COMS 4995-3: Advanced Algorithms — March 27th, 2017

Lecture 17 – Introduction to Linear Programming

Instructor: *Alex Andoni* — Scribes: *Ji Xu, Xuecheng Sun*

1 Introduction

Today's lecture is about introduction to Linear Programming (Optimization). In general, optimization problem is considered as the following:

$$\begin{aligned} \text{Obj}: &\quad \min f(x) \\ \text{s.t.} &\quad x \in \mathbb{R}^n, \text{ some constraints on } x \ (e.g.\ x \in \{0,1\}^n) \end{aligned}$$

Here is an example of the optimization problem:

Example 1. *The min conductance problem in the graph $G = (V, E)$ we discussed before is the following:*

$$\begin{aligned} \min &\quad \frac{|\partial S|}{\sum_{i \in S} d_i} \\ \text{s.t.} &\quad S \neq \emptyset, \quad \sum_{i \in S} d_i \leq \frac{1}{2} \sum_{i \in V} d_i, \end{aligned}$$

where d_i is the degree of node i. We can regard this problem as:

$$\begin{aligned} \textit{unknown variables:} &\quad x_i, \quad i = 1, 2 \cdots, n \\ &\quad x_i \in \{0,1\} \left(\Leftrightarrow \begin{cases} x_i \in \mathbb{R} \\ x_i(1 - x_i) = 0 \end{cases} \right) \\ \min &\quad f(x) = \frac{x^T L x}{\sum_{i \in V} d_i x_i} \\ \text{s.t.} &\quad \sum_{i \in V} x_i > 0, \quad \sum_{i \in V} d_i x_i \leq \frac{1}{2} \sum_{i \in V} d_i. \end{aligned}$$

In general, optimization problem is possible to formulate. But solving a problem with $f(x)$ and all constraints = degree-2 polynomials is NP-hard.

2 Linear Programming:

Definition 2. *LP: $f(x)$ is linear in x and all constraints are also linear (i.e, $ax \gtreqless b$):*

$$\begin{aligned} \text{Obj}: &\quad \min f(x) = c \cdot x \\ \text{s.t.} &\quad Ax \geq b \end{aligned}$$

Note that for maximization problems, we can convert the objective $\max f(x)$ *into* $\min -f(x) = -c \cdot x$*. For equality constraints* $Ax = b$*, we can convert it into* $Ax \geq b, -Ax \geq -b$*. For constraints* $Ax \leq b$*, we can convert it into* $-Ax \geq -b$*.*

Example 3. *Convert max-flows into a Linear Programming problem: Given* $G = (V, E)$*,* $(i, j) \in E$*,* $c_{ij} > 0$*, we solve the following LP problem:*

$$\begin{aligned} \textit{unknown variables:} \quad & f_{i,j}, \forall (i,j) \in E \\ \max \quad & \sum_{(s,j)\in E} f_{s,j} - \sum_{(j,s)\in E} f_{j,s} \\ \text{s.t.} \quad & \forall (i,j) \in E,\ 0 \leq f_{i,j} \leq c_{ij} \\ & \forall i \in V \backslash \{s,t\} \underbrace{\sum_{j:(j,i)\in E} f_{j,i}}_{\textit{flow in}} = \underbrace{\sum_{j:(i,j)\in E} f_{i,j}}_{\textit{flow out}} \end{aligned}$$

The main goal of this module will be: *How to solve a general LP?*

2.1 General form to Standard form:

Definition 4. *Any LP can be equivalently written in the following "standard form":*

$$\begin{aligned} \min \quad & c \cdot x \\ \text{s.t.} \quad & Ax = b \\ & x_i \geq 0\ \forall i. \end{aligned}$$

For any LP problem, we can convert it into the "standard form" by doing the following two steps:

- For $\forall x_i \in \mathbb{R}$, we replace x_i with $x_i^+ - x_i^-$, where $x_i^+ \geq 0, x_i^- \geq 0$ are the new unknown variables.
- Any constraint $A_i x \geq b_i$ is replaced with the constraint $\xi_i = A_i x - b_i$, where $\xi_i \geq 0$ is a new unknown. We call ξ_i as slack variables.

2.2 Structure of Solutions to Linear Programming:

Definition 5. *Define* x *is a feasible solution if it satisfies all constraints. Define* x *is optimal if it satisfies all constraints and there is no better solution for the objective.*

Note that each constraint can be considered as separating the space by a hyperplane. In other words,

$$\begin{aligned} P &= \text{set of feasible solutions} \\ &= \text{intersection of half-spaces (space on a side of a half-space)} \\ &= \text{polytope/ polyhedron} \end{aligned}$$

We call P is bounded if it is inside a box and P is unbounded if otherwise. See Figure 1 for an illustration of P.

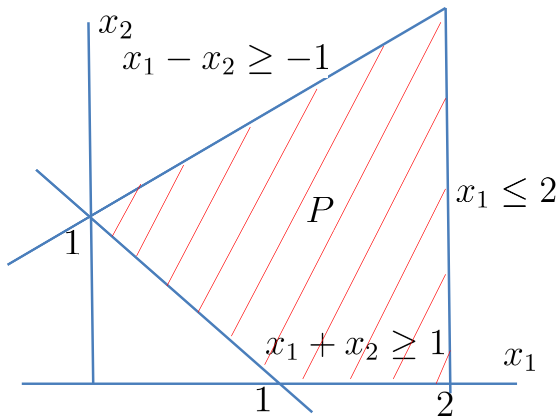

Figure 1: The red area is the polytope P defined by constraints $x_1 \leq 2$, $x_2 \geq 0$, $x_1 + x_2 \geq 1$ and $x_1 - x_2 \geq -1$.

2.3 Finding the solution for LP:

Let the optimal solution be x^*, then we know the optimal value of the objective will be on the line $c \cdot x = cx^*$ which represents a hyperplane as well. Therefore one strategy of finding the solution for LP is the following: Assume we are finding minimum of $x_1 + 2x_2$ over P represented in Figure 1. We do the following:

- test if the optimal value of objective can be -1000 $\Rightarrow$ no feasible solution s.t. $c \cdot x = -1000$.
- test if the optimal value of objective can be -1000 $+ \epsilon \cdots$

$\vdots$

See Figure 2 for illustration.

2.4 cases for solutions:

In general, the solution of LP falls into one of the following three options:

- There is a solution
- No solution $P = \emptyset$ (e.g. Having constraints $x_1 \geq 2$ and $x_1 \leq 1$)
- Unbounded (e.g. $\min x_1$, $x_1 \leq 1$)

3 Simpler case: solving system of linear equations

For simple case that there is no inequalities i.e, $Ax = b$ and A is a square matrix, we can use Gaussian Elimination process to solve the solution for $Ax = b$. The Gaussian Elimination eliminates one variable

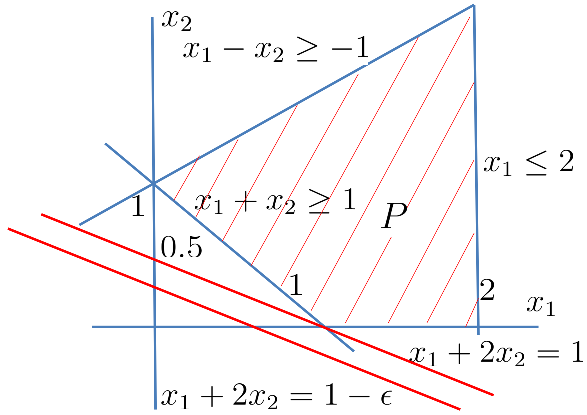

Figure 2: There is no feasible solution for $c \cdot x = x_1 + 2x_2 = 1 - \epsilon$. For $c \cdot x = x_1 + 2x_2 = 1$, we can find one.

at a time like the following example.

$$\begin{cases} 2x_1 + x_3 = 6 \\ x_1 - x_2 + x_3 = 2 \\ 2x_1 - x_4 = 0 \\ \vdots \end{cases}$$

Eliminate x_1 using $x_1 = 3 - x_3/2$, we have previous constraints become

$$\begin{cases} 3 - x_3/2 - x_2 + x_3 = 2 \\ 6 - x_3 - x_4 = 0 \\ \vdots \end{cases}$$

Here, we review some facts about linear algebra.

Fact 6. *The following statements are equivalent:*

- *A is invertible*
- $\det(\mathrm{A}) \neq 0$
- *A has linearly independent columns*
- *A has linearly independent rows*
- *$Ax = b$ has a unique solution for $\forall b$.*

Now we wonder what's the size of the solution for $Ax = b$ if there is a solution.

Fact 7. *The solution for $Ax = b$ has polynomial description.*

We'll starting proving this now (and finish in the next lecture). First assume that A is a square matrix.

- If all entries of A are integers, then x_i = multiple of $\frac{1}{\det(\mathrm{A})}$, furthermore these multiples are determinates of minors of A.
- If an entry A_{ij} requires at most b bits to represent, then $\det(A)$ can be represented with $O(n \log n + bn)$ bits. (since $\det(A) \leq n! \cdot 2^{bn}$)

If A is not square, then with some changes, we can turn it into a square matrix.

In the next lecture, we will consider the cases when matrix is non-square, $\det(\mathrm{A}) = 0$, and when there is no solution.