Compressed Sensing

$Ax = y$, $A \in \mathbb{C}^{m \times n}$ measurement matrix

interested in undetermined case $m \ll n$. ~~#~~ signal over time

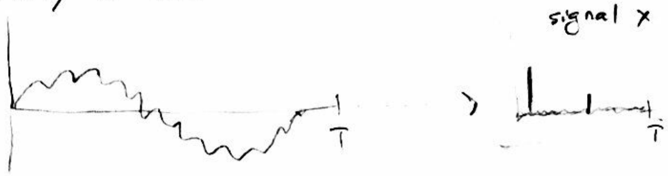
Shannon sampling: $x(t)$ contains no frequencies higher than B hertz, can completely characterize by sampling at $1/2B$ intervals

solution:

k-sparse: $x \approx 0$ outside of k coord.

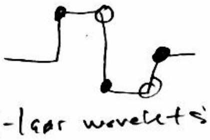
difficulty is location of nonzeroes is unknown. ~~the~~

observe



intro: related to HH. reconstruct signal using # of measurements $\ll$ length of signal. useful in image compression, MRI. one solution is application of JL.

in applications, need sparsity w.r.t. correct basis. Change of basis is linear.



observation: $\Pi \cdot \mathcal{F}^{-1}(x) = Ax$

applications: JPEG compression (discrete cosine transform, Haar wavelets)

shorten MRI scanning sessions

sparse vector is ~~still~~ still very powerful in descriptive capability

algorithmic goal agnostic learning paradigm. Signal x, recover $\tilde{x}$ s.t.

$$\|x - \tilde{x}\|_p \le C \min_{\|y\|_0 \le k} \|y - x\|_q \qquad \ell_p / \ell_q$$

if x is actually k-sparse, get perfect answer (RHS = 0). otherwise do as well as best k-sparse approx.

model for k-sparse signal + noise

~~can't~~

exact recovery $\min_{z \in \mathbb{R}^n} \|z\|_0$ s.t. $\Pi z = y$

NP-hard. Can solve w/ additional constraint on Π and using ℓ_1 as convex relax. of ℓ_0

RIP matrix $\Pi \in \mathbb{R}^{m \times n}$ is (ε, k)-RIP if for all k-sparse vectors

$$(1-\varepsilon)\|x\|_2^2 \le \|\Pi x\|_2^2 \le (1+\varepsilon)\|x\|_2^2$$

nearly orthonormal

homework: JL works for k-dimensional subspaces

apply JL to $\binom{n}{k}$ subspaces of k-sparse vectors

~~dimens~~ # measurements $m \lesssim \frac{1}{\varepsilon^2} k \log(n/k)$

open problem: good deterministic constructions

Theorem Π is $(\varepsilon_{2k}, 2k)$-RIP with $\varepsilon_{2k} \le \sqrt{2} - 1$ and $\tilde{x} = x + h$ is the solution to

$$\min_{z \in \mathbb{R}^n} \|z\|_1 \quad \text{s.t. } \Pi z = \Pi x$$

then $\|h\|_2 \le O\left(\frac{1}{\sqrt{k}}\right) \|x_{tail(k)}\|_1$

where $x_{tail(k)}$ is x with top k elements removed

special case of $\min_{\|y\|_0 \le k} \|y - x\|_q$ for $q = 1$ "lasso"

Proof x has support on a set S:

notation: x_S s.t. $(x_S)_i = 0$ if $i \notin S$, $(x_S)_i = x_i$ if $i \in S$

sets: $T_0 \subseteq [n]$ indices of largest k coordinates of $|x|$ (in abs value)

T_1: indices of largest k coord of $h_{T_0^c} = h_{tail(k)}$

T_2: indices of second largest k coord. of $h_{T_0^c}$

T_3: " " third " "

h_{T_j} additive for different j

strategy: $\|h\|_2 \le \|h_{T_0 \cup T_1}\|_2 + \|h_{(T_0 \cup T_1)^c}\|_2$

"noise" RIP not needed (1) $\|h_{(T_0 \cup T_1)^c}\|_2 \le \|h_{T_0 \cup T_1}\|_2 + O\left(\frac{1}{\sqrt{k}}\right) \|x_{tail(k)}\|_1$

"signal" (2) $\|h_{T_0 \cup T_1}\|_2 \le O\left(\frac{1}{\sqrt{k}}\right) \|x_{tail(k)}\|_1$

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lemma 1 $\sum_{j \ge 2} \|h_{T_j}\|_2 \le \frac{2}{\sqrt{k}} \|x_{T_0^c}\|_1 + \|h_{T_0 \cup T_1}\|_2$

proof "shelling trick"

$$\sum_{j \ge 2} \|h_{T_j}\|_2 \le \sqrt{k} \sum_{j \ge 2} \|h_{T_j}\|_\infty$$

each vector h_{T_j} has at most k nonzero components. $\sqrt{\ }$ comes from norm.

$$\le \frac{1}{\sqrt{k}} \sum_{j \ge 2} \|h_{T_{j-1}}\|_1$$

every term of h_{T_j} is bounded from above by every term of $h_{T_{j-1}}$ (in particular the average $\frac{1}{k} \|h_{T_{j-1}}\|_1$)

$$= \frac{1}{\sqrt{k}} \|h_{T_0^c}\|_1$$

$\tilde{x} = x+h$ minimizes $\|z\|_1$ and $z=x$ satisfies constraints so

$$\|x\|_1 \geq \|x+h\|_1$$
$$= \|(x+h)_{T_0}\|_1 + \|(x+h)_{T_0^c}\|_1$$
$$\geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1 - \|x_{T_0^c}\|_1$$

$\|x\| = \|x'+h-h\|$
$\leq \|x+h\| + \|h\|$

rearrange:

$$\|h_{T_0^c}\| \leq \|x\|_1 - \|x_{T_0}\|_1 + \|h_{T_0}\|_1 + \|x_{T_0^c}\|_1$$
$$= 2\|x_{T_0^c}\|_1 + \|h_{T_0}\|_1 \quad (\|h_{T_0}\cdot \mathbb{1}\|_1)$$
$$\leq 2\|x_{T_0^c}\|_1 + \sqrt{k}\|h_{T_0}\|_2 \quad \text{(Cauchy-Schwartz)}$$
$$\leq 2\|x_{T_0^c}\|_1 + \sqrt{k}\|h_{T_0\cup T_1}\|_2 .$$

$$\therefore \sum_{j\geq 2}\|h_{T_j}\|_2 \leq \frac{1}{\sqrt{k}}\|h_{T_0^c}\|_1 \leq \frac{2}{\sqrt{k}}\|x_{T_0^c}\|_1 + \|h_{T_0\cup T_1}\|_2 \quad \square$$

upper bound of $\|h_{(T_0\cup T_1)^c}\|_2$: ~~ineq~~

$$\|h_{(T_0\cup T_1)^c}\|_2 = \Big\|\sum_{j\geq 2} h_{T_j}\Big\|_2$$
$$\leq \sum_{j\geq 2}\|h_{T_j}\|_2$$
$$\leq \|h_{T_0\cup T_1}\|_2 + \frac{2}{\sqrt{k}}\|x_{T_0^c}\|_1 \quad \text{by lemma}$$

proves ~~l~~ inequality (1)

upper bound of $\|h_{(T_0\cup T_1)}\|_2$: first prove lemma

RIP property used here

<u>lemma 2</u> x, x' have support on disjoint sets T, T' where $|T| = k$, $|T'| = k'$, then

$$|\langle \Pi x, \Pi x'\rangle| \leq \varepsilon_{k+k'}\|x\|_2\|x'\|_2 . \quad \text{where } \Pi \text{ i}$$
$(\varepsilon_{k+k'}, k+k')$-

<u>proof</u> assume wlog $\|x\|_2 = \|x'\|_2 = 1$ (divide through)

$$\|\Pi x + \Pi x'\|_2^2 = \|\Pi x\|_2^2 + \|\Pi x'\|_2^2 + 2\langle \Pi x, \Pi x'\rangle$$

$$\|\Pi x - \Pi x'\|_2^2 = \|\Pi x\|_2^2 + \|\Pi x'\|_2^2 - 2\langle \Pi x, \Pi x'\rangle$$

Subtracting gives

$$|\langle \Pi x, \Pi x' \rangle| = \frac{1}{4} \left| \|\Pi(x+x')\|_2^2 - \|\Pi(x-x')\|_2^2 \right|$$

$$\leq \frac{1}{4}\left((1+\varepsilon_{k+k'})\|x+x'\|_2^2 - (1-\varepsilon_{k+k'})\|x-x'\|_2^2 \right)$$

$$= \frac{1}{4}\left((1+\varepsilon_{k+k'}) \cdot 2 - (1-\varepsilon_{k+k'}) \cdot 2 \right)$$

since x, x' are disjointly supported, sum of squares is 2

$$= \varepsilon_{k+k'} \quad \square \text{ end proof of lemma} \qquad k = k'$$

bounding $\|h_{T_0 \cup T_1}\|_2$:

observe $\Pi h_{T_0 \cup T_1} \underset{\uparrow}{=} \Pi h - \sum_{j \geq 2} \Pi h_{T_j} \underset{\uparrow}{=} - \sum_{j \geq 2} \Pi h_{T_j}$

Π linear, $h = h_{T_0 \cup T_1} + \sum_{j \geq 2} h_{T_j}$

$\Pi(x+h) = \Pi x$ solution to LP

therefore inner product

distribute summation

$$\|\Pi h_{T_0 \cup T_1}\|_2^2 = - \sum_{j \geq 2} \langle \Pi h_{T_0 \cup T_1}, \Pi h_{T_j} \rangle$$

$$\leq \sum_{j \geq 2} \left(|\langle \Pi h_{T_0}, \Pi h_{T_j} \rangle| + |\langle \Pi h_{T_1}, \Pi h_{T_j} \rangle| \right)$$

since $\Pi h_{T_0 \cup T_1} = \Pi(h_{T_0} + h_{T_1})$

by lemma 2 each summand is bounded by

$$\varepsilon_{2k} \left(\|h_{T_0}\|_2 + \|h_{T_1}\|_2 \right) \|h_{T_j}\|_2$$

$$\leq \varepsilon_{2k} \sqrt{2} \, \|h_{T_0 \cup T_1}\|_2 \, \|h_{T_j}\|_2$$

since $\|h_{T_0 \cup T_1}\|_2 = \sqrt{2}\sqrt{\frac{1}{2}\|h_{T_0}\|_2^2 + \frac{1}{2}\|h_{T_1}\|_2^2}$ and concavity of $\sqrt{\ }$

by RIP property

$$(1-\varepsilon_{2k})\|h_{T_0\cup T_1}\|_2^2 \le \|\Pi h_{T_0\cup T_1}\|_2^2$$ prev expression

$$\le \varepsilon_{2k}\sqrt{2}\,\|h_{T_0\cup T_1}\|_2 \sum_{j\ge 2}\|h_{T_j}\|_2$$

lemma (1) $$\le \varepsilon_{2k}\sqrt{2}\,\|h_{T_0\cup T_1}\|_2\left(\frac{2}{\sqrt{k}}\|x_{T_0^c}\|_1 + \|h_{T_0\cup T_1}\|_2\right)$$

$$\therefore\ \|h_{T_0\cup T_1}\|_2 \le \frac{\varepsilon_{2k}\,2\sqrt{2}}{(1-\varepsilon_{2k}-\varepsilon_{2k}\sqrt{2})\sqrt{k}}\|x_{T_0^c}\|_1 = O\!\left(\frac{1}{\sqrt{k}}\right)\|x_{T_0^c}$$

divide by $\|h_{T_0\cup T_1}\|_2$ & rearrange :

∎ proof of inequality (2)

problem LP expensive to solve. polynomial in n

Solution greedy optimization → matching pursuit / basis pursuit

orthogonal matching pursuit:

- input: measurement matrix A, measurement vector y

initialize: $S^0 = \phi$, $x^0 = 0$

repeat until stopping criteria:

$$S^{n+1} = S^n \cup \{j_{n+1}\} \quad \text{where} \quad j_{n+1} = \underset{j \in [N]}{\operatorname{argmax}} \left| \left(A^*(y - Ax^n)\right)_j \right|$$

$$x^{n+1} = \underset{z \in \mathbb{C}^N}{\operatorname{argmin}} \left\{ \|y - Az\|_2,\ \operatorname{supp}(z) \subset S^{n+1} \right\}$$

$y - Ax^n$: diff between measurement and current reconstruction in measurement space

$A^* \cdot (\ \ )$ brings coord. space back to signal space of x

pick index j with biggest difference

other matching pursuit algorithms based on bipartite expanders, etc.