Bell–Kochen–Specker Thm:

Recall if observables $A$, $B$ that don’t commute
not compatible
not simultaneously measurable

$A: X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B: Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$XZ = -ZX$  \text{ anticommuting}
1. Value definiteness

Outcome of measuring $A$

$\nu(A, \lambda) \in \mathbb{E} \pm \beta$

2. Functional consistency for compatible observables

If $A, B$ commute

$C = AB$

$\nu(C, \lambda) = \nu(A, \lambda) \cdot \nu(B, \lambda)$

This is true with certainty in QM.
**Thm (Bell-Kochen-Specker):**

If dimension $\geq 3$, then

$\exists$ observables s.t. no HVT

obeying $1 \leq 2$ is consistent w/ them

**Historical aside:**

v. Neumann showed this

1) (2) assumed even for

non-compatible observables
$x, z = \frac{x^2 + z^2}{\sqrt{2}}$

$\sqrt{(x^2 + z^2)} = \sqrt{x} \pm \sqrt{z}$

Bell [early 60s]

Showed that

K S [later 60s] independently

Merrin–Peres 80s

Much simpler proof

d $\geq$ 4

Merrin–Peres Magic Square

$x = (0,1) \quad z = (1,0)$

$x^2 = z^2 - I$

$x^2 \geq -z^2$
We'll work over $C^2 \otimes C^2$

<table>
<thead>
<tr>
<th>$I \otimes X$</th>
<th>$X \otimes I$</th>
<th>$X \otimes X$</th>
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<tbody>
<tr>
<td>$2 \otimes I$</td>
<td>$I \otimes 2$</td>
<td>$2 \otimes 2$</td>
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<tr>
<td>$2 \otimes X$</td>
<td>$X \otimes 2$</td>
<td>$2 \otimes X \otimes 2$</td>
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</tbody>
</table>

- Every row
- Every column is compatible
- Even & can $\times$ odd = odd
- $I \otimes I$
- $(X \otimes X) (2 \otimes 2)$
- $(X \otimes 2) (2 \otimes X)$
- $= X^2 \otimes X^2$
- $= - 2 \otimes 2 \otimes X^2$
- $= + 2 \otimes 2 \otimes X^2$
- $= (2 \otimes 2) (X^2 \otimes X)$

Example column 3

- $X \otimes X$
- $2 \otimes X$
- $2 \otimes 0 \otimes X$
- $= (X \otimes 2) (2 \otimes X)$
\[(X \circ X)(2X \circ X2)\]
\[= X \circ X \circ X \circ X \circ X \circ X\]
\[= -2X \circ X \circ X \circ X \circ X\]
\[= -7X \circ X \circ X \circ X \circ X\]
\[= + (2X \circ X2)(X \circ X)\]

\[
\begin{array}{c|c|c}
10X & X01 & X0X \\
\hline
201 & 1\circ 2 & 2\circ 2 = 1\circ 2 \\
\hline
2\circ X & 1\circ 2 & 2X \circ X2 = 1\circ 1 \\
\hline
\end{array}
\]

Suppose \(HVT = (A, \tau)\) \[X_0 \ V_1 \ V_2 \ V_3 \ V_4 \ V_5 \ V_6 \ V_7 \ V_8\]

\[(1) \Rightarrow \frac{\sqrt{3}}{V_6} \frac{\sqrt{4}}{V_7} \frac{\sqrt{8}}{V_8}\]
(2) \[ A_2 = A_0 A_1 \]

There is no possible choice of \( \nu_i \)'s to make this work.

"Contextuality"

To assign a value to \( A_0 \), we need to know whether it's being measured with \( A_1, A_2 \), or \( \{ A_3, A_4, A_5 \} \) or \( \{ A_6 \} \).
State-independent contextuality

Contextuality vs. Non-locality

Bell-1<5

Assume some measurements are compatible \( \Rightarrow \) non-contextual HVM

\[ \text{Bell, CHSH} \]

Use spatial nonlocality to enforce Alice measurements are compatible with Bob measurements

MS game [Aravind late 90s, early '00s]

Referee

Alice \( \sim \{0, 3, 4, 5, 6, 7, 8\} \) \( \searrow \) Bob
Win if valid row & col assignment and consistent.

Classical value:
\[ w_{\text{classical}} < 1 \]
\[ w_{\text{classical}} = \frac{8}{9} \]

Quantum value:
\[ w^* = 1 \]
\[ w^* = 1 \]
Regardless of $|\Psi\rangle$, $A \cap B$ satisfy row \& col. conditions.

$|\Psi\rangle = |EPR\rangle^{\otimes 2} \Rightarrow A \cap B$ satisfy consistency conditions.

3) MS game has "perfect completion" "pseudo-telepathy" ($\omega^* = 1$)

Moreover, MS is a self-to-

for $|EPR\rangle^{\otimes 2}$

$\omega^*(S) \geq 1 - \varepsilon$
∃ local isometries $V_A, V_B$ s.t.

$$V_A \otimes V_B |\psi\rangle \cong \sigma_{\epsilon} |\psi\rangle_{EPR} \otimes |\text{out}\rangle$$

$$(V_A \otimes V_B) (A_0 \otimes I) |\psi\rangle$$

$$\cong \sigma_{\epsilon} (I_X \otimes I) |\psi\rangle_{EPR} \otimes |\text{out}\rangle$$

**Pf.**

$$\co (s) 2 \rightarrow 3$$

$$A_0 \quad \quad \quad \quad \quad \quad \text{out}$$

$$\Rightarrow A_0 A_0 \otimes I |\psi\rangle$$

$$\Rightarrow -A_0 A_0 \otimes I |\psi\rangle$$

$$\Rightarrow$$

Diagram: [Diagram representation of quantum operations]
Generalizations:

- Robert Spekkens et al.
  operational, noise-robust versions
  "measurement contextuality"
  "preparation contextuality"

- Graph-based contextuality
Cabello Severini Winter '10

Say you have "events" represented by projectors \( \{P_i\}_i \)

\[ P_i \sim P_j \text{ if } P_i + P_j \leq I \]

\[ \Rightarrow P_i P_j = 0 \]
\[ \Rightarrow P_j P_i = 0 \]
compatible, mutually exclusive
\[ \langle \sum \Pi_i \rangle \leq \chi G \]

classical, non-contextual model

\[ \langle \sum \Pi_i \rangle \leq \chi G \quad \text{independent set} \]

contextuality inequalities if

\[ \exists \{1, 2, \ldots, n\} \text{ st.} \]

\[ \langle \Psi_i \angle \sum \Pi_i \angle \Psi_j \rangle > \chi G \]

E.g. For each row or column, \( \Pi_i \)

for every "valid" assignment

\[
\begin{array}{cccc}
\# & 000 & 011 & 101 \\
\# & 110 \\
24 \Pi_i \text{'s in total}
\end{array}
\]
Connections to relaxations of \( \alpha \) Lovász theta function

**Next time:**
- Show that this doesn't work mod \( p \), \( p \neq 2 \)
- \( 3 \) non-contextual HVM for mod \( p \) versions of \( X \) and \( Z \)