Analyzing reverse middle-ear transmission: Noninvasive Gedankenexperiments

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The phenomenological framework outlined in the companion paper [C. A. Shera and G. Zweig, J. Acoust. Soc. Am. 92, 1356–1370 (1992)] characterizes both forward and reverse transmission through the middle ear. This paper illustrates its use in the analysis of noninvasive measurements of middle-ear and cochlear mechanics. A cochlear scattering framework is developed for the analysis of combination-tone and other experiments in which acoustic distortion products are used to drive the middle ear "in reverse." The framework is illustrated with a simple psychophysical Gedankenexperiment analogous to the neurophysiological experiments of P. F. Fahey and J. B. Allen [J. Acoust. Soc. Am. 77, 599–612 (1985)].

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INTRODUCTION

Otoacoustic emissions offer a promising acoustic window on the mechanics of the cochlea. That window is clouded, however, by an incomplete knowledge of the transmission properties of the middle ear. The potential for using otoacoustic emissions as a noninvasive probe of cochlear mechanics is nicely illustrated by the work of Allen and Fahey, who have recently proposed using measurements of acoustic distortion products to measure the gain of the "cochlear amplifier" (Allen and Fahey, 1992; Fahey and Allen, 1985). Cogent analysis of that and other combination-tone experiments depends critically, however, on understanding the reflection and transmission of retrograde waves by the middle ear. Models of otoacoustic emissions often ignore the considerable interference effects arising from middle-ear reflection, making it difficult to compare their predictions with experiment.

The framework outlined in the companion paper (Shera and Zweig, 1992a) characterizes both forward and reverse transmission through the middle ear. This paper—the third in a series (Shera and Zweig, 1991c, 1992a–c) devoted to problems of middle-ear mechanics—illustrates, largely by means of concrete example, how that framework can be applied to address problems of cochlear and middle-ear mechanics involving the reflection and transmission of cochlear waves by and through the middle ear.

The paper consists of two parts. In the first, the phenomenological framework outlined in the companion paper (Shera and Zweig, 1992a) is re-expressed in the equivalent language of middle-ear scattering coefficients. In the second, a cochlear scattering formalism suited to the analysis of combination- and cancellation-tone experiments is developed within the context of a simple psychophysical Gedankenexperiment similar to the neurophysiological experiments of Fahey and Allen (1985). Finally, examples are provided that illustrate the considerable interference effects that arise due to the reflection of retrograde waves from the stapes. It is shown, for example, that the ear can generate tones that are considerably louder outside the cochlea than they are within it.

I. SCATTERING REPRESENTATION OF THE MIDDLE EAR

Earlier papers (Shera and Zweig, 1991c; Shera and Zweig, 1992a) adopted the transfer-matrix formalism as a natural framework for the systematic "deconstruction" of middle-ear transduction characteristics into a product of separately measurable component transformations. The overall middle-ear transformation can, however, also be described using another representation of two-port networks suited to the analysis of wave reflection and transmission on either side of the middle ear: the scattering matrix (e.g., Carlin, 1956; Kuo, 1962). With separate knowledge of the wave impedances at the two ports, the two representations are equivalent and interconvertible. Although not as computationally convenient as transfer matrices for the description of cascades of systems, scattering matrices provide a more intuitive framework for the analysis of wave reflection and interference effects.

The middle ear converts air-borne waves into hydromechanical waves that travel along the organ of Corti. Let \( P^+ \) and \( P^- \) be the pressure waves at the eardrum propagating in directions, respectively, towards and away from the drum. On the other side of the middle ear, let the forward- and backward-traveling waves at the basal end of the organ of Corti near the stapes be denoted \( P_{o+} \) and \( P_{o-} \), respectively. The scattering matrix for the middle ear \( S_0 \) is then defined by the equation

\[
\begin{pmatrix}
P_{o+}
\end{pmatrix} = S_0 \begin{pmatrix}
P^+
\end{pmatrix},
\]

which characterizes the middle-ear transformation by expressing the two outgoing waves (i.e., the two waves propagating away from the middle ear) as linear combinations of the two incoming waves. The four matrix elements of \( S_0 \), denoted...
are thus the forward and reverse transmission and reflection coefficients for the middle ear (defined here as the part of the ear lying between the end of the ear canal and the beginning of the organ of Corti). The waves traveling in the two directions are related to the total pressure and volume velocity through the wave impedances \( Z_{\pm} \), which generalize the concept of characteristic impedance to nonuniform media (Shera and Zweig, 1991b). For example, so long as the cross-sectional area of the ear canal does not vary too rapidly, the wave impedances for the ear canal are independent of direction and equal to the local impedance to nonuniform media (Shera and Zweig, 1991b). 

To find the cochlear waves \( P_+^+ \), recall that the cochlea of the cat manifests a tapering symmetry (Shera and Zweig, 1991a) that guarantees that the wavelength or characteristic impedance \( \lambda \) changes slowly with position \( \chi \) in the basal turns of the cochlea. The cochlear wave impedances are therefore also approximately independent of direction (despite the rapid variation of scalae area in the basal turn), and

\[
P_\pm = \frac{1}{2}(P_\pm + \lambda U_\pm).
\]

When evaluated at the basal end of the organ of Corti near the stapes (i.e., at \( \chi = 0 \)), the wavelength \( \lambda_0 \) is simply the cochlear input impedance (Shera and Zweig, 1991a). 

Appendix A summarizes the formulas for obtaining the scattering matrix of a system given its transfer matrix and the wave impedances at its ports. In addition, Appendix A shows how to combine scattering matrices representing the individual networks in a cascade to find the matrix representing the cascaded system.

**II. ANALYZING COMBINATION-TONE EXPERIMENTS**

This section develops, largely by means of concrete example, a framework for the analysis of combination-tone and similar experiments in which cochlear nonlinearities create distortion products that propagate in both directions along the organ of Corti. The effects of the middle ear are explicitly included. The framework allows the convenient incorporation of cochlear reflection and interference phenomena into analytic approximations of the responses of cochlear models to multiple pure-tone stimuli.

Although the techniques outlined here are useful for the analysis of a number of experiments (e.g., Furst et al., 1988), the discussion uses as its principle example an experiment similar to that of Fahey and Allen (1985). That experiment uses distortion-product otoacoustic emissions to drive the middle ear "in reverse," and compares the amplitude of the combination tone measured in the ear canal at threshold (defined, for example, by a constant basilar membrane velocity at CF) with the pressure recorded when a tone at the distortion-product frequency—its loudness also adjusted to threshold—is played directly.

More specifically, the experiment consists of measuring the ratio \( \rho \) of ear-canal pressures \( P_{\text{ec}} \) defined by

\[
\rho = \frac{P_{\text{th}}}{P_{\text{th}}} | \frac{P_{\text{ec}}}{P_{\text{th}}} | \frac{P_{\text{th}}}{P_{\text{th}}}
\]

where the superscripted "th" indicates that the pressure is measured at psychophysical or neurophysiological threshold. The pressures \( P_{\text{th}} \) represent the complex Fourier components of the ear-canal pressure \( P_{\text{ec}} \) at the distortion-product frequency. The qualifier "incochlear source" denotes a stimulus condition in which the measured tone is generated within the cochlea as a distortion product; "external source" indicates that the tone is produced in the ear canal with an earphone.

**A. Assumptions and analysis**

Figure 1 provides a schematic diagram of the peripheral auditory system for each of the two stimulus conditions. An earpiece containing a miniature earphone and microphone is sealed into the ear canal; its Norton-equivalent source impedance is denoted \( Z_{\text{se}} \). The residual ear-canal space and the middle ear are represented, respectively, by the two-port networks \( \mathcal{N}_{\text{se}} \) and \( \mathcal{N}_{\text{m}} \). Appendix A shows how the matrix \( \mathcal{S}_{\text{se}} \), with elements denoted

\[
\mathcal{S}_{\text{se}} = \begin{pmatrix} R & T^- \\ T^+ & R^- \end{pmatrix}
\]

can be obtained by combining the elements of \( \mathcal{S}_{\text{se}} \) and \( \mathcal{S}_{\text{m}} \).

The generation of distortion products by the nonlinear interaction of primary tones at frequencies \( f_1 \) and \( f_2 \) (with \( f_2 > f_1 \)) is assumed to occur within some region \( [a,b] \) of the
cochlea, presumably near the \( f_2 \) place where the product of the envelopes of the responses to the two primaries is large. The "source region" \([a,b]\) is defined so that outside its boundaries the primaries are small enough that the cochlear response to those tones is linear. Outside the region \([a,b]\) the response to the combination tone (e.g., at a frequency \( f_{ca} = 2f_1 - f_2 \)) is therefore assumed to superpose linearly with the responses to the primaries; any nonlinear interactions—either between the primaries themselves (e.g., those generating the combination tone) or, subsequently, between the combination tone and its primaries (e.g., two-tone suppression)—thus occur, by definition, entirely within \([a,b]\).

The details of those nonlinear interactions, however, are unimportant for the phenomenological analysis presented here.

The source region is regarded as a nonlinear "glass box" (denoted \( \Phi_{bp} \)) described by level-dependent boundary conditions at its borders. The box is glass, its walls transparent, because its behavior, unlike that of a linear system, cannot be characterized completely in the frequency domain without detailed knowledge of its components. The following analysis focuses exclusively on wave propagation at the combination-tone frequency \( f_{ca} \), regarding the primaries simply as parametric "knobs" controlling the amplitude and phase of the distortion products generated within \([a,b]\). Unless otherwise noted, all equations are therefore tacitly understood to apply only at the frequency \( f_{ca} \).

1. The basal region \([0,a]\)

The basal region of the cochlea \([0,a]\) is assumed analogous to a linear, one-dimensional hydromechanical transmission line (Peterson and Bogert, 1950; Zweig et al., 1976; Zweig, 1991). That is, the analysis assumes that the wavelengths of the waves in the basal turns of the cochlea (i.e., within and basal to the source region) are long relative to the heights of the scalae and, hence, that wave propagation is one-dimensional. This so-called "long-wavelength approximation," while valid for the lower-frequency combination tone, is presumably violated within \([a,b]\) for the primaries. At the combination-tone frequency the region \([0,a]\) can thus be represented by a two-port network \( \Phi_{a} \) and has a corresponding scattering-matrix description \( S_a \).

Throughout this paper the elements of the cochlear scattering matrix \( S_y \) representing the interval \([x,y]\) are denoted

\[
S_y = \begin{pmatrix}
    r_{xy}^+ & t_{xy}^-
    t_{xy}^+ & r_{xy}^-
\end{pmatrix},
\]

(7)

For simplicity, and in accord with measurements of the cochlear input impedance in cat (Shera and Zweig, 1991a), waves are assumed to propagate without reflection through the basal turn, despite the rapid secular variation of the stiffness of the basilar membrane (see also Shera and Zweig, 1991b). The reflection coefficients are therefore assumed negligible:

\[
r_{xy}^\pm = 0 \quad (x,y < a).
\]

(8)

As an example, the cochlear transmission-line equations in Appendix C imply that the matrix \( S_y \) has the approximate form

\[
xS_y \approx \begin{pmatrix}
    0 & \sqrt{\lambda_x/\lambda_y} e^{-i\phi}
    \sqrt{\lambda_y/\lambda_x} e^{i\phi} & 0
\end{pmatrix},
\]

(9)

where

\[
\phi = \int_x^y \frac{d\chi}{\chi}.
\]

(10)

The subscripts "\( x \)" and "\( y \)" indicate that the wavelengths \( \lambda \) have been evaluated at the corresponding "port" or boundary.

2. The source region \([a,b]\)

At the combination-tone frequency, the source region \([a,b]\) has two effects: first, it acts as a source of wave energy and, second, it transmits from one boundary to the other waves originating outside the region.

Consider first its role as a source of energy at the combination-tone frequency. Imagine for the moment the region \([a,b]\) embedded in an infinite, reflectionless cochlea without other combination-tone sources. Two waves then propagate away from the source region. At the apical border one measures a net forward-traveling wave \( \bar{P}_b^+ \) and at the basal boundary a net backward-traveling wave \( \bar{P}_a^- \). Those wave amplitudes are, of course, functions of the primaries: \( \bar{P}_b^+ = \bar{P}_b^+ [\mathcal{P}_1, \mathcal{P}_2] \) and \( \bar{P}_a^- = \bar{P}_a^- [\mathcal{P}_1, \mathcal{P}_2] \),

(11)

where \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) represent the complex amplitudes of the primaries at some reference location (e.g., the stapes). The primaries are written in a calligraphic font, rather than the standard italic, to remind the reader that they represent complex amplitudes at frequencies, namely \( f_1 \) and \( f_2 \), other than the combination-tone frequency \( f_{ca} \) for which the response is sought.

Cochlear waves created outside the source region are transmitted or reflected as they propagate through \([a,b]\). In the absence of the primaries, that propagation would be described by a standard scattering matrix \( S_b \). When the primaries are present, however, possible nonlinear interactions between the primaries and waves at the combination-tone frequency may make things more complicated. A useful phenomenological approach is to regard the primaries as modifying the effective scattering matrix for the region. This approach enables one to obtain an approximate initial solution to the nonlinear problem. The procedure can then be iterated, if necessary, and, with the help of this bootstrap, higher accuracy obtained (see below).

At the frequency \( f_{ca} \) wave propagation through the region is then described by the matrix \( \tilde{S}_b \), whose elements—the reflection and transmission coefficients at \( f_{ca} \) in the presence of the primaries—are, in general, functions of the primaries. When wave propagation at \( f_{ca} \) is linear and unaffected by the presence of the primaries (e.g., when the primaries are sufficiently higher in frequency), the matrix \( \tilde{S}_b \) reduces to \( S_b \).

Descriptions of the region's wave-generation and wave-transmission characteristics can now be combined. As viewed from its boundaries, the source region is assumed to be described, at the combination-tone frequency, by the equation...
The first term in Eq. (12) describes wave propagation across the region; the second, wave generation within it. In the infinite, reflectionless cochlea described above, the vector functions of the primaries \( \mathbf{P}_a \) and \( \mathbf{P}_b \): \[
abla \mathbf{S}_b = \mathbf{S}_b [\mathbf{P}_a, \mathbf{P}_2, \mathbf{P}_a^\perp, \mathbf{P}_b^\perp].
\] (13)

The first term in Eq. (12) describes wave propagation across the region; the second, wave generation within it. In the infinite, reflectionless cochlea described above, the vector vanishes, and the region acts only as a source:

\[
\begin{pmatrix} \mathbf{P}_a^- \\ \mathbf{P}_b^- \end{pmatrix} = \begin{pmatrix} \mathbf{P}_a^- \\ \mathbf{P}_b^- \end{pmatrix} + \begin{pmatrix} \mathbf{P}_a^+ \\ \mathbf{P}_b^+ \end{pmatrix},
\] (12)

where the matrix \( \mathbf{S}_b \) and source waves \( \mathbf{P}_a^\perp \) and \( \mathbf{P}_b^\perp \) are functions of the primaries \( \mathbf{P}_a \) and \( \mathbf{P}_b \) (and, in general, of the incoming and outgoing wave amplitudes \( \mathbf{P}_a^\pm \) and \( \mathbf{P}_b^\pm \)).

3. The apical region \([b, \infty]\)

As seen from the boundary \( b \), the apical region of the cochlea can be characterized by an equivalent reflection coefficient \( R^f \) for waves of frequency \( f_a \). (The superscripted \( \triangleright \) indicates that the reflection coefficient is defined with the primary wave traveling apically to the right.) A nonzero reflection coefficient can arise, for example, from scattering of the forward-traveling wave by mechanical inhomogeneities in the organ of Corti near the \( f_a \) place. Reflected wavelets combine and propagate back toward the stapes, ultimately giving rise to stimulus-frequency emissions measurable in the ear canal (Shera and Zweig, 1992e). Since the amplitude of stimulus-frequency emissions varies with stimulus level, the reflection coefficient \( R^f \) depends on the amplitude of the combination tone. In the experiment analyzed here, however, the combination tone is held at threshold where the response is assumed linear; \( R^f \) is therefore independent of \( \mathbf{P}_a^\perp \) (see Sec. 1).

Detection of the tone at \( f_a \)—whether that tone is generated externally or within the cochlea—is assumed to occur when the velocity of the basilar membrane at the \( f_a \) place (denoted \( z_{4a} \), the characteristic-frequency point) reaches some threshold value \( V_{th} \) or, equivalently, when the forward-traveling pressure wave at the point \( b \) reaches the threshold value \( P_{b}^{th} \). Note, however, that because the pressures \( P_{b}^{th} \) (or, equivalently, the velocities \( V_{th}^{th} \)) cancel when forming the ratio, \( \rho \), given by Eq. (5), the measurement remains independent of the assumed detection criterion.

4. Iterating for self-consistency

Because of the nonlinearities, the properties of the source region (summarized at its boundaries, for the frequency \( f_a \), by the amplitudes \( \mathbf{P}_a^\perp \) and \( \mathbf{P}_b^- \) and the matrix \( \mathbf{S}_b \) depend, in general on all characteristics of the system (e.g., on boundary conditions in the ear canal). For example, the original source waves \( \mathbf{P}_a^\perp \) and \( \mathbf{P}_b^- \) are partially reflected by the cochlear boundary with the middle ear. Some fraction of their energy therefore returns to the source region, where it may interact with the primaries, perhaps changing, in turn, the amplitudes of the outgoing source waves. In general, the region \([a,b]\) cannot, therefore, be isolated and characterized independently, with exact solutions for the cochlear response at \( f_a \) obtained simply by superposition.

Such solutions can, however, be obtained by iteration. One assumes some reasonable initial values for \( \mathbf{P}_b^\perp \), \( \mathbf{P}_a^- \), and \( \mathbf{S}_b \) and "solves" the system assuming superposition at the frequency \( f_a \) (see below). The resulting approximate solution is then used to adjust the characteristics of the source region (according to equations determined by the underlying model of cochlear mechanics), and the procedure iterated until a self-consistent solution is obtained. A sample iteration algorithm is given in Appendix B.

B. Solutions assuming superposition

This section solves for the ear canal pressures \( P_{ec}^\perp \) for the two stimulus conditions by assuming that the principle of superposition holds throughout the cochlea for waves at the distortion-product frequency. Superposition implies that the cochlear response at \( f_a \) can be obtained simply by summing contributions from all combination-tone wavelets (e.g., those originating in the ear canal, within the source region, or by reflection). Depending on the nature of the cochlear nonlinearities, additional iteration may then be required to accurately approximate the behavior of the system.

Many of the equations obtained in this section represent variations on the cascading formulae derived in Appendix A. Familiarity with those derivations may be helpful in understanding the physical content of equations whose origin may otherwise appear somewhat obscure.

1. External source

To solve for the ear canal pressure \( P_{ec} \) in terms of the pressure \( P_{b}^\perp \), note that

\[
P_{ec} = P_{ec}^\perp + P_{ec} = P_{ec}^\perp (1 + R^f_{ec}).
\] (15)

Similarly,

\[
P_{b} = P_{b}^\perp + P_{b} = P_{b}^\perp (1 + R^f_{b}).
\] (16)

When the source \( \tilde{U}_s \) is used to generate the tone externally (Fig. 1, top),

\[
P_{b}^\perp = \frac{t_{ob}^\perp t_{tb}^\perp P_{ec}^\perp}{1 - R_{b}^f t_{ob}^\perp t_{tb}^\perp R_{b}^{-}}.
\] (17)

The numerator represents the wave produced by direct transmission of the forward-traveling wave \( P_{ec}^\perp \) in the ear canal; that wave is then multiply reflected (with reflection coefficient \( R_{b}^f \) in the forward direction and \( t_{ob}^\perp \) in the backward). Summing those multiple reflections introduces an overall multiplicative factor, given by the reciprocal of the denomenator (see Appendix A), and yields Eq. (17) for the net forward-traveling wave at \( b \).

Combining Eqs. (15) and (17) yields the relation

\[
P_{ec}^\perp / P_{b}^\perp = \frac{1 + R^f_{ec}}{1 - R_{b}^f t_{ob}^\perp t_{tb}^\perp R_{b}^{-}}.
\] (18)

The cascading formulae in Appendix A imply that
\[ R_{\text{det}} = \frac{R^+ - R \det S_{\text{det}}}{1 - RR^-}, \]  
(19)

where \( R = R_{\text{det}} \) is the traveling-wave ratio evaluated at the basal end of the cochlear spiral (Shera and Zweig, 1992d). Since waves are assumed to propagate through the basal region \([0,b]\) without reflection, \( R \) and \( R_{\text{tot}}^\circ \) are related by

\[ R = t_{\text{tot}}^a t_{\text{det}}^b R_{\text{tot}}^\circ. \]  
(20)

Equation (18) can therefore be simplified to yield

\[ P_{\text{tot}}^a / P_{\text{det}}^b = \frac{(I + R_{\text{tot}}^\circ) (1 - RR^-)}{t_{\text{tot}}^b T^+}. \]  
(21)

2. Intracochlear source

When the tone is generated within the cochlea as a distortion product (Fig. 1, bottom), the total forward-traveling wave at \( b \) is

\[ P_{\text{tot}}^+ = \frac{P_{\text{det}}^+ + P_{\text{det}}^- R_{\text{tot}}^\circ t_{\text{tot}}^+}{1 - R_{\text{tot}}^\circ R_{\text{det}}^\circ}. \]  
(22)

In this case, the numerator contains contributions from both source waves \( P_{\text{det}}^+ \) and \( P_{\text{det}}^- \); the latter, after partial reflection at \( a \), propagates across \([a,b]\) as a forward-traveling wave before joining with \( P_{\text{det}}^+ \). As before, the net forward-traveling wave is obtained by summing the effects of multiple reflection. Note that we assume \( R_{\text{det}} = 0 \), for simplicity.

Similarly, the total backward-traveling wave at \( a \) is

\[ P_{\text{tot}}^- = \frac{P_{\text{det}}^- + P_{\text{det}}^+ R_{\text{tot}}^\circ t_{\text{tot}}^-}{1 - R_{\text{tot}}^\circ R_{\text{det}}^\circ}. \]  
(23)

For future reference, note that the relations

\[ R_{\text{det}}^\circ R_{\text{tot}}^\circ = R_{\text{tot}}^\circ R_{\text{det}}^\circ \]  
(24)

imply that

\[ R_{\text{det}}^\circ R_{\text{tot}}^\circ = R_{\text{tot}}^\circ R_{\text{det}}^\circ. \]  
(25)

Waves traveling outwards from the eardrum are reflected by the transducer assembly with reflection coefficient \( R_s \), given in terms of the Norton-equivalent source impedance by the equation

\[ R_s = \frac{Z_s - Z_0}{Z_s + Z_0}. \]  
(26)

where \( Z_s \) is the characteristic impedance of the ear canal. Consequently, the total ear-canal pressure can be written

\[ P_{\text{ec}} = P_{\text{tot}}^- (1 + R_s), \]  
(27)

where the outward-traveling wave

\[ P_{\text{tot}}^- = \frac{t_{\text{tot}}^b T^- P_{\text{tot}}^-}{1 - R_{\text{tot}}^\circ R_s T^+}. \]  
(28)

Combining Eqs. (22)-(28), one obtains

\[ P_{\text{tot}}^a / P_{\text{tot}}^b = \frac{t_{\text{tot}}^b T^- (1 + R_s)}{1 - R_{\text{tot}}^\circ R_s + \frac{\tilde{\alpha} + t_{\text{tot}}^- R_{\text{tot}}^\circ}{1 + \tilde{\alpha} R_{\text{tot}}^\circ}}. \]  
(29)

where \( \tilde{\alpha} \) is the ratio of source-wave amplitudes:

\[ \tilde{\alpha} = \frac{\tilde{P}_{\text{det}}^-}{\tilde{P}_{\text{det}}^b}. \]  
(30)

3. The pressure ratio \( \rho \)

Combining the two pressure ratios [Eqs. (21) and (29)] yields

\[ \rho = \frac{t_{\text{tot}}^b T^+ T^- (1 + R_s)}{(1 + R_{\text{tot}}^\circ) (1 - R_{\text{tot}}^\circ R_s)} \frac{\tilde{\alpha} + t_{\text{tot}}^- R_{\text{tot}}^\circ}{1 + \tilde{\alpha} R_{\text{tot}}^\circ} \]  
(31)

Note that the pressures \( P_{\text{tot}}^a \) and \( P_{\text{tot}}^b \) have cancelled in the ratio; \( \rho \) is therefore independent of the assumed detection criterion. Section II C evaluates Eq. (31) in several simple limiting cases.

C. Some simple examples

This section illustrates the formalism by calculating the ratio of ear-canal pressures \( \rho \) for the simple case in which the distortion product is assumed to originate from a distribution of ideal point sources and the middle ear is regarded as a simple mechanical transformer.

1. A distribution of ideal sources

For this example, assume that \( S_{\text{det}} = S_{\text{det}}^\circ \) and let the generation of distortion products in \([a,b]\) be modeled by a distribution of ideal "current" sources (each with negligible Norton-equivalent source admittance) operating at the distortion-product frequency. Each source is assumed to launch waves of equal amplitude in the two directions. Although cochlear nonlinearities are responsible for creating the combination tone (i.e., for introducing the sources), once created the source wavelets are assumed, in this example, to superpose linearly with the primaries and with each other.

Sources at different locations may interfere with one another in complicated ways. The assumed superposition of source wavelets implies, however, that the waves \( P_{\text{tot}}^+ \) and \( P_{\text{tot}}^- \) measured at the boundaries can be written as an integral over the source region. The relative amplitude \( \tilde{\alpha} \) of the waves generated by the distribution of sources is then summarized by the simple formula

\[ \tilde{\alpha} = \int_{a}^{b} \sigma(\chi) i - (a,\chi) d\chi \int_{a}^{b} \sigma(\chi) i + (\chi, b) d\chi \]  
(32)

where \( i ^ \pm (x, y) \equiv i ^ \pm (x, y) \) and the complex source density function \( \sigma(\chi) \) represents the source strength per unit length \( d\chi \). The transmission coefficients \( i ^ \pm (a, \chi) \) and \( i ^ + (\chi, b) \) simply propagate wavelets from their site of generation to the appropriate boundary.

In this simple model, the source density function \( \sigma \) can be expected to depend, for example, on the amplitude of the primaries, particularly that of \( f_2 \). As \( f_2 \) increases in amplitude, the region of overlap between the two primaries increases and \( \sigma \) broadens. Consider, however, the limiting case of a single ideal point source located at \( p \):

\[ \sigma(\chi) = \sigma_0 \delta(\chi - p), \]  
(33)

where \( \delta \) is the Dirac \( \delta \) function. Then,

\[ \tilde{\alpha} = i ^ \pm p / t ^ \pm p. \]  
(34)
A transformer middle ear

To simplify the calculation of middle-ear transmission, let the residual ear-canal space be negligible (so that $S_0 = S_0^-$—see footnote 2) and imagine that the middle ear acts like a mechanical transformer, albeit with a possibly complex and frequency-dependent transformer ratio $N_{me}$ (e.g., Allen and Fahey, 1992). The action of the middle ear is that of a transformer if its transfer matrix has the form (e.g., Shera and Zweig, 1992a)

$$e_{T_0} \approx \begin{pmatrix} 1/N_{me} & 0 \\ 0 & N_{me} \end{pmatrix}. \quad (35)$$

The interconversion formulae in Appendix A imply that the elements of $S_0^-$ then satisfy the equations

$$r^- = -r^+, \quad (36)$$

and

$$\det S_0 = -1. \quad (37)$$

A noninvasive experiment that exploits cochlear nonlinearities to test the transformer assumption is described elsewhere (Shera and Zweig, 1992f).

The pressure ratio

If evoked emissions are ignored (so that $R = 0$), Eq. (31) for $\rho$ then reduces to

$$\rho = \frac{\gamma_p^2(1 + r^-)(1 + R_s)}{1 + R_s r^- + \gamma_p^2(r^- + R_s)}, \quad (38)$$

where

$$\gamma_p^2 = \frac{1}{\omega_{0p}^2 \omega_{0p}^2}. \quad (39)$$

Equation (9) implies that

$$\gamma_p^2 \approx \exp \left( -2i \int_{0}^{\infty} \frac{d\gamma_p}{\lambda} \right). \quad (40)$$

(a) Limiting special cases. Two special cases are of interest. If the transducer source impedance $Z_s$ is perfectly matched to the characteristic impedance of the ear canal, then $R_s = 0$, and

$$\rho = \frac{\gamma_p^2(1 + r^-)}{1 + \gamma_p^2 r^-} (R_s = 0). \quad (41)$$

An equation equivalent to Eq. (41) has been obtained independently by Allen and Fahey (1992), who use it to estimate the gain of the "cochlear amplifier" (i.e., $|\gamma_p|$). Note that when the gain is large (i.e., $|\gamma_p| > 1$), the ratio $\rho$, in this example, depends only on the basal reflection coefficient $r^-$:

$$\lim_{|\gamma_p| \to \infty} \rho = (1 + r^-)/r^- . \quad (42)$$

If the source impedance $Z_s$ is infinite, then $R_s = 1$, and

$$\rho = \frac{2\gamma_p^2}{1 + \gamma_p^2} (R_s = 1). \quad (43)$$

In this case, $\rho$ is, remarkably, independent of the middle ear, with

$$\lim_{|\gamma_p| \to \infty} \rho = 2. \quad (44)$$

A demonstrated example

This section demonstrates the considerable, and often underappreciated, effects that can arise due to cochlear reflection and interference phenomena by computing the ratio $\rho$ using a more realistic model of the middle ear. The generation of distortion products is modeled by a single, ideal point source located at $p \in \{a, b\}$, with $S_0^+ = S_0^-$, neglecting the concomitant generation of stimulus-frequency emissions at the distortion-product frequency (i.e., $R_p^+ = 0$ is assumed zero). These simplifying assumptions are adopted purely for the purposes of illustration; the situation in the real cochlea (or any nonlinear model) will presumably be more complicated. Middle-ear scattering coefficients are computed using published models of the human middle ear. This simple Gedankenexperiment is first analyzed heuristically and then with the help of the cochlear scattering formalism outlined above.

Figure 2(a) plots $[\rho_1(f_s)]$ computed using the middle-ear models of Zwislocki (1962) and Kringlebotn (1988). (The subscript "1" denotes the value of $R_s$ and indicates that the calculations assume that the Norton-equivalent source impedance $Z_s$ of the transducer is infinite and hence that the ear-canal reflection coefficient is $+1$.) Shown for comparison in Fig. 2(b) are corresponding calculations for the cat.

![FIG. 2. Amplitude of the pressure ratio $\rho_1(f_s)$ for the human (panel a) and feline (panel b) ears. In this idealized example, distortion products at the frequency $f_s$ are produced by a single, ideal point source located at $p$. For the human, the predictions use the middle-ear models of Zwislocki (---) and Kringlebotn (---); for the cat, they use the models of Carr and Zweig (-----) and Puria (----) in which the cavity impedances have been set to zero to simulate open-cavity recording conditions. Note that the scales along the axes differ in the two panels. The frequency $f_s$ was fixed at 10 kHz in the human and at 20 kHz in the cat. Other parameter values and matrix representations for $\gamma_p$ and $\gamma_p'$ are given in Appendix C. The calculations assume that the Norton-equivalent source impedance $Z_s$ is infinite. Note that $|\rho|$ can become much greater than unity ($\cdots \cdots$) when wave cancellation occurs within the cochlea.](image-url)
eter values and matrix characterizations for the networks
corresponding wave has either been generated directly by
the abscissa. [This paradigm, chosen for the simplicity of its
analysis, differs from that of Fahey and Allen (1985), who
held \( f_2 \) fixed and varied the primaries.] The figures indicate
that \( |\phi| \) has considerable structure, including the presence of
"resonance peaks" due to interference effects within the
cochlea (see below). Although details of the predictions
such as the locations and widths of the peaks depend on
characteristics of the middle and inner ears not known with
certainty, the qualitative features of the curves are robust.

1. Interpretation as an interference phenomenon

Although one might naively expect \( |\phi| \) always to be less
than unity—the idea being that because all measurements
are performed at the \( f_{ct} \) threshold, a supra-threshold pres-
cure can never appear in the ear canal (Fahey and Allen,
1985)—no such constraint is apparent in Eq. (31) for \( \rho \).
Indeed, Fig. 2 clearly indicates that \( |\phi| \) can become quite
large. To understand this (and the origin of the "resonance
peaks"), note that the distortion-product source generates
two waves: a wave that travels apically towards its character-
stic place and another that travels basally towards the
stapes, where it undergoes partial reflection (cf. Shera and
Zweig, 1991b). Large values of \( |\phi| \) occur when the two for-
ward-traveling waves (i.e., the forward-traveling wave or-
iginally produced by the source and that subsequently gener-
ated from the backward-traveling wave by reflection off the
stapes) interfere with one another and nearly cancel at the
site of detection. The total phase difference between the
waves, which determines the extent to which cancellation
occurs, depends both on propagation delays in traveling to
and from the stapes and on the phase of the stapes reflection
coefficient.

Figure 3 illustrates how phase shifts due to propagation
and reflection sum to yield the total phase difference by plot-
ting the ratio of the two forward-traveling waves at the \( f_2 \)
place \( p \):

\[
\psi = P^-_p \left| \text{reflected} \right| / P^-_p \left| \text{generated} \right| .
\]  

The \( P^\pm \) denote the wave components and the subscripts \( \pm \)
indicate the direction of travel (the + indicates that the
wave is traveling towards the helicotrema). The qualifiers
"generated" and "reflected" indicate, respectively, that the
corresponding wave has either been generated directly by
the distortion-product source at \( p \) or created by reflection of
the retrograde wave \( P^-_p \left| \text{generated} \right| \) from the cochlear
boundary with the middle ear. The ratio \( \psi \), computed at \( p \), is main-
tained as both waves propagate towards their common char-
acteristic place.

An approximate expression for \( \psi \) can be obtained by
noting that measurements of the cochlear input impedance
in cat (Lynch et al., 1982) imply that the wavelength \( \lambda \)
of the traveling wave (or, equivalently, the characteristic im-
pedance of the transmission line) changes slowly in the basal
turns of the cochlea (Shera and Zweig, 1991a). Since
\[ |\chi| \ll 1, \] the wave impedances \( Z_\pm \) seen by the source \( U_{ct} \)
are approximately equal in the two directions (Shera and
Zweig, 1991b):

\[
Z_\pm \approx Z_\pm .
\]  

Hence, the backward- and forward-traveling waves pro-
duced by the source are nearly equal at \( p \):

\[
P^-_p \mid \text{generated} \approx P^-_p \mid \text{generated} .
\]  

Consequently, \( \psi \) can be re-expressed as

\[
\psi \approx R_{p^-}^2 \equiv P^-_p \mid \text{reflected} / P^-_p \mid \text{generated} .
\]  

The ratio \( \psi \) of the two forward-traveling waves is thus approxi-
pately equal to the traveling-wave ratio \( R_{p^-}^2 \), measured
at \( p \), when the cochlea is driven "in reverse." The magnitude of
that traveling-wave ratio is determined principally by any
impedance mismatch at the cochlear boundary with the
middle ear. Were that boundary not present, but the cochlea
instead extended infinitely in the negative-x direction, the
traveling-wave ratio \( R_{p^-}^2 \) would be zero (cf. Shera and Zweig,
1991b).

The traveling-wave ratio \( R_{p^-}^2 \) measured at \( p \) will differ
from its value \( R_{p^-}^2 \) at \( x = 0 \) principally because of phase
shifts introduced by traveling to and from the stapes. Those
phase changes can be computed using the WKB approxima-
tion to solve for the wave components \( P^\pm \). The resulting
WKB waves—which constitute accurate approximate solu-
tions to the transmission-line equations describing the basal turn of the cat cochlea at low frequencies (Shera and Zweig, 1991a)—are defined in Shera and Zweig (1991b). That paper also computes the basal reflection coefficient $R_0^{\circ}$ for the cat (with intact cavities), under both normal, physiological conditions and the simulated recording conditions used here.

Equation (48) therefore implies that $\psi$ has the approximate value (see Appendix C)

$$\psi \approx R_0^{\circ} e^{-\frac{8N_0(p_0-\beta_0)}{\beta_0}} \quad (\beta_0 < 1),$$

(49)

where

$$\beta_0 = f_a/f_2 \quad \text{and} \quad \beta_0 = f_a/f_{ca}.$$  

(50)

The frequency $f_{ca}$ represents the maximum frequency of hearing, and $N$ represents the approximate number of wavelengths of the pressure wave in the cochlea in response to sinusoidal stimulation. Thus, $\psi$ is simply the product of two factors: the basal reflection coefficient $R_0^{\circ}$ and a phase shift $e^{-\frac{8N_0(p_0-\beta_0)}{\beta_0}}$ due to round-trip wave propagation through the basal turn. Peaks in $|\psi|$ occur at frequencies $f_{peak}$ for which

$$L\psi(f_{peak}) = -(2n+1)\pi \quad (n = 0, 1, \ldots);$$

(51)

that is, when the two waves are out of phase at the $f_{ca}$ place.

The results and interpretation presented heuristically above can be obtained directly from the cochlear scattering formalism outlined in Sec. II B. In the idealized case considered here (i.e., point source at $p_1$; superposition at $f_{ca}$; $R_p^{\circ} = 0$; and $R_s = +1$), Eq. (31) for $p_1$ reduces to

$$p_1 = \frac{2\gamma^2 T^+ T^-}{(1 - R^+ R^+)(1 + R^\circ)}.$$  

(52)

Poles in $p_1$ occur at zeroes of the denominator. In our case $|R^+|^2 < 1$ (e.g., Puria and Allen, 1991), and the second factor in the denominator dominates the behavior. Thus, $p_1$ has a pole whenever

$$R^\circ = -1.$$  

(53)

Since $\psi \approx R_0^{\circ}$ [Eq. (48)], peaks in $|p_1|$ occur whenever

$$L R^\circ \approx L\psi = -(2n+1)\pi \quad (n = 0, 1, \ldots),$$

(54)

in agreement with Eq. (51) derived above.

Note that $R_0^{\circ}$ can be written

$$R_0^{\circ} = R_{p}^\circ R_0^\circ$$

(55)

where

$$R_0^\circ = \frac{R - R_s \det S_0}{1 - R_s R^\circ},$$

(56)

with, in this case, $R_s = +1$. Comparison with Eq. (49) for $\psi$ shows the round-trip phase shift $e^{-\frac{8N_0(p_0-\beta_0)}{\beta_0}}$ is simply $\gamma^2$, the product of the one-way transmission coefficients $\gamma_0^{\circ}$ and $\gamma_p^{\circ}$.

Note that at frequencies for which $|\psi| > 1$, the intracochlear source is producing a sound that is, by this measure, much louder outside the cochlea than it is within it. The interference effects seen here differ from those that may underly the microstructure observed in the threshold hearing curve (Elliot, 1958; Thomas, 1975; Kemp, 1980; Zweig, 1991). Recall that, for simplicity in the example, the scattering of traveling waves and their possible amplification at low levels by the “lasing” action of the cochlea (Zweig, 1991) have here been ignored (i.e., $R_p^{\circ} = 0$). In the more realistic case—or an actual measurement—one expects an additional fine structure, cognate to the microstructure in the threshold hearing curve, superimposed on the features found here.

III. SUMMARY

This paper has shown how the phenomenological framework outlined in the companion paper (Shera and Zweig, 1992a) can be employed in the analysis of noninvasive measurements of middle-ear and cochlear mechanics. A cochlear scattering framework has been developed for the analysis of combination-tone experiments in which acoustic distortion products are used to drive the middle ear “in reverse.” The framework—applied to the analysis of a simple noninvasive Gedankenexperiment—was used to demonstrate that the ear can generate sounds that are considerably louder outside the cochlea than they are within it.

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APPENDIX A: TRANSFER AND SCATTERING MATRICES

This Appendix provides a succinct outline of basic properties of transfer and scattering matrices, including rules for their mutual interconversion. In addition, formulæ for computing the overall transfer or scattering matrix characterizing a cascade of networks each individually so characterized are given.

1. Definitions and Interconversion

Consider a two-port network $\mathbf{\mathbb{I}_2}$ with ports labeled “1” and “2.” Let $V_i$, $I_i$, $V^+$, and $V^-$ be, respectively, the Fourier transforms of the total “voltage” and the total “current” (positive flowing to the right) at the two ports. The transfer matrix, $T_{ij} \equiv \langle V_j | I_i \rangle$, for the system is defined by

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = T_{ij} \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}.$$  

(51)

Define right- and leftward traveling waves (denoted $V^+$ and $V^-$, respectively) by the equations

$$V_i^\pm = \hat{\mathbf{\mathbb{I}_2}} (V_i \pm Z_i I_i),$$

(52)

where $Z_i$ and $Z_2$ are the wave (or characteristic) impedances at the two ports (assumed to be independent of the

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The corresponding scattering matrix,
\[ \mathbf{S}_1 \equiv \begin{pmatrix} R^+ & T^- \\ T^+ & R^- \end{pmatrix}, \]  
(A3)
is then defined by the equation
\[ \mathbf{S}_1 = Y_{12} \begin{pmatrix} B - DZ_1 + AZ_2 - CZ_2 \\ 2Z_2 \end{pmatrix} \begin{pmatrix} 2Z_2 \det \mathbf{T}_2 \\ B + DZ_1 - AZ_2 - CZ_2 \end{pmatrix}, \]  
(A4)

Simple algebraic manipulation allows the matrix elements of \( \mathbf{S}_2 \) to be found from the elements of \( \mathbf{T}_2 \), and vice versa. In terms of the elements of \( \mathbf{T}_2 \),
\[ \begin{pmatrix} V_1^- \\ V_2^- \end{pmatrix} \equiv \mathbf{S}_2 \begin{pmatrix} V_1^+ \\ V_2^+ \end{pmatrix}. \]  
(A4)

where
\[ 1/Y_{12} = B + DZ_1 + AZ_2 + CZ_2. \]  
(A6)

And in terms of the elements of \( \mathbf{S}_2 \), the elements of \( \mathbf{T}_2 \) are found to be:
\[ A = (1 + R^+ - R^- - \det \mathbf{S}_2)/2T^+; \]  
(A7)
\[ B = (1 + R^+ + R^- + \det \mathbf{S}_2)Z_2/2T^+; \]  
(A8)
\[ C = (1 - R^+ - R^- + \det \mathbf{S}_2)/2Z_1T^+; \]  
(A9)

and
\[ D = (1 - R^+ + R^- - \det \mathbf{S}_2)Z_2/2Z_1T^+. \]  
(A10)

If the network \( \mathbf{H}_2 \) is reciprocal,
\[ \det \mathbf{T}_2 = 1 \text{ and } Z_1T^+ = Z_2T^- \]  
(A11)

2. Cascades

Consider a cascade (\( \mathcal{O} \)) of two networks:
\[ \mathbf{H}_3 = \mathbf{H}_2 \mathcal{O} \mathbf{H}_3. \]  
(A12)
The transfer matrix \( \mathbf{T}_3 \) for the cascade is simply the product of the corresponding matrices:
\[ \mathbf{T}_3 = \mathbf{T}_2 \mathbf{T}_3. \]  
(A13)
The rules for cascading scattering matrices are more elaborate. Let the two scattering matrices have elements
\[ \begin{pmatrix} r^+ + t^- + r^- R + t^+ T + R^- T^- \end{pmatrix} \]  
(A14)

Then,
\[ \begin{pmatrix} r^+ + t^- + r^- R + t^+ T + R^- T^- \end{pmatrix} \]  
(A15)

Although Eq. (A15) can be tediously proved by algebra, a more physically illuminating derivation goes as follows. Let a rightward-traveling wave of unit amplitude be incident from the left on \( \mathbf{H}_2 \). The matrix element \( (\mathbf{S}_3)_{11} \) is then simply equal to the amplitude of the net reflected wave, which is easily obtained by superposition; that is, by summing that fraction \( r^+ \) of the incident wave initially reflected by \( \mathbf{H}_2 \) with the series of wavelets that "rattle around," (i.e., are reflected back and forth between the two systems \( \mathbf{H}_2 \) and \( \mathbf{H}_3 \), all possible number of times) before "escaping" and contributing to the sum:
\[ \begin{pmatrix} r^+ + t^- + r^- R + t^+ T + R^- T^- \end{pmatrix} \]  
(A15)

Analogous arguments yield expressions for the other matrix elements.

The net reflection coefficient for a two-port terminated with a one-port (e.g., an impedance) follows trivially from Eq. (A16). Let \( \mathbf{H}_3 \) be terminated at port 2 in a one-port characterized by the reflection coefficient \( R \). The reflection coefficient \( r \) measured at port 1 is then
\[ r = \frac{r^+ + t^- + r^- R + t^+ T + R^- T^-}{1 - r^- R^-}. \]  
(A17)

When the termination is attached to port 1 and \( r \) measured at port 2, symmetry implies that
\[ r = \frac{r^+ - R^+ \det \mathbf{S}_2}{1 - r^- R^-}. \]  
(A18)

APPENDIX B: AN ITERATION ALGORITHM

This Appendix illustrates the use of the equations derived in Sec. II by providing an example algorithm for obtaining a self-consistent solution for the pressure ratio \( P_b/P_e \) by iteration. For simplicity, we assume that forward-traveling waves passing the apical boundary of the source region are not subsequently reflected (i.e., that \( R_b^+ = 0 \)). The steps in the algorithm are as follows.

1. Fix the values of the primaries, \( \mathbf{P}_1 \) and \( \mathbf{P}_2 \), at the stapes.

2. Assume, initially, that the cochlear response at the frequency \( f_{\chi} \) is zero (i.e., that \( P_{\mathbf{d}} = 0 \) and \( P_{\mathbf{d}} = 0 \)).

3. Determine the source characteristics (that is, find the waves \( \mathbf{P}_b^- \) and \( \mathbf{P}_b^+ \)), and the matrix \( \mathbf{S}_b \) using the equations defining the cochlear model and the current values of \( P_{\mathbf{d}}^+ \) and \( P_{\mathbf{d}}^\pm \).
(4) Assume superposition and compute the distortion-product pressures \( P_+ \) and \( P_- \) using the following equations:

\[
P_+ = P_+ + \bar{P}_- R_+ \quad \text{[from Eq. (22)];}
\]
\[
P_- = 0 \quad \text{[from Eq. (16)];}
\]
\[
P_- = \bar{P}_- \quad \text{[from Eq. (23)];}
\]

and

\[
P_+ = \bar{P}_- R_+ \quad \text{[inferred from Eq. (22)].}
\]

Recall that \( R_+ \) and \( R_- \) have all been assumed zero.

(5) Repeat steps #3 and #4 until satisfactory convergence for \( P_+ \) and \( P_- \) is obtained.

(6) Compute \( P_+/P_\infty \) using Eq. (29).

APPENDIX C: REPRESENTATIONS OF THE EAR CANAL AND COCHLEA

This Appendix presents simple scattering-matrix descriptions of the two ports \( \mathbf{\Sigma}_p \) and \( \mathbf{\Sigma}_s \), representing, respectively, the residual ear-canal space and the basal portion of the cochlea. The corresponding transfer matrices can be found using the interconversion formulae summarized in Appendix A.

1. The ear canal

The residual ear-canal space between the transducers and the eardrum is modeled as a rigid-walled cylindrical tube of constant cross section. At frequencies low enough that the pressure is uniform in any cross section, the scattering matrix \( \mathbf{S}_o \) then has the approximate form (e.g., Pierce, 1981)

\[
\mathbf{S}_o \approx \begin{pmatrix} 0 & e^{-ikL} \\ e^{ikL} & 0 \end{pmatrix},
\]

where the wave number \( k = \omega/c \) and \( L \) is the length of the tube. The characteristic acoustic impedance of the tube is given by

\[
Z_o = \rho c / S
\]

where \( \rho \) is the density of air, \( c \) is the speed of sound, and \( S \) is the cross-sectional area of the tube. At 34 °C the constants have the approximate values: \( \rho = 1.15 \times 10^{-3} \text{ g/cm}^3 \); \( c \approx 3.52 \times 10^4 \text{ cm/s} \). The parameter values used in the calculations are \( L = 1.5 \text{ cm} \) and \( S = 0.4 \text{ cm}^2 \) for the human; and \( L = 1.5 \text{ cm} \) and \( S = 0.2 \text{ cm}^2 \) for the cat. The parameter values for the cat are the same as those used to compute the stapes reflection coefficient under simulated recording conditions (Shera and Zweig, 1991b).

2. The cochlea

The basal portion of the cochlea (i.e., between the stapes and the distortion-product source) is modeled as a linear, one-dimensional, hydromechanical transmission line of the type used in an earlier paper to discuss the reflection of retrograde waves (Shera and Zweig, 1991b). The model is assumed scaling-symmetric, in accord with measurements of the cochlear input impedance (Shera and Zweig, 1991a).

For the purposes of the examples, the wavelength (or characteristic impedance) \( \lambda \) was taken to have the form

\[
\lambda = \lambda_0 \sqrt{1 - \beta^2 + i\delta^2},
\]

where \( \beta = f / f_c \) and \( f_c(x) \) is the cochlear frequency-position map. The real constant \( \lambda_0 \) was set equal to the value \( R_0 \) of the cochlear input impedance (resistance) adopted by the middle-ear model used for the calculation. [For the feline middle-ear model of Puria (1991), the resistance measured by Lynch et al. (1982) was used.] Note that when the distance \( x \) between two points along the cochlear transmission line is defined to be \( -i \) times the total series impedance between them, then \( \lambda \) is simply \( 1/2\pi \) times the wavelength of the traveling pressure wave.

Solving the transmission-line equations using the WKB approximation yields an expression for the wave components \( P \pm \) (Zweig et al., 1976):

\[
P \pm \approx \sqrt{\lambda} \exp \left( \mp i \int_0^x \frac{dx}{\lambda} \right),
\]

The scattering matrix \( \mathbf{S}_p \) follows immediately:

\[
\mathbf{S}_p \approx \begin{pmatrix} 0 & e^{-ikx} \sqrt{\lambda / \lambda_0} e^{-i\phi} \\ e^{ikx} \sqrt{\lambda_0 / \lambda} & 0 \end{pmatrix},
\]

where

\[
\phi = \int_0^x \frac{dx}{\lambda}.
\]

Since the integrals cover only the basal, small-\( \beta \) region of the cochlea, \( \lambda \) is essentially real, and (Zweig et al., 1976)

\[
\int_0^x \frac{dx}{\lambda} \approx 4N(\beta_+ - \beta_0) \quad (\beta_0 \ll 1).
\]

Here, the variables \( \beta_+ = f_c / f_c \) and \( \beta_0 = f_\text{st} / f_c \), assuming that \( f_\text{st}(\rho) = f_c \). For the human, \( f_\text{st} \approx 20 \text{ kHz} \); for the cat, \( f_\text{st} \approx 57 \text{ kHz} \) (Liberman, 1982). The constant \( N \) represents the approximate number of wavelengths of the traveling wave in the cochlea in response to sinusoidal stimulation. The value \( N = 5 \) was assumed in all calculations.

For simplicity, the calculations shown in Figs. 2 and 3 assume that the Norton-equivalent source impedance \( Z \) is infinite. The input impedance of the apical portion of the cochlea seen from \( p \) is approximated as

\[
Z_{\infty} \approx \lambda_0.
\]

Note that \( Z_{\infty} \) is a smooth function of frequency and does not include oscillations arising from the interference of wavelets reflected from mechanical inhomogeneities expected in the more realistic case.

1 In the notation of the companion paper (Shera and Zweig, 1992a), which was concerned only with the sum and difference pressures just inside the cochlear windows, the two-port \( \mathbf{\Sigma}_s \) would be written as the cascade \( \mathbf{\Sigma}_s \circ \mathbf{\Sigma}_a \) where \( \mathbf{\Sigma}_a \) represents the vestibular space between the cochlear windows and the beginning of the organ of Corti.

2 If the residual ear-canal space between the transducers and the eardrum is made small enough, the matrices \( \mathbf{S}_o \) and \( \mathbf{S}_s \) become equivalent. The residual ear-canal space is negligible if \( 2aL/c < 1 \), where \( L \) is the length of the residual space and \( c \) is the speed of sound (see Appendix C). The factor of 2 arises because phase differences due to round-trip travel are important here. As an example, the inequality requires \( L \approx 3 \) cm at a frequency of 1 kHz. The residual ear-canal space is therefore not negligible in the examples shown in Figs. 2 and 3.


