Hofweber's Philosophy of Mathematics

Agustín Rayo
MIT and the University of Oslo

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Are there any numbers? Yes and no, says Thomas Hofweber. In his new book, *Ontology and the Ambitions of Metaphysics* (Oxford University Press, 2016), he argues that "something is a number" has two different readings, one true and the other false.

On its *external* reading "something is a number" is used to make claims about the "domain of objects" (p. 60). Hofweber is somewhat uncommittal about what the "domain of objects" is supposed to be, but he makes clear that he takes it to be the focus of ontological inquiry: "[t]he ontological question about numbers is "Are there numbers?" on the external reading." (p. 168) He argues, moreover, that "something is a number" is false on its external reading, and therefore that the "ontological question" about numbers should be answered in the negative.

One of the chief aims of the book, however, is to argue that "something is a number" also has an *internal* reading, and that this reading is true. Hofweber thinks, moreover, that the standard reading of numerical quantifiers is the internal reading. So he thinks that, unless one is speaking non-standardly, one can truthfully say that something is a number, and therefore that there are numbers—even though there are no numbers in the "domain of objects".

In this note I would like to make a few comments about Hofweber's internalist proposal.

1 The internalist program

In postulating internal readings of numerical quantifiers, Hofweber joins a venerable tradition in the philosophy of mathematics: the tradition of proposing ontologically innocent analyses of arithmetical sentences. (Other contributions include Putnam (1967), Hodes (1984), Hellman (1989) and Yablo (2002).)

Not every member of the tradition thinks that their proposal can be motivated on purely linguistic grounds. Hofweber does: he sees his analysis as the output of natural language semantics for English number statements. But he also thinks that the resulting analysis can be used to address what he describes as the "central tension" in the philosophy of arithmetic: the problem of reconciling (*i*) the fact that we are able to uncover arithmetical facts by thinking alone, and (*ii*) the fact that arithmetic appears to be concerned with a distinctive (and fully objective) subject-matter (Section 6.1.1).

The reason Hofweber sees a tension here is that he thinks it's hard to understand how one could uncover arithmetical truths by thinking alone, if the distinctive subject-matter of arithmetic concerns a particular domain of objects. On his "internalist" analysis of number statements, however, number words are not referring expressions; they are *determiners*: "determiners that [...] can nonetheless appear in singular term position." (p. 168)

A determiner is a word like "some", "many", or "the", which conjoins with a noun to form a noun phrase. (For instance, one constructs the noun-phrase "some elephants" by combining the determiner "some" with the noun "elephant".) It is uncontroversial that number words sometimes occur as determiners in English: "two" in "two elephants walked together", for example. What is distinctive about Hofweber's position is the view that English number words should be thought of as determiners even when they occur in singular term positions.

To describe the proposal in further detail, I will focus on a particular sentence with number words occurring in singular term positions:

(1) "two plus two is four".

Hofweber's internalist analysis of (1) proceeds by setting forth a plural variant,

(2) "two and two are four",

and making four claims, all of them non-obvious:

Sameness of content (1) and (2) have the same content. When we move between them "we merely change the form of the representation. We do not replace one representation with another one that has a different content." (p. 137)

Bare determiners As they occur in (2), "two" and "four" are determiners. More specifically, they are *bare determiners*: determiners that are not explicitly accompanied by a noun. (p. 126)

Generalization The bare determiners in (2) are of a special kind. When bare determiners are used in English, they are often *elliptical*. (For instance, the occurrence of "one" in "most of the elephants walked together, but one was left behind" is elliptical for the complete noun phrase "one elephant"). But the occurrences of "two" and "four" in (2) are not of this kind. Instead, they are such as to make sentences like (2) "express generalizations" (p. 126).

Ontological Innocence The truth of a bare determiner statement like (2) does not depend what objects the world contains. (p. 165)

By bringing these claims together, Hofweber argues that the truth of the original sentence, "two plus two is four", does not depend on what objects the world contains—a conclusion he sees as offering relief from the "central tension".

Hofweber then extends the proposal in a number of interesting ways. Most importantly, he suggests an internalist reading of arithmetical quantifiers, according to which the truth conditions of "something is F" are equivalent to the truth conditions of the (infinite) disjunction of its instances. So, for example, the truth conditions of "something is prime" are equivalent to the truth conditions of the infinitary disjunction "zero is prime, or one is prime, or two is prime, or ..." (Section 5.6).

Hofweber's internalist analysis of arithmetical statements is an important contribution to the debate, which deserves careful study. In the next couple of sections, I will highlight two of its distinctive features, and mention an advantage and a disadvantage of each.

2 First feature: bare determiners

The first issue I would like to highlight is that unlike other instances of arithmetical analysis aimed at ontological innocence, Hofweber's proposal is not explicitly couched in a first- or higher-order language. As we have seen, it instead relies on bare determiners of a special non-elliptical, generalizing kind. Hofweber believes, moreover, that such determiners cannot be captured using first- or higher-order vocabulary, or even modal resources such as conditionals.[1] He sees them as a fresh addition to our expressive toolkit.

This departure from familiar resources has a definite advantage. As Hofweber points out (p. 175), it allows his internalist program to escape a family of technical results, which limit the effectiveness of other proposals. One such result shows that, given certain reasonable assumptions, it is impossible to give ontologically innocent analysis of arbitrary arithmetical sentences in a first- or higher-order language.[2]

It seems to me, however, that there is also a disadvantage. I think the use of unfamiliar resources renders the foundation of Hofweber's program somewhat obscure. The internalist analysis of number statements is ultimately based on sentences like "two and two are four", and its multiplicative cousin, "two times two are four". But usage of such sentences in ordinary English is not particularly robust. It is worth noting, moreover, that their grammaticality is somewhat marginal—especially in the case of "two times two are four". (For example, Alexander Baine's *An English Grammar*, which was once orthodoxy, declares strings of either kind "wrong".[3]) In the absence of a more robust pattern of

[1]Some of Hofweber's arguments strike me as a little hasty. For example, he argues that "five and seven are tweleve" is not equivalent to the conditional "if there were five Fs and seven more Fs then there would be twelve Fs", on the grounds that "[i]t might well be that if you add seven more to the five there are, then you get an explosion, not twelve things. Balls of uranium are an example, in a world that is small enough". It seems to me, however, that there is an equivocation here. In discussing the small world example, Hofweber considers a conditional whose antecedent is equivalent to "there are five Fs and you (attempt to) add seven more". Hofweber is certainly right that this antecedent might be satisfied in his small world, since it can be satisfied in a world with only five Fs. But note that this is not the antecedent of the conditional we should be considering. Our conditional is "if there were five Fs and seven more Fs then there would be twelve Fs". Its antecedent is "there are five Fs and seven more Fs", which can only be satisfied at worlds which (unlike Hofweber's small world) contain twelve Fs.

[2]For more on this sort of result, including its application to modal languages, see Rayo, 2014.

[3]Here is the relevant passage:

> We hear sometimes 'two and two *are* four;' 'three times four *are* twelve;' but the 'are' is scarcely defensible in either case. It would be correct to say 'two pounds and five pounds are (or make) seven pounds;' but with numbers in the abstract what we mean is that the numerical combination of 'two and two' is the same as four So 'twice one *are* two' must be wrong because there is no plurality in the strict sense; and 'three times four' should be regarded as a combination or unity made up in a particular way. (Bain, 1866, p. 175)

This does not, of course, settle the issue. But it does make clear that it is not obvious that usage, it seems to me that it would be a mistake to lean too heavily on our pre-theoretic intuitions about the truth-conditions of such sentences. (For what it's worth, my own intuitions are pretty thin. When I try to interpret "two times two are four", I find it hard to be sure whether I'm hearing the sentence's truth-conditions, or the result of implementing some sort of pragmatic repair strategy.)

One way of clarifying the foundations of the internalist program would be to give a compositional semantics for sentences like "two and two are four" and "two times two are four", and thereby explain how the proposed readings of these sentences are supposed to be generated from the semantic values of determiners like "two" and "four". Unfortunately, Hofweber shies away from supplying such a semantics. It is worth pointing out, moreover, that there is some initial reason to doubt that supplying a semantics would be routine. This is because on orthodox accounts of determiners (e.g. Barwise and Cooper, 1981), the semantic values of numerical determiners like "two" and "four" can be represented using higher-order resources. And a variant of the technical result mentioned above suggests that such semantic values shouldn't be able to generate an assignment of truth-conditions of the kind Hofweber's program demands.[4]

The worry here is not that assigning semantic values to determiners like "two" and "four" would somehow threaten the ontological innocence of "two and two are four" or "two times two are four". Such a worry would be misplaced. As Hofweber points out (Section 8.3), one should be careful not to confuse the ontological commitments of a sentence with the ontological commitments of a semantic theory that is used to assign truth-conditions to that sentence. The worry is to do with complexity considerations. The upshot of the relevant technical result is that the set of arithmetical truths is too complex to be captured using higher-order resources (unless one gives up on ontological innocence by assuming an infinite domain). So the worry is that a compositional semantics based on orthodox semantic values for determiners shouldn't be able to produce the kind of complexity that is needed to generate Hofweber's preferred reading of arithmetical sentences.

I have suggested that a cost of using unfamiliar expressive resources is that the foundations of Hofweber's program are not as clear as they might have been. Our understanding of expressive resources of other kinds (e.g. counterfactual conditionals) has been greatly enhanced by compositional semantics. It seems to me that our understanding of the special style of bare determiner that the internalist program requires would also benefit from such a semantics.

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sentences like "two and two are four" and "three times four are twelve" are good English.

(*A historical note*: Alexander Bain was en eminent Scottish scholar. He was Lord Rector, and professor of logic, at the University of Aberdeen—and he founded the journal *Mind*.)

[4]Let me be a little more specific. On a Barwise and Cooper style semantics for determiner statements, the truth-conditions of, e.g. "many Fs are Gs" are equivalent to those of the higher-order statement "Many(F-and-Gs,Fs)" (read: "the F-and-Gs are many of the Fs"), where "Many" is an atomic second-order predicate. Accordingly, the semantic value of the determiner "many" can be captured using a third-order variable. The claim I'm making here is that Hofweber cannot preserve this feature of a Barwise and Cooper style semantics. For if he did, one would expect it to be possible to model the semantic values of his numerical quantifiers using third-order quantifiers. So it would follow from the formal result mentioned above that the resulting assignment of truth-conditions to arithmetical sentences can't be of the kind his program demands.

3 Second feature: linguistics

The second feature of the internalist program I would like to highlight is that Hofweber regards the *philosophical* project of addressing the "central tension" as continuous with the *linguistic* project of analyzing the semantic structure of arithmetical sentences as they occur in natural language.

This approach has the advantage of making Hofweber's proposal doubly illuminating, if successful. Not only would he be in a position to address the "central tension" of the philosophy of arithmetic. He would be able to do so in a way that deepens our understanding of natural language. Hofweber suggests a specific linguistic mechanism to explain the transition from "two and two are four" to "two plus two is four": a form of cognition-oriented type lowering. He thinks that in order to reduce the cognitive load of processing arithmetical sentences, the semantic type of determiners gets lowered to the type of objects, and the semantic type of operations on determiners gets lowered to the type of operations on objects. (Section 5.4.3.)

It seems to me, however, that there is also a disadvantage to treating the philosophical project of addressing the "central tension" as continuous with the linguistic project of natural language semantics.

The first thing to note is that Hofweber's internalist account of arithmetical sentences is a proposal about the semantics of a particular natural language: English. As far as Hofweber's arguments go, there might be a language other than English with occurrences of number words that shouldn't be understood as determiners. Hofweber seems to think that the speakers of such a language wouldn't really be talking about numbers (p. 170). In other words: they wouldn't be doing arithmetic even if they were concerned with structures isomorphic to the natural numbers. This is a little too parochial for my taste.

Notice, moreover, that if Hofweber's "central tension" is a problem for arithmetic, it is also a problem for other branches of mathematics. Set theory and real analysis, in particular, are akin to arithmetic in that we seem to have a distinctive (and fully objective) subject-matter, discoverable by thinking alone. There are, however, principled reasons for thinking that Hofweber's internalist program cannot be extended to set theory or real analysis. Most prominently, the internalist account of quantification relies on the assumption that our language contains an term for every instance of the relevant quantifiers. English satisfies this assumption in the case of arithmetic, but not in the case of set theory or real analysis.[5] And it is no accident that this is so: English is generated from a finite lexicon, and both set theory and real analyses have uncountable domains.

Hofweber is careful to point out that the "central tension" might call for different answers with respect to different branches of mathematics (Section 6.7). I myself would find it hard to believe that we'd truly gotten to the heart the "central tension" unless we had an answer general enough to apply beyond arithmetic.

As I mentioned earlier, Hofweber thinks that his analysis of number statements can be defended purely on the basis of linguistic considerations. I am

[5] Hofweber's internalist account of properties faces a related worry. Hofweber attempts to address the problem by tweaking the internalist account of quantification (Chapter 9). As far as I can tell, however, the tweak is not helpful in the present context.

not a linguist, and do not feel qualified to assess this claim.[6] The point I am making here is that even if Hofweber is right about natural language semantics, it is not clear that the resulting analysis of arithmetical statements delivers a satisfying answer to the "central question". For the answer one gets is more parochial and less general than one might have hoped. It is parochial because it entails that one can only do arithmetic by speaking a language like English. It lacks generality because it only addresses a special case of the "central tension". If this is the best we can do, we should learn to live with it. Fortunately, I think it is possible to articulate a more general, and less parochial, answer to the "central tension". The trick is to allow for a certain separation between one's philosophy of mathematics and one's natural language semantics. I will end this note with a few words about how I think the story might go.

In *The Foundations of Arithmetic*, Frege argued that "the number of the Fs is the number of the Gs" and "the Fs are in one-one correspondence with the Gs" have the same content. But Frege did not justify this claim by doing natural language semantics of the sort Hofweber is interested in. He based his view on a philosophical conception of concepts and objects, according to which different "carvings" of a single content can be built from different concepts and objects. This allowed him to identify the contents of different sentences regardless of whether he was in a position to move from one sentence to the other by linguistic analysis.

Some philosophers—myself included—believe that Frege's content-identification strategy succeeded in the case of arithmetic.[7] Unfortunately, his efforts to apply the strategy in the case of set theory led to contradiction. The main source of the problem, it seems to me, is that Frege was presupposing a naive conception of sets, according to which arbitrary predicates characterize sets. But there is no reason one couldn't implement a version of Frege's content-identification idea based on a different conception of sets. On an iterative conception of sets, for example, one would proceed in stages. At the initial stage, one identifies the content of $\ulcorner x \in \{z : \Phi(z)\}\urcorner$ with the content of $\ulcorner \Phi(x) \urcorner$, where Φ is a predicate of non-sets. At each successive stage, one repeats the idea, using predicates that apply to objects introduced at earlier stages.[8]

If a research program of this kind is to be successful, we need a principled way of deciding when it is legitimate to identify the contents of different sentences and when it is not. On one proposal—the one I take to be most promising—scientific inquiry involves two interrelated tasks. On the one hand, one needs to decide which sentences to count as true; on the other, one needs to decide when to identify the contents of different sentences. In both cases, the decision should be made on the basis of its ability to deliver a simple and fruitful overall theory of the world. In the case of content-identifications this is done by determining whether the relevant identification would eliminate a distinction that does useful explanatory work. For instance, one should decide whether to identify the contents of "there is water on Mars" and "there is H_2O on Mars" on the basis of whether our simplest, most fruitful overall theory of the world would find interesting explanatory work for a distinction between something's

[6] For linguistically-informed criticism, see Jackson, 2013 and Moltmann, 2013.

[7] See, for instance, Hale and Wright, 2003.

[8] For a fully developed version of the proposal, see Chapter 3 of Rayo, 2013. For more on the iterative conception of sets, see Boolos, 1971. For an especially interesting implementation of the iterative conception of set, see Linnebo, 2010.

being composed of water and something's being composed of H_2O. On the resulting picture, there is room for identifying the contents of $\ulcorner x \in \{z : \Phi(z)\}\urcorner$ and $\ulcorner \Phi(x) \urcorner$ not on the basis of semantic analysis, but on the basis of an assessment of the explanatory needs of our simplest, most fruitful overall theory of the world.

It seems to me that a picture of this kind can be used to articulate a general answer to the "central tension": one that applies not just to the special case of arithmetic, but also to set theory (and therefore real analysis).[9] The resulting philosophy of mathematics is also less parochial than Hofweber's, since it is not restricted to the mathematical statements of English speakers and their kin. And it has the additional advantage of not calling for two distinct kinds of quantification, one for doing mathematics and one for doing ontology.

I am certainly not expecting you to buy into my preferred philosophy of mathematics on the basis of a few cursory remarks. I mention it to illustrate the fact that there might be advantages to maintaining a certain separation between one's philosophy of mathematics and one's natural language semantics.

Hofweber has written a first-rate book, chock-full of insight. It sets forth an ambitious research program, aimed at producing a linguistically informed philosophy of arithmetic. If the project succeeds, it will deliver unusually illuminating answers to longstanding philosophical questions. But it is worth keeping in mind that Hofweber's methodology carries a certain kind of risk: there is a price to be paid for combining one's philosophy of mathematics and one's natural language semantics.

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[9]For details, see Rayo, 2013.

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