Wave propagation patterns in a “classical” three-dimensional model of the cochlea

Egbert de Boer  
Room D2-226, Academic Medical Center, University of Amsterdam, Meibergdreef 9, 1105 AZ, Amsterdam, The Netherlands

Alfred L. Nuttall  
Oregon Hearing Research Center, NRC04, Oregon Health & Science University, 3181 SW Sam Jackson Park Road, Portland, Oregon 97239-3098, and Kresge Hearing Research Institute, University of Michigan, 1301 E. Ann Street, Ann Arbor, Michigan 48109-0506

Christopher A. Shera  
Eaton-Peabody Laboratory, Massachusetts Eye and Ear Infirmary, 243 Charles Street, Boston Massachusetts 02114

(Received 29 June 2006; revised 6 October 2006; accepted 11 October 2006)

The generation mechanisms of cochlear waves, in particular those that give rise to otoacoustic emissions (OAEs), are often complex. This makes it difficult to analyze wave propagation. In this paper two unusual excitation methods are applied to a three-dimensional stylized classical nonlinear model of the cochlea. The model used is constructed on the basis of data from an experimental animal selected to yield a smooth basilar-membrane impedance function. Waves going in two directions can be elicited by exciting the model locally instead of via the stapes. Production of DPOAEs was simulated by presenting the model with two relatively strong primary tones, with frequencies $f_1$ and $f_2$, estimating the driving pressure for the distortion product (DP) with frequency $2f_1-f_2$, and computing the resulting DP response pattern – as a function of distance along the basilar membrane. For wide as well as narrow frequency separations the resulting DP wave pattern in the model invariably showed that a reverse wave is dominant in nearly the entire region from the peak of the $f_2$-tone to the stapes. The computed DP wave pattern was further analyzed as to its constituent components with the aim to isolate their properties. © 2007 Acoustical Society of America. [DOI: 10.1121/1.2385068]

PACS number(s): 43.64.Kc, 43.64.Bt [WPS]  
Pages: 352–362

I. INTRODUCTION

Experimental data on the responses of animal cochleae to complex stimuli often present a bewildering picture. One common example is the situation where two pure tones are presented simultaneously, and the response of the cochlea is investigated as to the properties of distortion products (DPs). The response can be observed in the form of oscillations of sure or as otoacoustic emissions investigated as to the properties of distortion products presented simultaneously, and the response of the cochlea is common example is the situation where two pure tones are to complex stimuli often present a bewildering picture. One constitutes a good explanation of the spectral irregularities experimentally observed. The second report is a good example of the use of a model of the cochlea as a device for obtaining deeper insight. It should be stressed at this point that the particular model was constructed directly from the measured response of a particular animal. In the past we have often utilized this method. For that special purpose the experimental data were acquired by presenting wide-band periodic noise signals to the animal, recording the waveforms of the responses, and analyzing those waveforms with cross-correlation functions (de Boer and Nuttall, 1997, 1999, 2000a, b). The underlying theory, the EQ-NL theorem, was described by the first author in 1997.

A brief description of this theorem follows. The cochlea is modeled as a nonlinear system of which the elements are smoothly distributed along the length of the basilar membrane (BM). For excitation with a noise stimulus, the frequency components of the signals inside the model are so close together that distortion products (combination tones) are swamped, so that only overall nonlinearity (in particular, compression) remains. The nonlinear model is compared to a linear one, in which the elements are given parameters that are modified by compression. It is shown that, with the proper choice of the latter elements and for the same stimulus signal used, the input-output cross-correlation functions are the same for the two models. Using this theorem to de-
scribe cochlear responses allows us to use the linear version of the model, for forward and inverse solutions.

That linear model is constructed in such a way that it reproduces or simulates the data obtained in a particular animal. Stimulation with signals of different levels results in different parameters. With this technique it has proven possible to explain global and refined properties of linear as well as nonlinear effects observed in the cochlea; see the aforementioned reports on effects of stimulus intensity on the response and de Boer and Nuttall (2002) on tone versus noise responses. A different, equally universal but more conventional use of a model of the cochlea is amply illustrated by the work of Talmadge et al. (1998a, b) and Tubis et al. (2000a, b). In this case the model used is to be considered as a generic one, capable of explaining the most general properties of responses on the basis of mathematical elaborations.

In a cochlear model two types of waves propagating along the basilar membrane are, in general, possible: one in the direction of the apex (the forward-traveling wave) and one in the direction of the stapes (the backward-traveling or reverse wave). In a “classical” model (this term is defined in the Appendix) these two waves are equivalent in their modes of propagation. Whether or not both waves are observable depends on the circumstances (type of stimulation, local mechanical properties, and presence of internal reflections). Ever since otoacoustic emissions were detected (Kemp, 1978), attention has been given to reverse waves, their origin, and their properties, because otoacoustic emissions are generally assumed to arise from reverse waves reaching the stapes.

In particular, the question has been raised whether the total delay (latency) of an otoacoustic emission signal can be decomposed into a forward-wave delay and a reverse-wave delay. In the simplest case, that of a stimulus-frequency emission (SFOAE), the two delays would be, in a crude approximation, the same. However, the source of the forward wave is a simple, localized, one—the stapes—but in contrast, the source of the reverse wave is distributed over a certain range—the region of “activity.” Therefore, even in this relatively simple case the interpretation is not simple. For a distortion-product emission (DPOAE) three frequencies are involved, and it is much more difficult to unravel the actual delay of the emission signal. Many authors have held the view that indeed the round-trip delay is composed of a forward and a backward delay, and can approximately be decomposed in this way (among others, Schneider et al., 1999; Prijs et al., 2000; Goodman et al., 2003, 2004; Shera and Guinan, 2003).

Recently, findings by Tianing Ren (2004a,b, Ren et al., 2006) have cast doubt on this opinion. Ren performed measurements in the gerbil at several locations along the basal part of the basilar membrane. In analyzing evidence of distortion products (DPs), he did not detect a reverse DP wave in situations where the source of the DP was located apically from the measurement location. And a new interpretation of the data (e.g., Ren, op. cit.; Ruggero, 2004; Siegel et al., 2005; Ren and Nuttall, 2006) has appeared.

The present paper has been written with the intention of analyzing wave-traveling problems into some more detail and generality. As earlier in our work, a (classical) model of the cochlea is constructed on the basis of data obtained from one particular experiment. That model is subsequently analyzed with respect to several possible subtle effects associated with wave propagation. To wit, the model is stimulated with various types of stimulus and analyzed as to its frequency and space-domain responses. In Sec. II briefly we describe the technique used in obtaining and analyzing the original data, and the genesis of the model on that basis. In Sec. III describe specific operations carried out with the model, including the use of artificial forms of stimulation. In Sec. IV, a fundamental—and successful—test of the extended computation method is described. In Sec. V it is attempted to simulate excitation as it occurs in the genesis of Distortion Products (DPs) and DPOAEs. It is shown that invariably two waves are generated: one traveling in the direction of the stapes and one in the direction of the base. In these model exercises, the reverse wave is always found to be dominant in the region near the stapes. In Sec. VI a method is developed to approximately separate these forward and reverse waves into two component waves. The result shows that the two component waves are of a different nature, but that their amplitudes are never more than 20 dB different. In a further test, “basis waves” for the model were created, and it was shown that the component waves behave essentially as these basis waves. In Sec. VII, finally, we summarize and discuss the results.

II. DATA AND MODEL

We first describe the acquisition of experimental data in general terms. Data on movements of the basilar membrane (BM) were collected from the basal turn of the guinea-pig cochlea with a laser velocimeter, for details see the papers cited earlier. The present study, on deeply anesthetized animals, was consistent with NIH guidelines for humane treatment of animals and was approved by the Oregon Health & Science Committee on the Use and Care of Animals. The best frequency (BF) at the recording site was between 15 and 18 kHz. As judged by the Compound Action Potential (CAP), the hearing loss due to surgery was between 7 and 25 dB at 18 kHz.

We have collected responses to sounds consisting of wide bands of flat-spectrum pseudorandom noise. The velocity of the BM was measured as a function of time, and input-output cross-correlation functions (ccfs) were computed from stimulus and response signals (see de Boer and Nuttall, 1997). Finally, the ccf spectrum was derived from the ccf waveform. For greater accuracy we constructed the “composite ccf spectrum” from parts of the frequency range measured with different bandwidths and stimulus levels—for low-frequency bands we used data with stimuli presented at higher levels—see de Boer and Nuttall (1999) for more details. Such “composite ccf spectra” will convey all the infor-
mation needed to develop a model of the cochlea adapted to a particular experiment. The last-cited paper also gives details on how the measured cf spectrum is converted into a *cochlear response pattern*, a complex function of location variable $x$. The so-derived response pattern will be used as a “template.”

Given a certain cochlear pattern, BM velocity as a function of distance $x$ along the length of the BM (the template), we can apply the inverse-solution method. With that method the *BM impedance*—which again is a function of $x$—is determined in such a way that a cochlear model with this impedance produces a cochlear-pattern response that is (almost) identical to the original pattern. In view of the aforementioned EQ-NL theorem, this model is a linear model. The model has been made three dimensional so that it can accommodate long as well as short waves. A set of convenient approximate solution methods for this model has been published by de Boer (1997), but for the present work the Green’s function method—as used in earlier work—was selected. The advantage is greater accuracy for uncommon situations (such as we will employ here). It should be noted at this point that the Green’s function describes only the hydrodynamics of the fluid, constrained as it is by the geometry of the model. Details of the inverse-solution technique are found in de Boer and Nuttall, 1999 and details on the variations of that technique used for the present paper are given in the Appendix.

The EQ-NL theorem helps to interpret results where the original stimulus conditions force the cochlea into nonlinearity. In point of fact, only in the case of production of distortion products (DPs) will we use strong (tonal) stimuli and adapt our parameters appropriately, but the propagation of DP waves will be treated by linear perturbation theory. In all other cases treated in this paper we consider only linear systems.

Figure 1 illustrates the “template response” on which our model will be based. The particular experiment, 19922, was selected from our database because it yielded a BM impedance function that is fairly smooth. The upper panel shows the original response pattern $\nu_{\text{int}}(x)$, plotted over a length of 6 mm (sufficient to accommodate the range of frequencies around 17 kHz—the best frequency for the location at which we made the original recording), in the form of a thick continuous curve for the amplitude and a somewhat thinner dotted curve for the phase. The response shown is derived from the measured response of the cochlea for a noise stimulus presented at 20 dB SPL per octave, and has been converted from the frequency to the location $x$ domain. The curves shown should be interpreted as the basilar membrane velocity pattern, amplitude, and phase, for the frequency of 17 kHz. Although the figure shows response and impedance data over an interval of 6 mm, the actual model had a length of 12 mm, and that length was subdivided into 1024 points. The thin continuous line in this panel shows the amplitude of the response of the same animal recorded post-mortem. The difference between the maximal values of the two amplitude curves (approximately 30 dB) illustrates the signal amplification that takes place in the viable cochlea. The very thin dotted line shows the post-mortem phase. The

![Figure 1](image-url)
in this case). To reduce reflection at the stapes, which would complicate the figure, we have modified the model equation, by a procedure that is also described in the Appendix. As regards interpretation, we must take into account the artificiality of the situation. In addition, we depend on the accuracy of the Green’s function solution method for this unusual stimulus situation. Figures 2 and 3 present the results of the computations for two placements of the source, and these figures clearly show the behavior of the generated waves in these two cases.

Figure 2 shows the wave pattern produced when a constant pressure (operating at 17 kHz) is prescribed (“injected”) at five consecutive points centered at location $x$ = 4.3 mm, beyond the peak of the response. Again two waves are observed, but the forward-going one is soon quenched in the region where the BM impedance has become constant. The reverse wave undergoes amplification in the same region where the template response increases in power. Internal interaction and residual reflection from the stapes cause small amplitude and phase undulations to the left of the injection region.

Figure 3 shows the corresponding situation when the same type of “injection” occurs just to the right of the response peak, at $x$ = 4.3 mm. Remember that in Figs. 2 and 3 the plotted patterns are solely due to “injection;” no stimulus is applied to the stapes, and stapes reflection is minimized. In Fig. 3 the left-going wave is more extended and shows the effect of local amplification occurring along its course. In contrast, the rightgoing wave covers only a small region of the $x$ axis before it starts to behave like an evanescent wave—showing almost constant phase. Note, also, that the phase slopes near the injection region are steeper in Fig. 3 than in Fig. 2: the waves originate closer to the peak of the template response. Both Figs. 2 and 3 demonstrate that forward and reverse waves are created together, starting from a space-confined region. That region seems wider than the actual injection region, because cochlear activity contributes to local stimulation and is not spatially confined. The interference pattern shown in Fig. 3 in the 4 mm region may be due to interference of the waves in this extended region. There may be a contribution from interference of the reverse wave and a forward wave caused by residual reflection at the stapes. Finally, there may also be an influence of irregularity of the BM impedance in this region—remember that both forward and reverse waves will undergo amplification in the 4 mm region—and this makes it difficult to analyze the situation.

IV. A DISTRIBUTED SOURCE

In the next step we critically test the extended solution procedure by approaching an actual and realistic situation very closely. We stimulate the cochlea with a pure tone, via the stapes. Using the terminology defined in the Appendix, we decompose the BM impedance—as determined from the inverse solution—as the sum of $Z_{BM}^{pass}$, the impedance associated with the response measured post-mortem, and $Z_{BM}^{act}$, the impedance specifically associated with the cochlear amplification system. Clearly, $Z_{BM}^{act}$ is determined by subtracting $Z_{BM}^{pass}$ from $Z_{BM}$:

$$Z_{BM}^{act} = Z_{BM} - Z_{BM}^{pass}.$$  

Note that that both terms in the right member of this equation are found from experiments (in the same animal). Then, the pressure $p_{ac}$, defined by

FIG. 4. Distributed “injection.” Resynthesis with a “passive” model and injection with the “active” pressure. See the text. The resynthesized response is plotted 1 dB lower than it is computed, and as a thinner curve. The associated phase curve is so similar to that of the template that it has been omitted from the figure. This figure demonstrates the accuracy of the computation procedure.

\[
p_{\text{act}} = -1/2 \nu_{\text{BM}} \cdot Z_{\text{BM}}^{\text{act}},
\]

is the equivalent pressure generated by the Outer Hair Cells (OHCs), responsible for cochlear amplification. The minus sign and the factor \(1/2\) are due to sign conventions, see the Appendix.

We next construct a model with the “passive” BM impedance \(Z_{\text{BM}}^{\text{pass}}\), and stimulate it with both a given velocity of the stapes and the distributed pressure \(p_{\text{act}}\) from Eq. (2). Figure 4 shows the result. The so-computed response is virtually identical to the original response of the model (i.e., the template response) computed with the full BM impedance \(Z_{BM}\). In order to show the individual curves, the amplitude curves have been displaced by 1 dB with respect to one another. It is seen that the correspondence extends to a point where the response is approximately 40 dB below its peak. The phase curves are indistinguishable over this range, and only the original function has been displayed. The conclusion is obvious: with a distributed form of (pressure) excitation the computation technique is virtually flawless.

One remark needs to be made with respect to Figs. 2–4. In the former two figures reflection at the stapes has been minimized with the aim of showing the reverse wave in its purest form. In Fig. 4 the stapes boundary condition has been returned to the original form—with a prescribed stapes velocity—in order to create a condition with distributed injection that is equivalent to the one used for the template response.

V. SIMULATION OF A DP CONDITION

We now attempt to realize a condition that is comparable to the one where two stimulus tones (“primaries”), with frequencies \(f_1\) and \(f_2\), are given and the principal DP is generated. We first need to simulate the response patterns of these primaries. For that purpose we construct a second template, from data obtained at a stronger stimulation (the primary stimuli are generally presented at a fairly high level) from the same animal, but still corresponding to the original frequency. As a result of nonlinearity, this template will, in general, have a lower and wider peak than the template used earlier. Next, two copies of that second template are made, these are displaced with respect to the original template and are made to correspond to the two primary frequencies \(f_1\) and \(f_2\). In doing this, we keep the DP frequency \(f_{DP}\), given by

\[
f_{DP} = 2f_1 - f_2 \tag{3}
\]

contant (equal to 17 kHz in this case), and assume the ratio \(f_2/f_1\) to be given. Call the respective template functions \(\nu_{\text{prim1}}(x;f_1)\) and \(\nu_{\text{prim2}}(x;f_2)\), keeping in mind their respective frequencies, \(f_1\) and \(f_2\). These shifted templates represent the excitation provoked by the two primary tones. Note that they reflect the compression that the primary tones undergo.

We now need to obtain an estimate of how the OHCs are stimulated by them. Consider the presentation of one of these primaries, say, the first. Let \(Z_{BM}^{\text{act}}(x;f_1)\) be the BM impedance corresponding to this second template response, displaced according to the frequency \(f_1\). Then, let \(Z_{BM}^{\text{act}}(x;f_1)\) be the active component of this impedance [cf. Eq. (1)]. As a result of nonlinearity the function \(Z_{BM}^{\text{act}}(x;f_1)\) differs from \(Z_{BM}\) of Eq. (1), evaluated for the frequency \(f_1\), especially in its real part. The model thus reflects the nonlinearity that this primary tone undergoes. The active component \(p_{\text{act}}(x;f_1)\) of the pressure generated by the OHCs is given by

\[
p_{\text{act}}(x;f_1) = -1/2 \nu_{\text{prim1}}(x;f_1) \cdot Z_{BM}^{\text{act}}(x;f_1). \tag{4}
\]

In our simple conception of the OHCs, we have assumed that these cells are simple no-memory transducers, all frequency dependence is absorbed in the complex nature of \(Z_{BM}^{\text{act}}(x;f_1)\) (see Fig. 1 and related description in de Boer and Nuttall, 2002). This means that \(p_{\text{act}}(x;f_1)\) in Eq. (4) expresses, apart from a factor, the excitation of the OHCs at the frequency \(f_1\), let us call that excitation \(\psi_1(x;f_1)\). Similarly, \(Z_{BM}^{\text{act}}(x;f_2)\) is the active component of the BM impedance valid for the frequency \(f_2\) for the secondary tone. The model thus also reflects the nonlinearity that the second primary tone undergoes. The active component \(p_{\text{act}}(x;f_2)\) of the OHC pressure at the frequency \(f_2\) is given by

\[
p_{\text{act}}(x;f_2) = -1/2 \nu_{\text{prim2}}(x;f_2) \cdot Z_{BM}^{\text{act}}(x;f_2) \tag{5}
\]

The excitation associated with \(p_{\text{act}}(x;f_2)\)—which also is a complex function of \(x\)—will be called \(\psi_2(x;f_2)\). Now, let both primary tones be presented together. Inside the hair cells the sum signal is subjected to nonlinear distortion. We assume that the third-order term in the polynomial expansion of this distortion term is the most important one. Expanding the expression

\[
[\text{Re}(\psi_1(x;f_1)e^{2\pi i f_1 t} + \psi_2(x;f_2)e^{2\pi i f_2 t})]^3,
\]

in terms with various frequencies, gives as the principal DP with the frequency \(2f_1 - f_2\), a term of which the effective (complex) amplitude \(A_{DP}(x)\) is given by

\[
A_{DP}(x) = 3/4A_{\psi_1(x;f_1)} \cdot \psi_2(x;f_2). \tag{7}
\]

The square arises because the DP frequency \(f_{DP}\) contains the term \(2f_1\), the asterisk (*) denotes the complex conjugate
and is needed because the DP frequency contains the term $-f_2$. We have included a constant of proportionality $A$ that at the moment is left unspecified, we will come back to this factor presently. A pressure proportional to $A_{DP}(x)$, having the frequency $f_{DP}$ (equal to 17 kHz), will be active as the injection source for the DP, it will be denoted by $p_{act}(x)$. The pressure $p_{act}(x)$ or, abbreviated, $p_{act}$, can be included in the model computation in the same sense as in the preceding sections; see more later. In evaluating expression (7), the actual template functions $v_{prim1}(x;f_1)$ and $v_{prim2}(x;f_2)$ and the associated impedance functions $Z_{BM}^{act1}(x;f_1)$ and $Z_{BM}^{act2}(x;f_2)$ have to be substituted. In summary, this implies that nonlinearity of the response to the primary tones is properly taken care of. Although in experimental practice the $f_1$ tone is generally presented at a 10 dB higher level than the $f_2$ tone, we have neglected this difference and have used one level (60 dB in this case) for the template functions of both primary tones. The reason is that the difference in the shape of the response patterns would be small and that the actual stimulus amplitude would only appear as a constant factor in Eqs. (4) and (7). In addition, we omit suppression of the propagating DP wave by the primary tones.

We now solve the model equation for the wave with the DP frequency. In this solution the model is equipped with the BM impedance function that corresponds to the low-level template of Fig. 1 because we consider propagation of the DP as a linear perturbation. As in Sec. III, we remove excitation with the DP frequency at the stapes in this case because no DP component is present in the actual stimulus signal at the stapes. Furthermore, we minimize the reflection of reverse waves at the stapes to simplify interpretation. The insertion of Eq. (7) in the model equation (A9) yields the DP component of the BM velocity, to be called $v_{DP}(x)$. The proportionality factor $A$ in Eq. (7) has been chosen to make the amplitude of $v_{DP}(x)$ at the location $x=4$ (mm) equal for the three figures to follow, in this way following the Allen-Fahey paradigm (cf. de Boer et al., 2005a). Figure 5 shows the result for the frequency ratio $f_2/f_1$ equal to 1.05. In this case the overlap in the excitation patterns is quite large, as the function labeled $p_{act}$ in the figure demonstrates. The DP velocity has a large and pronounced peak in its amplitude, somewhat more apically than the overlap region (thickest curve).

Two waves emerge from this peak: one going to the left and one going to the right. Both waves undergo amplification by the cochlear amplification mechanism, an effect that is included in our computation paradigm. Note that in this case the reverse wave is relatively small in its amplitude. This is caused by wave interference in the sense described by Shera (2003) and by de Boer and Nuttall (2006). In Fig. 6, where $f_2/f_1$ is equal to 1.2, the response situation is somewhat easier to interpret. Especially from the course of the phase curve, the two waves are clearly discernible. Figure 7, finally, shows the case where $f_2/f_1$ is larger, equal to 1.4; here the peak in $v_{DP}(x)$ is closer to the stapes; and it is evident that the right-going wave is undergoing appreciable amplification. This effect is to be expected, of course. On the other hand, from the phase pattern it is evident that a substantial leftgoing wave is traveling toward the stapes. In an actual experiment the level of the primary tones has to be increased for increasing frequency ratio $f_2/f_1$. This is indicated in the figures by the increased level of the templates for the primary tones. The increase shown is only symbolical, of course, because the actual level increases would be much larger. Fur-
thermore, we have abstained from using templates adapted to each \( f_2/f_1 \) condition—wider and lower for higher primary-tone levels—because in our experience the effect of wider primary-tone responses is small with this type of computation.

Let us return to Fig. 5. With \( f_2/f_1 \) near 1, a good deal of wave interference takes place; this is the reason why the DP velocity at the stapes is fairly small. With increasing frequency ratio (Fig. 6), this velocity increases, in the same manner as the data in de Boer et al. (2005a). A decreasing degree of wave interference would constitute an explanation of this property. Consider next Fig. 7. When the level of the primaries were the same as before, the wave arriving at the stapes would be smaller than in Fig. 6. That it is actually larger is, first, due to the fact that the level of the primaries has to be higher in order to achieve a constant DP level at the DP place, and, second, due to the nearness of the peak evoked by the \( f_2 \) tone to the stapes—a corresponding situation is not usually encountered in experiments on DPOAEs. For these reasons we observe only half of the familiar “band-pass” character of DPOAEs.

VI. SEPARATION OF WAVES

In principle it seems simple to separate the two waves that are visible in Figs. 5 to 7. The function \( \bar{\nu}_{\text{DP}}(x) \), the DP component in the cochlear response pattern, can be Fourier transformed to the wavenumber domain \( \bar{\nu}_{\text{DP}}(k) \). Wave components with \( k > 0 \) would refer to rightgoing and components with \( k < 0 \) to leftgoing waves. This is, however, a too simplistic attitude. Consider, for a moment, the “template” response \( \nu_{\text{tem}}(x) \) of Fig. 1. In the figure it appears as a monotonic rightgoing wave, yet its Fourier transform \( \bar{\nu}_{\text{tem}}(k) \) proves to have substantial nonzero terms for \( k < 0 \). It is easy to understand where these come from: the wave is monotonic in its phase course, but it does not have a constant amplitude nor a constant wave-propagation velocity. Therefore, its wavenumber spectrum is not a single peak (delta function) but appears to be convolved with some kind of modulation function. The convolution causes the spectrum to spread out, and even to cross the zero line. Some degree of internal reflection could add to the same effect.

It appears unavoidable, then, that the wave spectrum \( \bar{\nu}_{\text{DP}}(k) \) should be decomposed into two parts that are both two sided in the wavenumber domain. One part will have its major contribution in the positive and the other in the negative \( k \) domain. This type of decomposition is not unique, however, and we have few rules and reasons to guide us. We have decided to use the following procedure. Figure 8 shows the wavenumber spectrum \( \bar{\nu}_{\text{DP}}(k) \) corresponding to \( \nu_{\text{DP}}(x) \) of Fig. 6, on a logarithmic ordinate scale (amplitude only). We have isolated the part with \( k > 0 \) and extended it into the region \( k < 0 \) with a fixed slope (actually, that slope is a complex number); see the thick dashed line. In this way we believe to have obtained a good estimate of the complete spectrum of the rightgoing component wave. We have carried out the corresponding procedure with the isolated part of \( \bar{\nu}_{\text{DP}}(k) \) with negative wavenumbers, extending it into the domain of positive wavenumbers (the thin dashed line). This leads to an estimate of the spectrum of the leftgoing component wave. The two resulting spectra have the property that they are continuous in value and slope at and around the point \( k = 0 \). In the \( x \) domain the resulting two wave-component patterns can be expected to be smooth and “well behaving.”

Figure 9 shows, for the value of \( f_2/f_1 \) (1.2) of Fig. 6, the decomposition of the \( \nu_{\text{DP}}(x) \) wave. The figure is a simplified version of Fig. 6 and shows the extracted rightgoing and leftgoing component waves by thin curves—solid curves for the amplitude and dotted curves for the phase. Indeed, the right-going component wave, to be called \( \nu_{\text{DP}}^{\text{right}}(x) \), approximates the right-hand part of the \( \nu_{\text{DP}}(x) \) function quite well. Note that over the entire \( x \) domain its phase curve has an almost monotonic slope. The course of the leftgoing component wave, to be called \( \nu_{\text{DP}}^{\text{left}}(x) \), is somewhat less regular. It should be noted that the separation between the original \( \nu_{\text{DP}}(x) \) amplitude curve and the amplitudes of the two com-
Component waves is never very large, averaging 15 dB for the leftgoing component and being somewhat larger for the rightgoing component. This reveals that our separation method may be an approximate one, but it may well be optimal in the sense that a “better” separation cannot be achieved. We have applied the same separation method to the results shown by Figs. 2 and 3, and we found the separation of the two wave components to have the same properties as shown by Fig. 9. In summary, it should be noted that the separation of waves shown by Fig. 9 is based on Fourier transforms, that is, on average properties of the actual wave \( v_{\text{DP}}(x) \), averaged over \( x \).

For one- and two-dimensional cochlear models the theoretical study of internal reflection has been facilitated by the definition of “basis waves” (Shera and Zweig, 1991; Talmadge et al., 1998a; Shera et al., 2005). Usable forms of such basis waves can be obtained from specialized resynthesis results of our three-dimensional model. Consider Fig. 4, the resynthesized response is a wave that appears to be solely consisting of a rightgoing wave, a few small ripples signify an additional component. Applying a moderate amount of smoothing to the BM impedance functions and doing resynthesis produces a wave function that is smoother and unidirectional in a more pronounced way. We will call that wave the first basis wave, \( v_{\text{basis}}^{\text{right}}(x) \). Consider next Fig. 3; injection occurs here far to the right and the resulting wave is dominated by a leftgoing wave. Putting the injection point still more to the right will yield a wave that can serve as the second basis function, \( v_{\text{basis}}^{\text{left}}(x) \). We may consider these two responses as due to “extreme” types of stimulation. Let us now return to the component waves \( v_{\text{DP}}^{\text{right}}(x) \) and \( v_{\text{DP}}^{\text{left}}(x) \) of Fig. 9, obtained from simulation of DP generation. Write each of these wave components in the form of a slowly varying coefficient times one of the basis waves:

\[
\begin{align*}
\psi_{\text{DP}}^{\text{right}}(x) &= A(x) \cdot v_{\text{basis}}^{\text{right}}(x), \\
\psi_{\text{DP}}^{\text{left}}(x) &= B(x) \cdot v_{\text{basis}}^{\text{left}}(x).
\end{align*}
\] (8a) (8b)

The functions \( A(x) \) and \( B(x) \) play a part similar to that of the “osculating parameters” (Shera and Zweig, 1991). Figure 10 illustrates the character of this decomposition. The figure shows, for orientation, the main curves of Fig. 9 and, in the top part of the picture, amplitude (solid curves) and phase (dotted curves) of the coefficients \( A(x) \) and \( B(x) \). Each of these is shown only over the range where a basis function is expected to approximate a component wave function. The amplitudes of the coefficients \( A(x) \) and \( B(x) \) have been shifted so that their maximal levels lie at +60 dB, the phase curves have been shifted by integer multiples of \( 2\pi \) to make them come closest to the zero line (dotted horizontal line in the figure). From this figure it is clear that over a certain range the amplitudes of the two coefficients are nearly constant, the phase of \( A(x) \) is almost \( \pi \), and the phase of \( B(x) \) has a small slope. Indeed, the coefficient functions are varying more slowly than the basis functions, and the two extracted component waves therefore have an intrinsic behavior that resembles very much that of the “basis waves.” If the coefficients were constant, the component waves could be thought to originate from one point (as the basis waves do). Apparently, the distribution of excitation as involved in the generation of a DP is to a certain degree equivalent with pointwise excitation. There appears to be relatively little interaction between the two waves and little evidence of internal reflection. On a final note, it is stressed that in the analysis demonstrated in Fig. 9 the creation of the component waves is based on wavenumber spectra, whereas the present description of basis waves is founded on similarity of wave-pattern shapes, as indicated by the slow variations of the factors \( A(x) \) and \( B(x) \).
VII. CONCLUDING COMMENTS

In this study we have shown that a simulation of the process of creation of a DP [in this case always the cubic difference tone, i.e., the DP with the frequency given by Eq. (3)] invariably produces two waves: one, the forward wave, traveling in the direction of the apex and the other one, the reverse wave, traveling in the direction of the stapes. It proved possible to isolate these two waves to a certain extent (see Fig. 9). In view of our type of modeling the cochlea, it is the reverse wave, which, having arrived at the stapes, actually gives rise to a DPOAE observable in the external ear canal. The existence of the two waves is inextricably connected with the physical-mathematical nature of the hydrodynamics of the system. In terms of the variable x all expressions are symmetrical. This does not necessarily mean that under all circumstances such a pair of symmetrical waves arises. This depends on the type of excitation.

In the case treated in the present paper the excitation that simulates DP generation is distributed over only a part of the length of the model. To be more precise the injection is effective (i.e., produces DPs) only inside the strip, where the model is “active” for both of the primary frequencies. From every point in this strip two DP waves emerge: a forward and a reverse wave. Therefore, distributed (Figs. 5 to 7) as well as pointwise injections (Figs. 2 to 4) have the same basic property: they excite both of the wave components. In searching through our database, and applying the analysis described in this paper, we have found nine experiments where the BM impedance function was sufficiently smooth to allow the analysis of this paper. In all these the results were equivalent to those shown here. In a further six experiments the results were somewhat less clear but still easily interpretable. We have not found a case where the reverse wave component dominates in the stapes region.

In Sec. VI it was attempted to separate the composite DP wave into a forward-propagating and a backward-propagating component wave. That attempt was successful, but the amplitudes of the two component waves generally remain at a limited distance (generally, less than 25 dB). This result may indicate that the component waves can never be separated completely. Internal reflection due to irregularity of the BM impedance function may constrain this distance, too. By way of Fig. 10 we have shown that each of the two component waves can be considered as a “basis function” that is only slowly modulated (in amplitude and phase). Starting from the central region (the region of overlap) each wave gradually increases its amplitude, and there is relatively little evidence of interference and internal reflection. It should be noted, finally, that all our conclusions are valid for the type of model used, i.e., a classical model—defined in the Appendix as a model governed by a driving point impedance \(Z_{BM}(x, \omega)\). An extension of our work for nonclassical models is in progress.

In our work we have artificially minimized reflection at the stapes. If we would re-introduce reflection at this point, the amplitude of the forward component wave could be increased. Theoretically, it might thus be possible that the forward component wave dominates the reverse component wave. Such a case we have not been able to find in our data material, however.

ACKNOWLEDGMENTS

Most of the experiments from which we selected the response used as the “template” in this work were executed in close collaboration with Jiefu Zheng. We acknowledge with great pleasure his extreme expertise and diligence in this task. From the many datasets collected, we have selected a “template” response that seems to show the smallest degree of pollution by data errors and a minimum amount of internal reflection. This study received support from NIH NIDCD R01 DC00141 (ALN) and R01 DC003687 (CAS).

APPENDIX

The matrix formulation has earlier been described in de Boer and Nuttall (1999, 2000b). We recapitulate the description here. We will use the following symbols: \(x\) is the longitudinal coordinate, \(\omega\) the radian frequency, \(\rho\) the density of the fluid, \(u_{BM}(x, \omega)\) the BM velocity, and \(Z_{BM}(x, \omega)\) is the BM impedance. The height and width of the model are \(h\) and \(b\), respectively. The parameter \(e\) is the width of the BM divided by the width \(b\) of the model. To prepare for greater generality we have reformulated the solution in the way described by Allen (1977), Allen and Sondhi (1978), and Mammano and Nobili (1993). First, divide the \(x\) axis into \(N\) discrete points \(x_i\) \((i=1, \ldots, N)\). The pressure in the fluid \(p(x, \omega)\)—close to the BM in the upper channel—and the BM velocity \(u_{BM}(x, \omega)\) are represented by column vectors \(p\) and \(v\), both of length \(N\). The hydrodynamics of the fluid inside the model is described by a matrix \(G\) of size \(N \times N\), which represents the Green’s function—which function relates pressure to acceleration but is here used in a form to relate pressure to velocity. The pressure \(p\) can be expressed as

\[
p = i \omega p (Gv + s v_s).
\]  

(A1)

Here \(s\) is a column vector (called the stapes propagator), which represents the way the stapes boundary condition expresses itself in the pressure \(p(x, \omega)\), and \(v_s\) is the stapes velocity. In this approach, \(G\) and \(s\) express the hydrodynamics of the fluid constrained as it is by the geometry of the model and the presence of the stapes and the helicotrema. Furthermore, \(G\) and \(s\) automatically incorporate boundary conditions. Assume further that pressure and velocity are related via the driving-point BM impedance \(Z_{BM}(x, \omega)\). This implies that, mechanically, the BM receives only stimulation from its own local velocity, all other stimulation comes via the fluid, this defines our model as a “classical model.” Write the relation between pressure and velocity as

\[
p = -1/2Zv.
\]  

(A2)

where \(Z\) is a diagonal \(N \times N\) matrix, which has \(Z_{BM}(x_i, \omega)\) \((i=1, \ldots, N)\) in its main diagonal. Substitute \(p\) from Eq. (A2) in Eq. (A1):

\[
(i \omega p G + 1/2Z)v = -i \omega p s v_s.
\]  

(A3)

In the forward solution, Eq. (A3) is solved for \(v\) with
given $Z$ and $v_{a0}$. In the inverse solution the pressure $p$ is computed directly from Eq. (A1), $v$ and $v_{a0}$ being given, and Eq. (A2) is applied to find the impedance. By the nature of the problem formulation the matrix $G$ is a “full” matrix. We have derived $G$ and $s$ in the manner described by Mammano and Nobili (1993), but have given our “stylized” model a constant width $b$, a constant BM width $eb$ ($e=0.2$), and a constant height $h$. For simplicity, we have made $b$ equal to $h$, and equal to 1 (mm). The number $N$ of sections is 1024. The length of the model was chosen to be 12 mm of which the first 6 mm are shown in the figures.

The model is normally driven by a given velocity, the stapes velocity, $v_{a0}$. In order to minimize reflection from the stapes we consider the stapes to be driven via an impedance $Z_0$, from a source with strength $v_0$. In that case the following relation holds:

$$v_{a0} = v_0 - p(1)/Z_0,$$

(A4)

where $v_0$ is the driving velocity and $p(1)$ is the pressure at the stapes location. The matrix equation (A3) is formulated in terms of velocity, hence Eq. (A5) has to be rewritten as

$$v_{a0} = v_0 + 1/2[Z_{BM}(1)/Z_0] p(1).$$

(A5)

The term with $p(1)$ has to be included in the first term of Eq. (A3). To this aim a matrix $G_{01}$ is created that contains all zeros, except for the first column, where it contains the elements of the vector $s$. When used in a matrix equation, $G_{01}$ would operate only on the unknown variable $v(1)$. This leads to the revised model equation,

$$[i\omega p(G + 1/2[Z_{BM}(1)/Z_0]G_{01})]v = -i\omega p s v_0.$$  

(A6)

For a long-wave model with given BM impedance $Z_{BM}(0)$, the characteristic impedance $Z_0$ is given by

$$Z_0 = \left[1/2i\omega p h_{eff}Z_{BM}(0, 0)\right]^{1/2},$$

(A7)

where $h_{eff}$ is the “effective height” of the model, the cross-sectional area of the channels divided by the width of the BM. Note that expression (A8) has the same dimension as the BM impedance $Z_{BM}(0)$. As said above, this modification of the model equation will be used to minimize reflection at the stapes. Where there is no stimulation at the stapes we put $v_0$ equal to zero.

Equation (A3) is in reality an equation in which each term represents a pressure. When $p_{act}$ is the extra pressure (injected or generated by OHCs), expressed as a column vector, the equation can be rewritten as

$$(i\omega p G + 1/2Z)v = -i\omega p s v_{a0} + p_{act}.$$  

(A8)

To test the extended solution method (see Fig. 4), $p_{act}$ has been given as the actual “active” pressure component generated by the OHCs (computed from the BM velocity and the active component of the BM impedance), for $Z$ is taken the passive impedance component, and the equation is solved. The resulting response pattern is almost indistinguishable from the “active” resynthesized response of the model—as Fig. 4 demonstrates. Equation (A8) combined with Eq. (A6) is used to compute the distribution of wave components generated by a distributed set of pressure sources with minimized reflection at the stapes (Figs. 2–7).