\delta x + D_\delta w$ with a left invertible D_δ.

  EXAMPLE: `g==f;`

- **h2hi** H2 and H-Infinity optimization, with four abstract signal input arguments representing disturbance, control, cost, and measurement.

  EXAMPLE: `[k2,ki,g2,gi]=h2hi(w,u,e,y);`

- **slap** n-dimensional "Laplace" s, introduced for compatibility with SCILAB.

  EXAMPLE: `s=slap(1);`

- **minlti** Minimal realization of LTI system, introduced for compatibility with SCILAB.

  EXAMPLE: `H=minlti(G);`

In particular, the MATLAB code

```
newmodel;f=newsignal(1);e=newsignal(1);e==2*f-(1/s)*e;tf(e<=f)
```

should return transfer function $2s/(s+1)$.

Question 1.

FOR ALL VALUES OF PARAMETER $a \in \mathbf{R}$, FIND L2 GAIN OF SYSTEM

$$f(t) \mapsto y(t) = f(t) - |f(t-a)|.$$

Answer: for $a \geq 0$, the L2 gain equals 2; for $a < 0$, the L2 gain is $+\infty$.

Proof. Since $(a+b)^2 \leq 2(a^2+b^2)$ for all a, b,

$$|f(t) - |f(t-a)||^2 \leq |f(t)|^2 + |f(t-a)|^2.$$

for $a \geq 0$ this yields

$$\int_0^T \{4|f(t)|^2 - |y(t)|^2\}dt \geq \int_0^T \{2|f(t)|^2 - 2|f(t-a)|^2\}dt \geq -\int_{-a}^0 |f(t)|^2 dt > -\infty.$$

Hence L2 gain does not exceed 2 for $a \geq 0$. On the other hand, input $f(t) \equiv -1$ produces output $y(t) \equiv -2$, which shows that L2 gain is not smaller than 2.

For $a < 0$, input $f(t) = e^{rt}$ produces output $y(t) = (1 - e^{ra})e^{rt}$. Since $|1 - e^{ra}| \to \infty$ as $r \to +\infty$, the L2 gain is infinite. ∎

Question 2.

Systems G and Δ on block diagram on Figure 1 have L2 gains not larger than $a \geq 0$. Depending on the value of a, what is the largest possible L2 gain from f to y?

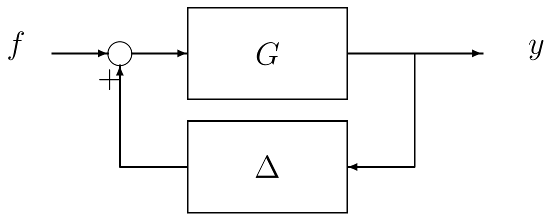

Figure 1: Block diagram for Problems 2 and 6.

Answer: when $a < 1$, the largest possible gain is $a(1-a^2)^{-1}$; otherwise, the largest gain is infinity.

Proof. For $a < 1$, considering the case when G and Δ are LTI systems with constant transfer functions $G(s) = \Delta(s) = a$ shows that $a(1-a^2)^{-1}$ is a *lower bound* for the answer. When $a \geq 1$, using $G(s) = \Delta(s) = 1 - \epsilon$, where $\epsilon > 0$ and $\epsilon \to 0$ shows that the only upper bound for the gain is plus infinity.

To show that the gain cannot be larger than $a(1-a^2)^{-1}$ consider systems G_1 and Δ_1, where G_1 maps inputs $[f_1; w]$ to outputs $[y_1; v] = [c_3G(c_1f_1 + c_2w); c_4G(c_1f_1 + c_2w)]$, and Δ_1 maps v to $w = c_2^{-1}\Delta(c_1^{-1}v)$ (the dashed boxes on Figure 2), where $c_i > 0$ are parameters.

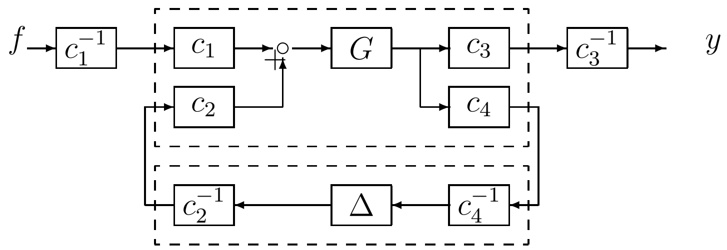

Figure 2: Re-scaling in Problem 1.2

It is easy to see that L2 power gain of G_1 is not larger than $(c_1^2 + c_2^2)^{1/2}(c_1^3 + c_4^2)^{1/2}a$, and power gain of Δ_1 is not larger than $(c_2c_4)^{-1}a$. Therefore, according to the small gain theorem, L2 power gain in the original system is not larger than $(c_1 c_3)^{-1}$, provided that

$$c_2 c_4 > a, \quad (c_1^2 + c_2^2)^{1/2} (c_1^3 + c_4^2)^{1/2} < a^{-1}.$$

Using $c_2 = c_4 = b^{1/2}$ and $c_1 = c_3 = (b^{-1} - b)^{1/2}$, where $b > a$, shows that the L2 power gain is not larger than $(b^{-1} - b)^{-1}$. As $b \to a + 0$, this yields the desired upper bound. ■

Question 3.

MATRICES A, B, C ARE SUCH THAT

$$C(sI - A)^{-1} B = (s + 2)^{-200},$$

AND A IS A HURWITZ MATRIX. FOR WHICH VALUES OF $\gamma > 0$ DOES THERE EXIST A STABILIZING SOLUTION $P = P'$ OF THE RICCATI EQUATION

$$C'C + PA + A'P + \gamma^{-1} PBB'P = 0?$$

Answer: $\gamma > 2^{-400}$.

Proof. According to the KYP Lemma, the Riccati equation, which can be re-written as

$$-C'C + P(-A) + (-A)'P = PB\gamma^{-1} B'P,$$

has a stabilizing solution if and only if for all $\omega \in \mathbf{R}$ the Hermitian form

$$\sigma(x, w) = \gamma |w|^2 - |Cx|^2$$

is positive definite on the linear subspace defined by

$$j\omega x = -Ax + Bw.$$

Equivalently, the inequality $|G(j\omega)|^2 < \gamma$ must be satisfied for all ω. Since H-Infinity norm of G is 2^{-200}, a stabilizing solution P exists if and only if $\gamma > 2^{-400}$. ■

Question 4.

FOR ALL VALUES OF PARAMETER $a \in \mathbf{R}$ SUCH THAT THE STANDARD FEEDBACK OPTIMIZATION SETUP

$$\begin{aligned} \dot{x} &= ax + u + w_1, \\ z &= x + u, \\ y &= x + w_1 + w_2 \end{aligned}$$

IS NOT SINGULAR, FIND AN EXPLICIT FORMULA FOR THE H2 OPTIMAL FEEDBACK CONTROLLER.

Answer:

$$u=-x_f,\ \dot{x}_f=ax_f+u-(a+\sqrt{a^2-a+0.5})(x_f-y)$$

for $a<1$, and

$$u=(1-2a)x_f,\ \dot{x}_f=ax_f+u-(a+\sqrt{a^2-a+0.5})(x_f-y)$$

for $a>1$; the problem has a control singularity at $\omega=0$ when $a=1$.

Proof. The abstract H2 optimization problem associated with the full information control part of the setup has the form

$$\int_0^\infty |x+u|^2dt\to\min,\quad\text{subject to}$$

$$\dot{x}=ax+u,\ x(0)=1,\ \lim_{t\to\infty}x(t)=0.$$

The corresponding completion of squares takes the form

$$(x+u)^2+2px(ax+u)=(u-Fx)^2,$$

which yields $F=-p-1$ and the Riccati equation

$$p^2=2p(a-1).$$

For $a<1$, its stabilizing solution is $p_u=0$, with $F=-1$. For $a>1$, its stabilizing solution is $p_u=2a-2$, with $F=1-2a$. For $a=1$, its only solution $p=0$ is not a stabilizing one.

The abstract H2 optimization problem associated with the state estimation part of the setup has the form

$$\int_0^\infty\{(\psi+v)^2+v^2\}dt\to\min\quad\text{subject to}$$

$$\dot{\psi}=a\psi+v,\ \psi(0)=F.$$

The corresponding completion of squares takes the form

$$(\psi+v)^2+v^2+2p\psi(a\psi+v)=2(v-L\psi)^2,$$

which yields $L=-0.5(p+1)$ and the Riccati equation

$$1+2(2a-1)p=p^2.$$

Its stabilizing solution is

$$p_y=2a-1+\sqrt{4a^2-4a+2},\quad L=-a-\sqrt{a^2-a+0.5}.$$

The H2 optimal controller has the form

$$u = Fx_f, \quad \dot{x}_f = ax_f + u + L(x_f - y).$$

■

Numerical verification. The following MATLAB code (file `q4leuven.m` uses the standard numerical optimization to check correctness of the analytical solution.

```
function q4leuven(a)
% function q4leuven(a)
%
% checks correctness of analytical solution of Question 4/Leuven 2005

B0=a+sqrt(a^2-a+0.5);
if a<1,
    C0=-1;
    A0=-1-sqrt(a^2-a+0.5);
elseif a>1,
    C0=1-2*a;
    A0=1-2*a-sqrt(a^2-a+0.5);
else
    fprintf('\nq4leuven: the problem is singular\n');
end
s=slap(1);
newmodel;
u=newsignal(1);
w1=newsignal(1);
w2=newsignal(1);
x=(1/(s-a))*(u+w1);
k2=h2hi([w1;w2],u,x+u,x+w1+w2);
fprintf('\nq4leuven: analytical')
tf(ss(A0,B0,C0,0))
fprintf('\nq4leuven: numerical')
tf(k2)
```

Question 5.

For all values of parameter $a > 0$ find Hankel norm of $G(s) = (s + a)^{-1}$.

Answer: $(2a)^{-1}$.

Proof. A state space model for G has $A = -a$, $B = C = 1$. Accordingly, the controllability and observability Gramians satisfy $-2aW = -1$, i.e. $W_c = W_o = (2a)^{-1}$. Hence the Hankel norm of G, being the square root of the largest eigenvalue of $W_c W_o$, equals $(2a)^{-1}$. ■

Question 6.

For the block diagram on Figure 1, where

$$\Delta(s) = \frac{s-2}{(s-1)(s-3)},$$

find a stabilizing $G = G(s)$ such that the variance of y, assuming f is the normalized white noise, is minimal.

Answer: the optimal controller

$$G(s) = \frac{-320s + 384}{s^2 + 12s - 253}$$

can be found using the following MATLAB script (available in file `q6leuven.m`):

```
% numerical solution of Question 6/Leuven 2005
s=slap(1);
D=(s-2)/((s-1)*(s-3));
newmodel;
f=newsignal(1);
y=newsignal(1);
[k2,ki,g2]=h2hi(f,y,y,D*y+f);
fprintf('\nq6leuven: minimal variance %f at',norm(g2)^2);
tf(k2)
```

The resulting variance is quite large, which is not surprising since the plant has an unpleasant combination of unstable zeros and poles.

Question 7.

For the block diagram on Figure 3, where

$$H(s) = \frac{s-1}{(s+1)^2},$$

FIND A STABILIZING CONTROLLER $K = K(s)$ SUCH THAT L2 GAIN FROM r AND f TO u AND q IS NOT LARGER THAN 100, AND THE CLOSED LOOP BANDWIDTH, DEFINED AS THE MAXIMAL ω_0 SUCH THAT $|G(j\omega) - 1| \leq 0.1$ FOR ALL $\omega \in [0, \omega_0]$, IS AS LARGE AS POSSIBLE. HERE G IS THE CLOSED LOOP TRANSFER FUNCTION FROM r TO q.

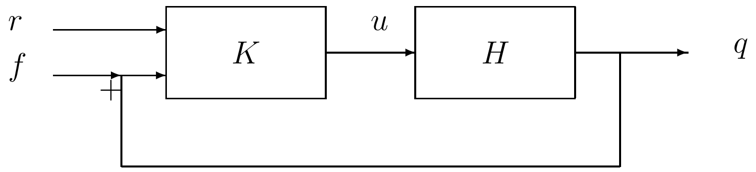

Figure 3: Block diagram for Question 7

Answer: a bandwidth of $\omega_0 = 0.22$ is achieved by a controller produced by the MATLAB function `q7leuven.m`:

```
function K=q7leuven(w0,d)
% function K=q7leuven(w0,d)
%
% tries to achieve bandwidth w0 in Question 7/Leuven 2005
if nargin<2, d=5; end
s=slap(1);
w1=9*w0;
k0=sqrt(2)/abs(1+sqrt(2)*j*(w0/w1)-(w0/w1)^2);
L=k0*(1+sqrt(2)*(s/w1)+(s/w1)^2)/(1+sqrt(2)*(s/w0)+(s/w0)^2);
P=(s-1)/(s+2)^2;
newmodel;
w1=newsignal(1);
w2=newsignal(1);
u=newsignal(1);
q=P*u;
r=L*w1;
e1=10*(q-r);
e2=(0.1/d)*u;
e3=(0.1/d)*q;
y=[r;q+(d/10)*w2];
[k2,K,g2,gi]=h2hi([w1;w2],u,[e1;e2;e3],y);
fprintf('\nq7leuven: bandwidth %f achieved if %f<1\n',w0,norm(gi,Inf));
```

```
u==K*y;
w2==0*w1;
E=(e1<=w1);
close(gcf);bode(0.1*E/L);grid
```

Note that the system specifications practically disallow the use of q measurements in feedback, so that the controller simply pre-conditions the reference input. When the plant equations are changed to unstable, as in $P(s) = (s-1)/(s-0.1)^2$, the achievable bandwidth remains essentially the same, but the output measurement is actively used in the feedback.

Question 8.

FOR

$$G(s) = \sum_{k=1}^{100} \frac{1}{s^2 + 2s + k/10},$$

FIND A GOOD LOW BOUND OF H-INFINITY NORM OF APPROXIMATION ERROR $G - \hat{G}$, WHERE THE ORDER OF $\hat{G}$ IS NOT LARGER THAN 10. ALSO, USE HANKEL OPTIMAL MODEL REDUCTION TO FIND A GOOD 10-TH ORDER APPROXIMATION $\hat{G}$ OF G.

11-th Hankel singular number of G provides a lower bound for the quality of such approximation. Numerical calculations (file `q8leuven.m` show that the 10th order Hankel optimal reduced model is very close to H-Infinity optimality.

```
% script for Question 8/Leuven 2005Gbal,sig]
N=100;
A=zeros(2*N);
B=zeros(2*N,1);
C=zeros(1,2*N);
for k=1:N,
    k2=2*k;
    k1=k2-1;
    A(k1:k2,k1:k2)=[0 1;-k/10 -2];
    B(k1:k2)=[0;1];
    C(k1:k2)=[1 0];
end
G=ss(A,B,C,0);
Gnom=pck(A,B,C,0);
[Gbal,sig]=sysbal(Gnom);
```

```
Gred=hankmr(Gbal,sig,10,'d');
[Ar,Br,Cr,Dr]=unpck(Gred);
Gr=ss(Ar,Br,Cr,Dr);
fprintf('\nq8leuven: lower bound %d, actual %d\n',sig(11),norm(G-Gr,Inf));
ww=linspace(0,10,300);
g=squeeze(freqresp(G,ww));
gr=squeeze(freqresp(Gr,ww));
close(gcf);
subplot(2,1,1); plot(ww,real(g),ww,real(gr));grid
subplot(2,1,2); plot(ww,imag(g),ww,imag(gr));grid
```